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Boundary Values in Ultradistribution Spaces Related to Extended Gevrey Regularity

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Abstract: Following the well-known theory of Beurling and Roumieu ultradistributions, we investigate new spaces of ultradistributions as dual spaces of test functions which correspond to associated functions of logarithmic-type growth at infinity. In the given framework we prove that boundary values of analytic functions with the corresponding logarithmic growth rate towards the real domain are ultradistributions. The essential condition for that purpose, known as stability under ultradifferential operators in the classical ultradistribution theory, is replaced by a weaker condition, in which the growth properties are controlled by an additional parameter. For that reason, new techniques were used in the proofs. As an application, we discuss the corresponding wave front sets.

1. Introduction

In this paper we describe certain intermediate spaces between the spaces of Schwartz distributions and any space of Gevrey ultradistributions as boundary values of analytic functions. More precisely, we continue to investigate a new class of ultradifferentiable functions and their duals ([1–4]) following Komatsu's approach [5,6]. We refer to [7] and the references therein for another equally interesting approach.

The derivatives of such ultradifferentiable functions are controlled by the two-parameter sequences of the form $M_p^{\tau,\sigma} = p^{\tau p^\sigma}$, $p \in \mathbf{N}$, $\tau > 0$, $\sigma > 1$. For that reason we call them extended Gevrey functions. It turns out that such functions can be used in the study of a class of strictly hyperbolic equations and systems. In particular, the extended Gevrey class associated with the sequence $M_p^{1,2} = p^{p^2}$ is used in the analysis of the regularity of the corresponding Cauchy problem in [8]. It captures the regularity of the coefficients in the space variable (with low regularity in time), so that the corresponding Cauchy problem is well posed in appropriate solution spaces.

Actually, the growth rate of sequence $M_p^{\tau,\sigma}$ implied a change in the growth of the expression h^p in the classical definition (see [5]). Hence, instead of that expression, we use h^{p^σ}, which essentially changes the corresponding proofs in the analysis of new ultradistribution spaces. Indeed, the extra exponent σ which appears in the power of term h implies that the extended Gevrey classes are different from any Carleman class C^L; cf. [9]. This difference is essential for many calculations—for example, in the proof of the inverse closedness property; cf. [10].

We especially emphasize the role of the Lambert W function that appears in the theory of new ultradistribution spaces for the first time. This is the essential contribution

of our approach. The properties of new ultradistribution spaces described in terms of the Lambert function and its asymptotic properties show that our approach is naturally included in the general theory of ultradistributions positioning the new spaces; let us call them extended Gevrey ultradistributions, between classical distributions and Komatsu type ultradistributions.

Distributions as boundary values of analytic functions are investigated in many papers; see [11] for the historical background and the relevant references therein. We point out a nice survey for distribution and ultradistribution boundary values given in the book [9]. The essence of the existence of a boundary value is the determination of the growth condition under which an analytic function $F(x + iy)$, observed on a certain tube domain with respect to y, defines an (ultra)distribution as y tends to 0. The classical result can be roughly interpreted as follows: if $F(x + iy) \leq C|y|^{-M}$ for some $C, M > 0$ then $F(x + i0)$ is in the Schwartz space $\mathcal{D}'(U)$ in a neighborhood U of x. (see Theorem 3.1.15 in [9]). For Gevrey ultradistributions, sub-exponential growth rate of analytic function F of the form $|F(x + iy)| \leq Ce^{k|y|^{-1/(t-1)}}$ for some $C, k > 0$ and $t > 1$ implies the boundary value result. The function in the exponent precisely describes the asymptotic behavior of the associated function to the Gevrey sequence $p!^t$, $p \in \mathbf{N}$; cf. [6,12]. In general, such representations are provided if test functions admit almost analytic extensions in the non-quasianalytic case related to Komatsu's condition $(M.2)$ (see [13]).

Different results concerning boundary values in the spaces of ultradistributions can be found in [5,6,11,13,14]. Even now this topic for ultradistribution spaces is interesting (cf. [15–18]). Especially, we have to mention [19]. At the end of this introduction we will briefly comment on the approach in this paper and our approach.

Extended Gevrey classes $\mathcal{E}_{\tau,\sigma}(U)$ and $\mathcal{D}_{\tau,\sigma}(U)$, $\tau > 0$, $\sigma > 1$, are introduced and investigated in [1–4,10,20]. The derivatives of functions in such classes are controlled by sequences of the form $M_p^{\tau,\sigma} = p^{\tau p^{\sigma}}$, $p \in \mathbf{N}$. Although such sequences do not satisfy Komatsu's condition $(M.2)$, the corresponding spaces consist of ultradifferentiable functions; that is, it is possible to construct differential operators of infinite order and prove their continuity properties on the test and dual spaces.

Our main intention in this paper is to establish the sufficient condition when the elements of dual spaces can be represented as boundary values of analytic functions. We follow the classical approach to boundary values given in [11] and carry out necessary modifications in order to use it in the analysis of spaces developed in [1–4]. Here, for such spaces, plenty of non-trivial constructions are established. In particular, we analyze the corresponding associated functions as a main tool in our investigations.

Moreover, we apply these results in the description of related wave front sets. The wave front set $\mathrm{WF}_{\tau,\sigma}(u)$, $\tau > 0$, $\sigma > 1$, of a Schwarz distribution u is analyzed in [2–4,10,20]. In particular, it is proved that they are related to the classes $\mathcal{E}_{\tau,\sigma}(U)$. We extend the definition of $\mathrm{WF}_{\tau,\sigma}(u)$ to a larger space of ultradistributions by using their boundary value representations. This allows us to describe intersections and unions of $\mathrm{WF}_{\tau,\sigma}(u)$ (with respect to τ) by using specific functions with logarithmic type behavior.

Let us comment on another very interesting concept of construction of a large class of ultradistribution spaces. In [19,21,22] and several other papers the authors consider sequences of the form $k!M_k$, where they presume a fair number of conditions on M_k and discuss in details their relations. For example, consequences of the composition of ultradifferentiable functions determined by different classes of such sequences are discussed. Moreover, they consider weighted matrices, that is, a family of sequences of the form $k!M_k^{\lambda}$, $k \in \mathbb{N}$, $\lambda \in \Lambda$ (partially ordered and directed set), and make the unions, again considering various properties such as compositions and boundary values. Their analysis follows the approach of [7,23]. In essence, an old question of ultradistribution theory was the analysis of unions and intersections of ultradifferentiable function spaces. This is very well elaborated in quoted papers. The main reason why our classes are not covered by the quoted papers is the factor $h^{|\alpha|^{\sigma}}$, $\sigma > 1$, in the seminorm (4). For that reason our conditions on the weight sequence $(\widetilde{(M.2)}'$ and $\widetilde{(M.2)}$ given below) differ from the corresponding

ones in the quoted papers. As we already explained, our growth rate is not just another point of view, since the basic facts used in the proofs are related to a new investigations involved by the Lambert W function. Actually, the precise estimates of our paper can be used for the further extensions in weighted matrix approach, since the original idea for our approach is quite different and based on the relation between $[n^s]!$ and $n!^s$ in the estimate of derivatives ($[n^s]$ means integer value not exceeding n^s, $s \in (0,1)$; cf. [1,2]).

