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Sturm–Liouville Differential Equations Involving Kurzweil–Henstock Integrable Functions

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Abstract: In this paper, we give sufficient conditions for the existence and uniqueness of the solution of Sturm–Liouville equations subject to Dirichlet boundary value conditions and involving Kurzweil–Henstock integrable functions on unbounded intervals. We also present a finite element method scheme for Kurzweil–Henstock integrable functions.

Keywords: Kurzweil–Henstock integral; KH–Sobolev space; Sturm–Liouville equation; finite element method



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1. Introduction

The Sturm–Liouville equation appears in certain practical areas, such as heat flow and vibration problems, electroencephalography applications, and other areas of physics. It has a relevant role in quantum mechanics, and some of these problems are formulated in unbounded intervals. On occasion, these problems are described by differential equations with highly oscillatory coefficients. A particular characteristic of these coefficients is that they are not square Lebesgue integrable. The study of differential equations involving integrable Henstock–Kurzweil functions has been developed by several authors, for example, [1–7]. In [8], Pérez et al. introduced the KH–Sobolev space on bounded intervals and guaranteed the existence and uniqueness of the solution to some boundary value problems involving Kurzweil–Henstock integrable functions on $[0, 1]$. In this paper, particularly in Section 3, we introduce the KH–Sobolev space for unbounded intervals, and then we apply these spaces and the Fredholm alternative theorem to establish the existence and uniqueness of the solution to the Sturm–Liouville differential equation

$$-[\rho\dot{u}]' + qu = f \quad \text{a.e. on } [a, \infty)$$

subject to the Dirichlet boundary value conditions

$$u(a) = u(a+) = 0, \quad u(\infty) = u(\infty-) = 0,$$

where the derivative is in the weak sense, q is a Lebesgue integrable function, f is Kurzweil–Henstock integrable, ρ is a function of bounded variation on all compact intervals $J \subseteq [a, \infty)$, $\frac{1}{\rho}$ is Lebesgue integrable, and ACG^* on $[a, \infty]$. The solution is proven to be stable under small variations of f . See Section 4.

The Finite Element method (FEM) has been used to give approximations of the solution of a differential equation when the functions involved are continuous or square

integrable. León-Velasco et al. in [9] used the FEM to find numerical approximations of the solution of certain Sturm–Liouville-type differential equations involving Henstock–Kurzweil integrable functions. The existence and uniqueness of the problems given in [9], as well as the convergence of the FEM are not studied in that paper. In Section 5, we give conditions for the existence and uniqueness of elliptic problems on a bounded interval. In Section 6, we show a scheme for the convergence of the Finite Element method.

2. Preliminaries

Throughout this paper, I will be an interval of the form $[a, \infty)$, $(-\infty, \infty)$, $(-\infty, b]$, or $[a, b]$. Positive functions δ defined on I will be called gauge functions. These functions will control the refinements of the partitions in the Kurzweil–Henstock integral. Next, we give the definition of this integral when I is a bounded interval. Let $\delta : I \rightarrow \mathbb{R}^+$ be a gauge function. A tagged partition $P = \{([s_{k-1}, s_k], t_k)\}_{k=1}^n$ of I is said to be δ -fine, when for every $k = 1, \dots, n$,

$$[s_{k-1}, s_k] \subset (t_k - \delta(t_k), t_k + \delta(t_k)).$$

Definition 1. A function $f : I \rightarrow \mathbb{R}$ is said to be Kurzweil–Henstock integrable (in abbreviation, KH-integrable) if there exists a number $A \in \mathbb{R}$ with the property that for every $\varepsilon > 0$, there exists a gauge $\delta_\varepsilon : I \rightarrow \mathbb{R}^+$ such that for every tagged partition $P = \{([s_{k-1}, s_k], t_k)\}_{k=1}^n$ of $[a, b]$, if P is δ_ε -fine, then

$$\left| \sum_{k=1}^n f(t_k)(s_k - s_{k-1}) - A \right| < \varepsilon.$$

The Kurzweil–Henstock integral of f on $[a, b]$ is denoted and defined by $\int_a^b f = A$.

Now we define the Kurzweil–Henstock integral for non-bounded intervals. Let $a \in \mathbb{R}$ and $\delta : [a, \infty) \rightarrow \mathbb{R}^+$ be a gauge function. A tagged partition $P = \{([s_{k-1}, s_k], t_k)\}_{k=1}^{n+1}$ of $[a, \infty)$ is said to be δ -fine, if $s_{n+1} = t_{n+1} = \infty$, $1/\delta(t_{n+1}) < s_n$ and for each $k = 1, 2, \dots, n$, $[s_{k-1}, s_k] \subset (t_k - \delta(t_k), t_k + \delta(t_k))$.

Definition 2. A function f defined on $[a, \infty)$ is said to be Kurzweil–Henstock integrable (KH-integrable) if there exists a number $A \in \mathbb{R}$ with the property that for every $\varepsilon > 0$, there exists a gauge $\delta_\varepsilon : [a, \infty) \rightarrow \mathbb{R}^+$ such that

$$\left| \sum_{k=1}^n f(t_k)(s_k - s_{k-1}) - A \right| < \varepsilon$$

for all tagged partitions P of $[a, \infty)$ which is δ_ε -fine.

For functions defined over the intervals $[-\infty, a]$ and $[-\infty, \infty]$, we have similar definitions. In the case where f is defined on $[a, \infty)$, we can extend f on $[a, \infty]$, assuming that $f(\infty) = 0$. In this situation, we say that f is Kurzweil–Henstock integrable on $[a, \infty)$ if f extended to $[a, \infty]$ is KH-integrable. Similar considerations are given for intervals of the form $(-\infty, b]$, $(-\infty, \infty)$.

The space of functions which are Kurzweil–Henstock integrable on I is denoted by $KH(I)$. The Alexiewicz seminorm for this space is denoted and defined as

$$\|f\|_A = \sup \left\{ \left| \int_J f \right| : J \text{ is an interval contained in } I \right\}.$$

The Lebesgue space $L^p(I)$, for $1 \leq p < \infty$, is defined as the set of Lebesgue-measurable functions f on I for which $\int_I |f|^p < \infty$. The seminorm of this space is given by

$$\|f\|_p = \left[\int_I |f|^p \right]^{\frac{1}{p}}.$$

It is well-known that $L^1(I) \subset KH(I)$. See ([10], Corollary 4.80). This inclusion is strict, see, for example, ([11], Example 3.12). In particular, if $f \in L^1(I)$, then $\|f\|_A = \|f\|_1$. Moreover, when I is a bounded interval, it follows that $L^p(I) \subseteq L^1(I) \subseteq KH(I)$. Unfortunately, the space $(KH(I), \|\cdot\|_A)$ is not a complete space.

The variation of a function h on the interval I is denoted by $V_I h$. If $V_I h < \infty$, then h is of a bounded variation on I , and we write $h \in BV(I)$. The functions of bounded variations are the multipliers of the KH-integrable functions. This allows the following Hölder-type theorem to be established:

Theorem 1. ([12], Lemma 24) *If $f \in KH(I)$ and $h \in BV(I)$, then $fh \in KH(I)$ and*

$$\left| \int_I fh \right| \leq \left| \int_I f \right| \inf_I |h| + \|f\|_A V_I h.$$

A function F is AC_* on a set E , if for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that for every collection $\{J_k\}_{k=1}^n$ of non-overlapping closed intervals with endpoints in E , it follows that if $\sum_{k=1}^n l(J_k) < \delta_\varepsilon$, then $\sum_{k=1}^n \sup\{|F(v) - F(u)| : u, v \in J_k\} < \varepsilon$. Additionally, the function F is ACG_* on a bounded interval J , if F is continuous on J and $J = \cup_{n \in \mathbb{N}} E_n$, where (E_n) is a sequence of subsets of J and F is AC_* on each E_n . In the case of unbounded intervals, we say that f is ACG_* on $[a, \infty]$ if f is ACG_* on each compact interval $J \subseteq [a, \infty)$ and f is continuous at ∞ . For other intervals, we have similar definitions.

