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Generalization of Quantum Ostrowski-Type Integral Inequalities

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Abstract: In this paper, we prove some new Ostrowski-type integral inequalities for q-differentiable bounded functions. It is also shown that the results presented in this paper are a generalization of know results in the literarure. Applications to special means are also discussed.

Keywords: Ostrowski inequality; q-integral; quantum calculus

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1. Introduction

Quantum calculus, or q-calculus, is a modern term for the study of calculus without limits. It has been studied since the early eighteenth century. Euler, a prominent mathematician, invented q-calculus, and F. H. Jackson [1] discovered the definite q-integral known as the q-Jackson integral in 1910. Orthogonal polynomials, combinatorics, number theory, quantum theory, simple hypergeometric functions, dynamics, and theory of relativity are the applications of quantum calculus in mathematics and physics; see [2–4] and refernces cited there. Kac and Cheung's book [5] discusses the fundamentals of quantum calculus as well as the basic theoretical terms.

Because of its enormous importance in a wide range of applied and pure sciences, in recent decades, the definition of convex and bounded functions has received much attention. Since the theory of inequalities and the concept of convex and bounded functions are closely related, various inequalities for convex, differentiable convex and differentiable bounded functions can be found in the literatur; see [6–22]. Inspired by this study, we prove some new quantum Ostrowski's inequalities to expand the relationship between differentiable bounded functions and quantum integral inequalities. We prove some new quantum Ostrowski's inequalities to expand the relationship between differentiable bounded functions and quantum integral inequalities, generalizing existing results in the literature [23].

2. Basics of q-Calculus

In this portion, we recall some formerly developed concepts. We also use the following notation in this paper (see [5]):

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \ldots + q^{n-1}, \quad q \in (0,1).$$

In [1], the q-Jackson integral of a function $\mathcal{F}$ from 0 to π_2 and $0 < q < 1$ is defined as follows:

$$\int_0^{\pi_2} \mathcal{F}(x) \ d_q x \ = (1-q)\pi_2 \sum_{n=0}^{\infty} q^n \mathcal{F}(\pi_2 q^n) \tag{1}$$

provided the sum converges absolutely.

Definition 1. *Reference [4]: the quantum q_{π_1}-derivative for a mapping $\mathcal{F} : [\pi_1, \pi_2] \to \mathbb{R}$ at $x \in [\pi_1, \pi_2]$ is defined as:*

$$_{\pi_1}D_q\mathcal{F}(x) = \frac{\mathcal{F}(x) - \mathcal{F}(qx + (1-q)\pi_1)}{(1-q)(x-\pi_1)}, \ x \neq \pi_1. \tag{2}$$

If $x = \pi_1$, we define $_{\pi_1}D_q\mathcal{F}(\pi_1) = \lim_{x\to\pi_1} {}_{\pi_1}D_q\mathcal{F}(x)$ if it exists and it is finite.

Definition 2. *Reference [13] The quantum q^{π_2}-derivative for a mapping $\mathcal{F} : [\pi_1, \pi_2] \to \mathbb{R}$ at $x \in [\pi_1, \pi_2]$ is defined as:*

$$^{\pi_2}D_q\mathcal{F}(x) = \frac{\mathcal{F}(qx + (1-q)\pi_2) - \mathcal{F}(x)}{(1-q)(\pi_2 - x)}, \ x \neq \pi_2.$$

If $x = \pi_2$, we define $^{\pi_2}D_q\mathcal{F}(\pi_2) = \lim_{x\to\pi_2} {}^{\pi_2}D_q\mathcal{F}(x)$ if it exists and is finite.