The paper is organized as follows: We end the introduction with some notation. In Section 2 we introduce the necessary background on the spaces of extended Gevrey functions and their duals, spaces of ultradistributions. Our main result, Theorem 1, is given in Section 3. Wave front sets in the framework of our theory are discussed in Section 4. Finally, in Appendix A we prove a technical result concerning the associated functions $T_{\tau,\sigma,h}(k)$ and recall the basic continuity properties of ultradifferentiable operators on extended Gevrey classes, in a certain sense analogous to stability under the ultradifferentiable operators in the classical theory.

Notation

We denote by $\mathbf{N}$, $\mathbf{Z}_+$, $\mathbf{R}$ and $\mathbf{C}$ the sets of nonnegative integers, positive integers, real numbers and complex numbers, respectively. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbf{N}^d$, we write $\partial^\alpha = \partial^{\alpha_1} \ldots \partial^{\alpha_d}$, $D^\alpha = (-i)^{|\alpha|}\partial^\alpha$ and $|\alpha| = |\alpha_1| + \cdots + |\alpha_d|$. The open ball $B_r(x_0)$ has radius $r > 0$ and center at $x_0 \in \mathbf{R}^d$; $\partial_{\bar{z}} = (\partial_{\bar{z}_1}, \ldots, \partial_{\bar{z}_n})$ where $\partial_{\bar{z}_j} = \frac{1}{2}(\partial_{x_j} + i\partial_{y_j})$, $j = 1, \ldots, d$, $z = x + iy \in \mathbf{C}^d$. By Hartogs's theorem, $f(z), z \in \Omega$, Ω is open in $\mathbf{C}^d$, and is analytic if it is analytic with respect to every coordinate variable z_i.

Throughout the paper we always assume $\tau > 0$ and $\sigma > 1$.

2. Test Spaces and Duals

We are interested in $M_p^{\tau,\sigma}$, $p \in \mathbf{N}$, sequences of positive numbers such that, for some $C > 1$, the following conditions are satisfied:

$$(M.1) \qquad (M_p^{\tau,\sigma})^2 \leq M_{p-1}^{\tau,\sigma} M_{p+1}^{\tau,\sigma}, \qquad\qquad p \in \mathbf{N};$$

$$\widetilde{(M.2)} \qquad M_{p+q}^{\tau,\sigma} \leq C^{p^\sigma + q^\sigma + 1} M_p^{2^{\sigma-1}\tau,\sigma} M_q^{2^{\sigma-1}\tau,\sigma}, \qquad p,q \in \mathbf{N};$$

$$\widetilde{(M.2)}' \qquad M_{p+1}^{\tau,\sigma} \leq C^{p^\sigma + 1} M_p^{\tau,\sigma}, \qquad\qquad p \in \mathbf{N};$$

$$(M.3)' \qquad \sum_{p=1}^{\infty} \frac{M_{p-1}^{\tau,\sigma}}{M_p^{\tau,\sigma}} < \infty.$$

We notice that $(M.1)$ and $(M.3)'$ are usual conditions of logarithmic convexity and non-quasianalyticity, respectively, and when $\sigma = 1$ and $\tau > 0$ the conditions $\widetilde{(M.2)}'$ and $\widetilde{(M.2)}$ become the standard conditions of stability under differential and ultradifferential operators, $(M.2)'$ and $(M.2)$, respectively (see [5]). In the sequel we consider the sequence $M_p^{\tau,\sigma} = p^{\tau p^\sigma}$, $p \in \mathbf{N}$, which fulfills the above mentioned conditions (see Lemma 2.2 in [1]). This particular choice slightly simplifies our exposition. Clearly, by choosing $\sigma = 1$ and $\tau > 1$ we recover the well known Gevrey sequence $p!^\tau$.

Recall [4], the associated function related to the sequence $p^{\tau p^\sigma}$ is defined by

$$T_{\tau,\sigma,h}(k) = \sup_{p \in \mathbf{N}} \ln \frac{h^{p^\sigma} k^p}{p^{\tau p^\sigma}}, \quad k > 0. \tag{1}$$

For $h = \sigma = 1$ and $\tau > 1$, $T_{\tau,1,1}(k)$ is the associated function of the Gevrey sequence $p!^\tau$.

In the next lemma we derive the precise asymptotic behavior of the function $T_{\tau,\sigma,h}$ associated with the sequence $p^{\tau p^\sigma}$. This in turn highlights the essential difference between $T_{\tau,\sigma,h}$ and the associated functions determined by Gevrey type sequences.

We first introduce some notation. The Lambert W function is defined as the inverse function of ze^z, $z \in \mathbf{C}$, wherefrom

$$x = W(x)e^{W(x)}, \quad x \geq 0.$$

We denote its principal (real) branch by $W(x)$, $x \geq 0$ (see [24,25]). It is a continuous, increasing and concave function on $[0, \infty)$, $W(0) = 0$, $W(e) = 1$, and $W(x) > 0$, $x > 0$. It can be shown that W can be represented in the form of the absolutely convergent series

$$W(x) = \ln x - \ln(\ln x) + \sum_{k=0}^{\infty} \sum_{m=1}^{\infty} c_{km} \frac{(\ln(\ln x))^m}{(\ln x)^{k+m}}, \quad x \geq x_0 > e,$$

with suitable constants c_{km} and x_0. Thus the following estimates hold:

$$\ln x - \ln(\ln x) \leq W(x) \leq \ln x - \frac{1}{2}\ln(\ln x), \quad x \geq e, \tag{2}$$

with the equality in (2) if and only if $x = e$.

For given $\sigma > 1$, $\tau, h > 0$, let

$$\mathcal{R}(h, k) := h^{-\frac{\sigma-1}{\tau}} e^{\frac{\sigma-1}{\sigma}} \frac{\sigma-1}{\tau\sigma} \ln k = h^{-\frac{\sigma}{\tau\sigma'}} e^{\frac{1}{\sigma'}} \frac{1}{\tau\sigma'} \ln k, \quad k > e,$$

where

$$\frac{1}{\sigma} + \frac{1}{\sigma'} = 1, \quad \text{i.e.} \quad \sigma' = \frac{\sigma}{\sigma-1}.$$

Lemma 1. *Let $h > 0$, and let $T_{\tau,\sigma,h}$ be given by (1). Then there exist constants $B_1, B_2, b_1, b_2 > 0$ such that*

$$B_1 \, k^{b_1 \left(\frac{\ln k}{\ln(\ln k)}\right)^{\frac{1}{\sigma-1}}} \leq \exp\{T_{\tau,\sigma,h}(k)\} \leq B_2 \, k^{b_2 \left(\frac{\ln k}{\ln(\ln k)}\right)^{\frac{1}{\sigma-1}}}, \quad k > e.$$

More precisely, if

$$c_1 = \left(\frac{\sigma-1}{\tau\sigma}\right)^{\frac{1}{\sigma-1}}, \quad \text{and} \quad c_2 = h^{-\frac{\sigma-1}{\tau}} e^{\frac{\sigma-1}{\sigma}} \frac{\sigma-1}{\tau\sigma},$$

then there exist constants $A_1, A_2 > 0$ such that

$$A_1 \, k^{\frac{1}{2}\frac{\sigma-1}{\sigma} c_1 \left(\frac{\ln k}{\ln(c_2 \ln k)}\right)^{\frac{1}{\sigma-1}}} \leq \exp\{T_{\tau,\sigma,h}(k)\} \leq A_2 \, k^{c_1 \left(\frac{\ln k}{\ln(c_2 \ln k)}\right)^{\frac{1}{\sigma-1}}}, \quad k > e.$$

