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Some New Extensions of Multivalued Contractions in a b-metric Space and Its Applications

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Abstract: The H^{β}-Hausdorff–Pompeiu b-metric for $\beta \in [0,1]$ is introduced as a new variant of the Hausdorff–Pompeiu b-metric H. Various types of multi-valued H^{β}-contractions are introduced and fixed point theorems are proved for such contractions in a b-metric space. The multi-valued Nadler contraction, Czervik contraction, q-quasi contraction, Hardy Rogers contraction, weak quasi contraction and Ciric contraction existing in literature are all one or the other type of multi-valued H^{β}-contraction but the converse is not necessarily true. Proper examples are given in support of our claim. As applications of our results, we have proved the existence of a unique multi-valued fractal of an iterated multifunction system defined on a b-metric space and an existence theorem of Filippov type for an integral inclusion problem by introducing a generalized norm on the space of selections of the multifunction.

Keywords: b-metric space; H^{β}-Hausdorff–Pompeiu b-metric; multi-valued fractal; iterated multifunction system; integral inclusion

MSC: 47H10; 47H20; 54H25; 34A60

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1. Introduction

Romanian mathematician D. Pompeiu in [1] initiated the study of distance between two sets and introduced the Pompeiu metric. Hausdorff [2] further studied this concept and thereby introduced the Hausdorff–Pompeiu metric H induced by the metric d of a metric space (X, d), as follows:

For any two subsets A and B of X, the function H given by $H(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{x \in B} d(x, A)\}$ is a metric for the set of compact subsets of X. Note that

$$\begin{aligned} H(A,B) &= \max\{\beta \sup_{x\in A} d(x,B) + (1-\beta)\sup_{x\in B} d(x,A), \beta \sup_{x\in B} d(x,A) \\ &+ (1-\beta)\sup_{x\in A} d(x,B)\} \text{ for } \beta = 0 \text{ or } 1. \end{aligned} \tag{1}$$

Nadler [3] extending the Banach contraction principle introduced multi-valued contraction principle in a metric space using the Hausdorff–Pompieu metric H. Thereafter many extensions and generalizations of multi-valued contraction appeared (see [4–7]). In 1998, Czerwik [8] introduced the Hausdorff–Pompeiu b-metric H_b as a generalization of Hausdorff–Pompeiu metric H and proved the b-metric space version of Nadler contraction principle. Czervik's result drew attention of many researchers who further obtained many generalized multi-valued contractions, named q-quasi contraction [9], Hardy Rogers contraction [10], weak quasi contraction [11], Ciric contraction [12], etc. and proved the existence theorem for such contraction mappings in a b-metric space. The aim of this work is to introduce new variants of the Hausdorff–Pompeiu b-metric and thereby introduce various types of multi-valued H^{β}-contraction and prove fixed point theorems for such types of contractions in a b-metric space. It is shown that for any b-metric space (X, d_s) and $\beta \in [0,1]$, the function given in (1) defines a b-metric for the set of closed and bounded subsets of X. We call this metric H^{β}-Hausdorff–Pompeiu b-metric induced by the b-metric d_s. Thereafter, using this H^{β}-Hausdorff–Pompeiu b-metric, we have introduced various types of multi-valued H^{β}-contraction and proved fixed point theorems for such types of contractions in a b-metric space. The multi-valued Nadler contraction [3], Czervik contraction [8], q-quasi contraction [9], Hardy Rogers contraction [10], Ciric contraction [12], weak quasi contraction [11] existing in literature are all one or the other type of multi-valued H^{β}-contraction; however, it is shown with proper examples that the converse is not necessarily true. Finally to demonstrate the applications of our results, we prove the existence of a unique multi-valued fractal of an iterated multifunction system defined on a b-metric space and also an existence theorem of Filippov type for an integral inclusion problem by introducing a generalized norm on the space of selections of the multifunction.

2. Preliminaries

Bakhtin [13] introduced b-metric space as follows:

Definition 1 ([13])**.** *Let X be a nonempty set and $d_s\colon X \times X \to [0, \infty)$ satisfies:*

1. *$d_s(x,y) = 0$ if and only if $x = y$ for all $x, y \in X$;*
2. *$d_s(x,y) = d(y,x)$ for all $x, y \in X$;*
3. *there exist a real number $s \geq 1$ such that $d(x,y) \leq s[d_s(x,z) + d_s(z,y)]$ for all $x, y, z \in X$.*

Then, d_s is called a b-metric on X and (X, d_s) is called a b-metric space with coefficient s.

Example 1. *Let $X = R$ and $d:\ X \times X \to [0, \infty)$ be given by $d(x,y) = |x-y|^2$, for all $x, y \in X$. Then (X, d) is a b-metric space with coefficient $s = 2$.*

Definition 2 ([13])**.** *Let (X, d_s) is a b-metric space with coefficient s.*

(i) *A sequence $\{x_n\}$ in X, converges to $x \in X$, if $lim_{n\to\infty} d_s(x_n, x) = 0$.*
(ii) *A sequence $\{x_n\}$ in X is a Cauchy sequence if for all $\epsilon > 0$, there exist a positive integer $n(\epsilon)$ such that $d_s(x_n, x_m) < \epsilon$ for all $n, m \geq n(\epsilon)$.*
(iii) *(X, d_s) is complete if every Cauchy sequence in X is convergent.*

For some recent fixed point results of single valued and multi-valued mappings in a b-metric space, see [9,14–18]. Throughout this paper, (X, d_s) will denote a complete b-metric space with coefficient s and $CB^{d_s}(X)$ the collection of all nonempty closed and bounded subsets of X with respect to d_s.

