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Exact Solutions and Conserved Vectors of the Two-Dimensional Generalized Shallow Water Wave Equation

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Abstract: In this article, we investigate a two-dimensional generalized shallow water wave equation. Lie symmetries of the equation are computed first and then used to perform symmetry reductions. By utilizing the three translation symmetries of the equation, a fourth-order ordinary differential equation is obtained and solved in terms of an incomplete elliptic integral. Moreover, with the aid of Kudryashov's approach, more closed-form solutions are constructed. In addition, energy and linear momentum conservation laws for the underlying equation are computed by engaging the multiplier approach as well as Noether's theorem.

Keywords: two-dimensional generalized shallow water wave equation; Lie point symmetries; Kudryashov's method; conservation laws; Noether's theorem

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1. Introduction

It is widely known that most physical phenomena of the natural world are governed by nonlinear partial differential equations (NPDEs). Such equations arise in several physical and other problems such as in the study of electromagnetism, electrodynamics, fluid flow, elasticity, propagation of heat or sound, quantum mechanics, meteorology, and oceanography, just to mention a few. Some of these models include the modified Zakharov–Kuznetsov model [1] that recounts the ion-acoustic drift solitary waves existing in a magnetoplasma with electron–positron ions, which are found in a primordial universe. A generalized system of three-dimensional modified Kadomtsev–Petviashvili–Burgers-type equation, which describes ion-acoustic and dust-magneto-acoustic waves in one of cosmic or laboratory dusty plasmas, was studied in [2]. Moreover, in [3], the vector bright solitons of the Fokas–Lenells system, which models the femtosecond pulses in an optical fibre, was investigated. In addition, the Boussinesq–Burgers-type system of equations, which delineates shallow water waves appearing close to lakes or ocean beaches, was studied in [4]. The list continues; see also [5–15].

To obtain a good understanding of physical phenomena that are governed by NPDEs, it is imperative to look for their closed-form solutions. However, this is an arduous exercise, and for several years, researchers and scientists have been working on developing methods for finding closed-form solutions.

A good number of methods for obtaining closed-form solutions of NPDEs have been proposed. We have, amongst others, the following various special methods for finding closed-form solutions of NPDEs: the inverse scattering transform technique [16], the bifurcation method [17], the simplest equation technique [18], the extended simplest equation technique [19], Kudryashov's technique [20], Hirota's technique [21], Bäcklund transformation [22], Darboux transformation [23], the homogeneous balance method [24], (G'/G)—expansion technique [25], Lie symmetry method [26–28], etc.

It was during the halfway point of the nineteenth century that Sophus Lie, a Norwegian mathematician, endowed the study of continuous transformation groups, presently called Lie groups. These groups have had an intense impact on pure and applied mathematics in addition to engineering, physics, and other applied sciences. It provides the most effective and powerful techniques for obtaining closed-form solutions to NPDEs. For example, see [26–33]. It is worth noting here that the notion of a Galois group had influenced Lie's work on differential equations (DEs).

In the study of DEs, conservation laws play a crucial part. Conservation laws can be used to check whether a given differential equation is integrable. The accuracy of numerical solution methods can be checked by invoking conservation laws. They can be utilized in constructing solutions of partial differential equations (PDEs) by reducing their order. Conservation laws refer to certain physical properties that do not change subsequently within an isolated physical system. For example, conservation of energy, linear momentum, angular momentum, and electric charge. Recently, exact solutions were obtained for some PDEs using conservation laws. For differential equations that arise from the variational principle, there exists a connection between the symmetries of differential equations and conservation laws. This fact was established by Emmy Noether, a German mathematician, in 1918 and is stated in Noether's theorem. For more details, see for example [34–45] and the references therein.

In this work, we study the generalized two-dimensional generalized shallow water wave equation, given by [46]

$$u_{tx} + \alpha u_y u_{xx} + 2\alpha u_x u_{xy} + \beta u_{xy} + \nu u_{xxxy} = 0, \tag{1}$$

where α, β, and ν are nonzero constants. Here, t, x, and y represent, respectively, time and two scaled space coordinates whereas $u(t, x, y)$ represents the height of the Riemann wave.

In [46], the authors presented painlévé analysis, nonlocal symmetries, and exact solutions of a variable coefficients form of (1), whereas in [47], the binary Bell polynomials technique was employed to a variable coefficient form of Equation (1). We note that, in the study of fluids, this equation decomposes to well-known equations when taking special cases of constants α, β, and ν. For example, when $\alpha = -2$, $\beta = 0$, and $\nu = -1$, it becomes a $(2+1)$-dimensional breaking soliton [48]:

$$u_{tx} - 2u_y u_{xx} - 4u_x u_{xy} - u_{xxxy} = 0, \tag{2}$$

which is one of the models used in describing the two-dimensional interplay of a Riemann wave distributed along the y-axis with a long wave generated along the x-axis. Wazwaz [49] obtained solutions of this equation using the modified form of Hirota's bilinear method. The authors in [48] presented solutions of Equation (2) using the symmetry method. In the case where $\alpha = 2$, $\beta = 0$, and $\nu = 1$, Equation (1) transforms to a $(2+1)$-dimensional Calogero–Bogoyavlenskii–Schiff [50]

$$u_{tx} + 2u_y u_{xx} + 4u_x u_{xy} + u_{xxxy} = 0, \tag{3}$$

which describes the exchange of a long generating wave in the x direction with a Riemann generating wave in the y direction. Moreover, the researcher indicated that Equation (3) possesses soliton as well as N-soliton solutions, which are smooth in one coordinate. In [51], Wazwaz invoked Hirota's bilinear approach to secure solutions of (3) and further utilized the tanh-coth method to gain its travelling wave solutions in [52]. The authors in [53] derived a Bäcklund transformation alongside a Lax pair of the form in the Calogero–Bogoyavlenskii–Schiff Equation (3) via singular manifold method. Furthermore, in [54], the authors invoked a tanh function as well as improved the (G'/G)—expansion methods to construct travelling wave solutions of (3). It was also revealed that some analytic solutions secured via these methods are analogous. However, a generalized version of (3), namely

$$u_{tx} + a u_x u_{xy} + b u_y u_{xx} + u_{xxxy} = 0, \tag{4}$$

has been studied by many researchers by simply replacing 4 by a and 2 by b in (3). For instance, the (G'/G)—expansion technique was invoked to achieve some travelling wave solutions of (4) in [55]. Moreover, in [56], the authors obtained breather-type and analytic soliton solutions of (4) by utilizing Hirota's bilinear method and the extended homoclinic test technique. Using the idea of a tanh-coth approach, some closed-form solutions of (4) were obtained in [57]. Furthermore, sine-cosine and complex techniques were engaged to secure, respectively, analytic and meromorphic exact solutions of (4) in [58,59].

The formation of this paper is as follows: In Section 2, first, we employ Lie's theory to reduce (1) to an ordinary differential equation (ODE). Closed-form solutions are then derived by an integration of the ODE and by invoking Kudryashov's method. Thereafter, in Section 3, we construct conserved densities and spatial fluxes for (1) by invoking the multiplier method and Noether's theorem. Finally, we present concluding remarks in Section 4.

2. Exact Solutions

In this section, we derive exact solutions of the two-dimensional generalized shallow water wave Equation (1). We begin by deriving symmetries of (1) and thereafter use them to obtain an ODE. We then employ the direct integration procedure together with Kudryashov's method to construct closed-form solutions of (1).