Proof. Lemma 1 can be proved by following the arguments used in the proof of Theorem 2.1 in [4]. There it is shown that for given $h > 0$, $\tau > 0$ and $\sigma > 1$ the following inequalities hold:

$$\tilde{A}_{\tau,\sigma,h} \exp\left\{\left(\frac{\ln k}{(2^{\sigma-1}\tau W(\mathcal{R}(h,k)))^{\frac{1}{\sigma}}\sigma'}\right)^{\sigma'}\right\} \leq e^{T_{\tau,\sigma,h}(k)}$$

$$\leq A_{\tau,\sigma,h} \exp\left\{\left(\frac{\ln k}{(\tau\sigma' W(\mathcal{R}(h,k)))^{\frac{1}{\sigma}}}\right)^{\sigma'}\right\}, \quad k > e,$$

for some $A_{\tau,\sigma,h}, \tilde{A}_{\tau,\sigma,h} > 0$. Moreover, in the view of (2), it follows that

$$W^{-\frac{\sigma'}{\sigma}}(\mathcal{R}(h,k)) (\ln k)^{\sigma'} \asymp \left(\frac{\ln k}{\ln(C_h \ln k)}\right)^{\frac{\sigma'}{\sigma}} \ln k, \quad k \to \infty,$$

with $C_h := h^{-\frac{\sigma-1}{\tau}} e^{\frac{\sigma-1}{\sigma}} \frac{\sigma-1}{\tau\sigma} = h^{-\frac{\sigma}{\tau\sigma'}} e^{\frac{1}{\sigma'}} (\tau\sigma')^{-1}$.

Details are left for the reader. □

We define (following the classical approach [5]):

$$T^*_{\tau,\sigma,h}(k) = \sup_{p\in\mathbf{N}} \ln \frac{h^{p^\sigma}k^p}{p^{p(\tau p^{\sigma-1}-1)}}, \quad k > 0. \tag{3}$$

It turns out that $T^*_{\tau,\sigma,h}(k)$ enjoys the same asymptotic behavior as $T_{\tau,\sigma,h}$; cf. Lemma A1 (a) in Appendix A. This is another difference between our approach and the classical ultradistribution theory, where T^* plays an important role.

Next we recall the definition of spaces $\mathcal{E}_{\tau,\sigma}(U)$ and $\mathcal{D}_{\tau,\sigma}(U)$, where U is an open set in $\mathbf{R}^d$ ([1]).

Let $K \subset\subset \mathbf{R}^d$ be a regular compact set. Then, $\mathcal{E}_{\tau,\sigma,h}(K)$ is the Banach space of functions $\phi \in C^\infty(K)$ such that

$$\|\phi\|_{\mathcal{E}_{\tau,\sigma,h}(K)} = \sup_{\alpha\in\mathbf{N}^d} \sup_{x\in K} \frac{|\partial^\alpha\phi(x)|}{h^{|\alpha|^\sigma}|\alpha|^{\tau|\alpha|^\sigma}} < \infty. \tag{4}$$

We have

$$\mathcal{E}_{\tau_1,\sigma_1,h_1}(K) \hookrightarrow \mathcal{E}_{\tau_2,\sigma_2,h_2}(K), \quad 0 < h_1 < h_2, \, 0 < \tau_1 < \tau_2, \, 1 < \sigma_1 < \sigma_2,$$

where $\hookrightarrow$ denotes the strict and dense inclusion.

The set of functions from $\mathcal{E}_{\tau,\sigma,h}(K)$ supported by K is denoted by $\mathcal{D}^K_{\tau,\sigma,h}$. Next,

$$\mathcal{E}_{\{\tau,\sigma\}}(U) = \varprojlim_{K\subset\subset U} \varinjlim_{h\to\infty} \mathcal{E}_{\tau,\sigma,h}(K), \tag{5}$$

$$\mathcal{E}_{(\tau,\sigma)}(U) = \varprojlim_{K\subset\subset U} \varinjlim_{h\to 0} \mathcal{E}_{\tau,\sigma,h}(K), \tag{6}$$

$$\mathcal{D}_{\{\tau,\sigma\}}(U) = \varinjlim_{K\subset\subset U} \mathcal{D}^K_{\{\tau,\sigma\}} = \varinjlim_{K\subset\subset U} \left(\varinjlim_{h\to\infty} \mathcal{D}^K_{\tau,\sigma,h} \right), \tag{7}$$

$$\mathcal{D}_{(\tau,\sigma)}(U) = \varinjlim_{K\subset\subset U} \mathcal{D}^K_{(\tau,\sigma)} = \varinjlim_{K\subset\subset U} \left(\varprojlim_{h\to 0} \mathcal{D}^K_{\tau,\sigma,h} \right). \tag{8}$$

Spaces in (5) and (7) are called Roumieu type spaces, and (6) and (8) are Beurling type spaces. Note that all the spaces of ultradifferentiable functions defined by Gevrey type sequences are contained in the corresponding spaces defined above.

For the corresponding spaces of ultradistributions we have:

$$\mathcal{D}'_{\{\tau,\sigma\}}(U) = \varprojlim_{K\subset\subset U} \varprojlim_{h\to 0} (\mathcal{D}^K_{\tau,\sigma,h})', \quad \mathcal{D}'_{(\tau,\sigma)}(U) = \varprojlim_{K\subset\subset U} \varinjlim_{h\to\infty} (\mathcal{D}^K_{\tau,\sigma,h})'.$$

Topological properties of all those spaces are the same as in the case of Beurling and Roumieu type spaces given in [5].

We will use abbreviated notation τ,σ for $\{\tau,\sigma\}$ or (τ,σ). Clearly,

$$\mathcal{D}'(U) \hookrightarrow \mathcal{D}'_{\tau,\sigma}(U) \hookrightarrow \varprojlim_{t\to 1} \mathcal{D}'_t(U),$$

where $\mathcal{D}'_t(U) = \mathcal{D}'_{t,1}(U)$ denotes the space of Gevrey ultradistributions with index $t > 1$. More precisely, if (for $\sigma > 1$) we put

$$\mathcal{D}^{(\sigma)}(U) = \varprojlim_{\tau\to 0} \mathcal{D}_{\tau,\sigma}(U), \quad \text{and} \quad \mathcal{D}^{\{\sigma\}}(U) = \varinjlim_{\tau\to\infty} \mathcal{D}_{\tau,\sigma}(U),$$

then

$$\mathcal{D}'(U) \hookrightarrow \mathcal{D}'^{\{\sigma\}}(U) \hookrightarrow \mathcal{D}'^{(\sigma)}(U) \hookrightarrow \varprojlim_{t\to 1} \mathcal{D}'_t(U),$$

where $\mathcal{D}'^{(\sigma)}(U)$ and $\mathcal{D}'^{\{\sigma\}}(U)$ are dual spaces of $\mathcal{D}^{(\sigma)}(U)$ and $\mathcal{D}^{\{\sigma\}}(U)$, respectively.

Thus we are dealing with intermediate spaces between the space of Schwartz distributions and spaces of Gevrey ultradistributions. In the next section we show the boundary value result in the given framework. This, however, asks for the use of new techniques.

3. Main Result

The condition $(M.2)$ (also known as the stability under the ultradifferentiable operators), essential for the boundary value theorems in the framework of ultradistribution spaces [5,13], is in our approach replaced by the condition $\widetilde{(M.2)}$. We note that in [19] a more general condition than $(M.2)$ is considered. In the case of the sequence $M_p^{\tau,\sigma} = p^{\tau p^{\sigma}}$, $p \in \mathbf{N}$, the asymptotic behaviour given in Lemma 1 is essentially used to prove our main result as follows.