Theorem 2. ([13]) *Let q_1 be the left end of the interval I . The following properties hold:*

1. *If $f \in KH(I)$ and $F_f = \int_{q_1}^{(\cdot)} f$, then F_f is ACG_* on I and $F'_f = f$ a.e. on I . Moreover, if f is continuous at $s \in I$ then $F'_f(s) = f(s)$.*
2. *F is ACG_* on I if, and only if F' exists a.e. on I and $\int_{q_1}^s F' = F(s) - F(q_1)$ for all $s \in I$.*

Theorem 3. ([12], Lemma 25) *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ and $\omega : \mathbb{R}^2 \rightarrow \mathbb{R}$ be functions such that $f \in KH(\mathbb{R})$ and for each compact interval $J \subseteq \mathbb{R}$,*

1. *$\int_{-\infty}^{\infty} V_J \omega(\cdot, t) dt$ exists,*
2. *there exists $K_J > 0$ such that $\|\omega(s, \cdot)\|_1 \leq K_J$ for all $s \in J$.*

Then

$$\int_a^b \int_{-\infty}^{\infty} f(s) \omega(s, t) dt ds = \int_{-\infty}^{\infty} \int_a^b f(s) \omega(s, t) ds dt$$

for all $a, b \in \mathbb{R}$ with $a < b$. Moreover, if

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(s) \omega(s, t) dt ds$$

exists, then

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(s) \omega(s, t) dt ds = \lim_{\substack{b \rightarrow \infty \\ a \rightarrow -\infty}} \int_{-\infty}^{\infty} \int_a^b f(s) \omega(s, t) ds dt.$$

3. The Kurzweil–Henstock–Sobolev Space for Unbounded Intervals

Let $C_P^2([a, b])$ be the space of functions $v \in C([a, b])$ for which there exists $\{[s_{k-1}, s_k]\}_{k=1}^n$ a partition of $[a, b]$ such that for every $k = 1, \dots, n$, $v \in C^2((s_{k-1}, s_k))$, and $v^{(i)}(s_0+)$, $v^{(i)}(s_1-)$, $v^{(i)}(s_1+)$, \dots , $v^{(i)}(s_{n-1}-)$, $v^{(i)}(s_{n-1}+)$, $v^{(i)}(s_n-)$ exist, for $i = 1, 2$. Now, we define the space \mathcal{V}_I for an interval I as follows:

$$\mathcal{V}_I = \{v \in C_P^2([a, b]) : v(a) = v(b) = 0\}, \text{ when } I = [a, b],$$

$$\mathcal{V}_I = \{v \in C(I) : \text{there exists } a, b \in \mathbb{R} \text{ such that } \text{supp } v \subseteq [a, b] \text{ and } v \in C_P^2([a, b])\},$$

when $I = \mathbb{R}$,

$$\mathcal{V}_I = \{v \in C(I) : v(a) = 0, \text{ there exists } b > a \text{ such that } \text{supp } v \subseteq [a, b] \text{ and } v \in C_P^2([a, b])\}, \text{ when } I = [a, \infty),$$

$\mathcal{V}_I = \{v \in C(I) : v(b) = 0, \text{ there exists } a < b \text{ such that } \text{supp } v \subseteq [a, b] \text{ and } v \in C_P^2([a, b])\}$, when $I = (-\infty, b]$.

It is clear that if $v \in \mathcal{V}_I$, then v and v' belong to $BV(I)$. Throughout this section, we will only consider the interval $I = \mathbb{R}$. We denote by $KH_{loc}(\mathbb{R})$ the space of functions defined on \mathbb{R} that are KH -integrable on every compact interval.

Lemma 1. Let $f \in KH_{loc}(\mathbb{R})$ and suppose that $\int_{-\infty}^{\infty} f v = 0$ for all $v \in \mathcal{V}_{\mathbb{R}}$. Then the function f is zero a.e. on \mathbb{R} .

Proof. Let $\alpha \in \mathbb{R}$. Then we show that

$$F_{\alpha}(s) := \int_{\alpha}^s f = 0$$

for all $s \in \mathbb{R}$. Let $s^* > \alpha$, choose $\gamma > 0$ such that $\alpha < \alpha + \gamma < s^* - \gamma < s^*$, and define the function

$$v(s) = \begin{cases} 0, & \text{if } s \leq \alpha; \\ \frac{s-\alpha}{\gamma}, & \text{if } \alpha \leq s \leq \alpha + \gamma; \\ 1, & \text{if } \alpha + \gamma \leq s \leq s^* - \gamma; \\ \frac{s^*-s}{\gamma}, & \text{if } s^* - \gamma \leq s \leq s^*; \\ 0, & \text{if } s^* \leq s. \end{cases}$$

Then $v \in \mathcal{V}_{\mathbb{R}}$ and so

$$\begin{aligned} 0 &= \int_{-\infty}^{\infty} f v = \int_{\alpha}^{s^*} f v = \int_{\alpha}^{\alpha+\gamma} f v + \int_{\alpha+\gamma}^{s^*-\gamma} f v + \int_{s^*-\gamma}^{s^*} f v \\ &= \frac{1}{\gamma} \int_{\alpha}^{\alpha+\gamma} f(s)(s-\alpha) ds + \int_{\alpha+\gamma}^{s^*-\gamma} f + \frac{1}{\gamma} \int_{s^*-\gamma}^{s^*} f(s)(s^*-s) ds. \end{aligned} \quad (1)$$

From ([14], Theorem 12.5), there exist $\theta_1 \in [\alpha, \alpha + \gamma]$ and $\theta_2 \in [s^* - \gamma, s^*]$ such that

$$\int_{\alpha}^{\alpha+\gamma} f(s)(s-\alpha) ds = (\alpha - \alpha) \int_{\alpha}^{\theta_1} f + (\alpha + \gamma - \alpha) \int_{\theta_1}^{\alpha+\gamma} f = \gamma \int_{\theta_1}^{\alpha+\gamma} f$$

and

$$\int_{s^*-\gamma}^{s^*} f(s)(s^*-s) ds = (s^* - (s^* - \gamma)) \int_{s^*-\gamma}^{\theta_2} f + (s^* - s^*) \int_{\theta_2}^{s^*} f = \gamma \int_{s^*-\gamma}^{\theta_2} f.$$

Thus, by (1),

$$0 = \int_{\theta_1}^{\alpha+\gamma} f + \int_{\alpha+\gamma}^{s^*-\gamma} f + \int_{s^*-\gamma}^{\theta_2} f.$$

Therefore,

$$\int_{\alpha}^{s^*} f = \lim_{\gamma \rightarrow 0} \left[\int_{\theta_1}^{\alpha+\gamma} f + \int_{\alpha+\gamma}^{s^*-\gamma} f + \int_{s^*-\gamma}^{\theta_2} f \right] = 0.$$

The case when $s^* < c$ is proved in a similar way. Consequently, $F'_{\alpha} = 0$. On the other hand, $F'_{\alpha} = f$ a.e. on \mathbb{R} , thus, $f = 0$ a.e. on \mathbb{R} . \square

Corollary 1. Let $f \in KH_{loc}(\mathbb{R})$ and suppose that $\int_{-\infty}^{\infty} f v' = 0$ for all $v \in \mathcal{V}_{\mathbb{R}}$. Then there is $K \in \mathbb{R}$ such that $f = K$ a.e. on \mathbb{R} .

Proof. Take $z \in \mathcal{V}_{\mathbb{R}}$ satisfying $\int_{-\infty}^{\infty} z = 1$. Let $y \in \mathcal{V}_{\mathbb{R}}$ and define

$$v(s) = \int_{-\infty}^s \left[y - \left(\int_{-\infty}^{\infty} y \right) z \right].$$

Then, $v \in \mathcal{V}_{\mathbb{R}}$ and so

$$\begin{aligned} 0 &= \int_{-\infty}^{\infty} f v' \\ &= \int_{-\infty}^{\infty} f(s) \left[y(s) - \left(\int_{-\infty}^{\infty} y(t) dt \right) z(s) \right] ds \\ &= \int_{-\infty}^{\infty} f(s) y(s) ds - \int_a^b \int_{-\infty}^{\infty} f(s) z(s) y(t) dt ds, \end{aligned}$$

where $a, b \in \mathbb{R}$ are such that $\text{supp}(z) \subseteq [a, b]$. Let $f^* = fz$ and $\omega(s, t) = y(t)$, then $f^* \in KH(\mathbb{R})$ and for every compact interval J , $\int_{-\infty}^{\infty} V_J \omega(\cdot, t) dt < \infty$ and $\|\omega(s, \cdot)\|_1 = \int_{-\infty}^{\infty} |y(t)| dt$ for all $s \in J$. Therefore, by Theorem 3,

$$\int_a^b \int_{-\infty}^{\infty} f^*(s) \omega(s, t) dt ds = \int_{-\infty}^{\infty} \int_a^b f^*(s) \omega(s, t) ds dt.$$

Consequently,

$$\begin{aligned} 0 &= \int_{-\infty}^{\infty} f(t) y(t) dt - \int_{-\infty}^{\infty} \int_a^b f(s) z(s) y(t) ds dt \\ &= \int_{-\infty}^{\infty} \left(f(t) - \int_{-\infty}^{\infty} f(s) z(s) ds \right) y(t) dt. \end{aligned}$$

Therefore, by Lemma 1,

$$f = \int_{-\infty}^{\infty} f(s) z(s) ds$$

a.e. on \mathbb{R} . \square

Lemma 2. Let $f \in KH(\mathbb{R})$ and define

$$\sigma(s) = \int_{-\infty}^s f, \quad s \in \mathbb{R}.$$

Then, $\sigma \in C(\mathbb{R})$ and

$$\int_{-\infty}^{\infty} \sigma v' = - \int_{-\infty}^{\infty} f v,$$

for all $v \in \mathcal{V}_{\mathbb{R}}$.