2.1. Symmetries and Symmetry Reductions

Consider the symmetry group of (1) brought about by

$$U = T\frac{\partial}{\partial t} + X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial u},$$

where all T, X, Y, and η are functions of variables t, x, y, and u. Using the fourth prolongation $\mathrm{pr}^{(4)}U$ in (1) and with the help of Maple, we obtain an overdetermined system of linear homogeneous PDEs:

$$\begin{aligned}&T_u = 0,\ T_x = 0,\ T_y = 0,\ X_y = 0,\ Y_u = 0,\ Y_x = 0,\ Y_{tt} = 0,\ T_{ttt} = 0,\ X_u = 0,\ Y_{yy} = 0,\\ &2\eta_u + T_t - Y_y = 0,\ 2\alpha\eta_x + \beta T_t - \beta Y_y - Y_t = 0,\ \alpha\eta_y + X_t = 0,\ 2Y_{ty} - T_{tt} = 0,\\ &2X_x - T_t + Y_y = 0.\end{aligned}$$

The above linear PDEs may be easily solved and yield the following eight Lie point symmetries:

$$\begin{aligned}&U_1 = \frac{\partial}{\partial t},\ U_2 = \frac{\partial}{\partial x},\ U_3 = \frac{\partial}{\partial y},\ U_4 = \frac{\partial}{\partial u},\\ &U_5 = 2\alpha t\frac{\partial}{\partial y} + x\frac{\partial}{\partial u},\ U_6 = 2\alpha t\frac{\partial}{\partial t} + \alpha x\frac{\partial}{\partial x} - (\beta x + \alpha u)\frac{\partial}{\partial u},\\ &U_7 = \alpha x\frac{\partial}{\partial x} - 2\alpha y\frac{\partial}{\partial y} - (\alpha u + \beta x)\frac{\partial}{\partial u},\\ &U_8 = 2\alpha t^2\frac{\partial}{\partial t} + \alpha tx\frac{\partial}{\partial x} + 2\alpha ty\frac{\partial}{\partial y} - (\beta tx - xy + \alpha tu)\frac{\partial}{\partial u}.\end{aligned} \tag{5}$$

Consider the symmetry $U = U_1 + aU_2 + bU_3$, where a and b are constants. Using this symmetry U, we reduce (1) to a PDE with one less independent variable, that is, two independent variables. The symmetry U gives three invariants:

$$f = x - at,\ g = y - bt,\ \theta = u. \tag{6}$$

Using these invariants, Equation (1) reduces to

$$\nu\theta_{fffg} + \beta\theta_{fg} + \alpha\theta_g\theta_{ff} + 2\alpha\theta_f\theta_{fg} - b\theta_{fg} - a\theta_{ff} = 0, \tag{7}$$

which has

$$R_1 = \frac{\partial}{\partial f},\ R_2 = \frac{\partial}{\partial g},\ R_3 = \alpha H(g)\frac{\partial}{\partial g} + aH(g)\frac{\partial}{\partial \theta},$$
$$R_4 = \alpha f\frac{\partial}{\partial f} - \alpha g\frac{\partial}{\partial g} + (bf - \beta f - \alpha\theta)\frac{\partial}{\partial \theta}$$

as its symmetries. The symmetry $R = R_1 + cR_2$, with c a constant, provides two invariants $q = g - cf$ and $\mathcal{U} = \theta$. These invariants transform (7) into a nonlinear ordinary differential equation (NLODE)

$$k_1\,\mathcal{U}''''(q) - k_2\,\mathcal{U}'(q)\,\mathcal{U}''(q) + k_3\,\mathcal{U}''(q) = 0, \tag{8}$$

where $k_1 = b - ac^2 - \beta$, $k_2 = -3\alpha c$, $k_3 = -\nu c^2$ and $q = (ac - b)t - cx + y$.

2.2. Solution Using Direct Integration

In this subsection, we derive a solution of the two-dimensional generalized shallow water Equation (1) by direct integration of the ODE (8). Taking $\mathcal{U}'(q) = \mathcal{V}(q)$, Equation (8) becomes

$$k_1\,\mathcal{V}'' - k_2\,\mathcal{V}\,\mathcal{V}' + k_3\,\mathcal{V}' = 0. \tag{9}$$

Twice integration of (9) with respect to q gives

$$\frac{1}{2}k_1\,\mathcal{V}'^2 - \frac{1}{6}k_2\mathcal{V}^3 + \frac{1}{2}k_3\mathcal{V}^2 + r_1\,\mathcal{V} + r_2 = 0 \tag{10}$$

with r_1 and r_2 constants. Equation (10) can be rewritten as

$$\mathcal{V}'^2 = \frac{k_2}{3k_1}\mathcal{V}^3 - \frac{k_3}{k_1}\mathcal{V}^2 - \frac{2r_1}{k_1}\mathcal{V} - \frac{2r_2}{k_1}. \tag{11}$$

Assume that n_1, n_2, and n_3 are roots of

$$\mathcal{V}^3 - \frac{3k_3}{k_2}\mathcal{V}^2 - \frac{6r_1}{k_2}\mathcal{V} - \frac{6r_2}{k_2} = 0$$

with $n_1 > n_2 > n_3$. Equation (11) now becomes

$$\mathcal{V}'^2 = \frac{k_2}{3k_1}(\mathcal{V} - n_1)(\mathcal{V} - n_2)(\mathcal{V} - n_3)$$

and its solution could be written in terms of the Jacobi elliptic function [60–62]

$$\mathcal{V}(q) = n_2 + (n_1 - n_2)\mathrm{cn}^2\left\{\sqrt{\frac{k_2(n_1 - n_3)}{12k_1}}q, R^2\right\},\ R^2 = \frac{n_1 - n_2}{n_1 - n_3}, \tag{12}$$

where cn is the elliptic cosine function. Integrating Equation (12) with respect to q and returning to the original variables, we accomplish the solution of (1) as

$$u(t,x,y) = \sqrt{\frac{12k_1(n_1-n_2)^2}{k_2(n_1-n_3)R^8}}\left\{\text{EllipticE}\left[\text{sn}\left(\sqrt{\frac{k_2(n_1-n_3)}{12k_1}}q, R^2\right), R^2\right]\right\} + \left\{n_2 - (n_1-n_2)\frac{1-R^4}{R^4}\right\}q + K, \tag{13}$$

with $q = (ac-b)t - cx + y$, K being a constant, and $\text{EllipticE}[g,k]$ being the incomplete elliptic integral given as [60]

$$\text{EllipticE}[g,k] = \int_0^g \sqrt{\frac{1-k^2s^2}{1-s^2}}ds. \tag{14}$$

The wave profile of the periodic solution (13) for parametric values $k_1 = 70$, $k_2 = 10$, $a = -4$, $b = 0.2$, $c = 0.6$, $K = 1$, $n_1 = 100$, $n_2 = 50.05$, and $n_3 = -60$ at $t = -14$, can be seen in Figure 1.

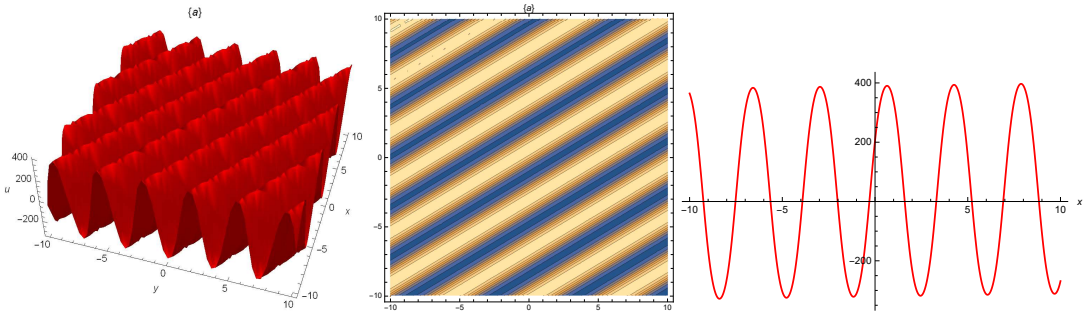

Figure 1. The 3D and 2D solution profiles of (13).