Theorem 1. *Let $\sigma > 1$, U be an open set in $\mathbf{R}^d$, Γ an open cone in $\mathbf{R}^d$ and $\gamma > 0$. Assume that $F(z)$, $z \in Z$ is an analytic function, where*

$$Z = \{z \in \mathbf{C}^d \mid \operatorname{Re} z \in U, \operatorname{Im} z \in \Gamma, |\operatorname{Im} z| < \gamma\},$$

and such that

$$|F(z)| \leq A|y|^{-H\left(\frac{\ln(1/|y|)}{\ln(\ln(1/|y|))}\right)^{\frac{1}{\sigma-1}}}, \quad z = x + iy \in Z,$$

for some $A, H > 0$ (resp. for every $H > 0$ there exists $A > 0$). Then

$$F(x + iy) \to F(x + i0), \quad y \to 0, y \in \Gamma, \tag{9}$$

in $\mathcal{D}'^{(\sigma)}(U)$ (resp. $\mathcal{D}'^{\{\sigma\}}(U)$).

More precisely, if

$$|F(z)| \leq A \exp\{T_{(2^{\sigma}-1)\tau,\sigma,H}(1/|y|)\} \quad z = x + iy \in Z, \tag{10}$$

for some $A, H > 0$ (resp. for every $H > 0$ there exists $A > 0$) then (9) holds in $\mathcal{D}'_{(\tau/2^{\sigma-1},\sigma)}(U)$ (resp. $\mathcal{D}'_{\{\tau/2^{\sigma-1},\sigma\}}(U)$).

Proof. Let $K \subset\subset U$ and $\varphi \in \mathcal{D}^K_{\tau/2^{\sigma-1},\sigma}$. Moreover, let $\kappa \in \mathcal{D}_{\tau/2^{\sigma-1},\sigma}(\mathbf{R}^d)$ be such that $\operatorname{supp}\kappa \subseteq \overline{B(0,2)}$, $\kappa = 1$ on $B(0,1)$.

In the sequel we denote $m_p = p^{\tau((2p)^{\sigma-1}-1)}$, $p \in \mathbf{N}$. Clearly, m_p is an increasing sequence and $m_p \to \infty$ as $p \to \infty$.

Fix $h > 0$, and let

$$\kappa_\alpha(y) = \kappa(4hm_{|\alpha|}y), \quad \alpha \in \mathbf{N}^d, y \in \mathbf{R}^d.$$

Note that

$$\operatorname{supp}\kappa_\alpha \subseteq \{y \in \mathbf{R}^d \mid |y| \leq 1/(2hm_{|\alpha|})\}, \tag{11}$$

and for $j = 1, \ldots, d$,

$$\operatorname{supp}\partial_{y_j}\kappa_\alpha \subseteq \{y \in \mathbf{R}^d \mid 1/(4hm_{|\alpha|}) \leq |y| \leq 1/(2hm_{|\alpha|})\}, \quad \alpha \in \mathbf{N}^d. \tag{12}$$

Let

$$\Phi(z) = \sum_{\alpha \in \mathbf{N}^d} \frac{\partial^\alpha \varphi(x)}{|\alpha|^{\tau|\alpha|}}(iy)^\alpha \kappa_\alpha(y), \quad z = x + iy \in \mathbf{C}^d. \tag{13}$$

Clearly, Φ is a smooth function in $\mathbb{R}^{2d}$ and $\Phi(x) = \varphi(x)$ for $x \in K$.

Fix $Y = (Y_1, \ldots, Y_d) \in \Gamma$, $Y \neq 0$, $|Y| < \gamma$, and set

$$Z_Y = \{x + itY \mid x \in K, t \in (0,1]\}. \tag{14}$$

In order to use Stoke's formula (see [13]) we need to estimate Φ and its derivatives on Z_Y. To that end we had to adjust the standard technique in a nontrivial manner.

Let us show that there exists $A_h > 0$ such that

$$|\Phi(z)| \leq A_h \|\varphi\|_{\mathcal{E}_{\tau/2^{\sigma-1},\sigma,h}}, \quad h > 0, \; z \in Z_Y. \tag{15}$$

Note that (11) implies

$$|tY|^{|\alpha|}|\kappa_\alpha(tY)| \leq \frac{1}{(2hm_{|\alpha|})^{|\alpha|}} = \frac{|\alpha|^{\tau|\alpha|}}{(2h)^{|\alpha|}|\alpha|^{2^{\sigma-1}\tau|\alpha|^\sigma}}, \quad t \in (0,1], \; \alpha \in \mathbf{N}^d,$$

and therefore we obtain

$$|\Phi(z)| \leq \sum_{\alpha \in \mathbf{N}^d} \frac{|\partial^\alpha \varphi(x)|}{|\alpha|^{\tau|\alpha|}}|tY|^\alpha|\kappa_\alpha(tY)| \leq \sum_{\alpha \in \mathbf{N}^d} \frac{|\partial^\alpha \varphi(x)|}{(2h)^{|\alpha|}|\alpha|^{2^{\sigma-1}\tau|\alpha|^\sigma}}$$

$$\leq \|\varphi\|_{\mathcal{E}_{\tau/2^{\sigma-1},\sigma,h}} \sum_{\alpha \in \mathbf{N}^d} \frac{h^{|\alpha|^\sigma}|\alpha|^{(\tau/2^{\sigma-1})|\alpha|^\sigma}}{(2h)^{|\alpha|}|\alpha|^{2^{\sigma-1}\tau|\alpha|^\sigma}} = A_h\|\varphi\|_{\mathcal{E}_{\tau/2^{\sigma-1},\sigma,h}},$$

where $A_h = \sum_{\alpha \in \mathbf{N}^d} \frac{h^{|\alpha|^\sigma-|\alpha|}}{2^{|\alpha|}|\alpha|^{\tau_0|\alpha|^\sigma}} < \infty$ for $\tau_0 = \tau(2^{\sigma-1} - \frac{1}{2^{\sigma-1}}) > 0$. Hence (15) follows.

Next we estimate $\partial_{\overline{z_j}}\Phi(z)$, $j \in \{1,\dots,d\}$, when $z \in Z_Y$. More precisely, we show that for a given $h > 0$, there exists $B_h > 0$ such that

$$|\partial_{\overline{z_j}}\Phi(z)| \leq B_h\|\varphi\|_{\mathcal{E}_{\tau/2^{\sigma-1},\sigma,h}} \exp\{-T_{(2^\sigma-1)\tau,\sigma,h}(1/|tY|)\}, \quad z \in Z_Y. \tag{16}$$

By (11) and (12) it is sufficient to prove (16) for

$$1/(4hm_{|\alpha|}) \leq |tY| \leq 1/(2hm_{|\alpha|}), \quad 0 < t \leq 1, \; \alpha \in \mathbf{N}^d. \tag{17}$$

Note that for $z \in Z_Y$ we have

$$\partial_{\overline{z_j}}\Phi(z) = \frac{1}{2}\Big(\sum_{\alpha \in \mathbf{N}^d} \frac{\partial^{\alpha+e_j}\varphi(x)}{|\alpha|^{\tau|\alpha|}}i^{|\alpha|}(tY)^\alpha\kappa_\alpha(tY)$$

$$+ \sum_{\alpha \in \mathbf{N}^d} \frac{\partial^\alpha\varphi(x)}{|\alpha|^{\tau|\alpha|}}\alpha_j i^{|\alpha|+1}t^{|\alpha|}Y^{\alpha-e_j}\kappa_\alpha(tY)$$

$$+ \sum_{\alpha \in \mathbf{N}^d} \frac{\partial^\alpha\varphi(x)}{|\alpha|^{\tau|\alpha|}}i^{|\alpha|+1}t^{|\alpha|+1}Y^\alpha 4hm_{|\alpha|} \cdot (\partial_{y_j}\kappa)(4hm_{|\alpha|}tY)\Big) = \frac{1}{2}(S_1 + S_2 + S_3)(z).$$

We will show that there exists a constant $B_h > 0$ such that

$$\exp\{T_{(2^\sigma-1)\tau,\sigma,h}(1/|tY|)\}|S_1(z)| \leq B_h\|\varphi\|_{\mathcal{E}_{\tau/2^{\sigma-1},\sigma,h}}, \quad z \in Z_Y.$$

The estimates for S_2 and S_3 can be obtained in a similar way.