Proof. First, observe that

$$\begin{aligned} - \int_{-\infty}^{\infty} f(s) v(s) ds &= \int_{-\infty}^{\infty} f(s) \lim_{l \rightarrow \infty} [v(l) - v(s)] ds \\ &= \int_{-\infty}^{\infty} f(s) \lim_{l \rightarrow \infty} \left[\int_s^l v'(t) dt \right] ds \\ &= \int_{-\infty}^{\infty} f(s) \left[\int_s^{\infty} v'(t) dt \right] ds \\ &= \int_{-\infty}^{\infty} \int_s^{\infty} f(s) v'(t) dt ds. \end{aligned}$$

Define

$$\omega(s, t) = \begin{cases} v'(t) & \text{if } s \leq t \\ 0 & \text{if } t < s. \end{cases}$$

Then, for every compact interval $J = [a, b]$,

$$V_J \omega(\cdot, t) = \begin{cases} 0, & \text{if } t < a; \\ |v'(t)|, & \text{if } a \leq t \leq b; \\ 0, & \text{if } b < t. \end{cases} \quad (2)$$

Thus, $\int_{-\infty}^{\infty} V_J \omega(\cdot, t) dt$ exists; moreover,

$$\|\omega(s, \cdot)\|_1 \leq \|v'\|_1$$

for all $s \in J$. Therefore, by Theorem 3,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(s) \omega(s, t) dt ds = \lim_{\substack{b \rightarrow \infty \\ a \rightarrow -\infty}} \int_{-\infty}^{\infty} \int_a^b f(s) \omega(s, t) ds dt.$$

From (2), $\omega(\cdot, t) \in BV(\mathbb{R})$ for all $t \in \mathbb{R}$. Therefore, by Hake's Theorem,

$$\lim_{\substack{b \rightarrow \infty \\ a \rightarrow -\infty}} \int_a^b f(s) \omega(s, t) ds = \int_{-\infty}^{\infty} f(s) \omega(s, t) ds.$$

Now, since

$$\left| \int_a^b f(s) \omega(s, t) ds \right| \leq |v'(t)| \|f\|_A$$

and $v' \in L^1(\mathbb{R})$, it follows by the Dominated Convergence Theorem that

$$\begin{aligned} \int_{-\infty}^{\infty} \int_s^{\infty} f(s) v'(t) dt ds &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(s) \omega(s, t) dt ds \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(s) \omega(s, t) ds dt \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^y f(s) v'(t) ds dt \\ &= \int_{-\infty}^{\infty} \sigma(t) v'(t) dt. \end{aligned}$$

Therefore,

$$\int_{-\infty}^{\infty} \sigma v' = - \int_{-\infty}^{\infty} f v.$$

□

Definition 3. The KH-Sobolev space $W_{KH}(\mathbb{R})$ is defined as

$$W_{KH}(\mathbb{R}) = \left\{ u \in KH_{loc}(\mathbb{R}) : \text{there exists } w \in KH(\mathbb{R}) \text{ such that } \int_{-\infty}^{\infty} u \varphi' = - \int_{-\infty}^{\infty} w \varphi \text{ for all } \varphi \in \mathcal{V}_{\mathbb{R}} \right\}.$$

The weak derivative of $u \in W_{KH}(\mathbb{R})$ is denoted and defined by $\dot{u} = w$, where w is as in Definition 3. Lemma 1 implies the uniqueness of \dot{u} , except for sets of measure zero. It is clear that if $u_1, u_2 \in W_{KH}(\mathbb{R})$ and λ is a scalar, then $\lambda u_1 + u_2 \in W_{KH}(\mathbb{R})$ and $(\lambda u_1 + u_2)' = \lambda \dot{u}_1 + \dot{u}_2$.

Remark 1. 1. If $f \in KH(\mathbb{R})$ and $F = \int_{-\infty}^{(\cdot)} f$, then by Lemma 2 $F \in W_{KH}(\mathbb{R})$ and $\dot{F} = f$.
2. If $u_1 = u_2$ a.e. on \mathbb{R} and u_2 is ACG* on \mathbb{R} , then $u_1 \in W_{KH}(\mathbb{R})$ and $\dot{u}_1 = u_2'$ a.e. on \mathbb{R} .

Theorem 4. For each $u \in W_{KH}(\mathbb{R})$, there is $\chi_u \in C(\overline{\mathbb{R}})$ for which $u = \chi_u$ a.e. on \mathbb{R} and

$$\chi_u(b) - \chi_u(a) = \int_a^b \dot{u}, \text{ for all } a, b \in \overline{\mathbb{R}} \text{ with } a < b.$$

Proof. We set $\sigma(s) = \int_{-\infty}^s \dot{u}$. From Lemma 2, we obtain that $\int_{-\infty}^{\infty} \sigma v' = - \int_{-\infty}^{\infty} \dot{u} v$ for every $v \in \mathcal{V}_{\mathbb{R}}$, and since $u \in W_{KH}(\mathbb{R})$,

$$- \int_{-\infty}^{\infty} \dot{u} v = \int_{-\infty}^{\infty} u v'$$

for all $v \in \mathcal{V}_{\mathbb{R}}$.

Consequently, for each $v \in \mathcal{V}_{\mathbb{R}}$,

$$\int_{-\infty}^{\infty} (u - \sigma) v' = 0.$$

Therefore, by Corollary 1, there is $K \in \mathbb{R}$ for which $u - \sigma = K$ a.e. on \mathbb{R} . Putting $\chi_u = \sigma + K$, we obtain the conclusion of the theorem. \square

By ([6], Corollary 2.4) we have the following integration by a parts formula for the weak derivative.

Theorem 5. If u, w are in $W_{HK}(\mathbb{R})$, then uw is also in $W_{HK}(\mathbb{R})$ and $(uw)' = \dot{u}w + u\dot{w}$. Moreover, for every $c, d \in \overline{\mathbb{R}}$, if the product $\dot{u}w$ is in $KH([c, d])$ and $u(c+) = u(c)$, $w(c+) = w(c)$, $u(d-) = u(d)$ and $v(d-) = v(d)$, then

$$\int_c^d \dot{u}w = uw \Big|_c^d - \int_c^d u\dot{w}. \quad (3)$$

4. Sturm–Liouville Differential Equations for Unbounded Intervals

We denote by $W_{KH}^2(I)$ the space of functions $u \in W_{KH}(I)$ such that $\dot{u} \in W_{KH}(I)$, and by $\Omega_0(I)$ the space of functions $u \in W_{KH}(I)$ such that $\dot{u} \in L^\infty(I)$, $u(q_1) = u(q_1+) = 0$ and $u(q_2) = u(q_2-) = 0$, where q_1 is the left end of the interval I and q_2 is the right end.

Observe that if $u \in \Omega_0(I)$, then by Theorem 4, there is $\chi_u \in C(\overline{I})$ so that $u = \chi_u$ a.e. on I , therefore $u \in L^\infty(I)$. In this way, we can equip the space $\Omega_0(I)$ with the seminorm $\|u\|_W = \|u\|_\infty + \|\dot{u}\|_\infty + \|\dot{u}\|_A$.

In this section, we will only consider $I = [a, \infty)$; however, the results are also true for intervals of the form $(-\infty, b]$ or $(-\infty, \infty)$.