For $A, B \in CB^{d_s}(X)$, define $d_s(x, A) = \inf\{d_s(x,a) : a \in A\}$, $\delta_{d_s}(A,B) = \sup_{a\in A} d_s(a, B)$ and $H_{d_s}(A,B) = \max\{\delta_{d_s}(A,B), \delta_{d_s}(B,A)\}$. Czerwik [8] has shown that H_{d_s} is a b-metric in the set $CB^{d_s}(X)$ and is called the Hausdorff–Pompeiu b-metric induced by d_s.

Motivated by the fact that a b-metric is not necessarily continuous (as $\frac{1}{s^2} d_s(x,y) \leq \underline{lim}_{n\to\infty} d_s(x_n, y_n) \leq \overline{lim}_{n\to\infty} d_s(x_n, y_n) \leq s^2 d_s(x,y)$ and $\frac{1}{s} d_s(x,y) \leq \underline{lim}_{n\to\infty} d_s(x_n, y) \leq \overline{lim}_{n\to\infty} d_s(x_n, y) \leq s d_s(x,y)$ see [19–21]), Miculescu and Mihail [12] introduced the following concept of $*$-continuity.

Definition 3 ([12])**.** *The b-metric d_s is called $*$-continuous if for every $A \in CB^{d_s}(X)$, every $x \in X$ and every sequence $\{x_n\}$ of elements from X with $lim_{n\to\infty} x_n = x$, we have $lim_{n\to\infty} d_s(x_n, A) = d_s(x, A)$.*

Proposition 1 ([17])**.** *For any $A \subseteq X$,*

$$a \in \bar{A} \Longleftrightarrow d_s(a, A) = 0.$$

Lemma 1 ([12]). *Let $\{x_n\}$ be a sequence in (X, d_s). If there exists $\lambda \in [0,1)$ such that $d_s(x_n, x_{n+1}) \leq \lambda d_s(x_{n-1}, x_n)$ for all $n \in N$, then $\{x_n\}$ is a Cauchy sequence.*

The following lemma can also be proved using the same technique of proof of the above Lemma.

Lemma 2. *Let $\{x_n\}$ be a sequence in (X, d_s). If there exists $\lambda, \epsilon \in [0,1)$, with $\lambda < \epsilon$ such that $d_s(x_n, x_{n+1}) \leq \lambda d_s(x_{n-1}, x_n) + \epsilon^n$ for all $n \in N$, then $\{x_n\}$ is a Cauchy sequence.*

Czerwik [8] introduced multi-valued contraction in a b-metric space and proved that every multi-valued contraction mapping in a b-metric space has a fixed point.

Definition 4 ([8]). *A mapping $T: X \to CB^{d_s}(X)$ is a multi-valued contraction if there exists $\alpha \in (0, \frac{1}{s})$, such that $g^\imath, g^\jmath \in X$ implies $H_{d_s}(Tg^\imath, Tg^\jmath) \leq \alpha\, d_s(g^\imath, g^\jmath)$.*

Theorem 1 ([8]). *Every multi-valued contraction mapping defined on (X, d_s) has a fixed point.*

Thereafter using Hausdorff–Pompieu b-metric H_{d_s}, many authors introduced several generalized multi-valued contractions in a b-metric space (see Definitions 5 to 8 below) and proved the existence of fixed points for such generalized multi-valued contraction mappings.

Definition 5 ([9]). *A mapping $T: X \to CB^{d_s}(X)$ is a q-multi-valued quasi contraction if there exists $q \in (0, \frac{1}{s})$, such that $g^\imath, g^\jmath \in X$ implies*

$$H_{d_s}(Tg^\imath, Tg^\jmath) \leq q\, max\{d_s(g^\imath, g^\jmath), d_s(g^\imath, Tg^\imath), d_s(g^\jmath, Tg^\jmath), d_s(g^\imath, Tg^\jmath), d_s(g^\jmath, Tg^\imath)\}.$$

Definition 6 ([12]). *A mapping $T: X \to CB^{d_s}(X)$ is a q-multi-valued Ciric contraction if there exists $q, c, d \in (0,1)$, such that $g^\imath, g^\jmath \in X$ implies*

$$H_{d_s}(Tg^\imath, Tg^\jmath) \leq q\, max\{d_s(g^\imath, g^\jmath), c\, d_s(g^\imath, Tg^\imath), c\, d_s(g^\jmath, Tg^\jmath), \frac{d}{2}(d_s(g^\imath, Tg^\jmath) + d_s(g^\jmath, Tg^\imath))\}.$$

Definition 7 ([10]). *A mapping $T: X \to CB^{d_s}(X)$ is a multi-valued Hardy–Roger's contraction if there exists $a, b, c, e, f \in (0,1)$, $a + b + c + 2(e + f) < 1$, such that $g^\imath, g^\jmath \in X$ implies $H_{d_s}(Tg^\imath, Tg^\jmath) \leq a\, d_s(g^\imath, g^\jmath) + b\, d_s(g^\imath, Tg^\imath) + c\, d_s(g^\jmath, Tg^\jmath) + e\, d_s(g^\imath, Tg^\jmath) + f\, d_s(g^\jmath, Tg^\imath)$.*

Definition 8 ([11]). *A mapping $T: X \to CB^{d_s}(X)$ is a multi-valued weak quasi contraction if there exists $q \in (0,1)$ and $L \geq 0$ such that $g^\imath, g^\jmath \in X$ implies $H_{d_s}(Tg^\imath, Tg^\jmath) \leq q\, max\{d_s(g^\imath, g^\jmath), d_s(g^\imath, Tg^\imath), d_s(g^\jmath, Tg^\jmath)\} + L\, d_s(g^\jmath, Tg^\imath)$.*

3. Main Results

3.1. The H^β Hausdorff–Pompieu b-metric

Definition 9. *For $U, V \in CB^{d_s}(X)$, $\beta \in [0,1]$, we define*

$$R^\beta(U, V) = \beta \delta_{d_s}(U, V) + (1 - \beta)\delta_{d_s}(V, U)$$

and

$$H^\beta(U, V) = \max\{R^\beta(U, V), R^\beta(V, U)\}.$$

Proposition 2. *Let $U, V, W \in CB^{d_s}(X)$, we have*

(i) $H^\beta(U, V) = 0$ *if and only if* $U = V$.
(ii) $H^\beta(U, V) = H^\beta(V, U)$.
(iii) $H^\beta(U, V) \leq s[H^\beta(U, W) + H^\beta(W, V)]$.