Let $C_h = C\max\{h, h^{2^{\sigma-1}}\}$ where $C > 0$ is the constant from $\widetilde{(M.2)}'$. Using

$$(p+1)^\sigma \leq 2^{\sigma-1}(p^\sigma+1), \quad p \in \mathbf{N},$$

we obtain

$$\frac{h^{|\beta|^\sigma}}{|tY|^{|\beta|}|\beta|^{(2^\sigma-1)\tau|\beta|^\sigma}}|S_1(z)| \le C_h\|\varphi\|_{\mathcal{E}_{\tau/2^{\sigma-1},\sigma,h}}$$

$$\left(\sum_{\substack{\alpha\in\mathbf{N}^d\\|\alpha|\le|\beta|}}+\sum_{\substack{\alpha\in\mathbf{N}^d\\|\alpha|>|\beta|}}\right)\frac{h^{|\beta|^\sigma}C_h^{|\alpha|^\sigma}|\alpha|^{(\tau/2^{\sigma-1})|\alpha|^\sigma}}{|\beta|^{(2^\sigma-1)\tau|\beta|^\sigma}|\alpha|^{\tau|\alpha|}}|tY|^{|\alpha|-|\beta|}|\kappa_\alpha(tY)|$$

$$= C_h\|\varphi\|_{\mathcal{E}_{\tau/2^{\sigma-1},\sigma,h}}(I_{1,\beta}+I_{2,\beta}), \quad \beta\in\mathbf{N}^d, z\in Z_Y.$$

It remains to show that $\sup_{\beta\in\mathbf{N}^d}I_{1,\beta}$ and $\sup_{\beta\in\mathbf{N}^d}I_{2,\beta}$ are finite.

First we estimate $I_{1,\beta}$. Note that for $|\alpha|\le|\beta|$, the left-hand side in (17) implies

$$|tY|^{|\alpha|-|\beta|}|\kappa_\alpha(tY)| \le (4hm_{|\alpha|})^{|\beta|-|\alpha|} \le \frac{(4h)^{|\beta|}m_{|\beta|}^{|\beta|}}{h^{|\alpha|}m_{|\alpha|}^{|\alpha|}}$$

$$\le \frac{(4h)^{|\beta|}|\alpha|^{\tau|\alpha|}|\beta|^{2^{\sigma-1}\tau|\beta|^\sigma}}{h^{|\alpha|}|\alpha|^{2^{\sigma-1}\tau|\alpha|^\sigma}}, \quad t\in(0,1], \alpha,\beta\in\mathbf{N}^d. \quad (18)$$

Again, when $\tau_0 = \tau(2^{\sigma-1}-\frac{1}{2^{\sigma-1}})$, by (18) we have

$$I_{1,\beta} \le \frac{(4h)^{|\beta|}h^{|\beta|^\sigma}}{|\beta|^{(2^{\sigma-1}-1)\tau|\beta|^\sigma}}\sum_{\alpha\in\mathbf{N}^d}\frac{C_h^{|\alpha|^\sigma}}{h^{|\alpha|}|\alpha|^{\tau_0|\alpha|^\sigma}} = C_h'\frac{(4h)^{|\beta|}h^{|\beta|^\sigma}}{|\beta|^{(2^{\sigma-1}-1)\tau|\beta|^\sigma}}, \quad \beta\in\mathbf{N}^d.$$

Hence, we conclude $\sup_{\beta\in\mathbf{N}^d}I_{1,\beta} \le C_h'\exp\{T_{(2^{\sigma-1}-1)\tau,\sigma,h}(4h)\} < \infty$.

To estimate $I_{2,\beta}$ we first note that for $|\alpha|>|\beta|$ the right-hand side in (17) implies

$$|tY|^{|\alpha-\beta|}|\kappa_\alpha(tY)| \le (1/(2hm_{|\alpha|}))^{|\alpha-\beta|} \le 1/((2h)^{|\alpha-\beta|}m_{|\alpha-\beta|}^{|\alpha-\beta|})$$

$$\le \frac{|\alpha|^{\tau|\alpha|}}{(2h)^{|\alpha-\beta|}|\alpha-\beta|^{2^{\sigma-1}\tau|\alpha-\beta|^\sigma}}, \quad t\in(0,1], \ \alpha,\beta\in\mathbf{N}^d. \quad (19)$$

Set $C_h'' = C\max\{C_h, C_h^{2^{\sigma-1}}\}$. Using $\widetilde{(M.2)}$, (19) and (A4), for $\beta\in\mathbf{N}^d$, we have

$$I_{2,\beta} \le \sum_{\substack{\alpha\in\mathbf{N}^d\\|\alpha|>|\beta|}}\frac{h^{|\beta|^\sigma}C_h^{|\alpha|^\sigma}|\alpha|^{(\tau/2^{\sigma-1})|\alpha|^\sigma}}{|\beta|^{(2^\sigma-1)\tau|\beta|^\sigma}(2h)^{|\alpha-\beta|}|\alpha-\beta|^{2^{\sigma-1}\tau|\alpha-\beta|^\sigma}}$$

$$\le C\sum_{\substack{\alpha\in\mathbf{N}^d\\|\alpha|>|\beta|}}\frac{h^{|\beta|^\sigma}(C_h'')^{|\alpha-\beta|^\sigma}(C_h'')^{|\beta|^\sigma}|\alpha-\beta|^{\tau|\alpha-\beta|^\sigma}|\beta|^{\tau|\beta|^\sigma}}{|\beta|^{(2^\sigma-1)\tau|\beta|^\sigma}(2h)^{|\alpha-\beta|}|\alpha-\beta|^{2^{\sigma-1}\tau|\alpha-\beta|^\sigma}}$$

$$\le \frac{(C_h''h)^{|\beta|^\sigma}}{|\beta|^{\tau(2^\sigma-2)|\beta|^\sigma}}C\sum_{\delta\in\mathbf{N}^d}\frac{(C_h'')^{|\delta|^\sigma}}{(2h)^{|\delta|}|\delta|^{(2^{\sigma-1}-1)\tau|\delta|^\sigma}} = C_h'''\frac{(C_h''h)^{|\beta|^\sigma}}{|\beta|^{\tau(2^\sigma-2)|\beta|^\sigma}}.$$

In particular, $\sup_{\beta\in\mathbf{N}^d}I_{2,\beta} \le C_h'''\exp\{T_{(2^\sigma-2)\tau,\sigma,C_h''h}(1)\} < \infty$.