2.3. Solution via Kudryashov's Method

In this subsection, we present the solution of (1), which is obtained by employing Kudryashov's method [20]. This method is one of the most popular approaches for determining closed-form solutions of NPDEs. The first step is to reduce the NPDE (1) to a nonlinear ODE, which we already performed using the Lie symmetries in the previous section. Thus, we work with the ODE (8). We suppose that a solution of (8) can be expressed as

$$\mathcal{U}(q) = \sum_{n=0}^{N} A_n Y^n(q), \tag{15}$$

where $Y(q)$ satisfies the first-order nonlinear ODE

$$Y'(q) = Y^2(q) - Y(q). \tag{16}$$

We note that the solution of (16) is

$$Y(q) = \frac{1}{1+\exp q}. \tag{17}$$

For Equation (8), the balancing procedure yields $N = 1$. Thus, from (15), we have

$$\mathcal{U}(q) = A_0 + A_1 Y(q). \tag{18}$$

Now substituting (18) into (8) and using (16), we obtain

$$\begin{aligned}
&2b\,A_1cY^3(q) - 2a\,A_1c^2Y^3(q) + 3a\,A_1c^2Y^2(q) - 3b\,A_1cY^2(q)\\
&- A_1c^2Y(q)a + bA_1cY(q) + 6\,\alpha\,{A_1}^2c^2Y^5(q) - 15\,\alpha\,{A_1}^2c^2Y^4(q)\\
&+ 12\,\alpha\,{A_1}^2c^2Y^3(q) - 3\,\alpha\,{A_1}^2c^2Y^2(q) - 2\,\beta\,A_1cY^3(q) + 3\,\beta\,A_1cY^2(q)\\
&- \beta\,A_1cY(q) - 24\,\nu\,A_1c^3Y^5(q) + 60\,\nu\,A_1c^3Y^4(q) - 50\,\nu\,A_1c^3Y^3(q)\\
&+ 15\,\nu\,A_1c^3Y^2(q) - \nu\,A_1c^3Y(q) = 0.
\end{aligned}$$

Separating on the powers of $Y(q)$ gives the algebraic equations for the coefficients A_0 and A_1 as

$$\begin{aligned}
&\alpha\,{A_1}^2c^2 - 4\,\nu\,A_1c^3 = 0,\\
&bcA_1 - \nu\,A_1c^3 - ac^2A_1 - \beta\,A_1c = 0,\\
&\beta\,A_1c - \alpha\,{A_1}^2c^2 + 5\,\nu\,A_1c^3 + ac^2A_1 - bcA_1 = 0,\\
&6\,\alpha\,{A_1}^2c^2 - 25\,\nu\,A_1c^3 - ac^2A_1 + bcA_1 - \beta\,A_1c = 0.
\end{aligned}$$

The solution of these equations is

$$A_0 = A_0,\ A_1 = \frac{4\nu c}{\alpha},\ a = \frac{b - \beta - c^2\nu}{c}.$$

Thus, the solution of (1) reads

$$u(t,x,y) = A_0 + \frac{4\nu c}{\alpha(1 + \exp\{(-\beta - c^2\nu)t - cx + y\})}. \tag{19}$$

The wave profile of the solution (19) for parametric values $A_0 = 0$, $\alpha = 1$, $\beta = 0.05$, $c = 0.01$, $\nu = 1$, $t = 0$ is presented in Figure 2.

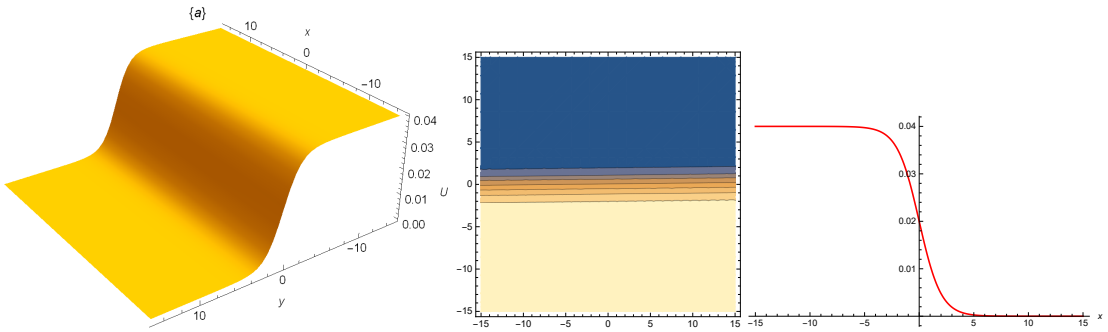

Figure 2. The 3D and 2D solution profile of (19).

Invariant solution using U_5

The Lie point symmetry U_5 produces the invariants $I_1 = t$, $I_2 = x$, and $I_3 = u - xy/(2\alpha t)$. The use of these invariants furnishes us with the group invariant solution of (1) given by

$$u(t,x,y) = \frac{xy}{2\alpha t} - \frac{\beta}{2\alpha}x - \frac{C}{x} + h(t), \tag{20}$$

where C is a constant and h is a function of t.

Invariant solution using U_6

The point symmetry U_6 of (1) yields the group invariant solution:

$$u(t,x,y) = \frac{1}{\sqrt{t}}G(f,g) - \frac{\beta}{2\alpha}x, \;\; f = \frac{x}{\sqrt{t}}, \;\; g = y. \tag{21}$$

The substitution of this expression for u into (1) gives the PDE

$$2\nu G_{fffg} + 4\alpha G_f G_{fg} + 2\alpha G_g G_{ff} - fG_{ff} - 2G_f = 0, \tag{22}$$

for which the symmetries are

$$\Gamma_1 = \frac{\partial}{\partial g}, \;\; \Gamma_2 = \frac{\partial}{\partial G}, \;\; \Gamma_3 = 2\alpha\frac{\partial}{\partial f} + g\frac{\partial}{\partial g}, \; \Gamma_4 = f\frac{\partial}{\partial f} - 2g\frac{\partial}{\partial g} - G\frac{\partial}{\partial G}. \tag{23}$$

The symmetry Γ_3 gives the invariant solution $G(f,g) = U(z) + fg/(2\alpha)$, where $z = g$. This expression for G satisfies the Equation (22) for arbitrary function U. Thus, the group-invariant solution under U_6 of (1) is

$$u(t,x,y) = \frac{xy}{2\alpha t} - \frac{\beta}{2\alpha}x - \frac{U(y)}{\sqrt{t}}, \tag{24}$$

where $U(y)$ is an arbitrary function of y.

Invariant solution using U_7

Here, we use the symmetry operator U_7 to obtain the invariant solution associated with this symmetry. The usual procedure gives

$$u(t,x,y) = \frac{1}{x}U(\xi) - \frac{\beta}{2\alpha}x, \;\; \xi = x^2 y. \tag{25}$$

Substituting this expression of u into Equation (1) gives the nonlinear ODE

$$\alpha U'^2 + 6\alpha\xi U'U'' - 2\alpha UU'' + 3\nu U'' + 4\nu\xi^2 U'''' + 12\nu\xi U''' = 0. \tag{26}$$

Invariant solution using U_8

Finally, the use of symmetry U_8 gives the invariant solution

$$u(t,x,y) = \frac{1}{\sqrt{t}}\Phi(f,g) + \frac{xy}{2\alpha t} - \frac{\beta x}{2\alpha}, \tag{27}$$

with $f = x/t^{1/2}$ and $g = y/t$, which when substituted in (1) gives the reduced PDE

$$\alpha\Phi_{ff}\Phi_g + 2\alpha\Phi_f\Phi_{fg} + \nu\phi_{fffg} = 0. \tag{28}$$

The above PDE (28) has three Lie symmetries

$$\Gamma_1 = \frac{\partial}{\partial f} + F_1(g)\frac{\partial}{\partial g}, \Gamma_2 = F_2(g)\frac{\partial}{\partial g} + \frac{\partial}{\partial \Phi}, \Gamma_3 = f\frac{\partial}{\partial f} + F_3(g)\frac{\partial}{\partial g} - \Phi\frac{\partial}{\partial \Phi}. \tag{29}$$

Now utilizing Γ_1 with $F_1(g) = 1$, we perform a reduction of Equation (28). We obtain the invariant solution $\Phi(f,g) = U(z)$, where $z = g - f$. This reduces Equation (28) to an ODE

$$\nu U'''' - 3\alpha U'U'' = 0, \tag{30}$$

for which the solution is

$$U(z) = -\sqrt{\frac{2^{8/3}\nu}{\alpha^{4/3}}}\,\zeta\left\{\sqrt{\frac{\alpha^{2/3}}{2^{4/3}\nu}}z + C_0; s_1, s_2\right\},\ s_1 = \frac{2^{5/2}}{\sqrt[3]{\alpha}}C_0,\ s_2 = 2C_1,$$

where C_0, C_1 are arbitrary constants and ζ is the Weierstrass zeta function [60]. Thus, the solution of (1) is

$$u(t,x,y) = -\sqrt{\frac{2^{8/3}\nu}{\alpha^{4/3}t}}\,\zeta\left\{\sqrt{\frac{\alpha^{2/3}}{2^{4/3}\nu}}\left(\frac{y}{t} - \frac{x}{\sqrt{t}}\right) + C_0; s_1, s_2\right\} + \frac{xy}{2\alpha t} - \frac{\beta x}{2\alpha}. \tag{31}$$

For the operator Γ_2 with $F_2(g) = 1$, we obtain the invariant solution $\Phi(f,g) = U(f) + g$, which reduces (28) to the ODE $U_{ff} = 0$, for which the solution is $U = C_1 f + C_2$, where C_1, C_2 are constants. Reverting to the original variables, we attain the solution of (1) given by

$$u(t,x,y) = \frac{C_1 x}{t} + \frac{C_2}{\sqrt{t}} + \frac{y}{t^{3/2}} + \frac{xy}{2\alpha t} - \frac{\beta x}{2\alpha}. \tag{32}$$

The dynamic behaviour of the solution (32), for parametric values $\alpha = 10$, $\beta = 5$, $C_1 = 0.3$, $C_2 = 0.01$, and $x = 0.5$, is sketched in Figure 3.