Proof. (i) By definition, $H^{\beta}(U,V)=0$ implies $\max\{\beta\delta_{d_s}(U,V)+(1-\beta)\delta_{d_s}(V,U),(1-\beta)\delta_{d_s}(U,V)+\beta\delta_{d_s}(V,U)\}=0$. This gives $\delta_{d_s}(U,V)=0$ and $\delta_{d_s}(V,U)=0$. Now, $\delta_{d_s}(U,V)=0$ implies $d_s(u,V)=0$ for all $u\in U$. By Proposition 1, we have $u\in\bar{V}=V$ for all $u\in U$ and so $U\subseteq V$. Similarly, $\delta_{d_s}(V,U)=0$ will imply $V\subseteq U$ and so $U=V$. The reverse implication is clear from the definition.

(ii) Follows from the definition of $H^{\beta}(U,V)$.

(iii) Let u,v,w be arbitrary elements of U,V,W, respectively. Then we have

$$d_s(u,V)\leq s[d_s(u,w)+d_s(w,V)].$$

Since w is arbitrary, we get

$$d_s(u,V)\leq s[d_s(u,w)+\delta_{d_s}(W,V)]\leq s[d_s(u,W)+\delta_{d_s}(W,V)].$$

Again, since u is arbitrary, we get

$$\delta_{d_s}(U,V)\leq s[\delta_{d_s}(U,W)+\delta_{d_s}(W,V)].$$

Similarly, we have

$$\delta_{d_s}(V,U)\leq s[\delta_{d_s}(V,W)+\delta_{d_s}(W,U)].$$

Therefore,

$$\begin{aligned} R^{\beta}(U,V) &= \beta\delta_{d_s}(U,V)+(1-\beta)\delta_{d_s}(V,U)\\ &\leq \beta s[\delta_{d_s}(U,W)+\delta_{d_s}(W,V)]+(1-\beta)s[\delta_{d_s}(V,W)+\delta_{d_s}(W,U)]\\ &\quad = s[\beta\delta_{d_s}(U,W)+(1-\beta)\delta_{d_s}(W,U)]+s[\beta\delta_{d_s}(W,V)+(1-\beta)\delta_{d_s}(V,W)]\\ &\quad = s[R^{\beta}(U,W)+R^{\beta}(W,V)]. \end{aligned}$$

Similarly

$$R^{\beta}(V,U)\quad\leq\quad s[R^{\beta}(V,W)+R^{\beta}(W,U)].$$

Then, we have

$$\begin{aligned} H^{\beta}(U,V) &= \max\{R^{\beta}(U,V),R^{\beta}(V,U)\}\\ &\leq \max\{s[R^{\beta}(U,W)+R^{\beta}(W,V)],s[R^{\beta}(V,W)+R^{\beta}(W,U)]\}\\ &\leq \max\{sR^{\beta}(U,W),sR^{\beta}(W,U)\}+\max\{sR^{\beta}(W,V),sR^{\beta}(V,W)\}\\ &= s[H^{\beta}(U,W)+H^{\beta}(W,V)]. \end{aligned}$$

□

Remark 1. *In view of Proposition 2, the function $H^{\beta}: CB^{d_s}(X)\times CB^{d_s}(X)\to[0,+\infty)$, is a b-metric in $CB^{d_s}(X)$ and we call it the H^{β}-Hausdorff–Pompeiu b-metric induced by d_s.*

Remark 2. *For $\beta\in[0,1]$ $H^{\beta}(A,B)\leq H_{d_s}(A,B)$ and for $\beta=0\vee 1$ $H^{\beta}(A,B)=H_{d_s}(A,B)$.*

Remark 3. *The Hausdorff–Pompeiu b-metric H^{β} is equivalent to the Hausdorff–Pompeiu b-metric H_{d_s} in the sense that for any two sets A and B, $H^{\beta}(A,B)\leq H_{d_s}(A,B)\leq 2H^{\beta}(A,B)$. However, the examples and applications provided in this paper illustrates the advantages of using H^{β}-Hausdorff–Pompeiu b-metric in fixed point theory and its applications.*

Theorem 2. *For all $u,v\in X$, $U,V\in CB^{d_s}(X)$ and $\beta\in[0,1]$, the following relations holds:*

(1) $d_s(u,v)=H^{\beta}(\{u\},\{v\})$,

(2) $U \subset \overline{S}(V, r_1), V \subset \overline{S}(U, r_2) \Rightarrow H^{\beta}(U, V) \leq r$ *where* $r = \max\{\beta r_1 + (1-\beta) r_2, \beta r_2 + (1-\beta) r_1\}$,

(3) $H^{\beta}(U, V) < r \Rightarrow \exists r_1, r_2 > 0$ *such that* $r = \max\{\beta r_1 + (1-\beta) r_2, \beta r_2 + (1-\beta) r_1\}$ *and* $U \subset S(V, r_1), V \subset S(U, r_2)$.