Definition 3. *Reference [4]: the quantum q_{π_1}-definite integral for a mapping $\mathcal{F} : [\pi_1, \pi_2] \to \mathbb{R}$ on $[\pi_1, \pi_2]$ is defined as:*

$$\begin{aligned}\int_{\pi_1}^{\pi_2} \mathcal{F}(x) \ {}_{\pi_1}d_qx &= (1-q)(\pi_2 - \pi_1)\sum_{n=0}^{\infty} q^n\mathcal{F}(q^n\pi_2 + (1-q^n)\pi_1) \\ &= (\pi_2 - \pi_1)\int_0^1 \mathcal{F}((1-t)\pi_1 + t\pi_2) \ d_qt.\end{aligned}$$

Definition 4. *Reference [13]: The quantum q^{π_2}-definite integral for a mapping $\mathcal{F} : [\pi_1, \pi_2] \to \mathbb{R}$ on $[\pi_1, \pi_2]$ is defined as:*

$$\begin{aligned}\int_{\pi_1}^{\pi_2} \mathcal{F}(x) \ {}^{\pi_2}d_qx &= (1-q)(\pi_2 - \pi_1)\sum_{n=0}^{\infty} q^n\mathcal{F}(q^n\pi_1 + (1-q^n)\pi_2) \\ &= (\pi_2 - \pi_1)\int_0^1 \mathcal{F}(t\pi_1 + (1-t)\pi_2) \ d_qt.\end{aligned}$$

Now, we present the classical Ostrowski inequality.

Theorem 1. *Let $\mathcal{F} : [\pi_1, \pi_2] \to \mathbb{R}$ be a continuous function that is differentiable on (π_1, π_2). If $|\mathcal{F}'(x)| \leq M$, then we have the following inequality for $x \in [\pi_1, \pi_2]$:*

$$\left|\mathcal{F}(x) - \frac{1}{\pi_2 - \pi_1}\int_{\pi_1}^{\pi_2} \mathcal{F}(t)dt\right| \leq \frac{M}{(\pi_2 - \pi_1)}\left[\frac{(x-\pi_1)^2 + (\pi_2 - x)^2}{2}\right]. \tag{3}$$

The quantum version of the inequality (3) given by Budak et al. can be stated as:

Theorem 2. *Reference [17]: Let $\mathcal{F} : [\pi_1, \pi_2] \subset \mathbb{R} \to \mathbb{R}$ be a function. If $\left|{}^{\pi_2}D_q\mathcal{F}(t)\right|$, $\left|{}_{\pi_1}D_q\mathcal{F}(t)\right| \leq M$ for all $t \in [\pi_1, \pi_2]$, then we have the following quantum Ostrowski-type inequality:*

$$\left|\mathcal{F}(x) - \frac{1}{\pi_2 - \pi_1}\left[\int_{\pi_1}^{x} \mathcal{F}(t)\ {}_{\pi_1}d_qt\ + \int_{x}^{\pi_2} \mathcal{F}(t)\ {}^{\pi_2}d_qt\ \right]\right| \tag{4}$$
$$\leq \frac{qM}{(\pi_2 - \pi_1)}\left[\frac{(x-\pi_1)^2 + (\pi_2 - x)^2}{1+q}\right]$$

for all $x \in [\pi_1, \pi_2]$ where $0 < q < 1$.

3. Quantum Ostrowski Type Inequalities

In this section, for the q-differentiable bounded functions, we prove some new Ostrowski-type inequalities. For this, we propose a new quantum integral identity that will be used as an aid in the development of new results.

Lemma 1. *Let $\mathcal{F} : [\pi_1, \pi_2] \to \mathbb{R}$ be a continuous and q-differentiable function on the given interval $[\pi_1, \pi_2]$. Then, the following equality holds for the quantum integrals:*

$$\begin{aligned}
&\int_{\pi_1}^{x}\left(t - \left(\pi_1 + h\frac{\pi_2 - \pi_1}{2}\right)\right) {}_{\pi_1}D_q\mathcal{F}(t)\ {}_{\pi_1}d_qt + \int_{x}^{\pi_2}\left(t - \left(\pi_2 - h\frac{\pi_2 - \pi_1}{2}\right)\right) {}^{\pi_2}D_q\mathcal{F}(t)\ {}^{\pi_2}d_qt \\
= \ &(\pi_2 - \pi_1)h\frac{\mathcal{F}(\pi_1) + \mathcal{F}(\pi_2)}{2} + (\pi_2 - \pi_1)(1-h)\mathcal{F}(x) \\
&-\left[\int_{\pi_1}^{x} \mathcal{F}(qt + (1-q)\pi_1)\ {}_{\pi_1}d_qt + \int_{x}^{\pi_2} \mathcal{F}(qt + (1-q)\pi_2)\ {}^{\pi_2}d_qt\right],
\end{aligned} \tag{5}$$

where $h \in [0,1]$ and $\pi_1 + h\frac{\pi_2-\pi_1}{2} \leq x \leq \pi_2 - h\frac{\pi_2-\pi_1}{2}$.