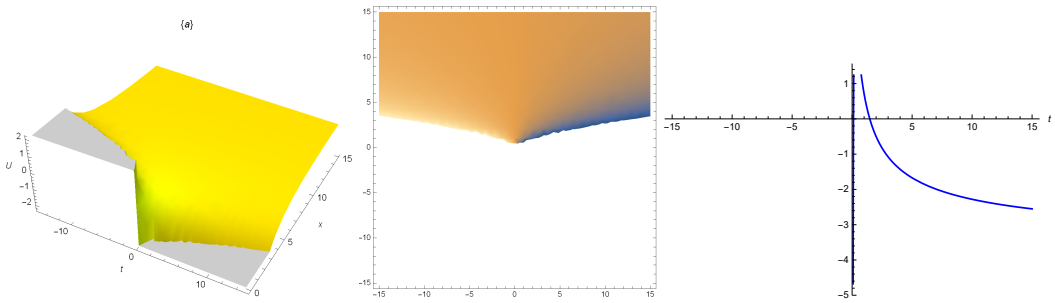

Figure 3. The 3D and 2D solution profiles of (32).

3. Conservation Laws

We now construct conservation laws for the two-dimensional generalized constant-coefficient shallow water Equation (1). To achieve this task, we utilize two methods; the first one is the multiplier method, and the second is classical Noether's theorem.

3.1. Conservation Laws Utilizing the Multiplier Method

We determine the conserved vectors of Equation (1) by employing the multiplier method [26]. Here, we seek first-order multipliers $\mathcal{Q} = \mathcal{Q}(t, x, y, u, u_t, u_x, u_y)$, which are established from

$$\frac{\delta}{\delta u}\left[\mathcal{Q}\left(u_{tx} + \alpha u_y u_{xx} + 2\alpha u_x u_{xy} + \beta u_{xy} + \nu u_{xxxy}\right)\right] = 0, \tag{33}$$

where $\delta/\delta u$ is the Euler operator [37]

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + \sum_{s=1}^{\infty}(-1)^s D_{i_1}\cdots D_{i_s}\frac{\partial}{\partial u_{i_1\cdots i_s}}$$

and for our purpose

$$\frac{\delta}{\delta u} = \partial_u - D_x\partial_{u_x} - D_y\partial_{u_y} + D_tD_x\partial_{u_{tx}} + D_x^2\partial_{u_{xx}} + D_xD_y\partial_{u_{xy}} + D_x^3D_y\partial_{u_{xxxy}}, \tag{34}$$

with D_t, D_x, and D_y being the total derivative operators given by

$$\begin{aligned} D_t &= \partial_t + u_t\partial_u + u_{tt}\partial_{u_t} + u_{tx}\partial_{u_x} + u_{ty}\partial_{u_y} + \cdots. \\ D_x &= \partial_x + u_x\partial_u + u_{tx}\partial_{u_t} + u_{xx}\partial_{u_x} + u_{xy}\partial_{u_y} + \cdots. \\ D_y &= \partial_y + u_y\partial_u + u_{ty}\partial_{u_t} + u_{xy}\partial_{u_x} + u_{yy}\partial_{u_y} + \cdots. \end{aligned} \tag{35}$$

Expanding (33) and separating on the appropriate derivatives of u, we obtain twenty-six multiplier-determining equations:

$$\begin{aligned} &\mathcal{Q}_{tx} + \alpha u_x\mathcal{Q}_{xy} + \beta\mathcal{Q}_{xy} = 0,\ \mathcal{Q}_{tu} + \alpha\mathcal{Q}_{xy} = 0,\ \mathcal{Q}_{tu_t} + 4\mathcal{Q}_u = 0,\ \mathcal{Q}_{xu_t} = 0,\ \mathcal{Q}_{xy} = 0,\\ &\mathcal{Q}_{xu_x} - \mathcal{Q}_u = 0,\ \mathcal{Q}_{tu_x} - 2\alpha u_y\mathcal{Q}_u + \alpha\mathcal{Q}_y = 0,\ \mathcal{Q}_{tu_y} - 2\alpha u_x\mathcal{Q}_u + 2\alpha\mathcal{Q}_x - 2\beta\mathcal{Q}_u = 0,\\ &\mathcal{Q}_{xx} = 0,\ \mathcal{Q}_{ux} = 0,\ \mathcal{Q}_{yy} = 0,\ \mathcal{Q}_{uy} = 0,\ \mathcal{Q}_{yu_t} = 0,\ \mathcal{Q}_{yu_x} = 0,\ \mathcal{Q}_{yu_y} - 2\mathcal{Q}_u = 0,\\ &\mathcal{Q}_{uu} = 0,\ \mathcal{Q}_{uu_t} = 0,\ \mathcal{Q}_{uu_x} = 0,\ \mathcal{Q}_{uu_y} = 0,\ \mathcal{Q}_{u_tu_t} = 0,\ \mathcal{Q}_{u_tu_x} = 0,\ \mathcal{Q}_{u_tu_y} = 0,\\ &\mathcal{Q}_{u_xu_x} = 0,\ \mathcal{Q}_{u_xu_y} = 0,\ \mathcal{Q}_{u_yu_y} = 0,\ \mathcal{Q}_{xu_y} = 0, \end{aligned}$$

for which the solution is

$$\begin{aligned} \mathcal{Q} = {} & yF'(t) + G(t) - \alpha u_xF(t) - 2K_5\alpha t^2u_t - K_5\alpha txu_x - 2K_5\alpha tyu_y \\ & - K_5\alpha tu + 2K_6\beta tu_y - 2K_1\alpha tu_y - K_5\beta tx + 4K_6tu_t + K_6xu_x \\ & + 2K_6yu_y + K_5xy + K_6u + K_2u_t + K_3u_x + K_4u_y + K_1x, \end{aligned}$$

where F and G are arbitrary functions of t and K_i, where $i = 1, \ldots, 6$, are constants of integration. The conserved quantities of Equation (1) are constructed using the divergence identity

$$D_tT^t + D_xT^x + D_yT^y = \mathcal{Q}\left(u_{tx} + \alpha u_yu_{xx} + 2\alpha u_xu_{xy} + \beta u_{xy} + \nu u_{xxxy}\right), \tag{36}$$

where T^t represents the conserved density and T^x and T^y are spatial fluxes. Thus, after some reckoning, we obtain low-order conservation laws corresponding to the eight multipliers and these are given below.