Proof. (1) This is immediate from the definition of H^{β}.
(2) Since $U \subset \overline{S}(V, r_1), V \subset \overline{S}(U, r_2)$, we have that

$$\forall u \in U, \exists v_u \in V \quad \text{satisfying} \quad d_s(u, v_u) \leq r_1$$

and

$$\forall v \in V, \exists u_v \in U \quad \text{satisfying} \quad d_s(u_v, v) \leq r_2$$

$$\Rightarrow \quad \inf_{v \in V} d_s(u, v) \leq r_1 \text{ for every } u \in U \text{ and } \inf_{u \in U} d_s(u, v) \leq r_2 \text{ for every } v \in V.$$

$$\Rightarrow \quad \sup_{u \in U} \Big(\inf_{v \in V} d_s(u, v) \Big) \leq r_1 \text{ and } \sup_{v \in V} \Big(\inf_{u \in U} d_s(u, v) \Big) \leq r_2.$$

Then, $H^{\beta}(U, V) \leq r$ where $r = \max\{\beta r_1 + (1-\beta) r_2, \beta r_2 + (1-\beta) r_1\}$.
(3) Let $H^{\beta}(U, V) = k < r$. Then, there is some $k_1, k_2 > 0$ satisfying

$$k = \max\{\beta k_1 + (1-\beta) k_2, \beta k_2 + (1-\beta) k_1\},$$

$$\delta(U, V) = \sup_{u \in U}(\inf_{v \in V} d_s(u, v)) = k_1,\ \delta(V, U) = \sup_{v \in V}(\inf_{u \in U} d_s(u, v)) = k_2.$$

Since $0 < k < r$, we can find $r_1, r_2 > 0$ such that $k_1 < r_1$, $k_2 < r_2$ and $r = \max\{\beta r_1 + (1-\beta) r_2, \beta r_2 + (1-\beta) r_1\}$. Thus,

$$\inf_{v \in V} d_s(u, v) \leq k_1 < r_1 \text{ for every } u \in U \text{ and } \inf_{u \in U} d_s(u, v)) \leq k_2 < r_2 \text{ for every } v \in V.$$

Then, for any $u \in U$ there is some $v_u \in V$ satisfying

$$d_s(u, v_u) < \inf_{v \in V} d_s(u, v) + r_1 - k_1 \leq r_1.$$

and, for any $v \in V$ there is some $u_v \in U$ satisfying

$$d_s(u_v, v) < \inf_{u \in U} d_s(u, v) + r_2 - k_2 \leq r_2.$$

Thus, for any $u \in U$ and $v \in V$ we have

$$u \in \bigcup_{v \in V} S(v; r_1) \text{ and } v \in \bigcup_{u \in U} S(u; r_2),$$

which implies

$$U \subset S(V, r_1) \text{ and } V \subset S(U, r_2).$$

□

Remark 4. *From Theorem 2 (2) and (3), it follows that the following statements also hold:*
$(2')$ $U \subset S(V, r_1), V \subset S(U, r_2) \Rightarrow H^{\beta}(U, V) \leq r$ *where* $r = \max\{\beta r_1 + (1-\beta) r_2, \beta r_2 + (1-\beta) r_1\}$
and
$(3')$ $H^{\beta}(A, B) < r \Rightarrow \exists r_1, r_2 > 0$ *such that* $r = \max\{\beta r_1 + (1-\beta) r_2, \beta r_2 + (1-\beta) r_1\}$ *and* $U \subset \overline{S}(V, r_1), V \subset \overline{S}(U, r_2)$.

Theorem 3. *Let* $U, V \in CB^{d_s}(X)$ *and* $\beta \in [0, 1]$. *Then the following equalities holds:*
(4) $H^{\beta}(U, V) = \inf\{r > 0 : U \subset S(V, r_1), V \subset S(U, r_2)\}$;

(5) $H^\beta(U,V) = \inf\{r > 0 : U \subset \overline{S}(V,r_1), U \subset \overline{S}(V,r_2)\}$,
where $r = \max\{\beta r_1 + (1-\beta)r_2, \beta r_2 + (1-\beta)r_1\}$.

Proof. By (2′), we have

$$H^\beta(U,V) \le \inf\{r > 0 : U \subset S(V,r_1), U \subset S(V,r_2)\}, r = \max\{\beta r_1 + (1-\beta)r_2, \beta r_2 + (1-\beta)r_1\}. \tag{2}$$

Now let $H^\beta(U,V) = k$, and let $t > 0$. Then $H^\beta(U,V) < k+t$. By Condition (3) of Theorem 2 we can find $t_1, t_2 > 0$ with $\max\{\beta t_1 + (1-\beta)t_2, \beta t_2 + (1-\beta)t_1\} = t$ such that $U \subset S(V;k+t_1)$ and $V \subset S(U;k+t_2)$. Thus,

$$\{r > 0 : U \subset S(V,r_1), B \subset S(U,r_2)\} \supset \{k+t : t > 0, U \subset S(V,k+t_1), V \subset S(U,k+t_2)\}.$$

This implies that

$$\inf\{r > 0 : U \subset S(V,r_1), V \subset S(U,r_2)\} \le \inf\{k+t : t > 0\} = k = H^\beta(U,V).$$

To conclude,

$$H^\beta(U,V) = \inf\{r > 0 : \ U \subset S(V,r_1), V \subset S(U,r_2)\}, r = \max\{\beta r_1 + (1-\beta)r_2, \beta r_2 + (1-\beta)r_1\}. \tag{3}$$
□

Theorem 4. *If (X,d_s) is a complete b-metric space, then $(CB^{d_s}(X), H^\beta)$ for any $\beta \in [0,1]$ is also complete. Moreover, $C(X)$ is a closed subspace of $(CB^{d_s}(X), H^\beta)$.*

Proof. Suppose (X,d_s) is complete and the sequence $\{A_n\}_{n\in\mathbf{N}}$ in $CB^{d_s}(X)$ is a Cauchy sequence. Let $B = \{x \in X : \ \forall\epsilon > 0, m \in \mathbf{N}, \exists\, n \ge m$ for which $S(x,\epsilon) \cap A_n \neq \emptyset\}$.