Problem 1. Let $f \in KH([a, \infty))$, $q \in L^1([a, \infty))$ and ρ be a function such that $\rho \in BV(J)$ for all compact intervals $J \subseteq [a, \infty)$, and $\frac{1}{\rho} \in ACG^*([a, \infty)) \cap L^1([a, \infty))$. We solve the following boundary value problem:

Find $u \in W_{HK}^2([a, \infty))$ that satisfies

$$\begin{cases} -[\rho \dot{u}]' + qu = f & \text{a.e. on } (a, \infty); \\ u(a) = u(a+) = 0, u(\infty) = u(\infty-) = 0. \end{cases} \quad (4)$$

This problem is equivalent to the following variational problem:

Find $u \in \Omega_0([a, \infty))$ such that

$$\int_a^\infty \rho \dot{u} v' + \int_a^\infty quv = \int_a^\infty f v, \text{ for all } v \in \mathcal{V}_{[a, \infty)}. \quad (5)$$

We will provide a solution to this variational problem. Define

$$x_f = \int_a^{(\cdot)} \frac{1}{\rho} (\beta_f - F), \quad (6)$$

where

$$F = \int_a^{(\cdot)} f \quad \text{and} \quad \beta_f = \frac{\int_a^\infty \frac{f}{\rho}}{\int_a^\infty \frac{1}{\rho}}.$$

Additionally, for every $u \in \Omega_0([a, \infty))$, define

$$h_u = -\frac{1}{\rho} \int_a^{(\cdot)} qu \quad (7)$$

and

$$z_u = \int_a^{(\cdot)} (h_u - \alpha_u \frac{1}{\rho}), \quad (8)$$

where

$$\alpha_u = \frac{\int_a^\infty h_u}{\int_a^\infty \frac{1}{\rho}}.$$

Then, $x_f, z_u \in \Omega_0([a, \infty))$. Define the operators $\Psi : \Omega_0([a, \infty)) \times \mathcal{V}_{[a, \infty)} \rightarrow \mathbb{R}$ and $\Gamma : \Omega_0([a, \infty)) \rightarrow \Omega_0([a, \infty))$ by

$$\Psi(u, v) = \int_a^\infty \rho uv' \quad \text{and} \quad \Gamma(u) = z_u \quad (9)$$

Then, Ψ is a bilinear operator and Γ is a linear operator that satisfies

$$\Psi(\Gamma u, v) = \int_a^\infty quv \quad \text{and} \quad \Psi(x_f, v) = \int_a^\infty fv$$

for all $v \in \mathcal{V}_{[a, \infty)}$. Therefore, equality (5) is represented by the equation

$$\Psi(u, v) + \Psi(\Gamma u, v) = \Psi(x_f, v).$$

If there were $u \in \Omega_0([a, \infty))$ such that

$$(I + \Gamma)u = x_f, \quad (10)$$

then the equality $\Psi(u, \varphi) + \Psi(\Gamma u, \varphi) = \Psi(x_f, \varphi)$ would be satisfied, for every $\varphi \in \mathcal{V}_{[a, \infty)}$. Therefore, u would be a solution to the variational problem (5). We will show, using Fredholm's alternative theorem, that under certain conditions there is indeed a solution to Equation (10).

Theorem 6. Suppose that Y is a compact Hausdorff topological space. A subset \mathcal{H} of $C(Y, \mathbb{R})$ is relatively compact in the topology induced by the uniform norm if, and only if:

- (i) $\sup_{v \in \mathcal{H}} |v(y)| < \infty$, for all $y \in Y$.
- (ii) For every $y^* \in Y$ and $\epsilon > 0$, there exists a neighborhood U_{y^*} of y^* such that

$$|v(y) - v(y^*)| < \epsilon,$$

for all $y \in U_{y^*}$ and $v \in \mathcal{H}$.

Theorem 7. The operator $\Gamma : (\Omega_0([a, \infty)), \|\cdot\|_W) \rightarrow (\Omega_0([a, \infty)), \|\cdot\|_W)$ is compact.

Proof. Let $E \subseteq \Omega_0([a, \infty))$ be such that $\|u\|_W \leq K_1$ for all $u \in E$ and some $K_1 > 0$. Consider the set

$$\mathcal{H} = \{h_u - \alpha_u \frac{1}{\rho} : u \in E\}.$$

We use Theorem 6 in order to prove that \mathcal{H} is relatively compact in $(C([0, \infty], \mathbb{R}), \|\cdot\|_\infty)$. Note

$$|\alpha_u| \leq \|q\|_1 K_1 \beta \text{ for all } u \in E,$$

$$\text{where } \beta = \frac{\int_0^\infty \frac{1}{|\rho|}}{\left| \int_0^\infty \frac{1}{\rho} \right|}.$$

(i) Let $s \in [a, \infty]$ and $u \in E$. Then

$$\left| h_u(s) - \alpha_u \frac{1}{\rho(s)} \right| \leq \frac{1}{|\rho(s)|} \|q\|_1 K_1 (1 + \beta).$$

(ii) Let $s \in [a, \infty]$ and $\epsilon > 0$. We suppose that $s = \infty$. Since $\frac{1}{\rho}$ is continuous at $s = \infty$ and $\frac{1}{\rho} \in L^1([a, \infty))$, it follows that similarly to ([15], Lemma 4.1) there is a number $\gamma > 0$ such that for every $t > \gamma$,

$$\left| \frac{1}{\rho(t)} \right| < \frac{\epsilon}{\|q\|_1 K_1 (1 + \beta)}.$$

Let $t \in (\gamma, \infty]$ and $u \in \Omega$. Then,

$$\left| h_u(t) - \alpha_u \frac{1}{\rho(t)} \right| \leq \left| \frac{1}{\rho(t)} \right| \left(\int_a^t |qu| + |\alpha_u| \right) \leq \left| \frac{1}{\rho(t)} \right| \|q\|_1 K_1 (1 + \beta) < \epsilon.$$

Suppose now that $s < \infty$. Since $Q := \int_a^{(\cdot)} q$ and $\frac{1}{\rho}$ are continuous at s , it follows that there is a number $\delta > 0$ which satisfies that if $t \in B_\delta(s)$, then

$$|Q(t) - Q(s)| < \frac{\epsilon |\rho(s)|}{2K_1} \quad \text{and} \quad \left| \frac{1}{\rho(t)} - \frac{1}{\rho(s)} \right| < \frac{\epsilon}{2\|q\|_1 K_1 (1 + \beta)}.$$

Let $t \in B_\delta(s)$ and $u \in E$. We suppose without loss of generality that $s < t$. Then,

$$\begin{aligned} & \left| h_u(t) - \alpha_u \frac{1}{\rho(t)} - \left(h_u(s) - \alpha_u \frac{1}{\rho(s)} \right) \right| \leq \left| \frac{1}{\rho(t)} - \frac{1}{\rho(s)} \right| \left(\int_a^t |qu| + |\alpha_u| \right) \\ & + \frac{1}{|\rho(s)|} \int_s^t |qu| \leq \left| \frac{1}{\rho(t)} - \frac{1}{\rho(s)} \right| \|q\|_1 K_1 (1 + \beta) + \frac{1}{|\rho(s)|} K_1 |Q(t) - Q(s)| < \epsilon. \end{aligned}$$

From Theorem 6, we have that $\overline{\mathcal{H}}$ is a compact set in $(C([0, \infty], \mathbb{R}), \|\cdot\|_\infty)$. Therefore, we have proved that $\overline{\mathcal{H}}$ is compact when E is a bounded subset of $\Omega_0([a, \infty))$. Take a bounded sequence (u_n) in $\Omega_0([a, \infty))$. Then, $\{h_{u_n} - \alpha_{u_n} \frac{1}{\rho} : n \in \mathbb{N}\}$ is a compact set in $(C([0, \infty], \mathbb{R}), \|\cdot\|_\infty)$; consequently, there is a subsequence (u_{n_k}) of (u_n) and $w \in C([a, \infty], \mathbb{R})$ such that $h_{u_{n_k}} - \alpha_{u_{n_k}} \frac{1}{\rho} \rightarrow w$ uniformly. Since $\left| h_{u_{n_k}} - \alpha_{u_{n_k}} \frac{1}{\rho} \right| \leq \frac{1}{|\rho|} \|q\|_1 K_1 (1 + \beta)$ and $\frac{1}{\rho} \in L^1([a, \infty))$ it follows by the Dominated Convergence Theorem that $w \in L^1([a, \infty))$ and

$$\int_a^\infty \left| h_{u_{n_k}} - \alpha_{u_{n_k}} \frac{1}{\rho} - w \right| \rightarrow 0.$$

We set $\hat{w}(s) = \int_a^s w$. Then, $\hat{w} \in W_{KH}$, $\hat{w}(a) = 0$ and

$$\hat{w}(\infty) = \int_a^\infty w = \lim_{k \rightarrow \infty} \int_a^\infty \left(h_{u_{n_k}} - \alpha_{u_{n_k}} \frac{1}{\rho} \right) = \lim_{k \rightarrow \infty} z_{u_{n_k}}(\infty) = 0.$$