Proof. Using the fundamental concepts of q integration and derivative [24], we have

$$\begin{aligned}
&\int_{\pi_1}^{x}\left(t - \left(\pi_1 + h\frac{\pi_2 - \pi_1}{2}\right)\right) {}_{\pi_1}D_q\mathcal{F}(t)\ {}_{\pi_1}d_qt \\
= \ &\left(x - \left(\pi_1 + h\frac{\pi_2 - \pi_1}{2}\right)\right)\mathcal{F}(x) + h\frac{\pi_2 - \pi_1}{2}\mathcal{F}(\pi_1) - \int_{\pi_1}^{x} \mathcal{F}(qt + (1-q)\pi_1)\ {}_{\pi_1}d_qt
\end{aligned} \tag{6}$$

and

$$\begin{aligned}
&\int_{x}^{\pi_2}\left(t - \left(\pi_2 - h\frac{\pi_2 - \pi_1}{2}\right)\right) {}^{\pi_2}D_q\mathcal{F}(t)\ {}^{\pi_2}d_qt \\
= \ &h\frac{\pi_2 - \pi_1}{2}\mathcal{F}(\pi_2) - \left(x - \left(\pi_2 - h\frac{\pi_2 - \pi_1}{2}\right)\right)\mathcal{F}(x) - \int_{x}^{\pi_2} \mathcal{F}(qt + (1-q)\pi_2)\ {}^{\pi_2}d_qt.
\end{aligned} \tag{7}$$

After the addition of equalities (6) and (7), we obtain the required equality (5). □

Remark 1. *By taking the limit as $q \to 1^-$ in Lemma 1, we have*

$$\begin{aligned}
&\int_{\pi_1}^{x}\left(t - \left(\pi_1 + h\frac{\pi_2 - \pi_1}{2}\right)\right)\mathcal{F}'(t)dt + \int_{x}^{\pi_2}\left(t - \left(\pi_2 - h\frac{\pi_2 - \pi_1}{2}\right)\right)\mathcal{F}'(t)dt \\
= \ &(\pi_2 - \pi_1)h\frac{\mathcal{F}(\pi_1) + \mathcal{F}(\pi_2)}{2} + (\pi_2 - \pi_1)(1-h)\mathcal{F}(x) - \int_{\pi_1}^{\pi_2} \mathcal{F}(t)dt
\end{aligned}$$

which is given by Dragomir et al. in [23] (Theorem 2).

Remark 2. *In Lemma 1, if we set $h = 0$, then we have*

$$\begin{aligned}
&\int_{\pi_1}^{x} (t-\pi_1)\ {}_{\pi_1}D_q\mathcal{F}(t)\ {}_{\pi_1}d_qt + \int_{x}^{\pi_2} (t-\pi_2)\ {}^{\pi_2}D_q\mathcal{F}(t)\ {}^{\pi_2}d_qt \\
= \quad &(\pi_2-\pi_1)\mathcal{F}(x) - \left[\int_{\pi_1}^{x} \mathcal{F}(qt+(1-q)\pi_1)\ {}_{\pi_1}d_qt + \int_{x}^{\pi_2} \mathcal{F}(qt+(1-q)\pi_2)\ {}^{\pi_2}d_qt\right].
\end{aligned}$$

Theorem 3. *Assume that the conditions of Lemma 1 hold. If $\left|{}_{\pi_1}D_q\mathcal{F}(t)\right|, \left|{}^{\pi_2}D_q\mathcal{F}(t)\right| \leq M$, then*

$$\begin{aligned}
&\Bigg|(\pi_2-\pi_1)h\frac{\mathcal{F}(\pi_1)+\mathcal{F}(\pi_2)}{2} + (\pi_2-\pi_1)(1-h)\mathcal{F}(x) \\
&-\left[\int_{\pi_1}^{x} \mathcal{F}(qt+(1-q)\pi_1)\ {}_{\pi_1}d_qt + \int_{x}^{\pi_2} \mathcal{F}(qt+(1-q)\pi_2)\ {}^{\pi_2}d_qt\right]\Bigg| \\
\leq \quad &M(P(\pi_1,\pi_2,h,x;q) + Q(\pi_1,\pi_2,h,x;q)),
\end{aligned}$$