Case 1. For $\mathcal{Q}_1 = x - 2\alpha t u_y$, we have the conserved vector

$$\begin{aligned}
T_1^t &= \frac{1}{2}\alpha t u u_{xy} - \frac{1}{2}\alpha t u_x u_y + \frac{1}{2}x u_x - \frac{1}{2}u,\\
T_1^x &= \frac{2}{3}\alpha^2 t u u_x u_{yy} + \frac{2}{3}\alpha^2 t u u_{xy} u_y - \frac{4}{3}\alpha^2 t u_x {u_y}^2 + \frac{3}{4}\alpha\nu t u u_{xxyy} + \frac{1}{2}\alpha\beta t u u_{yy} - \frac{1}{2}\alpha\nu t u_x u_{xyy}\\
&\quad + \frac{1}{4}\alpha\nu t u_{xx} u_{yy} - \alpha\nu t u_{xxy} u_y + \frac{1}{2}\alpha\nu t {u_{xy}}^2 - \frac{1}{2}\alpha\beta t {u_y}^2 + \frac{1}{2}\alpha t u u_{ty} - \frac{1}{2}\alpha t u_t u_y + \alpha x u_x u_y\\
&\quad + \frac{1}{2}\beta x u_y + \frac{3}{4}\nu x u_{xxy} - \frac{1}{2}\nu u_{xy} + \frac{1}{2}x u_t,\\
T_1^y &= -\frac{2}{3}\alpha^2 t u u_x u_{xy} - \frac{2}{3}\alpha^2 t u u_{xx} u_y - \frac{2}{3}\alpha^2 t {u_x}^2 u_y - \frac{3}{4}\alpha\nu t u u_{xxxy} - \frac{1}{2}\alpha\beta t u u_{xy} - \frac{1}{4}\alpha\nu t u_x u_{xxy}\\
&\quad - \frac{1}{2}\alpha\beta t u_x u_y + \frac{1}{4}\alpha\nu t u_{xx} u_{xy} - \frac{1}{4}\alpha\nu t u_{xxx} u_y - \alpha t u u_{tx} + \frac{1}{2}\alpha x {u_x}^2 - \frac{1}{2}\alpha u u_x + \frac{1}{2}\beta x u_x\\
&\quad - \frac{1}{2}\beta u + \frac{1}{4}\nu x u_{xxx} - \frac{1}{4}\nu u_{xx};
\end{aligned}$$

Case 2. For $\mathcal{Q}_2 = u_t$, we have the conserved vector

$$\begin{aligned}
T_2^t &= \frac{2}{3}\alpha u u_x u_{xy} + \frac{1}{3}\alpha u u_{xx} u_y + \frac{1}{2}\nu u u_{xxxy} + \frac{1}{2}\beta u u_{xy} + \frac{1}{4}u u_{tx} + \frac{1}{4}u_t u_x,\\
T_2^x &= -\frac{3}{8}\nu u u_{txxy} + \frac{3}{8}\nu u_t u_{xxy} - \frac{1}{4}\nu u_{tx} u_{xy} + \frac{1}{8}\nu u_{txx} u_y + \frac{1}{4}\nu u_{txy} u_x - \frac{1}{8}\nu u_{ty} u_{xx} - \frac{1}{3}\alpha u u_{tx} u_y\\
&\quad - \frac{1}{3}\alpha u u_{ty} u_x - \frac{1}{4}\beta u u_{ty} - \frac{1}{4}u u_{tt} + \frac{2}{3}\alpha u_t u_y u_x + \frac{1}{4}\beta u_t u_y + \frac{1}{4}{u_t}^2,\\
T_2^y &= \frac{1}{3}\alpha u_t {u_x}^2 - \frac{1}{3}\alpha u u_{tx} u_x + \frac{1}{4}\beta u_t u_x + \frac{1}{8}\nu u_{txx} u_x - \frac{1}{8}\nu u u_{txxx} + \frac{1}{8}\nu u_t u_{xxx} - \frac{1}{8}\nu u_{tx} u_{xx}\\
&\quad - \frac{1}{4}\beta u u_{tx};
\end{aligned}$$

Case 3. For $\mathcal{Q}_3 = u_x$, we have the conserved vector

$$\begin{aligned}
T_3^t &= \frac{1}{4}{u_x}^2 - \frac{1}{4}u u_{xx},\\
T_3^x &= \frac{2}{3}\alpha {u_x}^2 u_y + \frac{1}{3}\alpha u u_x u_{xy} + \frac{5}{8}\nu u_x u_{xxy} + \frac{1}{4}\beta u_x u_y + \frac{1}{4}u_t u_x + \frac{1}{8}\nu u u_{xxxy} - \frac{3}{8}\nu u_{xx} u_{xy}\\
&\quad + \frac{1}{8}\nu u_{xxx} u_y + \frac{1}{4}\beta u u_{xy} + \frac{1}{4}u u_{tx},\\
T_3^y &= \frac{1}{3}\alpha {u_x}^3 + \frac{1}{4}\beta {u_x}^2 - \frac{1}{3}\alpha u u_x u_{xx} + \frac{1}{4}\nu u_x u_{xxx} - \frac{1}{8}\nu u u_{xxxx} - \frac{1}{4}\beta u u_{xx} - \frac{1}{8}\nu {u_{xx}}^2;
\end{aligned}$$

Case 4. For $\mathcal{Q}_4 = u_y$, we have the conserved vector

$$\begin{aligned}
T_4^t &= \frac{1}{4}u_x u_y - \frac{1}{4}u u_{xy},\\
T_4^x &= \frac{2}{3}\alpha u_x {u_y}^2 + \frac{1}{4}\beta {u_y}^2 - \frac{1}{3}\alpha u u_{xy} u_y + \frac{1}{2}\nu u_{xxy} u_y + \frac{1}{4}u_t u_y - \frac{3}{8}\nu u u_{xxyy} + \frac{1}{4}\nu u_x u_{xyy}\\
&\quad - \frac{1}{8}\nu u_{xx} u_{yy} - \frac{1}{4}\nu {u_{xy}}^2 - \frac{1}{3}\alpha u u_x u_{yy} - \frac{1}{4}\beta u u_{yy} - \frac{1}{4}u u_{ty},\\
T_4^y &= \frac{1}{3}\alpha u u_x u_{xy} + \frac{1}{3}\alpha u u_{xx} u_y + \frac{3}{8}\nu u u_{xxxy} + \frac{1}{4}\beta u u_{xy} + \frac{1}{2}u u_{tx} + \frac{1}{3}\alpha {u_x}^2 u_y + \frac{1}{8}\nu u_x u_{xxy}\\
&\quad + \frac{1}{4}\beta u_x u_y - \frac{1}{8}\nu u_{xx} u_{xy} + \frac{1}{8}\nu u_{xxx} u_y;
\end{aligned}$$

Case 5. For $Q_5 = -2\alpha t^2 u_t - \alpha txu_x - 2\alpha tyu_y - \alpha tu - \beta tx + xy$, we have the conserved vector