Now, Stoke's formula gives

$$\langle F(x+i0),\varphi(x)\rangle = \int_K F(x+iY)\Phi(x+iY)dx$$

$$+ 2i\sum_{j=1}^d Y_j\int_0^1\int_K \partial_{\overline{z_j}}\Phi(x+itY)F(x+itY)dtdx, \quad (20)$$

and we have used the assumptions in Theorem 1, and inequalities (15) and (16).

Note that for $H = h$, (10) and (15) imply that there exists $A_h > 0$ such that

$$|F(x + iY)\Phi(x + iY)| \leq A_h \|\varphi\|_{\mathcal{E}_{\tau/2^{\sigma-1},\sigma,h}} \exp\{T_{(2^{\sigma}-1)\tau,\sigma,h}(1/|Y|)\}$$
$$= A'_h \|\varphi\|_{\mathcal{E}_{\tau/2^{\sigma-1},\sigma,h}}, \quad x \in K, \quad (21)$$

where $A'_h = A_h \exp\{T_{(2^{\sigma}-1)\tau,\sigma,h}(1/|Y|)\}$.

Moreover, (10) and (16) imply that there exists $B_h > 0$ such that

$$|\partial_{\overline{z_j}}\Phi(z)F(z)| \leq B_h \|\varphi\|_{\mathcal{E}_{\tau/2^{\sigma-1},\sigma,h}}, \quad 1 \leq j \leq d, z \in Z_Y. \quad (22)$$

Now (20)–(22) imply

$$|\langle F(x + i0), \varphi(x)\rangle| \leq B'_h \|\varphi\|_{\mathcal{E}_{\tau/2^{\sigma-1},\sigma,h}},$$

for suitable constant $B'_h > 0$. This completes the proof of the second part of theorem, and the first part follows immediately. $\square$

Remark 1. *In order to show that any ultradistribution f is locally (on a bounded open set U) the sum of boundary values of analytic functions defined in the corresponding cone domains Γ_j, $j = 1, \ldots, k$, one should proceed as in the classical theory. We multiply f by a cutoff test function κ_U equal to 1 over U, and obtain $f_0 = f\kappa_U$ equals f on U. Then we divide $\mathbb{R}^n \setminus \{0\}$ into regular non overlapping cones Γ_{j0}, $j = 1, \ldots, k$, dual cones of Γ_j, and define*

$$F_j(z) = \langle f_0(t), \int_{\Gamma_{j0}} \exp\{2\pi i(z - t)\eta\}d\eta\rangle, \quad z \in \mathbb{R}^n + i\Gamma_j, \quad j = 1, \ldots, k.$$

Now one can get the growth conditions for $F_j, j = 1, \ldots, k$, and show that

$$f_0 = \sum_{j=1}^{k} \lim_{y \to 0, y \in \Gamma_j} F_j(x + iy), \quad x \in \mathbb{R}^n.$$

The details will be given in a separate contribution where we will consider L^p versions of new ultradistributions spaces similar to the corresponding ones in [11].

4. Wave Front Sets

In this section we analyze wave front sets $\mathrm{WF}_{\tau,\sigma}(u)$ related to the classes $\mathcal{E}_{\tau,\sigma}(U)$ introduced in Section 2. We refer to [2–4,10,20] for properties of $\mathrm{WF}_{\tau,\sigma}(u)$ when u is a Schwartz distribution.

We begin with the definition.

Definition 1. *Let $\tau > 0$, $\sigma > 1$, U open set in $\mathbf{R}^d$ and $(x_0, \xi_0) \in U \times \mathbf{R}^d \setminus \{0\}$. Then for $u \in \mathcal{D}'_{\{\tau,\sigma\}}(U)$ (respectively $\mathcal{D}'_{(\tau,\sigma)}(U)$), $(x_0, \xi_0) \notin \mathrm{WF}_{\{\tau,\sigma\}}(u)$ (resp. $(x_0, \xi_0) \notin \mathrm{WF}_{(\tau,\sigma)}(u)$) if and only if there exists a conic neighborhood Γ of ξ_0; a compact neighborhood K of x_0; and $\phi \in \mathcal{D}_{\{\tau,\sigma\}}(U)$ (respectively $\phi \in \mathcal{D}_{(\tau,\sigma)}(U)$) such that $\mathrm{supp}\,\phi \subseteq K$, $\phi = 1$ on some neighborhood of x_0 and*

$$|\widehat{\phi u}(\xi)| \leq A \exp\{-T_{\tau,\sigma,h}(|\xi|)\}, \quad \xi \in \Gamma,$$

for some $A, h > 0$ (resp. for any $h > 0$ there exists $A > 0$).

We will write $\mathrm{WF}_{\tau,\sigma}(u)$ for $\mathrm{WF}_{(\tau,\sigma)}(u)$ or $\mathrm{WF}_{\{\tau,\sigma\}}(u)$.

Remark 2. *Note that $\mathrm{WF}_{\tau,1}(u) = \mathrm{WF}_\tau(u)$, $\tau > 1$, are Gevrey wave front sets investigated in [12].*

Moreover (cf. [3]), for $0 < \tau_1 < \tau_2$ and $\sigma > 1$ we have

$$\mathrm{WF}(u) \subset \mathrm{WF}_{\tau_2,\sigma}(u) \subset \mathrm{WF}_{\tau_1,\sigma}(u) \subset \bigcap_{t>1} \mathrm{WF}_t(u) \subset \mathrm{WF}_A(u), \quad u \in \mathcal{D}'(U),$$

where WF_A denotes the analytic wave front set.

Let

$$\mathrm{WF}^{\{\sigma\}}(u) = \bigcap_{\tau>0} \mathrm{WF}_{\tau,\sigma}(u), \quad u \in \mathcal{D}'^{\{\sigma\}}(U),$$

and

$$\mathrm{WF}^{(\sigma)}(u) = \bigcup_{\tau>0} \mathrm{WF}_{\tau,\sigma}(u), \quad u \in \mathcal{D}'^{(\sigma)}(U).$$

For such wave front sets we have the following corollary which is an immediate consequence of Lemma 1.

Corollary 1. *Let $u \in \mathcal{D}'^{\{\sigma\}}(U)$ (resp. $\mathcal{D}'^{(\sigma)}(U)$), $\sigma > 1$. Then $(x_0, \xi_0) \notin \mathrm{WF}^{\{\sigma\}}(u)$ (resp. $(x_0, \xi_0) \notin \mathrm{WF}^{(\sigma)}(u)$) if and only if there exists a conic neighborhood Γ of ξ_0; a compact neighborhood K of x_0; and $\phi \in \mathcal{D}^{\{\sigma\}}(U)$ (resp $\phi \in \mathcal{D}^{(\sigma)}(U)$) such that $\mathrm{supp}\,\phi \subseteq K$, $\phi = 1$ on some neighborhood of x_0 and*

$$|\widehat{\phi u}(\xi)| \leq A|\xi|^{-H\left(\frac{\ln|\xi|}{\ln(\ln|\xi|)}\right)^{\frac{1}{\sigma-1}}}, \quad \xi \in \Gamma,$$

for some $A, H > 0$ (resp. for any $H > 0$ there exists $A > 0$).

We write $u(x) = F(x + i\Gamma 0)$ if $u(x)$ is obtained as boundary value of the analytic function F as $y \to 0$ in Γ. Recall (cf. [9])

$$\Gamma_0 = \{\xi \in \mathbf{R}^d \,|\, y \cdot \xi \geq 0 \text{ for all } y \in \Gamma\}$$

denotes the dual cone of Γ.

To conclude the paper we prove the following theorem.