where

$$P(\pi_1,\pi_2,h,x;q) = \int_{\pi_1}^{x} \left|\left(t-\left(\pi_1+h\frac{\pi_2-\pi_1}{2}\right)\right)\right|\ {}_{\pi_1}d_qt$$

and

$$Q(\pi_1,\pi_2,h,x;q) = \int_{x}^{\pi_2} \left|\left(t-\left(\pi_2-h\frac{\pi_2-\pi_1}{2}\right)\right)\right|\ {}^{\pi_2}d_qt.$$

Proof. From Lemma 1 and properties of the modulus, we have

$$\begin{aligned}
&\Bigg|(\pi_2-\pi_1)h\frac{\mathcal{F}(\pi_1)+\mathcal{F}(\pi_2)}{2} + (\pi_2-\pi_1)(1-h)\mathcal{F}(x) \\
&-\left[\int_{\pi_1}^{x} \mathcal{F}(qt+(1-q)\pi_1)\ {}_{\pi_1}d_qt + \int_{x}^{\pi_2} \mathcal{F}(qt+(1-q)\pi_2)\ {}^{\pi_2}d_qt\right]\Bigg| \\
\leq \quad &\int_{\pi_1}^{x} \left|\left(t-\left(\pi_1+h\frac{\pi_2-\pi_1}{2}\right)\right)\right| \left|{}_{\pi_1}D_q\mathcal{F}(t)\right|\ {}_{\pi_1}d_qt \\
&+\int_{x}^{\pi_2} \left|\left(t-\left(\pi_2-h\frac{\pi_2-\pi_1}{2}\right)\right)\right| \left|{}^{\pi_2}D_q\mathcal{F}(t)\right|\ {}^{\pi_2}d_qt \\
\leq \quad &M\int_{\pi_1}^{x} \left|\left(t-\left(\pi_1+h\frac{\pi_2-\pi_1}{2}\right)\right)\right|\ {}_{\pi_1}d_qt + M\int_{x}^{\pi_2} \left|\left(t-\left(\pi_2-h\frac{\pi_2-\pi_1}{2}\right)\right)\right|\ {}^{\pi_2}d_qt \\
= \quad &M(P(\pi_1,\pi_2,h,x;q) + Q(\pi_1,\pi_2,h,x;q)).
\end{aligned}$$

□

Remark 3. *By taking the limit as $q \to 1^-$ in Theorem 3, we obtain the following inequality:*

$$\begin{aligned}
&\left|(\pi_2-\pi_1)h\frac{\mathcal{F}(\pi_1)+\mathcal{F}(\pi_2)}{2} + (\pi_2-\pi_1)(1-h)\mathcal{F}(x) - \int_{\pi_1}^{\pi_2} \mathcal{F}(t)dt\right| \\
\leq \quad &M\left[\frac{1}{4}(\pi_2-\pi_1)^2\left[h^2+(h-1)^2\right] + \left(x-\frac{\pi_1+\pi_2}{2}\right)^2\right]
\end{aligned} \tag{8}$$

which is given by Dragomir et al. in [23] (Theorem 2).

Remark 4. *In Theorem 3, if we put $h = 0$, then we have:*

$$\begin{aligned}
&\left|(\pi_2-\pi_1)\mathcal{F}(x) - \left[\int_{\pi_1}^{x} \mathcal{F}(qt+(1-q)\pi_1)\ {}_{\pi_1}d_qt + \int_{x}^{\pi_2} \mathcal{F}(qt+(1-q)\pi_2)\ {}^{\pi_2}d_qt\right]\right| \\
\leq \quad &M(P(\pi_1,\pi_2,0,x;q) + Q(\pi_1,\pi_2,0,x;q)).
\end{aligned}$$