$$\begin{aligned}
T_5^t =& -\frac{4}{3}\alpha^2 t^2 u u_x u_{xy} - \frac{2}{3}\alpha^2 t^2 u u_y u_{xx} - \alpha\beta t^2 u u_{xy} - \alpha\nu t^2 u u_{xxxy} - \frac{1}{2}\alpha t^2 u u_{tx} - \frac{1}{2}\alpha t^2 u_t u_x \\
&+ \frac{1}{4}\alpha txu u_{xx} + \frac{1}{2}\alpha tyu u_{xy} - \frac{1}{4}\alpha txu_x^2 - \frac{1}{2}\alpha tyu_x u_y + \frac{1}{4}\alpha tuu_x - \frac{1}{2}\beta txu_x + \frac{1}{2}\beta tu \\
&+ \frac{1}{2}xyu_x - \frac{1}{2}u_y, \\
T_5^x =& \frac{1}{2}\alpha t^2 u u_{tt} - \frac{1}{2}\beta^2 txu_y + \alpha tuu_t - \frac{1}{2}\beta txu_t + \frac{1}{2}\beta\nu tu_{xy} - \frac{1}{4}\alpha xuu_x + \frac{3}{4}\nu xyu_{xxy} \\
&+ \frac{1}{2}\beta xyu_y - \frac{1}{3}\alpha^2 txuu_x u_{xy} + \frac{2}{3}\alpha^2 tyuu_x u_{yy} + \frac{2}{3}\alpha^2 tyuu_{xy}u_y - \frac{1}{8}\alpha\nu txuu_{xxxy} \\
&+ \frac{3}{4}\alpha\nu tyuu_{xxyy} - \frac{1}{4}\alpha\beta txuu_{xy} + \frac{1}{2}\alpha\beta tyuu_{yy} - \frac{5}{8}\alpha\nu txu_x u_{xxy} - \frac{1}{2}\alpha\nu tyu_x u_{xyy} \\
&- \frac{5}{4}\alpha\beta txu_x u_y + \frac{3}{8}\alpha\nu txu_{xx}u_{xy} + \frac{1}{4}\alpha\nu tyu_{xx}u_{yy} - \frac{1}{8}\alpha\nu txu_{xxx}u_y - \alpha\nu tyu_{xxy}u_y \\
&- \frac{4}{3}\alpha^2 t^2 u_t u_x u_y + \frac{3}{4}\alpha t^2 \nu uu_{txxy} + \frac{1}{2}\alpha t^2 \nu u_{tx}u_{xy} + \frac{1}{4}\alpha t^2 \nu u_{ty}u_{xx} - \frac{1}{4}\alpha t^2 \nu u_{txx}u_y \\
&- \frac{1}{2}\alpha t^2 \nu u_{txy}u_x - \frac{3}{4}\alpha t^2 \nu u_{xxy}u_t + \frac{1}{2}\alpha\beta t^2 u u_{ty} - \frac{1}{2}\alpha\beta t^2 u_t u_y - \frac{2}{3}\alpha^2 txu_x{}^2 u_y \\
&- \frac{4}{3}\alpha^2 tyu_x u_y{}^2 + \alpha^2 tuu_x u_y + \frac{1}{2}\alpha\nu tyu_{xy}{}^2 - \frac{1}{2}\alpha\beta tyu_y{}^2 - \frac{1}{4}\alpha txuu_{tx} + \frac{1}{2}\alpha tyuu_{ty} \\
&+ \frac{3}{2}\alpha\nu tuu_{xxy} + \alpha\beta tuu_y - \frac{1}{4}\alpha txu_t u_x - \frac{1}{2}\alpha tyu_t u_y - \frac{1}{2}\alpha\nu tu_x u_{xy} - \frac{3}{4}\beta\nu txu_{xxy} \\
&+ \alpha xyu_x u_y + \frac{2}{3}\alpha^2 t^2 uu_{tx}u_y + \frac{2}{3}\alpha^2 t^2 uu_{ty}u_x + \frac{1}{4}u^2\alpha + \frac{1}{2}\nu u_x - \frac{1}{2}\alpha t^2 u_t{}^2 - \frac{1}{4}\nu xu_{xx} \\
&- \frac{1}{2}\nu yu_{xy} + \frac{1}{2}u_t xy, \\
T_5^y =& \frac{1}{3}\alpha^2 txuu_x u_{xx} - \frac{2}{3}\alpha^2 tyuu_x u_{xy} - \frac{2}{3}\alpha^2 tyuu_{xx}u_y + \frac{1}{4}\alpha\beta txuu_{xx} + \frac{1}{8}\alpha\nu txuu_{xxxx} \\
&- \frac{3}{4}\alpha\nu tyuu_{xxxy} - \frac{1}{2}\alpha\beta tyuu_{xy} - \frac{1}{4}\alpha\nu txu_x u_{xxx} - \frac{1}{4}\alpha\nu tyu_x u_{xxy} - \frac{1}{2}\alpha\beta tyu_x u_y \\
&+ \frac{1}{4}\alpha\nu tyu_{xx}u_{xy} - \frac{1}{4}\alpha\nu tyu_{xxx}u_y + \frac{2}{3}\alpha^2 t^2 uu_{tx}u_x - \frac{1}{4}\alpha\nu t^2 u_x u_{txx} - \frac{1}{2}\alpha\beta t^2 u_x u_t \\
&+ \frac{1}{4}\alpha t^2 \nu uu_{txxx} + \frac{1}{4}\alpha t^2 \nu u_{xx}u_{tx} - \frac{1}{4}\alpha t^2 \nu u_{xxx}u_t + \frac{1}{2}\alpha\beta ut^2 u_{tx} - \frac{2}{3}\alpha^2 tyu_x{}^2 u_y \\
&- \frac{3}{4}\alpha\beta txu_x{}^2 + \frac{1}{8}\alpha\nu txu_{xx}{}^2 - \alpha tyuu_{tx} + \frac{3}{4}\alpha\beta tuu_x + \frac{3}{8}\alpha\nu tuu_{xxx} - \frac{1}{8}\alpha\nu tu_x u_{xx} \\
&- \frac{1}{4}\beta\nu txu_{xxx} - \frac{1}{4}y\nu u_{xx} - \frac{1}{2}\beta yu + \frac{1}{2}\beta^2 tu - \frac{2}{3}\alpha^2 t^2 u_t u_x{}^2 + \frac{1}{2}\alpha xyu_x{}^2 - \frac{1}{2}\alpha yuu_x \\
&+ \frac{1}{2}y\beta xu_x + \frac{1}{4}y\nu u_{xxx}x - \frac{1}{3}\alpha^2 txu_x{}^3 + \frac{1}{3}\alpha^2 tuu_x{}^2 - \frac{1}{2}\beta^2 txu_x + \frac{1}{4}\beta\nu tu_{xx};
\end{aligned}$$

Case 6. For $\mathcal{Q}_6 = 2\beta t u_y + 2y u_y + 4t u_t + x u_x + u$, we have the conserved vector

$$
\begin{aligned}
T_6^t =& \frac{8}{3}uu_xu_{xy}\alpha t + \frac{4}{3}uu_{xx}u_y\alpha t + 2uu_{xxxy}\nu t + \frac{3}{2}uu_{xy}\beta t + uu_{tx}t - \frac{1}{4}uu_{xx}x - \frac{1}{2}uu_{xy}y \\
&- \frac{1}{4}uu_x + \frac{1}{2}u_xu_y\beta t + u_xu_tt + \frac{1}{4}{u_x}^2x + \frac{1}{2}u_xu_yy, \\
T_6^x =& \frac{1}{2}\beta y{u_y}^2 + \frac{1}{4}u_tu_xx + \frac{1}{2}u_tu_yy + \frac{1}{2}\beta^2t{u_y}^2 - uu_{tt}t - \frac{1}{2}\nu y{u_{xy}}^2 - \frac{3}{2}\nu uu_{xxy} + \frac{1}{2}\nu u_xu_{xy} \\
&+ \frac{1}{4}xuu_{tx} - \frac{1}{2}yuu_{ty} - \beta uu_y + {u_t}^2t - uu_t + \frac{4}{3}\alpha\beta tu_x{u_y}^2 - \frac{4}{3}\alpha tuu_{tx}u_y - \frac{4}{3}\alpha tuu_{ty}u_x \\
&- \frac{3}{4}\beta\nu tuu_{xxyy} + \frac{8}{3}\alpha tu_tu_xu_y + \frac{1}{2}\beta\nu tu_xu_{xyy} - \frac{1}{4}\beta\nu tu_{xx}u_{yy} + \beta\nu tu_{xxy}u_y + \frac{1}{3}\alpha xuu_xu_{xy} \\
&- \frac{2}{3}\alpha yuu_xu_{yy} - \frac{2}{3}\alpha yuu_{xy}u_y - \frac{2}{3}\alpha\beta tuu_xu_{yy} - \frac{2}{3}\alpha\beta tuu_{xy}u_y + \frac{2}{3}\alpha x{u_x}^2u_y + \frac{4}{3}\alpha yu_x{u_y}^2 \\
&+ \frac{1}{4}\beta xu_xu_y - \frac{1}{2}\beta^2tuu_{yy} - \frac{1}{2}\beta\nu t{u_{xy}}^2 - \frac{3}{2}\nu tuu_{txxy} - \frac{3}{2}\beta tuu_{ty} + \frac{3}{2}\nu tu_tu_{xxy} + \frac{3}{2}\beta tu_tu_y \\
&- \nu tu_{tx}u_{xy} + \frac{1}{2}\nu tu_{txx}u_y + \nu tu_{txy}u_x - \frac{1}{2}\nu tu_{ty}u_{xx} + \frac{1}{8}\nu xuu_{xxxy} - \frac{3}{4}\nu yuu_{xxyy} \\
&+ \frac{5}{8}\nu xu_xu_{xxy} + \frac{1}{2}\nu yu_xu_{xyy} - \frac{3}{8}\nu xu_{xx}u_{xy} - \frac{1}{4}\nu yu_{xx}u_{yy} + \frac{1}{8}\nu xu_{xxx}u_y + \nu yu_{xxy}u_y \\
&- \alpha uu_xu_y + \frac{1}{4}\beta xuu_{xy} - \frac{1}{2}\beta yuu_{yy}, \\
T_6^y =& \frac{2}{3}\alpha yuu_yu_{xx} + \frac{2}{3}\alpha\beta tu_x{}^2u_y - \frac{4}{3}\alpha tuu_{tx}u_x - \frac{1}{3}\alpha xuu_xu_{xx} + \frac{2}{3}\alpha yuu_xu_{xy} + \frac{1}{4}\beta\nu tu_xu_{xxy} \\
&+ \frac{3}{4}\beta\nu tuu_{xxxy} - \frac{1}{4}\beta\nu tu_{xx}u_{xy} + \frac{1}{4}\beta\nu tu_{xxx}u_y + \frac{2}{3}\alpha\beta tuu_yu_{xx} + \frac{2}{3}\alpha\beta tuu_xu_{xy} \\
&+ \frac{1}{4}\nu yu_{xxx}u_y + \frac{1}{2}u\beta^2tu_{xy} - \frac{1}{4}u\beta u_{xx}x + \frac{1}{2}u\beta u_{xy}y + \frac{4}{3}\alpha tu_t{u_x}^2 + \frac{2}{3}\alpha yu_x{}^2u_y \\
&+ \frac{1}{2}\beta^2tu_xu_y + \beta tu_tu_x + \frac{1}{2}\nu tu_{txx}u_x + \frac{1}{4}\nu xu_xu_{xxx} + \frac{1}{4}\nu yu_xu_{xxy} + \frac{1}{2}\beta yu_xu_y \\
&- \frac{1}{2}\nu tuu_{txxx} - \frac{1}{8}\nu xuu_{xxxx} + \frac{3}{4}\nu yuu_{xxxy} + \frac{1}{2}\nu tu_tu_{xxx} - \frac{1}{2}\nu tu_{tx}u_{xx} - \frac{1}{4}\nu yu_{xx}u_{xy} \\
&+ uyu_{tx} - \frac{1}{3}\alpha u{u_x}^2 + \frac{1}{4}\beta x{u_x}^2 - \frac{1}{4}\beta uu_x + \frac{1}{8}\nu u_xu_{xx} - \frac{1}{8}\nu x{u_{xx}}^2 - \frac{3}{8}\nu uu_{xxx} \\
&+ \frac{1}{3}\alpha x{u_x}^3;
\end{aligned}
$$