Let $\epsilon > 0$. By definition of Cauchy sequence, we can find $m(\epsilon) \in \mathbf{N}$ for which, $n \ge m(\epsilon)$ implies $H^\beta(A_n, A_{m(\epsilon)}) < \epsilon$. By Theorem 3 (4), $\exists\, \epsilon_1, \epsilon_2 > 0$ with $\epsilon = \max\{\beta\epsilon_1 + (1-\beta)\epsilon_2, \beta\epsilon_2 + (1-\beta)\epsilon_1\}$ and $m(\epsilon_1), m(\epsilon_2) \in \mathbf{N}$ such that $\min\{m(\epsilon_1), m(\epsilon_2)\} \ge m(\epsilon)$, $A_n \subset S(A_{m(\epsilon_1)}, \epsilon_1)$ for $n \ge m(\epsilon_1)$ and $A_{m(\epsilon_2)} \subset S(A_n, \epsilon_2)$ $n \ge m(\epsilon_2)$. Then we have $B \subset \overline{S}(A_{m(\epsilon_1)}, \epsilon_1)$, and so

(i) $B \subset \overline{S}(A_{m(\epsilon_1)}, 4\epsilon_1)$ holds.

Now set $\overline{\epsilon}_k = \dfrac{\epsilon_1}{2^k}, k \in \mathbf{N}$, and choose $n_k = m(\overline{\epsilon}_k) \in \mathbf{N}$ such that sequence $\{n_k\}_{k\in\mathbf{N}}$ is strictly increasing and

$$H^\beta(A_n, A_{n_k}) < \overline{\epsilon}_k, \ \forall n \ge n_k.$$

For some $p \in A_{n_0} = A_{m(\epsilon_1)}$, consider the sequence $\{p_{n_k}\}_{k\in\mathbf{N}}$ with $p_{n_0} = p$, $p_{n_k} \in A_{n_k}$ and $d_s(p_{n_k}, p_{n_{k-1}}) < \dfrac{\epsilon_1}{2^{k-2}}$. It follows that the sequence $\{p_{n_k}\}_{k\in\mathbf{N}}$ is a Cauchy sequence in the complete b-metric space (X,d_s) and so converges to some point $l \in X$.

Additionally, $d_s(p_{n_k}, p_{n_0}) < 4\epsilon_1$ implies $d_s(l,p) \le 4\epsilon_1$ and so $\inf_{y\in B} d_s(p,y) \le 4\epsilon_1$, that is, $p \in \overline{S}(B, 4\epsilon_1)$, from which we get

(ii) $A_{n_0} \subset \overline{S}(B, 4\epsilon_1)$.

Now, relations (i), (ii) from above and Theorem 2 (2) yields $H^\beta(A_{n_0}, B) \le 4\epsilon_1$. Since H^β is a b-metric on $CB^{d_s}(X)$, we have

$$H^\beta(A_n, B) \le s[H^\beta(A_n, A_{n_0}) + H^\beta(A_{n_0}, B)] < 5s\,\epsilon_1,$$

for any $n \ge m(\epsilon_1) = n_0$. Hence, sequence $\{A_n\}_{n\in\mathbf{N}}$ is convergent and $(CB^{d_s}(X), H^\beta)$ is complete. □

For the second part, consider the Cauchy sequence $\{A_n\}_{n\in\mathbf{N}}$ in $C(X)$ and consequently in $CB^{d_s}(X)$ and converging to some $A \in CB^{d_s}(X)$. Thus, if $\epsilon > 0$ is chosen, we can find $m(\epsilon) \in \mathbf{N}$ for which

$$H^\beta(A_n, A) < \frac{\epsilon}{2}\ \forall n \geq m(\epsilon), n \in \mathbf{N}.$$

Using (4) of Theorem 3, we get $\exists\, \epsilon_1, \epsilon_2 > 0$ with $\epsilon = \max\{\beta\epsilon_1 + (1-\beta)\epsilon_2, \beta\epsilon_2 + (1-\beta)\epsilon_1\}$ and $m(\epsilon_1), m(\epsilon_2) \in \mathbf{N}$ such that $\min\{m(\epsilon_1), m(\epsilon_2)\} \geq m(\epsilon)$, $A_n \subset S(A, \frac{\epsilon_1}{2})$ for $n \geq m(\epsilon_1)$ and $A \subset S(A_n, \frac{\epsilon_2}{2})$ for $n \geq m(\epsilon_2)$.

For any fixed $n_0 \geq m(\epsilon_2)$, we have, $A \subset S(A_{n_0}, \frac{\epsilon_2}{2})$ and the compactness of A_{n_0} in X (due to which it is also totally bounded) gives us $x_i^{\epsilon_2}, i \in \overline{1,p}$ such that $A_{n_0} \subset \bigcup_{i=1}^{p} S(x_i^{\epsilon_2}, \frac{\epsilon_2}{2})$, whence $A \subset \bigcup_{i=1}^{p} S(x_i^{\epsilon_2}, \epsilon_2)$. Therefore, $A \in C(X)$.