Theorem 4. *Assume that the conditions of Lemma 1 hold. If for $p > 1$, $\left|{}_{\pi_1}D_q\mathcal{F}(t)\right|^p$, $\left|{}^{\pi_2}D_q\mathcal{F}(t)\right|^p \leq M$, then*

$$\begin{aligned}
&\Bigg|(\pi_2-\pi_1)h\frac{\mathcal{F}(\pi_1)+\mathcal{F}(\pi_2)}{2}+(\pi_2-\pi_1)(1-h)\mathcal{F}(x) \\
&-\left[\int_{\pi_1}^{x}\mathcal{F}(qt+(1-q)\pi_1)\ {}_{\pi_1}d_qt+\int_{x}^{\pi_2}\mathcal{F}(qt+(1-q)\pi_2)\ {}^{\pi_2}d_qt\right]\Bigg| \\
\leq\ & M((x-\pi_1)A_1(\pi_1,\pi_2,h,x;q)+(\pi_2-x)A_2(\pi_1,\pi_2,h,x;q))
\end{aligned}$$

where

$$\begin{aligned}
A_1(\pi_1,\pi_2,h,x;q) &= \left(\int_{\pi_1}^{x}\left|\left(t-\left(\pi_1+h\frac{\pi_2-\pi_1}{2}\right)\right)\right|^s{}_{\pi_1}d_qt\right)^{\frac{1}{s}}, \\
A_2(\pi_1,\pi_2,h,x;q) &= \left(\int_{x}^{\pi_2}\left|\left(t-\left(\pi_2-h\frac{\pi_2-\pi_1}{2}\right)\right)\right|^s{}_{\pi_1}d_qt\right)^{\frac{1}{s}}
\end{aligned}$$

and $\frac{1}{p}+\frac{1}{s}=1$.

Proof. From Lemma 1 and Hölder's inequality, we have

$$\begin{aligned}
&\Bigg|(\pi_2-\pi_1)h\frac{\mathcal{F}(\pi_1)+\mathcal{F}(\pi_2)}{2}+(\pi_2-\pi_1)(1-h)\mathcal{F}(x) \\
&-\left[\int_{\pi_1}^{x}\mathcal{F}(qt+(1-q)\pi_1)\ {}_{\pi_1}d_qt+\int_{x}^{\pi_2}\mathcal{F}(qt+(1-q)\pi_2)\ {}^{\pi_2}d_qt\right]\Bigg| \\
\leq\ & \int_{\pi_1}^{x}\left|\left(t-\left(\pi_1+h\frac{\pi_2-\pi_1}{2}\right)\right)\right|\left|{}_{\pi_1}D_q\mathcal{F}(t)\right|\ {}_{\pi_1}d_qt \\
&+\int_{x}^{\pi_2}\left|\left(t-\left(\pi_2-h\frac{\pi_2-\pi_1}{2}\right)\right)\right|\left|{}^{\pi_2}D_q\mathcal{F}(t)\right|\ {}^{\pi_2}d_qt \\
\leq\ & \left(\int_{\pi_1}^{x}\left|\left(t-\left(\pi_1+h\frac{\pi_2-\pi_1}{2}\right)\right)\right|^s{}_{\pi_1}d_qt\right)^{\frac{1}{s}}\left(\int_{\pi_1}^{x}\left|{}_{\pi_1}D_q\mathcal{F}(t)\right|^p{}_{\pi_1}d_qt\right)^{\frac{1}{p}} \\
&+\left(\int_{x}^{\pi_2}\left|\left(t-\left(\pi_2-h\frac{\pi_2-\pi_1}{2}\right)\right)\right|^s{}_{\pi_1}d_qt\right)^{\frac{1}{s}}\left(\int_{x}^{\pi_2}\left|{}^{\pi_2}D_q\mathcal{F}(t)\right|^p{}^{\pi_2}d_qt\right)^{\frac{1}{p}} \\
\leq\ & M((x-\pi_1)A_1(\pi_1,\pi_2,h,x;q)+(\pi_2-x)A_2(\pi_1,\pi_2,h,x;q)).
\end{aligned}$$