Case 7. For $\mathcal{Q}_7 = yF'(t) - \alpha u_xF(t)$, we have the conserved vector

$$
\begin{aligned}
T_7^t =& -\frac{1}{2}yu_xF'(t) + \left(\frac{1}{4}\alpha uu_{xx} - \frac{1}{4}\alpha u_x^2\right)F(t), \\
T_7^x =& -\frac{1}{2}yuF''(t) + \left(\alpha yu_xu_y - \frac{1}{4}\alpha uu_x + \frac{3}{4}\nu yu_{xxy} + \frac{1}{2}\beta yu_y - \frac{1}{2}u\beta + \frac{1}{2}u_ty - \frac{1}{4}\nu u_{xx}\right)F'(t) \\
&- \left(\frac{1}{3}\alpha^2uu_xu_{xy} + \frac{2}{3}\alpha^2u_x^2u_y + \frac{5}{8}\alpha\nu u_xu_{xxy} + \frac{1}{4}\alpha\beta u_xu_y + \frac{1}{4}\alpha u_tu_x + \frac{1}{4}\alpha\beta uu_{xy}\right. \\
&\left. + \frac{1}{8}\alpha\nu uu_{xxxy} + \frac{1}{4}\alpha uu_{tx} - \frac{3}{8}\alpha\nu u_{xx}u_{xy} + \frac{1}{8}\alpha\nu u_{xxx}u_y\right)F(t) \\
T_7^y =& \left(\frac{1}{2}\alpha yu_x^2 + \frac{1}{2}\beta yu_x + \frac{1}{4}\nu yu_{xxxx}\right)F'(t) + \left(\frac{1}{3}\alpha^3uu_xu_{xx} - \frac{1}{3}\alpha^2u_x^3 - \frac{1}{4}\alpha\beta u_x^2 + \frac{1}{8}\alpha\nu u_{xx}^2\right. \\
&\left. - \frac{1}{4}\alpha\nu u_xu_{xxx} + \frac{1}{8}\alpha\nu uu_{xxxx} + \frac{1}{4}\alpha\beta uu_{xx}\right)F(t);
\end{aligned}
$$

Case 8. For $\mathcal{Q}_8 = G(t)$, we have the conserved vector

$$\begin{aligned}
T_8^t &= \frac{1}{2}u_x G(t),\\
T_8^x &= -\frac{1}{2}uG'(t) + \left(\alpha u_x u_y + \frac{1}{2}\beta u_y + \frac{3}{4}\nu u_{xxy} + \frac{1}{2}u_t\right)G(t),\\
T_8^y &= \left(\frac{1}{2}\alpha u_x^2 + \frac{1}{2}\beta u_x + \frac{1}{2}\nu u_{xxx}\right)G(t).
\end{aligned}$$

3.2. Conservation Laws Utilizing Noether's Theorem

We now construct conservation laws for Equation (1). We utilize the classical Noether's theorem [34,43] to achieve this task.

Equation (1) has a second-order Lagrangian

$$\mathcal{L} = \frac{1}{2}\left(\nu u_{xx}u_{xy} - u_t u_x - \alpha u_x^2 u_y - \beta u_x u_y\right), \tag{37}$$

since $\delta\mathcal{L}/\delta u = 0$ on (1). Here, $\delta/\delta u$ is the Euler operator. Thus, we can determine the Noether symmetries of (1) by employing the Lagrangian (37) on the determining equation

$$: X^{[2]}\mathcal{L} + \mathcal{L}\{D_t(\tau) + D_x(\xi) + D_y(\phi)\} - D_t(G^t) - D_x(G^x) - D_y(G^y) = 0, \tag{38}$$

with gauge functions G^t, G^x, and G^y. Expanding (38) and solving the resultant linear PDEs yield the following Noether symmetries and gauge functions:

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial t}, G^t = 0, G^x = 0, G^y = 0,\\
X_2 &= \frac{\partial}{\partial y}, G^t = 0, G^x = 0, G^y = 0,\\
X_3 &= g(t)\frac{\partial}{\partial u}, G^t = 0, G^x = -\frac{1}{2}ug'(t), G^y = 0,\\
X_4 &= 2\alpha t\frac{\partial}{\partial y} + x\frac{\partial}{\partial u}, G^t = -\frac{1}{2}u, G^x = 0, G^y = -\frac{1}{2}\beta u,\\
X_5 &= \alpha f(t)\frac{\partial}{\partial x} + yf'(t)\frac{\partial}{\partial u}, G^t = 0, G^x = -\frac{1}{2}uyf''(t) - \frac{1}{2}\beta u f'(t), G^y = 0,\\
X_6 &= 4\alpha t\frac{\partial}{\partial t} + \alpha x\frac{\partial}{\partial x} + 2\alpha y\frac{\partial}{\partial y} - (\beta x + \alpha u)\frac{\partial}{\partial u}, G^t = \frac{1}{2}\beta u, G^x = 0, G^y = \frac{1}{2}\beta^2 u.
\end{aligned}$$