3.2. Applications to Fixed Point Theory

We begin this section by introducing various classes of multi-valued H^β-contractions in a b-metric space:

Definition 10. *$T : X \to CB^{d_s}(X)$ is a multi-valued H^β-contraction if we can find $\beta \in [0,1]$ and $k \in (0,1)$, such that*

$$H^\beta(Tg^\imath, Tg^\jmath) \leq k \cdot d_s(g^\imath, g^\jmath) \text{ for all } g^\imath, g^\jmath \in X. \tag{4}$$

Definition 11. *$T : X \to CB^{d_s}(X)$ is a multi-valued H^β-Ciric contraction if we can find $\beta \in [0,1]$ and $k \in (0, \frac{1}{s})$, such that for all $g^\imath, g^\jmath \in X$,*

$$H^\beta(Tg^\imath, Tg^\jmath) \leq k \cdot \max\{d_s(g^\imath, g^\jmath), d_s(g^\imath, Tg^\imath), d_s(g^\jmath, Tg^\jmath), \frac{d_s(g^\imath, Tg^\jmath) + d_s(g^\jmath, Tg^\imath)}{2s}\}. \tag{5}$$

Definition 12. *$T : X \to CB^{d_s}(X)$ is a multi-valued H^β-Hardy–Rogers contraction if we can find $\beta \in [0,1]$ and $a, b, c, e, f \in (0,1)$ with $a + b + s(c+e) + f < 1$, $\min\{s(a+e), s(b+c)\} < 1$ such that for all $g^\imath, g^\jmath \in X$,*

$$H^\beta(Tg^\imath, Tg^\jmath) \leq a \cdot d_s(g^\imath, Tg^\imath) + b \cdot d_s(g^\jmath, Tg^\jmath) + c \cdot d_s(g^\imath, Tg^\jmath) + e \cdot d_s(g^\jmath, Tg^\imath) + f \cdot d_s(g^\imath, g^\jmath). \tag{6}$$

Definition 13. *We say that $T : X \to CB^{d_s}(X)$ is a multi-valued H^β-quasi contraction if we can find $\beta \in [0,1]$ and $k \in (0, \frac{1}{s})$, such that for all $g^\imath, g^\jmath \in X$,*

$$H^\beta(Tg^\imath, Tg^\jmath) \leq k \cdot \max\{d_s(g^\imath, g^\jmath), d_s(g^\imath, Tg^\imath), d_s(g^\jmath, Tg^\jmath), d_s(g^\imath, Tg^\jmath), d_s(g^\jmath, Tg^\imath)\}. \tag{7}$$

Definition 14. *We say that $T : X \to CB^{d_s}(X)$ is a multi-valued H^β-weak quasi contraction if we can find $\beta \in [0,1]$, $k \in (0, \frac{1}{s})$ and $L \geq 0$, such that for all $g^\imath, g^\jmath \in X$,*

$$H^\beta(Tg^\imath, Tg^\jmath) \leq k \cdot \max\{d_s(g^\imath, g^\jmath), d_s(g^\imath, Tg^\imath), d_s(g^\jmath, Tg^\jmath)\} + Ld_s(g^\imath, Tg^\jmath). \tag{8}$$

Example 2. *Let $X = [0, \frac{7}{9}] \bigcup \{1\}$ and $d_s(g^\imath, g^\jmath) = |g^\imath - g^\jmath|^2$ for all $g^\imath, g^\jmath \in X$.*

Then $\{X, d_s\}$ is a b-metric space. Define the mapping $T : X \to CB^{d_s}(X)$ by

$$T(g^\imath) = \begin{cases} \{\frac{g^\imath}{4}\}, & \text{for } g^\imath \in [0, \frac{7}{9}] \\ \{0, \frac{1}{3}, \frac{5}{12}\}, & \text{for } g^\imath = 1. \end{cases}$$

Then T is a multi-valued H^β-contraction with $\beta = \frac{3}{4}$ and $\frac{217}{256} \leq k < 1$ as shown below.

We will consider the following different cases for the elements of X.

(i) $g^\imath, g^\jmath \in [0, \frac{7}{9}]$.

By Theorem 2(1), we have $H^{\frac{3}{4}}(Tg^\imath, Tg^\jmath) = d_s(\frac{g^\imath}{4}, \frac{g^\jmath}{4}) \leq k\, d_s(g^\imath, g^\jmath), \quad k \geq \frac{1}{16}$.

(ii) $g^\imath \in [0, \frac{7}{9}], g^\jmath = 1$.

We have the following sub cases:

(ii)(a) $g^\imath \in [0, \frac{2}{3}], g^\jmath = 1$. Then $Tg^\imath = \{\frac{g^\imath}{4}\}$ and $0 \leq \frac{g^\imath}{4} \leq \frac{1}{6}$. Therefore, we have $\delta_{d_s}(Tg^\imath, T1) = \delta_{d_s}(\{\frac{g^\imath}{4}\}, \{0, \frac{1}{3}, \frac{5}{12}\})$ and $\delta_{d_s}(T1, Tg^\imath) = \delta_{d_s}(\{0, \frac{1}{3}, \frac{5}{12}\}, \{\frac{g^\imath}{4}\})$. Note that for $0 \leq \frac{g^\imath}{4} \leq \frac{1}{6}$, $\frac{g^\imath}{4}$ is nearest to 0 and farthest from $\frac{5}{12}$. Therefore, $\delta_{d_s}(Tg^\imath, T1) = |\frac{g^\imath}{4} - 0|^2 = \frac{g^{\imath 2}}{16}$ and $\delta_{d_s}(T1, Tg^\imath) = |\frac{5}{12} - \frac{g^\imath}{4}|^2 = \frac{9g^{\imath 2} - 30g^\imath + 25}{144}$

Therefore,

$$\begin{aligned} H^{\frac{3}{4}}(Tg^\imath, T1) &= \max\{\frac{3}{4}\delta_{d_s}(Tg^\imath, T1) + \frac{1}{4}\delta_{d_s}(T1, Tg^\imath), \frac{3}{4}\delta_{d_s}(T1, Tg^\imath) + \frac{1}{4}\delta_{d_s}(Tg^\imath, T1)\} \\ &= \max\{\frac{25}{576} - \frac{10g^\imath}{192} + \frac{4g^{\imath 2}}{64}, \frac{75}{576} - \frac{30g^\imath}{192} + \frac{4g^{\imath 2}}{64}\} \\ &= \frac{75}{576} - \frac{30g^\imath}{192} + \frac{4g^{\imath 2}}{64} \leq k\, d_s(g^\imath, 1), k \geq \frac{279}{576}. \end{aligned}$$

($\frac{279}{576}$ is the maximum value of k which satisfies the above inequality for different values of $g^\imath$ in $[0, \frac{2}{3}]$.)