Therefore, $\hat{w} \in \Omega_0([a, \infty))$. Finally,

$$\begin{aligned} \|z_{u_{n_k}} - \hat{w}\|_W &= \|z_{u_{n_k}} - \hat{w}\|_\infty + \|\dot{z}_{u_{n_k}} - \dot{\hat{w}}\|_\infty + \|\dot{z}_{u_{n_k}} - \dot{\hat{w}}\|_A \\ &\leq \|h_{u_{n_k}} - \alpha_{u_{n_k}} \frac{1}{\rho} - w\|_\infty + 2 \int_a^\infty \left| h_{u_{n_k}} - \alpha_{u_{n_k}} \frac{1}{\rho} - w \right| \rightarrow 0. \end{aligned}$$

Thus, Γ is a compact operator. \square

Theorem 8 (Fredholm's alternative theorem). *Let X be a normed space and consider a compact linear operator $T : X \rightarrow X$. Then, the transformation $T + I$ is injective if and only if $T + I$ is surjective. Therefore, $(T + I)^{-1} : X \rightarrow X$ is bounded, when $T + I$ is injective.*

Proposition 1. *If the homogeneous problem*

$$\begin{cases} -[\rho \dot{u}]' + qu = 0 & \text{a.e. on } (a, \infty); \\ u(a) = u(a+) = 0, u(\infty) = u(\infty-) = 0 \end{cases} \quad (11)$$

has only the trivial solution, then the operator $\Gamma + I$ is injective.

Proof. Take u in the kernel of $\Gamma + I$. Then, $z_u = -u$ a.e. on $[a, \infty)$. Since z_u is ACG* on $[a, \infty]$, it follows by Remark 1 that $-\dot{u} = z'_u = h_u - \alpha_u \frac{1}{\rho}$ a.e. on $[a, \infty)$, thus

$$\rho \dot{u} = -\rho h_u + \alpha_u = \int_a^{(\cdot)} qu + \alpha_u$$

a.e. on $[a, \infty)$. Since $\int_a^{(\cdot)} qu + \alpha_u$ is ACG* on $[a, \infty]$, it follows again by Remark 1 that $\rho \dot{u} \in W_{KH}([a, \infty))$ and

$$[\rho \dot{u}]' = \left[\int_a^{(\cdot)} qu + \alpha_u \right]' = qu$$

a.e. on $[a, \infty)$. Therefore, u is a solution of the homogeneous problem. Consequently, $u = 0$ a.e. on $[a, \infty)$. \square

Proposition 2. *If ρ and q are positive, then the homogeneous problem*

$$\begin{cases} -[\rho \dot{u}]' + qu = 0 & \text{a.e. on } (a, \infty); \\ u(a) = u(a+) = 0, u(\infty) = u(\infty-) = 0 \end{cases}$$

only has the trivial solution.

Proof. Let $u \in W_{KH}^2([a, \infty))$ be a solution of the homogeneous problem. Then

$$[\rho \dot{u}]' = qu, \quad \text{a.e. on } (a, \infty). \quad (12)$$

By Theorem 4, there exists $\sigma \in C([a, \infty))$ such that $\sigma = \rho \dot{u}$ a.e. on (a, ∞) and

$$\sigma(s) = \int_a^s [\rho \dot{u}]' + \sigma(a)$$

for all $s \in [a, \infty]$. From Remark 1, $\sigma \in W_{KH}([a, \infty))$ and $\dot{\sigma} = [\rho \dot{u}]$. Then by (12), $\dot{\sigma} = qu$ a.e. on (a, ∞) . Now, observe that $u(a+) = u(a) = u(\infty-) = u(\infty) = 0$, $\sigma(a+) = \sigma(a)$, $\sigma(\infty-) = \sigma(\infty)$ and $\dot{u}\sigma \in KH([a, \infty))$. Thus by Theorem 5, we have that

$$\begin{aligned} \int_a^\infty \dot{u}[\rho \dot{u}] &= \int_a^\infty \dot{u}\sigma = u\sigma \Big|_a^\infty - \int_a^\infty u\dot{\sigma} \\ &= - \int_a^\infty u\dot{\sigma} = - \int_a^\infty u[qu], \end{aligned}$$

which implies that

$$\int_a^\infty qu^2 + \int_a^\infty \rho(\dot{u})^2 = 0.$$

Since $\rho > 0$ and $q > 0$, we have that $u = 0$ a.e. on $[a, \infty)$. \square

Theorem 9. Let $f \in KH([a, \infty))$, $q \in L^1([a, \infty))$ and $\rho \in L^1([a, \infty))$ be such that $\rho \in BV(J)$ for all compact interval $J \subseteq [a, \infty)$, and $\frac{1}{\rho}$ is ACG* on $[a, \infty]$. Suppose that either of the two following conditions hold:

- (i) The functions ρ, q are positive;
- (ii) The function zero is the unique solution of the homogeneous problem (11).

Then the following problem

$$\begin{cases} -[\rho \dot{u}] + qu = f & \text{a.e. on } (a, \infty); \\ u(a) = u(a+) = 0, u(\infty) = u(\infty-) = 0 \end{cases}$$

has a unique solution u in the space $W_{KH}^2([a, \infty))$ and the solution u depends continuously on the data f .

Proof. By Theorem 7, the operator Γ is compact, and by Proposition 1, $\Gamma + I$ is injective. Thus, by Fredholm's alternative theorem, the transformation $\Gamma + I$ is surjective. Then, there exists $u \in \Omega_0([a, \infty))$ such that $(\Gamma + I)u = x_f$. Hence, $\Psi(u + \Gamma u - x_f, \varphi) = 0$ for all $\varphi \in \mathcal{V}_{[a, \infty)}$, consequently, u is a solution to the variational problem (5), and so u is a solution to the boundary problem (4). It is clear that if w is another solution to the boundary problem (4), then $u - w$ is a solution the homogeneous problem, which implies that $u = w$ a.e. on $[a, \infty)$.

Now, let (f_n) be a sequence in $KH([a, \infty))$ such that $\|f_n - f\|_A \rightarrow 0$, and for each $n \in \mathbb{N}$, let u_n be a solution to the boundary problem

$$\begin{cases} -[\rho \dot{u}] + qu = f_n & \text{a.e. on } (a, \infty); \\ u(a) = u(a+) = 0, u(\infty) = u(\infty-) = 0. \end{cases}$$

Observe that

$$\begin{aligned} \left| \frac{1}{\rho(s)}(F_n(s) - \beta_{f_n}) - \frac{1}{\rho(s)}(F(s) - \beta_f) \right| &\leq \\ &\leq \frac{1}{|\rho(s)|} (|F_n(s) - F(s)| + |\beta_{f_n} - \beta_f|) \\ &\leq \frac{1}{|\rho(s)|} \left(\left| \int_a^s (f_n - f) \right| + \frac{1}{\left| \int_a^\infty \frac{1}{\rho} \right|} \int_a^\infty \frac{1}{|\rho|} |F_n - F| \right) \\ &\leq \frac{1}{|\rho(s)|} \|f_n - f\|_A \left(1 + \frac{\int_a^\infty \frac{1}{|\rho|}}{\left| \int_a^\infty \frac{1}{\rho} \right|} \right). \end{aligned}$$

Therefore,

$$\begin{aligned}\|\dot{x}_{f_n} - \dot{x}_f\|_A &= \left\| \frac{1}{\rho}(F_n - \beta_{f_n}) - \frac{1}{\rho}(F - \beta_f) \right\|_1 \\ &= \int_a^\infty \left| \frac{1}{\rho}(F_n(s) - \beta_{f_n}) - \frac{1}{\rho}(F(s) - \beta_f) \right| ds \\ &\leq \|f_n - f\|_A \left(1 + \frac{\int_0^\infty \frac{1}{|\rho|}}{\int_0^1 \frac{1}{\rho}} \right) \int_0^\infty \frac{1}{|\rho|},\end{aligned}$$

$$\begin{aligned}\|\dot{x}_{f_n} - \dot{x}_f\|_\infty &= \left\| \frac{1}{\rho}(F_n - \beta_{f_n}) - \frac{1}{\rho}(F - \beta_f) \right\|_\infty \\ &= \max_{s \in [a, \infty]} \left| \frac{1}{\rho(s)}(F_n(s) - \beta_{f_n}) - \frac{1}{\rho(s)}(F(s) - \beta_f) \right| \\ &\leq \|f_n - f\|_A \left(1 + \frac{\int_a^\infty \frac{1}{|\rho|}}{\int_a^\infty \frac{1}{\rho}} \right) \max_{s \in [a, \infty]} \frac{1}{|\rho(s)|},\end{aligned}$$

and

$$\begin{aligned}\|x_{f_n} - x_f\|_\infty &= \max_{s \in [0, \infty]} |x_{f_n}(s) - x_f(s)| \\ &\leq \max_{s \in [0, \infty]} \int_0^s \left| \frac{1}{\rho}(F_n - \beta_{f_n}) - \frac{1}{\rho}(F - \beta_f) \right| \\ &\leq \int_0^\infty \left| \frac{1}{\rho}(F_n - \beta_{f_n}) - \frac{1}{\rho}(F - \beta_f) \right| \\ &\leq \|f_n - f\|_A \left(1 + \frac{\int_0^\infty \frac{1}{|\rho|}}{\int_0^1 \frac{1}{\rho}} \right) \int_0^\infty \frac{1}{|\rho|}.\end{aligned}$$