Thus, the conserved vectors corresponding to the above obtained Noether point symmetries, using [41]

$$T^k = \mathcal{L}\xi^k + \left(\eta^\alpha - u^\alpha_{x^j}\xi^j\right)\left(\frac{\partial\mathcal{L}}{\partial u^\alpha_{x^k}} - \sum_{l=1}^{k} D_{x^l}\left(\frac{\partial\mathcal{L}}{\partial u^\alpha_{x^l x^k}}\right)\right) + \sum_{l=k}^{n}\left(\zeta^\alpha_l - u^\alpha_{x^l x^j}\xi^j\right)\frac{\partial\mathcal{L}}{\partial u^\alpha_{x^k x^l}}$$

are given by, respectively,

$$\begin{aligned}
T_1^t &= -\frac{1}{2}\alpha u_x^2 u_y - \frac{1}{2}\beta u_x u_y + \frac{1}{2}\nu u_{xx}u_{xy},\\
T_1^x &= \alpha u_t u_x u_y + \frac{3}{4}\nu u_t u_{xxy} - \frac{1}{4}\nu u_{xx}u_{ty} - \frac{1}{2}\nu u_{tx}u_{xy} + \frac{1}{2}\beta u_t u_y + \frac{1}{2}u_t^2,\\
T_1^y &= \frac{1}{2}\alpha u_t u_x^2 + \frac{1}{2}\beta u_t u_x + \frac{1}{4}\nu u_t u_{xxx} - \frac{1}{4}\nu u_{xx}u_{tx};
\end{aligned}$$

$$
\begin{aligned}
T_2^t =& \frac{1}{2}u_x u_y,\\
T_2^x =& \frac{1}{2}u_t u_y + \alpha u_x u_y^2 + \frac{3}{4}\nu u_y u_{xxy} - \frac{1}{2}\nu u_{xy}^2 - \frac{1}{4}\nu u_{xx}u_{yy} + \frac{1}{2}\beta u_y^2,\\
T_2^y =& \frac{1}{4}\nu u_{xx}u_{xy} + \frac{1}{4}\nu u_{xxx}u_y - \frac{1}{2}u_t u_x;
\end{aligned}
$$

$$
\begin{aligned}
T_3^t =& -\frac{1}{2}u_x g(t),\\
T_3^x =& \left(-\alpha u_x u_y - \frac{3}{4}\nu u_{xxy} - \frac{1}{2}\beta u_y - \frac{1}{2}u_t\right)g(t) + \frac{1}{2}u g'(t),\\
T_3^y =& \left(-\frac{1}{2}\alpha u_x^2 - \frac{1}{2}\beta u_x - \frac{1}{4}\nu u_{xxx}\right)g(t);
\end{aligned}
$$

$$
\begin{aligned}
T_4^t =& \frac{1}{2}u - \frac{1}{2}x u_x + \alpha t u_x u_y,\\
T_4^x =& \frac{1}{2}\nu u_{xy} - \frac{1}{2}x u_t - \alpha x u_x u_y - \frac{1}{2}\beta x u_y - \frac{1}{2}\nu x u_{xxy} + \alpha t u_t u_y + 2\alpha^2 t u_x u_y^2 + \alpha\nu t u_y u_{xxy}\\
&- \alpha\nu t u_{xy}^2 - \alpha\nu t u_{xx}u_{yy} + \alpha\beta t u_y^2,\\
T_4^y =& \frac{1}{2}\beta u + \alpha\nu t u_y u_{xxx} - \frac{1}{2}\nu x u_{xxx} - \frac{1}{2}\beta x u_x - \frac{1}{2}\alpha x u_x^2 - \alpha t u_t u_x + \alpha\nu t u_{xx}u_{xy};
\end{aligned}
$$

$$
\begin{aligned}
T_5^t =& \frac{1}{2}\alpha u_x^2 f(t) - \frac{1}{2}y u_x f'(t),\\
T_5^x =& \left(\frac{1}{2}\alpha^2 u_x^2 u_y + \frac{3}{4}\alpha\nu u_x u_{xxy} - \frac{1}{4}\alpha\nu u_{xx}u_{xy}\right)f(t) + \left(-\frac{1}{2}y u_t - \alpha y u_x u_y - \frac{3}{4}\nu y u_{xxy}\right.\\
&\left.+\frac{1}{4}\nu u_{xx} - \frac{1}{2}\beta y u_y + \frac{1}{2}\beta u\right)f'(t) + \frac{1}{2}y u f''(t),\\
T_5^y =& \left(\frac{1}{2}\alpha^2 u_x^3 + \frac{1}{2}\alpha\beta u_x^2 + \frac{1}{4}\alpha\nu u_{xxx}u_x - \frac{1}{4}\alpha\nu u_{xx}^2\right)f(t) - \left(\frac{1}{2}\alpha y u_x^2 + \frac{1}{2}\beta y u_x\right.\\
&\left.+\frac{1}{4}\nu y u_{xxx}\right)f'(t);
\end{aligned}
$$

$$
\begin{aligned}
T_6^t =& \frac{1}{2}x\alpha u_x^2 - 2t\alpha^2 u_y u_x^2 + \frac{1}{2}x\beta u_x + \frac{1}{2}\alpha u u_x + y\alpha u_y u_x - 2t\alpha\beta u_y u_x + 2t\alpha\nu u_{xy}u_{xx}\\
&- \frac{1}{2}\beta u,\\
T_6^x =& \frac{1}{2}x u_y u_x^2\alpha^2 + 2y u_y^2 u_x\alpha^2 + u u_y u_x\alpha^2 + 4t u_y u_x u_t\alpha^2 + y\beta u_y^2\alpha - y\nu u_{xy}^2\alpha + 2t u_t^2\alpha\\
&+ \frac{1}{2}\beta u u_y\alpha + x\beta u_y u_x\alpha - \nu u_x u_{xy}\alpha - \frac{3}{4}\nu u_y u_{xx}\alpha - \frac{1}{2}y\nu u_{yy}u_{xx}\alpha - \frac{1}{4}x\nu u_{xy}u_{xx}\alpha\\
&+ \frac{3}{4}\nu u u_{xxy}\alpha + \frac{3}{2}y\nu u_y u_{xxy}\alpha + \frac{3}{4}x\nu u_x u_{xxy}\alpha + \frac{1}{2}u u_t\alpha + y u_y u_t\alpha + 2t\beta u_y u_t\alpha\\
&+ 3t\nu u_{xxy}u_t\alpha - t\nu u_{xx}u_{ty}\alpha - 2t\nu u_{xy}u_{tx}\alpha + \frac{1}{2}x\beta^2 u_y - \frac{1}{2}\beta\nu u_{xy} + \frac{3}{4}x\beta\nu u_{xxy}\\
&+ \frac{1}{2}x\beta u_t,
\end{aligned}
$$

$$\begin{aligned} T_6^y =& \frac{1}{2}x\alpha^2u_x^3 + x\alpha\beta u_x^2 + \frac{1}{2}\alpha^2 u u_x^2 + 2t\alpha^2 u_t u_x^2 + \frac{1}{2}x\beta^2 u_x + \frac{1}{2}\alpha\beta u u_x - \frac{1}{2}\alpha\nu u_{xx}u_x \\ &+ \frac{1}{4}x\alpha\nu u_{xxx}u_x - y\alpha u_t u_x + 2t\alpha\beta u_t u_x - \frac{1}{4}x\alpha\nu u_{xx}^2 - \frac{1}{4}\beta\nu u_{xx} + \frac{1}{2}y\alpha\nu u_{xy}u_{xx} \\ &+ \frac{1}{4}x\beta\nu u_{xxx} + \frac{1}{4}\alpha\nu u u_{xxx} + \frac{1}{2}y\alpha\nu u_y u_{xxx} + t\alpha\nu u_{xxx}u_t - t\alpha\nu u_{xx}u_{tx} - \frac{1}{2}\beta^2 u. \end{aligned}$$

It should be noted that we derived eight conservation laws of (1) by invoking the multiplier method whereas Noether's theorem yielded six conservation laws. However, we see that only one conservation law is common when using both methods. The conservation laws obtained with the Lagrange multiplier method could be obtained by means of Noether's theorem using contact transformations. Thus, we conclude that there are benefits and constraints of the different methods used when deriving the conservation laws of nonlinear partial differential equations.

4. Concluding Remarks

In this paper, we investigated the generalized two-dimensional generalized shallow water wave Equation (1). We determined the Lie point symmetries of this equation, performed symmetry reductions, and reduced it to an NLODE (8). This NLODE (8) was integrated directly and its solution was obtained in the form of an incomplete elliptic integral. Moreover, Kudrayshov's method was employed to obtain the solution of the NLODE (8). These solutions were presented graphically. Finally, we used the multiplier method and Noether's theorem to derive the conservation laws of (1). The multiplier method yielded eight multipliers, which gave eight local conservation laws for Equation (1), whereas Noether's theorem provided us with six local conservation laws, which included energy and linear momentum conservation laws.

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