(ii)(b) $g^\imath \in (\frac{2}{3}, \frac{7}{9}], g^\jmath = 1$.

Then $Tg^\imath = \{\frac{g^\imath}{4}\}$ and $\frac{6}{36} < \frac{g^\imath}{4} \leq \frac{7}{36}$.

Therefore, we have $\delta_{d_s}(Tg^\imath, T1) = \delta_{d_s}(\{\frac{g^\imath}{4}\}, \{0, \frac{1}{3}, \frac{5}{12}\})$ and $\delta_{d_s}(T1, Tg^\imath) = \delta_{d_s}(\{0, \frac{1}{3}, \frac{5}{12}\}, \{\frac{g^\imath}{4}\})$. Note that for $\frac{6}{36} < \frac{g^\imath}{4} \leq \frac{7}{36}$, $\frac{g^\imath}{4}$ is nearest to $\frac{1}{3}$ and farthest from $\frac{5}{12}$. Therefore, $\delta_{d_s}(Tg^\imath, T1) = |\frac{g^\imath}{4} - \frac{1}{3}|^2 = \frac{g^{\imath 2}}{16} - \frac{2g^\imath}{12} + \frac{1}{9}$ and $\delta_{d_s}(T1, Tg^\imath) = |\frac{g^\imath}{4} - \frac{5}{12}|^2 = \frac{g^{\imath 2}}{16} - \frac{10g^\imath}{48} + \frac{25}{144}$. Then, we have

$$\begin{aligned} H^{\frac{3}{4}}(Tg^\imath, T1) &= \max\{\frac{3}{4}\delta_{d_s}(Tg^\imath, T1) + \frac{1}{4}\delta_{d_s}(T1, Tg^\imath), \frac{3}{4}\delta_{d_s}(T1, Tg^\imath) + \frac{1}{4}\delta_{d_s}(Tg^\imath, T1)\} \\ &= \max\{\frac{73}{576} - \frac{34g^\imath}{192} + \frac{4g^{\imath 2}}{64}, \frac{91}{576} - \frac{38g^\imath}{192} + \frac{4g^{\imath 2}}{64}\} \\ &= \frac{91}{576} - \frac{38g^\imath}{192} + \frac{4g^{\imath 2}}{64} \leq k\, d_s(g^\imath, 1), k \geq \frac{217}{256}. \end{aligned}$$

However, we see that for $g^\imath = \frac{7}{9}, g^\jmath = 1$,

$$H(T(\frac{7}{9}), T(1)) = \frac{4}{81} = d_s(\frac{7}{9}, 1)$$

and hence T does not satisfy the contraction Condition of Nadler [3] and Czervic [8].

Example 3. *Let $X = \{0, \frac{1}{4}, 1\}$, $d_s(g^\imath, g^\jmath) = |g^\imath - g^\jmath|^2$ for all $g^\imath, g^\jmath \in X$ and $T: X \to CB(X)$ be as follows: $T(g^\imath) = \begin{cases} \{0\}, & \text{for } g^\imath \in \{0, \frac{1}{4}\} \\ \{0,1\}, & \text{for } g^\imath = 1, \end{cases}$*
We will show that T is a multi-valued H^β-contraction mapping with $\beta \in (\frac{7}{16}, \frac{9}{16})$. If $g^\imath, g^\jmath \in \{0, \frac{1}{4}\}$, then the result is clear. Suppose $g^\imath \in \{0, \frac{1}{4}\}$ and $g^\jmath = 1$. Then $\delta_{d_s}(Tg^\imath, T1) = 0$ and $\delta_{d_s}(T1, Tg^\imath) = 1$ so that $H^\beta(Tg^\imath, T1) = max\{\beta, 1-\beta\}$. In addition, we have $d_s(g^\imath, 1) = 1$ or $\frac{9}{16}$. If $\beta \in (\frac{7}{16}, \frac{1}{2}]$, then $H^\beta(Tg^\imath, T1) = 1-\beta$. Now $1-\beta \in [\frac{8}{16}, \frac{9}{16})$. Therefore, $1-\beta = \frac{16}{9}(1-\beta)\frac{9}{16}$ and $1-\beta < \frac{16}{9}(1-\beta)1$, that is $1-\beta \leq \frac{16}{9}(1-\beta)d_s(g^\imath, 1)$. Thus, we have $H^\beta(Tg^\imath, T1) = 1-\beta \leq kd_s(g^\imath, 1)$, where $k = \frac{16}{9}(1-\beta) < 1$. Similarly if $\beta \in [\frac{1}{2}, \frac{9}{16})$, we get $H^\beta(Tg^\imath, T1) = \beta \leq kd_s(g^\imath, 1)$ where $k = \frac{16}{9}\beta < 1$. Thus, T is a multi-valued H^β-contraction. However T is not a multi-valued quasi contraction mapping. Indeed, for $g^\imath = \frac{1}{4}$ and $g^\jmath = 1$, we have

$$\begin{aligned} H_{d_s}(T(\tfrac{1}{4}), T(1)) &= \max\{\delta_{d_s}(T(\tfrac{1}{4}), T1), \delta_{d_s}(T1, T(\tfrac{1}{4}))\} = 1 \\ &> k \cdot \max\{d_s(\tfrac{1}{4}, 1), d_s(\tfrac{1}{4}, T(\tfrac{1}{4})), d_s(1, T1), d_s(\tfrac{1}{4}, T1), d_s(1, T(\tfrac{1}{4}))\} \end{aligned}$$

for any $k \in (0,1)$. Therefore, T does not satisfy the contraction conditions given in Definitions 4–7.