On the other hand, by Fredholm's alternative theorem, $(\Gamma + I)^{-1}$ is a bounded operator. Then

$$\begin{aligned}\|u_n - u\|_W &= \|(\Gamma + I)^{-1}x_{f_n} - (\Gamma + I)^{-1}x_f\|_W \\ &= \|(\Gamma + I)^{-1}(x_{f_n} - x_f)\|_W \\ &\leq \|(\Gamma + I)^{-1}\| \|x_{f_n} - x_f\|_W \\ &= \|(\Gamma + I)^{-1}\| \left(\|x_{f_n} - x_f\|_\infty + \|\dot{x}_{f_n} - \dot{x}_f\|_\infty + \|\dot{x}_{f_n} - \dot{x}_f\|_A \right) \\ &\leq \|(\Gamma + I)^{-1}\| \left[2 \int_0^\infty \frac{1}{|\rho|} + \max_{s \in [a, \infty]} \frac{1}{|\rho(s)|} \right] \left(1 + \frac{\int_0^\infty \frac{1}{|\rho|}}{\int_0^1 \frac{1}{\rho}} \right) \|f_n - f\|_A. \quad (13)\end{aligned}$$

Consequently, $\|u_n - u\|_W \rightarrow 0$ when $\|f_n - f\|_A \rightarrow 0$. \square

Example 1. In order to derive the steady-state heat conduction model, consider a non-uniform bar of infinite length with cross-sectional area Λ . Let $u(t)$ be the temperature, $\phi(t)$ the heat flux and $f(t)$ the source term that models the generation or loss of heat at each point of the cross-section of the bar at position t , where $0 < a \leq t < \infty$. If $[t, t + dt]$ is a small and arbitrary portion of the bar, then by the law of conservation of energy, we have,

$$\Lambda\phi(t) - \Lambda\phi(t + dt) + f(t)\Lambda dt = 0.$$

Dividing by Δt and taking the limit as $dt \rightarrow 0$, we have

$$\phi'(t) = f(t).$$

If $\rho(t)$ is the thermal conductivity of the bar, then by Fourier's heat conduction law, $\phi(t) = -\rho(t)u'(t)$, we obtain the steady-state heat conduction equation

$$-[\rho(t)u'(t)]' = f(t), \quad a \leq t < \infty.$$

As a particular example, let us consider a non-uniform bar such that its property of conducting heat is greater as the position increases, then the thermal conductivity can be modeled by the function

$$\rho(t) = t^2 + 1, \quad a \leq t < \infty.$$

Furthermore, if we assume that heat is continuously lost in certain portions, and in others it is gained due to some source with null effect at distant locations, then one way to represent this behavior is by the function

$$f(t) = \frac{\sin(\sqrt{t})}{t}.$$

Setting the boundary conditions $u(a) = u(a+) = 0$ y $u(\infty) = u(\infty-) = 0$, we obtain the problem with boundary values for the temperature $u(t)$:

$$\begin{aligned} -[\rho u']' &= f \quad \text{a.e. on } [a, \infty); \\ u(a) &= u(a+) = 0, \quad u(\infty) = u(\infty-) = 0. \end{aligned} \quad (14)$$

As $f \in KH([a, \infty)) \setminus L^1([a, \infty))$, ρ is a function of bounded variation on every compact interval $J \subseteq [a, \infty)$, and the function $\frac{1}{\rho} \in ACG_*([a, \infty)) \cap L^1([a, \infty))$, and it follows by Theorem 9 that the problem (14) has a unique solution.

5. Sturm–Liouville Differential Equations for Bounded Intervals

Let us begin this section by showing that when I is a compact interval on \mathbb{R} we have

$$H^1(I) \subseteq W_{KH}(I) \quad (15)$$

where

$$H^1(I) = \left\{ u \in L^2(I) : \exists g \in L^2(I) \text{ such that } \int_I u \varphi' = \int_I g \varphi, \quad \forall \varphi \in C_c^1(I^\circ) \right\}$$

and I° is the interior of I . Let a be the left end of the interval I and b be the right end, and take $u \in H^1(I)$, then there exists $g \in L^2(I)$ such that $\int_I u \varphi' = \int_I g \varphi$ for all $\varphi \in C_c^1(a, b)$. Take an arbitrary $\varphi \in \mathcal{V}_I$, then $\varphi' \in L^1(I)$. Since $C_c^\infty(a, b)$ is dense in $L^1(I)$, there exists a sequence (ξ_n) in $C_c^\infty(a, b)$ such that $\|\xi_n - \varphi'\|_1 \rightarrow 0$. Consider $\varphi_0 \in C_c^\infty(a, b)$ such that $\int_I \varphi_0 = 1$ and define the functions ϕ_n, ψ_n by

$$\phi_n = \xi_n - \alpha_n \varphi_0,$$

where $\alpha_n = \int_I \xi_n$, and

$$\psi_n(s) = \int_a^s \phi_n.$$

Then (ψ_n) is a sequence in $C_c^\infty(a, b)$. We prove that

1. $\int_I u \psi_n' \rightarrow \int_I u \varphi'$
2. $\int_I g \psi_n \rightarrow \int_I g \varphi$.

For the first convergence, observe that

$$\begin{aligned} \left| \int_I u(\psi'_n - \varphi') \right| &= \left| \int_I u(\xi_n - \alpha_n \varphi_0 - \varphi') \right| \\ &\leq \int_I |u| |\xi_n - \varphi' - \alpha_n \varphi_0| \\ &\leq \|u\|_\infty \left(\|\xi_n - \varphi'\|_1 + |\alpha_n| \int_I |\varphi_0| \right). \end{aligned} \quad (16)$$

Now, since

$$\left| \int_I \xi_n - \int_I \varphi' \right| \leq \int_I |\xi_n - \varphi'| = \|\xi_n - \varphi'\|_1 \rightarrow 0,$$

it follows that

$$\alpha_n = \int_I \xi_n \rightarrow \int_I \varphi' = \varphi(b) - \varphi(a) = 0.$$

Thus, (16) tends to zero.

The second convergence is deduced by the following. From Theorem 1,

$$\begin{aligned} \left| \int_I g(\psi_n - \varphi) \right| &\leq \inf_{t \in I} |\psi_n(t) - \varphi(t)| \left| \int_I g \right| + \|g\|_1 V_I(\psi_n - \varphi) \\ &\leq |\psi_n(a) - \varphi(a)| \left| \int_I g \right| + \|g\|_1 V_I(\psi_n - \varphi) \\ &= \|g\|_1 V_I(\psi_n - \varphi) \\ &= \|g\|_1 \int_I |\psi'_n - \varphi'| \\ &= \|g\|_1 \int_I |\xi_n - \alpha_n \varphi_0 - \varphi'| \\ &\leq \|g\|_1 \left(\int_I |\xi_n - \varphi'| + |\alpha_n| \int_I \varphi_0 \right) \rightarrow 0. \end{aligned}$$

Finally, since each $\psi_n \in C_c^\infty(a, b)$, it follows that

$$\int_I u \psi'_n = - \int_I g \psi_n,$$

therefore

$$- \int_I g \psi_n = \int_I u \psi'_n \rightarrow \int_I u \varphi'.$$

However, it is also true that

$$- \int_I g \psi_n \rightarrow - \int_I g \varphi.$$

Consequently, due to the uniqueness of limits, it follows that

$$\int_I u \varphi' = - \int_I g \varphi.$$

Therefore, $u \in W_{KH}(I)$.