□

Remark 5. *By taking the limit as $q\to 1^-$ in Theorem 4, we have:*

$$\begin{aligned}
&\left|(\pi_2-\pi_1)h\frac{\mathcal{F}(\pi_1)+\mathcal{F}(\pi_2)}{2}+(\pi_2-\pi_1)(1-h)\mathcal{F}(x)-\int_{\pi_1}^{\pi_2}\mathcal{F}(t)dt\right| \\
\leq\ & M\Bigg[(x-\pi_1)\frac{\left(x-\left(\pi_1+h\frac{\pi_2-\pi_1}{2}\right)\right)\left|\left(x-\left(\pi_1+h\frac{\pi_2-\pi_1}{2}\right)\right)\right|^s+\left(h\frac{\pi_2-\pi_1}{2}\right)^{s+2}}{s+1} \\
&+(\pi_2-x)\frac{\left(x-\left(\pi_2-h\frac{\pi_2-\pi_1}{2}\right)\right)\left|\left(x-\left(\pi_2-h\frac{\pi_2-\pi_1}{2}\right)\right)\right|^s-\left(h\frac{\pi_2-\pi_1}{2}\right)^{s+2}}{s+1}\Bigg].
\end{aligned}$$

Remark 6. *In Theorem 4, if we put $h = 0$, we have:*

$$\begin{aligned} &\left|(\pi_2-\pi_1)\mathcal{F}(x) - \left[\int_{\pi_1}^{x} \mathcal{F}(qt+(1-q)\pi_1)\ {}_{\pi_1}d_qt + \int_{x}^{\pi_2} \mathcal{F}(qt+(1-q)\pi_2)\ {}^{\pi_2}d_qt\right]\right| \\ \leq\ & M((x-\pi_1)A_1(\pi_1,\pi_2,0,x;q) + (\pi_2-x)A_2(\pi_1,\pi_2,0,x;q)). \end{aligned}$$

4. Application to Special Means

For arbitrary positive numbers π_1, π_2 $(\pi_1 \neq \pi_2)$, we consider the means as follows:

1. The arithmetic mean
$$\mathcal{A} = \mathcal{A}(\pi_1,\pi_2) = \frac{\pi_1+\pi_2}{2}.$$
2. The harmonic mean
$$\mathcal{H} = \mathcal{H}(\pi_1,\pi_2) = \frac{2\pi_1\pi_2}{\pi_1+\pi_2}.$$
3. The logarithmic mean
$$\mathcal{L} = \mathcal{L}(\pi_1,\pi_2) = \frac{\pi_1-\pi_2}{\ln\pi_2 - \ln\pi_1}.$$
4. The p-logarithmic mean
$$\mathcal{L}_p = \mathcal{L}_p(\pi_1,\pi_2) = \begin{cases} \pi_1, & \text{if } \pi_1 = \pi_2 \\ \left[\frac{\pi_2^{p+1}-\pi_1^{p+1}}{(p+1)(\pi_2-\pi_1)}\right]^{\frac{1}{p}}, & \text{if } \pi_1 \neq \pi_2. \end{cases}$$

Proposition 1. *For $\pi_1, \pi_2 \in \mathbb{R}$, $\pi_1 < \pi_2$ and $p \in \mathbb{R}\backslash\{-1,0\}$, the following inequality is true:*

$$\begin{aligned} &\left|(1-h)x^p + h\mathcal{A}\left(\pi_1^p,\pi_2^p\right) - \mathcal{L}_p^p(\pi_1,\pi_2)\right| \\ \leq\ & \left\{(\pi_2-\pi_1)\left[\frac{h^2+(h-1)^2}{4}\right] + \frac{(x-\mathcal{A}(\pi_1,\pi_2))}{\pi_2-\pi_1}\right\}\epsilon_p(\pi_1,\pi_2) \end{aligned} \tag{9}$$

where

$$\epsilon_p(\pi_1,\pi_2) = \begin{cases} |p|\pi_2^{p-1}, & \textit{if } p > 1, \\ |p|\pi_1^{p-1}, & \textit{if } p \in (-\infty,1]\backslash\{-1,0\}. \end{cases}$$

Proof. The inequality (8) for the mapping $\mathcal{F} : (0,\infty) \to (0,\infty)$, $\mathcal{F}(x) = x^p$ leads to this conclusion. □