Theorem 2. *Let the assumptions of Theorem 1 hold, and let $u(x) = F(x + i\Gamma 0) \in \mathcal{D}'^{(\sigma)}(U)$ (resp. $\mathcal{D}'^{\{\sigma\}}(U)$). Then*

$$\mathrm{WF}^{(\sigma)}(u) \subseteq U \times \Gamma_0, \quad (resp. \ \mathrm{WF}^{\{\sigma\}}(u) \subseteq U \times \Gamma_0).$$

More precisely, if $u(x) = F(x + i\Gamma 0) \in \mathcal{D}'_{\{\tau/2^{\sigma-1},\sigma\}}(U)$ (resp. $\mathcal{D}'_{(\tau/2^{\sigma-1},\sigma)}(U)$) then

$$\mathrm{WF}_{\{(2^\sigma-1)\tau,\sigma\}}(u) \subseteq U \times \Gamma_0, \quad (resp. \ \mathrm{WF}_{((2^\sigma-1)\tau,\sigma)} \subseteq U \times \Gamma_0).$$

Proof. Fix $x_0 \in U$ and $\xi_0 \notin \Gamma_0 \backslash \{0\}$. Then there exists $Y = (Y_1, \ldots, Y_d) \in \Gamma$, $|Y| < \gamma$, such that $Y \cdot \xi_0 < 0$. Moreover, there exists conical neighborhood V of ξ_0 and constant $\gamma_1 > 0$ such that $Y \cdot \xi \leq -\gamma_1|\xi|$, for all $\xi \in V$. To see that, note that there exists $B_r(\xi_0)$ such that $Y \cdot \xi < 0$ for all $\xi \in B_r(\xi_0)$. The assertion follows for $V = \{s\xi \,|\, s > 0, \ \xi \in B_r(\xi_0)\}$ and $\gamma_1 = \inf\limits_{\xi \in V, |\xi|=1} (-Y) \cdot \xi$.

Let $\tau > 0$ and $\tau_0 = (2^\sigma - 1)\tau$. If $u(x) = F(x + i\Gamma 0) \in \mathcal{D}'_{\tau/2^{\sigma-1},\sigma}(U)$ as in Theorem 1, then

$$|F(z)| \leq A \exp\{T_{\tau_0,\sigma,h_1}(1/|y|)\}, \quad z = x + iy \in Z, \tag{23}$$

for suitable constants $A, h_1 > 0$.

Choose $\varphi \in \mathcal{D}^K_{\tau/4^{\sigma-1},\sigma}$ such that $\varphi = 1$ in a neighborhood of x_0 and let Z_Y be as in (14). Then there exists Φ (see (13)) such that

$$|\Phi(z)| \leq A_1, \quad \text{and} \quad |\partial_{\overline{z_j}}\Phi(z)| \leq A_2 \exp\{-T_{\tau_0/2^{\sigma-1},\sigma,h_2}(1/|tY|)\}, \tag{24}$$

$z \in Z_Y$, $1 \leq j \leq d$, for suitable constants $A_1, A_2, h_2 > 0$.

Note that formula (20) implies

$$\widehat{(\varphi u)}(\xi) = \langle u(x)e^{-ix\cdot\xi}, \varphi(x)\rangle = \int_K F(x+iY)e^{-i(x+iY)\cdot\xi}\Phi(x+iY)dx$$

$$+ 2i\sum_{j=1}^d Y_j \int_0^1 \int_K \partial_{\overline{z_j}}\Phi(x+itY)F(x+itY)e^{-i(x+itY)\cdot\xi}dtdx, \quad \xi \in V. \tag{25}$$

Using (23) and (24) we have

$$|F(x+iY)\Phi(x+iY)e^{-i(x+iY)\xi}| \leq B\,e^{-\gamma_1|\xi|}, \quad x \in K, \xi \in V, \tag{26}$$

for some $B > 0$.

Moreover, for $z \in Z_Y$ and $\xi \in V$ we have

$$|F(z)\partial_{\overline{z_j}}\Phi(z)e^{-iz\cdot\xi}|$$

$$\leq C\exp\{T_{\tau_0,\sigma,h_1}(1/|tY|) - T_{\tau_0/2^{\sigma-1},\sigma,h_2}(1/|tY|) - t\gamma_1|\xi|\}$$

$$\leq C_1\exp\{-T_{\tau_0,\sigma,c_{h_1,h_2}}(1/(t\gamma_1)) - t\gamma_1|\xi|\} \leq C_2\exp\{-T_{\tau_0,\sigma,c'_{h_1,h_2}}(|\xi|)\}, \tag{27}$$

for suitable constants $C_1, C_2, c_{h_1,h_2}, c'_{h_1,h_2} > 0$, where we have used inequalities (A2) and (A3).

Finally, using (25)–(27) we obtain

$$|\widehat{(\varphi u)}(\xi)| \leq B_1(e^{-\gamma_1|\xi|} + \exp\{-T_{\tau_0,\sigma,c'_{h_1,h_2}}(|\xi|)\}) \leq B_2\exp\{-T_{\tau_0,\sigma,c'_{h_1,h_2}}(|\xi|)\},$$

for $\xi \in V$ and for suitable constant $B_2 > 0$. This completes the proof. $\quad\square$

5. Conclusions

Various classes of (ultra)distributions are commonly introduced as topological duals of suitable test function spaces, or as equivalence classes of certain Cauchy sequences of smooth functions. Another approach is given through their representations as finite sums of boundary values of analytic functions. We refer to [11] for a history, motivation and a detailed study of the subject. In this paper, we give a characterization of analytic functions whose boundary values are elements of intermediate spaces between the spaces of Schwartz distributions and any space of Gevrey ultradistributions. Test function spaces for such spaces of ultradistributions are related to the so-called extended Gervey regularity studied by the authors in [1–4,10,20]. We note that the extended Gevrey classes are different from any Carleman class which appears to be essential for many calculations, leading to the use of novel tools and techniques. In particular, we have used asymptotic properties of the Lambert W function in order to describe appropriate logarithmic growth rate in our calculations. This tool appears in the theory of new ultradistribution spaces for the first time (see also [4]). Since we relaxed the condition of stability under ultradifferentiable operators, to prove our main result, Theorem 1, we had to preform nontrivial changes in proofs of related results of classical theory (cf. [13]). This methodology could be used in other situations as well. For example, in future research we will consider the Paley–Wiener theorem for the new spaces of ultradistributions. This, in turn, will be used to prove the structure theorems in terms of boundary values of analytic functions; cf. Remark 1. We will also study other classes of two parameter sequences $M_p^{\tau,\sigma}$, apart from $p^{\tau p^\sigma}$, $p \in \mathbf{N}$, $\tau > 0$, $\sigma > 1$. This will be done in the spirit of Komastu [5].

Appendix A

In the following Lemma we study $T_{\tau,\sigma,h}(k)$ in some detail.