We set

$$H_0^1(I) = \left\{ u \in H^1(I) : u = 0 \text{ on } \partial I \right\}.$$

From (15), we obtain the following sequence of inclusions:

$$\Omega_0(I) \subseteq H_0^1(I) \subseteq H^1(I) \subseteq W_{KH}(I) \subseteq L^\infty(I) \subseteq L^2(I) \subseteq L^1(I) \subseteq KH(I). \quad (17)$$

As in Section 4, we will consider $\Omega_0(I)$ with the seminorm $\|z\|_W = \|z\|_\infty + \|\dot{z}\|_\infty + \|\dot{z}\|_A$. The form of this semi-norm is required in the following section. Based on ([8], Theorem 4.3) and the results of the previous section, we state the following theorems:

Theorem 10. Let $f \in HK([a, b])$, $q \in L^1([a, b])$ and $\rho \in BV([a, b])$ be such that $\frac{1}{\rho}$ is ACG_* on $[a, b]$. Then, the following assertions are equivalent:

1. The following boundary problem

$$\begin{cases} -[\rho \dot{u}]' + qu = f & \text{a.e. on } (a, b); \\ u(a) = u(a+) = 0, u(b) = u(b-) = 0 \end{cases}$$

has a unique solution in $W_{HK}^2([a, b])$.

2. The following variational problem

$$\int_a^b \rho \dot{u} v' + \int_a^b quv = \int_a^b f v, \quad \text{for all } v \in \mathcal{V}_{[a,b]}$$

has a unique solution in $\Omega_0([a, b])$.

Theorem 11. Let $f \in HK([a, b])$, $q \in L^1([a, b])$ and $\rho \in BV([a, b])$ be such that $\frac{1}{\rho}$ is ACG_* on $[a, b]$. Suppose that either of the two following conditions hold:

- (i) The functions ρ, q are positive;
- (ii) The function zero is the unique solution of the homogeneous problem:

$$\begin{cases} -[\rho \dot{u}]' + qu = 0 & \text{a.e. on } (a, b); \\ u(a) = u(a+) = 0, u(b) = u(b-) = 0. \end{cases}$$

Then the following problem

$$\begin{cases} -[\rho \dot{u}]' + qu = f & \text{a.e. on } (a, b); \\ u(a) = u(a+) = 0, u(b) = u(b-) = 0 \end{cases}$$

has a unique solution u in the space $W_{HK}^2([a, b])$, and there is $K > 0$ for which

$$\|u - \tilde{u}\|_W \leq K \|f - \tilde{f}\|_A,$$

where $\tilde{f} \in KH([a, b])$ is arbitrary and \tilde{u} is the solution of the problem:

$$\begin{cases} -[\rho \dot{u}]' + qu = \tilde{f} & \text{a.e. on } (a, b); \\ u(a) = u(a+) = 0, u(b) = u(b-) = 0. \end{cases}$$

6. Finite Element Method

In this section, we give a finite element method scheme for KH -integrable functions. We consider ρ and f as in Section 5. Let $u \in W_{KH}^2([a, b])$ be the unique solution to the boundary problem

$$\begin{cases} -[\rho \dot{u}]' = f & \text{a.e. on } (a, b); \\ u(a) = u(a+) = 0, u(b) = u(b-) = 0. \end{cases}$$

Then by Theorem 10, $u \in \Omega_0([a, b])$ and

$$\int_a^b \rho \dot{u} \varphi' = \int_a^b f \varphi \quad \text{for all } \varphi \in \mathcal{V}_{[a,b]}. \quad (18)$$

Let $N \in \mathbb{N}$ and $a = s_0 < s_1 < \dots < s_N < x_{N+1} = b$ be a partition of $[a, b]$. We set $h = \max\{s_i - s_{i-1} : i = 1, 2, \dots, N+1\}$ and we consider the finite element space \mathcal{V}_h given by

$$\mathcal{V}_h = \{v \in C([a, b]) : v \text{ is linear on each subinterval } [s_{i-1}, s_i] \text{ for all } i = 1, \dots, N+1, \text{ and } v(a) = v(b) = 0\}.$$

Let $r_h f$ be an interpolate of f on $[a, b]$, that is, $r_h f(s_i) = f(s_i)$ for all $i = 0, \dots, N+1$. Then from Theorem 11, there exists $\tilde{u} \in \Omega_0([a, b])$ such that

$$\int_a^b \rho \tilde{u} \varphi' = \int_a^b \tilde{f} \varphi, \text{ for all } \varphi \in \mathcal{V}_{[a,b]}. \quad (19)$$

Now, we will find $u_h, \tilde{u}_h \in V_h$ such that they satisfy

$$\int_a^b \rho \dot{u}_h v' = \int_a^b f v, \text{ for all } v \in \mathcal{V}_h \quad (20)$$

$$\int_a^b \rho \dot{\tilde{u}}_h v' = \int_a^b \tilde{f} v, \text{ for all } v \in \mathcal{V}_h. \quad (21)$$

A basis for the space \mathcal{V}_h is given by the functions $\varphi_i, i = 1, \dots, N$, defined as

$$\varphi_i(s_j) = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i \neq j. \end{cases}$$

Then, for every $v \in \mathcal{V}_h$,

$$v(s) = \sum_{i=1}^N v(s_i) \varphi_i(s), \quad s \in [a, b]. \quad (22)$$

Observe that if u_h, \tilde{u}_h are defined by

$$u_h := \sum_{i=1}^N u_i \varphi_i \quad \text{and} \quad \tilde{u}_h := \sum_{i=1}^N \tilde{u}_i \varphi_i \quad (23)$$

then the Equations (20) and (21) are equivalent to

$$\sum_{i=1}^N \sum_{j=1}^N v(s_i) u_i \int_a^b \rho \varphi_i' \varphi_j' = \sum_{i=1}^N v(s_i) \int_a^b f \varphi_i. \quad (24)$$

$$\sum_{i=1}^N \sum_{j=1}^N v(s_i) \tilde{u}_i \int_a^b \rho \varphi_i' \varphi_j' = \sum_{i=1}^N v(s_i) \int_a^b \tilde{f} \varphi_i. \quad (25)$$

Let $\alpha_{ij} = \int_a^b \rho \varphi_i' \varphi_j'$ and define

$$M = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1N} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{N1} & \alpha_{N2} & \cdots & \alpha_{NN} \end{pmatrix}. \quad (26)$$

Then M is symmetric and tridiagonal. Additionally, M is positive-definitive, because if $\eta = (\eta_1, \eta_2, \dots, \eta_N) \in \mathbb{R}^N \setminus \{0\}$ then

$$\begin{aligned}\eta^T M \eta &= \sum_{i=1}^N \sum_{j=1}^N \eta_i \alpha_{ij} \eta_j = \int_a^b \rho \left(\sum_{i=1}^N \eta_i \varphi'_i \right) \left(\sum_{j=1}^N \eta_j \varphi'_j \right) \\ &= \int_a^b \rho \left(\sum_{i=1}^N \eta_i \varphi'_i \right)^2 \geq 0.\end{aligned}$$

If $\int_a^b \rho \left(\sum_{i=1}^N \eta_i \varphi'_i \right)^2 = 0$ then $\sum_{i=1}^N \eta_i \varphi'_i = 0$, therefore $\eta_1 = \eta_2 = \dots = \eta_N = 0$, which is a contradiction. Thus, $\eta^T M \eta > 0$. Consequently, M is invertible. Thus, there exist (u_1, u_2, \dots, u_N) and $(\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_N)$ unique solutions of the systems $Mz = y$ and $Mz = \tilde{y}$, where

$$y = \begin{pmatrix} \int_a^b f \varphi_1 \\ \int_a^b f \varphi_2 \\ \vdots \\ \int_a^b f \varphi_N \end{pmatrix} \quad \text{and} \quad \tilde{y} = \begin{pmatrix} \int_a^b \tilde{f} \varphi_1 \\ \int_a^b \tilde{f} \varphi_2 \\ \vdots \\ \int_a^b \tilde{f} \varphi_N \end{pmatrix}.$$

Consequently, $u_h = \sum_{i=1}^N u_i \phi_i$ and $\tilde{u}_h = \sum_{i=1}^N \tilde{u}_i \phi_i$ satisfy (24) and (25) for all $v = \sum_{i=1}^N v(s_i) \phi_i$.