Proposition 2. *For $\pi_1, \pi_2 \in \mathbb{R}$, $\pi_1 < \pi_2$, the following inequality is true:*

$$\begin{aligned} &|(1-h)\mathcal{H}(\pi_1,\pi_2)\mathcal{L}(\pi_1,\pi_2) + \mathcal{L}(\pi_1,\pi_2)xh - x\mathcal{H}(\pi_1,\pi_2)| \\ \leq\ & \frac{x\mathcal{H}(\pi_1,\pi_2)\mathcal{L}(\pi_1,\pi_2)}{\pi_1^2}\left\{(\pi_2-\pi_1)\left[\frac{h^2+(h-1)^2}{4}\right] + \frac{(x-\mathcal{A}(\pi_1,\pi_2))}{\pi_2-\pi_1}\right\}. \end{aligned}$$

Proof. The inequality (8) for the mapping $\mathcal{F} : (0,\infty) \to (0,\infty)$, $\mathcal{F}(x) = \frac{1}{x}$ leads to this conclusion. □

Proposition 3. *For $\pi_1, \pi_2 \in \mathbb{R}$, $\pi_1 < \pi_2$ and $p \in \mathbb{R}\backslash\{-1,0\}$, the following inequality is true:*

$$
\begin{aligned}
&\left|(1-h)x^p + h\mathcal{A}\left(\pi_1^p, \pi_2^p\right) - \mathcal{L}_p^p(\pi_1, \pi_2)\right| \\
\leq\ &\left[(x-\pi_1)\frac{\left(x-\left(\pi_1 + h\frac{\pi_2-\pi_1}{2}\right)\right)\left|\left(x-\left(\pi_1 + h\frac{\pi_2-\pi_1}{2}\right)\right)\right|^s + \left(h\frac{\pi_2-\pi_1}{2}\right)^{s+2}}{s+1}\right. \\
&\left. + (\pi_2 - x)\frac{\left(x-\left(\pi_2 - h\frac{\pi_2-\pi_1}{2}\right)\right)\left|\left(x-\left(\pi_2 - h\frac{\pi_2-\pi_1}{2}\right)\right)\right|^s - \left(h\frac{\pi_2-\pi_1}{2}\right)^{s+2}}{s+1}\right]\epsilon_p(\pi_1, \pi_2).
\end{aligned}
$$

Proof. The inequality in Remark 5, for the mapping $\mathcal{F} : (0, \infty) \to (0, \infty)$, $\mathcal{F}(x) = x^p$, leads to this conclusion. □

Proposition 4. *For $\pi_1, \pi_2 \in \mathbb{R}$, $\pi_1 < \pi_2$, the following inequality is true:*

$$
\begin{aligned}
&|(1-h)\mathcal{H}(\pi_1, \pi_2)\mathcal{L}(\pi_1, \pi_2) + \mathcal{L}(\pi_1, \pi_2)xh - x\mathcal{H}(\pi_1, \pi_2)| \\
\leq\ &\frac{x\mathcal{H}(\pi_1, \pi_2)\mathcal{L}(\pi_1, \pi_2)}{\pi_1^2}\left[(x-\pi_1)\frac{\left(x-\left(\pi_1 + h\frac{\pi_2-\pi_1}{2}\right)\right)\left|\left(x-\left(\pi_1 + h\frac{\pi_2-\pi_1}{2}\right)\right)\right|^s + \left(h\frac{\pi_2-\pi_1}{2}\right)^{s+2}}{s+1}\right. \\
&\left. + (\pi_2 - x)\frac{\left(x-\left(\pi_2 - h\frac{\pi_2-\pi_1}{2}\right)\right)\left|\left(x-\left(\pi_2 - h\frac{\pi_2-\pi_1}{2}\right)\right)\right|^s - \left(h\frac{\pi_2-\pi_1}{2}\right)^{s+2}}{s+1}\right].
\end{aligned}
$$

Proof. The inequality (9) for the mapping $\mathcal{F} : (0, \infty) \to (0, \infty)$, $\mathcal{F}(x) = \frac{1}{x}$, leads to this conclusion. □

5. Conclusions

Some new Ostrowski-type integral inequalities for q-differentiable bounded functions are established in the present research, generalizing existing results in the literature. Applications to special means are also discussed.

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