Lemma A1. *Let $h > 0$, and $T_{\tau,\sigma,h}$ be given by* (1)*, and let $T^*_{\tau,\sigma,h}$ be given by* (3)*. Then*

(*a*) *If $h_1 < h_2$ then $T_{\tau,\sigma,h_1}(k) < T_{\tau,\sigma,h_2}(k)$, $k > 0$. Moreover, for any $h > 0$ there exists $H > h$ such that*

$$T_{\tau,\sigma,h}(k) \leq T^*_{\tau,\sigma,h}(k) \leq T_{\tau,\sigma,H}(k), \quad k > 0. \tag{A1}$$

(*b*) *For $h_1, h_2 > 0$ there exists $C, c_{h_1,h_2} > 0$ such that*

$$T_{\tau,\sigma,h_1}(k) + T_{\tau,\sigma,h_2}(k) \leq T_{\tau/2^{\sigma-1},\sigma,c_{h_1,h_2}}(k) + \ln C \quad k > 0, \tag{A2}$$

(*c*) *For every $h > 0$ there exists $H > 0$ such that*

$$T_{\tau,\sigma,H}(l) \leq T_{\tau,\sigma,h}(1/k) + kl, \quad k, l > 0. \tag{A3}$$

Proof. (a) Notice that for arbitrary $h > 0$,

$$\ln \frac{h^{p^\sigma} k^p}{p^{\tau p^\sigma}} \leq \ln \frac{p^p h^{p^\sigma} k^p}{p^{\tau p^\sigma}} \leq \ln \frac{(Ch)^{p^\sigma} k^p}{p^{\tau p^\sigma}}, \quad k > 0,$$

where for the second inequality we use that for every $\sigma > 1$ there exists $C > 1$ such that $p^p \leq C^{p^\sigma}$, $p \in \mathbf{N}$ (see the proof of Proposition 2.1. in [1]). Now (A1) follows by putting $H = Ch$.

(b) Let $h_1, h_2 > 0$. We will use the following simple inequality

$$p^\sigma + q^\sigma \leq (p + q)^\sigma \leq 2^{\sigma-1}(p^\sigma + q^\sigma), \quad p, q \in \mathbf{N}. \tag{A4}$$

Since, $h_1^{p^\sigma} h_2^{q^\sigma} \leq (h_1 + h_2)^{p^\sigma + q^\sigma}$ we conclude that

$$h_1^{p^\sigma} h_2^{q^\sigma} \leq (h_1 + h_2)^{(p+q)^\sigma} \quad \text{when} \quad h_1 + h_2 \geq 1$$

and

$$h_1^{p^\sigma} h_2^{q^\sigma} \leq (h_1 + h_2)^{(1/2^{\sigma-1})(p+q)^\sigma} \quad \text{when} \quad 0 < h_1 + h_2 < 1.$$

Hence, there exists $0 < c_\sigma \leq 1$ such that

$$\ln \frac{h_1^{p^\sigma} k^p}{p^{\tau p^\sigma}} + \ln \frac{h_2^{q^\sigma} k^q}{q^{\tau q^\sigma}} \leq \ln \frac{(C(h_1 + h_2)^{c_\sigma})^{(p+q)^\sigma} k^{p+q}}{(p + q)^{(\tau/2^{\sigma-1})(p+q)^\sigma}} + \ln C, \quad p, q \in \mathbf{N},$$

where $C > 0$ is constant appearing in $\widetilde{(M.2)}$. Now (A2) follows after taking supremums over $p, q \in \mathbf{N}$.

(c) Recall (see [6]) that there exists $A > 0$ such that $kl = \sup\limits_{p \in \mathbf{N}} \ln \dfrac{A^p k^p l^p}{p^p}$. Note that for every $\sigma > 1$ there exists $0 < C < 1$ such that $\dfrac{1}{p^p} \geq C^{p^\sigma}$, $\quad p \in \mathbf{N}$.

Then for arbitrary $h > 0$ we have

$$T_{\tau,\sigma,h}(1/k) + kl = \sup_{p,q \in \mathbf{N}} \ln \frac{h^{p^\sigma}}{k^p p^{\tau p^\sigma}} \frac{A^q k^q l^q}{q^q} \geq \sup_{p,q \in \mathbf{N}, p = q} \ln \frac{(A'Ch)^{p^\sigma} l^p}{p^{\tau p^\sigma}} = T_{\tau,\sigma,H}(l), \quad k,l > 0,$$

where $A' = \min\{1, A\}$. This proves (A3). $\quad\square$

Finally, we discuss certain stability and embedding properties of $\mathcal{E}_{\tau,\sigma}(U)$ given by (5) and (6). Analogous considerations hold when the spaces $\mathcal{D}_{\tau,\sigma}(U)$ from (7) and (8) are considered instead.

Let $a_\alpha \in \mathcal{E}_{(\tau,\sigma)}(U)$ (resp. $a_\alpha \in \mathcal{E}_{\{\tau,\sigma\}}(U)$), where U is an open set in $\mathbf{R}^d$. Then we say that

$$P(x,\partial) = \sum_{|\alpha|=0}^{\infty} a_\alpha(x)\partial^\alpha$$

is an ultradifferential operator of class (τ,σ) (resp. $\{\tau,\sigma\}$) on U if for every $K \subset\subset U$ there exists constant $L > 0$ such that for any $h > 0$ there exists $A > 0$ (resp. for every $K \subset\subset U$ there exists $h > 0$ such that for any $L > 0$ there exists $A > 0$) such that

$$\sup_{x \in K} |\partial^\beta a_\alpha(x)| \leq Ah^{|\beta|^\sigma} |\beta|^{\tau|\beta|^\sigma} \frac{L^{|\alpha|^\sigma}}{|\alpha|^{\tau 2^{\sigma-1}|\alpha|^\sigma}}, \quad \alpha,\beta \in \mathbf{N}^d.$$

We refer to [2] for the proof of the following continuity and embedding properties.

Proposition A1. (*a*) *Let $P(x,\partial)$ be an ultradifferential operator of class (τ,σ) (resp. $\{\tau,\sigma\}$). Then $P(x,\partial): \quad \mathcal{E}_{\tau,\sigma}(U) \longrightarrow \mathcal{E}_{\tau 2^{\sigma-1},\sigma}(U)$ is a continuous linear mapping; the same holds for*

$$P(x,\partial): \quad \varinjlim_{\tau \to \infty} \mathcal{E}_{\tau,\sigma}(U) \longrightarrow \varinjlim_{\tau \to \infty} \mathcal{E}_{\tau,\sigma}(U).$$

(*b*) *Let $\sigma_1 \geq 1$. Then for every $\sigma_2 > \sigma_1$*

$$\varinjlim_{\tau \to \infty} \mathcal{E}_{\tau,\sigma_1}(U) \hookrightarrow \varprojlim_{\tau \to 0^+} \mathcal{E}_{\tau,\sigma_2}(U).$$

(*c*) *If $0 < \tau_1 < \tau_2$, then*

$$\mathcal{E}_{\{\tau_1,\sigma\}}(U) \hookrightarrow \mathcal{E}_{(\tau_2,\sigma)}(U) \hookrightarrow \mathcal{E}_{\{\tau_2,\sigma\}}(U), \quad \sigma > 1,$$

and

$$\varinjlim_{\tau \to \infty} \mathcal{E}_{\{\tau,\sigma\}}(U) = \varinjlim_{\tau \to \infty} \mathcal{E}_{(\tau,\sigma)}(U),$$

$$\varprojlim_{\tau \to 0^+} \mathcal{E}_{\{\tau,\sigma\}}(U) = \varprojlim_{\tau \to 0^+} \mathcal{E}_{(\tau,\sigma)}(U), \quad \sigma > 1.$$

Consequently we obtain that

$$\varinjlim_{t \to \infty} \mathcal{E}_t(U) \hookrightarrow \mathcal{E}_{\tau,\sigma}(U) \hookrightarrow C^\infty(U), \quad \tau > 0, \sigma > 1,$$

where $\mathcal{E}_t(U)$ is Gevrey space with index $t > 1$.

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