We will now estimate the error committed. First, observe that for every $z \in H^1([a, b])$ and $w \in \mathcal{V}_h$, if

$$\int_a^b \rho(z - w)v' = 0 \text{ for all } v \in \mathcal{V}_h,$$

then

$$\int_a^b \rho(z)^2 = \int_a^b \rho(z - w)^2 + \int_a^b \rho(w)^2, \quad (27)$$

$$\int_a^b \rho(z - v)^2 = \int_a^b \rho(z - w)^2 + \int_a^b \rho(w - v)^2 \quad \text{for all } v \in \mathcal{V}_h, \quad (28)$$

and

$$\int_a^b \rho(z - w)^2 = \int_a^b \rho(z - w)(z - v) \quad \text{for all } v \in \mathcal{V}_h. \quad (29)$$

Indeed, equality (27) is deduced by the following

$$\begin{aligned}\int_a^b \rho(z - w)^2 &= \int_a^b \rho(z - w)z - \int_a^b \rho(z - w)w \\ &= \int_a^b \rho(z - w)z = \int_a^b \rho z^2 - \int_a^b \rho z w \\ &= \int_a^b \rho z^2 - \int_a^b \rho(z - w)w - \int_a^b \rho w^2 \\ &= \int_a^b \rho z^2 - \int_a^b \rho w^2.\end{aligned}$$

Equality (28) is obtained from

$$\begin{aligned}\int_a^b \rho(z - v)^2 &= \int_a^b \rho(z - w + w - v)^2 \\ &= \int_a^b \rho(z - w)^2 + 2 \int_a^b \rho(z - w)(w - v) + \int_a^b \rho(w - v)^2 \\ &= \int_a^b \rho(z - w)^2 + \int_a^b \rho(w - v)^2,\end{aligned}$$

and equality (29) is deduced by

$$\begin{aligned}\int_a^b \rho(\dot{z} - \dot{w})(\dot{z} - \dot{v}) &= \int_a^b \rho(\dot{z} - \dot{w})\dot{z} - \int_a^b \rho(\dot{z} - \dot{w})\dot{v} \\ &= \int_a^b \rho(\dot{z} - \dot{w})\dot{z} \\ &= \int_a^b \rho(\dot{z} - \dot{w})\dot{z} - \int_a^b \rho(\dot{z} - \dot{w})\dot{w} \\ &= \int_a^b \rho(\dot{z} - \dot{w})(\dot{z} - \dot{w}) = \int_a^b \rho(\dot{z} - \dot{w})^2.\end{aligned}$$

From (17) we have that $u, \tilde{u} \in H^1([a, b])$. Then by (18) and (20), it follows that

$$\int_a^b \rho(\dot{u} - \dot{u}_h)v' = 0 \text{ for all } v \in \mathcal{V}_h.$$

Therefore by (28),

$$\int_a^b \rho(\dot{u} - \dot{u}_h)^2 = \int_a^b \rho(\dot{u} - \dot{u}_h)^2 + \int_a^b \rho(\dot{u}_h - \dot{u}_h)^2. \quad (30)$$

Observe that u_h is the optimal approximation for u , that is,

$$\int_a^b (\dot{u} - \dot{u}_h)^2 \leq \frac{L^2}{a_0^2} \int_a^b (\dot{u} - \dot{v})^2 \text{ for all } v \in \mathcal{V}_h, \quad (31)$$

where $a_0 \leq \rho(s) \leq L$ for all $s \in [a, b]$. Indeed, by (29),

$$\int_a^b \rho(\dot{u} - \dot{u}_h)^2 = \int_a^b \rho(\dot{u} - \dot{u}_h)(\dot{u} - \dot{v}) \text{ for all } v \in \mathcal{V}_h.$$

This implies that

$$\int_a^b (\dot{u} - \dot{u}_h)^2 \leq \frac{L}{a_0} \left(\int_a^b (\dot{u} - \dot{u}_h)^2 \right)^{\frac{1}{2}} \left(\int_a^b (\dot{u} - \dot{v})^2 \right)^{\frac{1}{2}} \text{ for all } v \in \mathcal{V}_h,$$

and so the inequality (31) is satisfied. Now, take an interpolate $r_h u \in \mathcal{V}_h$ of u on $[a, s_{N+1}]$, that is, $r_h u(s_i) = u(s_i)$ for all $i = 0, 1, \dots, N+1$. We take $z_i \in (s_i, s_{i+1})$ such that $\dot{u}(z_i) = (r_h u)'(z_i)$, for all $i = 0, \dots, N$. Then, for every $i = 0, \dots, N$, there exists c_i between s and z_i such that

$$\begin{aligned}|\dot{u}(s) - (r_h u)'(s)| &= |(\dot{u}(s) - (r_h u)'(s)) - (\dot{u}(z_i) - (r_h u)'(z_i))| \\ &= |\ddot{u}(c_i) - (r_h u)''(c_i)|(s - z_i) \\ &\leq \max_{t \in [a, b]} |\ddot{u}(t)| h,\end{aligned}$$

for all $s \in (s_i, s_{i+1})$. Consequently,

$$\int_a^b (\dot{u} - (r_h u)')^2 \leq (b - a) \left(\max_{t \in [a, b]} |\ddot{u}(t)| \right)^2 h^2.$$

Thus, by (31),

$$\begin{aligned}\int_a^b (\dot{u} - \dot{u}_h)^2 &\leq \frac{L^2}{a_0^2} \int_a^b (\dot{u} - (r_h u)')^2 \\ &\leq \frac{L^2}{a_0^2} (b-a) \left(\max_{t \in [a,b]} |\ddot{u}(t)| \right)^2 h^2.\end{aligned}\quad (32)$$

On the other hand, from (18)–(21) we have that

$$\int_a^b \rho[(\dot{u} - \dot{\tilde{u}}) - (\dot{u}_h - \dot{\tilde{u}}_h)]v' = 0 \text{ for all } v \in \mathcal{V}_h.$$

Thus by (27),

$$\begin{aligned}\int_a^b \rho(\dot{u}_h - \dot{\tilde{u}}_h)^2 &= \int_a^b \rho(\dot{u} - \dot{\tilde{u}})^2 - \int_a^b \rho[(\dot{u} - \dot{\tilde{u}}) - (\dot{u}_h - \dot{\tilde{u}}_h)]^2 \\ &\leq \int_a^b \rho(\dot{u} - \dot{\tilde{u}})^2,\end{aligned}$$

but, note that $\int_a^b \rho(\dot{u} - \dot{\tilde{u}})^2 \leq L \|\dot{u} - \dot{\tilde{u}}\|_\infty \int_a^b |\dot{u} - \dot{\tilde{u}}| = L \|\dot{u} - \dot{\tilde{u}}\|_\infty \|\dot{u} - \dot{\tilde{u}}\|_A$. Hence

$$\int_a^b (\dot{u}_h - \dot{\tilde{u}}_h)^2 \leq \frac{L}{a_0} \|\dot{u} - \dot{\tilde{u}}\|_\infty \|\dot{u} - \dot{\tilde{u}}\|_A.$$

Now, from Theorem 11, there exists a constant K such that $\|\dot{u} - \dot{\tilde{u}}\|_\infty \|\dot{u} - \dot{\tilde{u}}\|_A \leq K^2 \|f - \tilde{f}\|_A^2$. Then,

$$\int_a^b (\dot{u}_h - \dot{\tilde{u}}_h)^2 \leq \frac{L}{a_0} K^2 \|f - \tilde{f}\|_A^2. \quad (33)$$

Consequently, by (30), (32), and (33) we have that

$$\begin{aligned}\int_a^b (\dot{u} - \dot{\tilde{u}}_h)^2 &\leq \frac{L}{a_0} \left(\int_a^b (\dot{u} - \dot{u}_h)^2 + \int_a^b (\dot{u}_h - \dot{\tilde{u}}_h)^2 \right) \\ &\leq \frac{L^3}{a_0^3} (b-a) \left(\max_{t \in [a,b]} |\ddot{u}(t)| \right)^2 h^2 + \frac{L^2}{a_0^2} K^2 \|f - \tilde{f}\|_A^2.\end{aligned}$$

Finally,

$$\begin{aligned}|u(s) - \tilde{u}_h(s)|^2 &\leq \left(\int_a^s |\dot{u} - \dot{\tilde{u}}_h| \right)^2 \\ &\leq \int_a^b (\dot{u} - \dot{\tilde{u}}_h)^2 \\ &\leq \frac{L^3}{a_0^3} (b-a) \left(\max_{t \in [a,b]} |\ddot{u}(t)| \right)^2 h^2 + \frac{L^2}{a_0^2} K^2 \|f - \tilde{f}\|_A^2.\end{aligned}$$

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