Now we will present our main results in which we establish the existence of fixed points of generalized multi-valued contraction mappings using H^β Hausdorff–Pompeiu b-metric. Hereafter, $\mathcal{F}\{T\}$ will denote the fixed point set of T.

Theorem 5. *Suppose d_s is $*$-continuous and $T: X \to CB^{d_s}(X)$ is a multi-valued mapping satisfying the following conditions:*
(i) There exists $\beta \in [0,1]$, $a, b, c, e, f, h, j \geq 0$, $a + b + s(c + e + \frac{h}{2}) + f + j < 1$ and $\min\{s(a + e + \frac{h}{2}), s(b + c + \frac{h}{2})\} < 1$ such that for all $g^\imath, g^\jmath \in X$,

$$\begin{aligned} H^\beta(Tg^\imath, Tg^\jmath) &\leq a \cdot d_s(g^\imath, Tg^\imath) + b \cdot d_s(g^\jmath, Tg^\jmath) + c \cdot d_s(g^\imath, Tg^\jmath) + e \cdot d_s(g^\jmath, Tg^\imath) \\ &+ h \cdot \frac{d_s(g^\imath, Tg^\jmath) + d_s(g^\jmath, Tg^\imath)}{2} + j \cdot \frac{d_s(g^\imath, Tg^\imath) d_s(g^\jmath, Tg^\jmath)}{1 + d_s(g^\imath, g^\jmath)} + f \cdot d_s(g^\imath, g^\jmath). \end{aligned} \tag{9}$$

(ii) For every $g^\imath$ in X, $g^\jmath$ in $T(g^\imath)$ and $\epsilon > 0$, there exists g in $T(g^\jmath)$ satisfying

$$d_s(g^\jmath, g) \leq H^\beta(Tg^\imath, Tg^\jmath) + \epsilon. \tag{10}$$

Then $\mathcal{F}\{T\} \neq \phi$.

Proof. For some arbitrary $g_0^\imath \in X$, if $g_0^\imath \in Tg_0^\imath$ then $g_0^\imath \in \mathcal{F}\{T\}$. Suppose $g_0^\imath \notin Tg_0^\imath$. Let $g_1^\imath \in Tg_0^\imath$. Again, if $g_1^\imath \in Tg_1^\imath$ then $g_1^\imath \in \mathcal{F}\{T\}$. Suppose $g_1^\imath \notin Tg_1^\imath$. By (10), we can find

$g_2^\imath \in Tg_1^\imath$ such that

$$d_s(g_1^\imath, g_2^\imath) \leq H^\beta(Tg_0^\imath, Tg_1^\imath) + \epsilon.$$

If $g_2^\imath \in Tg_2^\imath$ then $g_2^\imath \in \mathcal{F}\{T\}$. Suppose $g_2^\imath \notin Tg_2^\imath$. By (10), we can find $g_3^\imath \in Tg_2^\imath$ such that

$$d_s(g_2^\imath, g_3^\imath) \leq H^\beta(Tg_1^\imath, Tg_2^\imath) + \epsilon^2.$$

In this way we construct the sequence $\{g_n^\imath\}$ such that $g_n^\imath \notin Tg_n^\imath$, $g_{n+1}^\imath \in Tg_n^\imath$ and

$$d_s(g_n^\imath, g_{n+1}^\imath) \leq H^\beta(Tg_{n-1}^\imath, Tg_n^\imath) + \epsilon^n.$$

Then, using (9), we have

$$\begin{aligned} d_s(g_n^\imath, g_{n+1}^\imath) &\leq H^\beta(Tg_{n-1}^\imath, Tg_n^\imath) + \epsilon^n \\ &\leq a \cdot d_s(g_{n-1}^\imath, Tg_{n-1}^\imath) + b \cdot d_s(g_n^\imath, Tg_n^\imath) + c \cdot d_s(g_{n-1}^\imath, Tg_n^\imath) + e \cdot d_s(g_n^\imath, Tg_{n-1}^\imath) \\ &+ h \cdot \frac{d_s(g_{n-1}^\imath, Tg_n^\imath) + d_s(g_n^\imath, Tg_{n-1}^\imath)}{2} + j \cdot \frac{d_s(g_{n-1}^\imath, Tg_{n-1}^\imath) d_s(g_n^\imath, Tg_n^\imath)}{1 + d_s(g_{n-1}^\imath, g_n^\imath)} + f \cdot d_s(g_{n-1}^\imath, g_n^\imath) + \epsilon^n, \end{aligned}$$

that is,

$$(1 - b - sc - j) \cdot d_s(g_n^\imath, g_{n+1}^\imath) \leq (a + sc + \frac{sh}{2} + f) \cdot d_s(g_{n-1}^\imath, g_n^\imath) + \epsilon^n. \tag{11}$$

Using symmetry of H^β, we also have

$$(1 - a - se - j) \cdot d_s(g_n^\imath, g_{n+1}^\imath) \leq (b + se + \frac{sh}{2} + f) \cdot d_s(g_{n-1}^\imath, g_n^\imath) + \epsilon^n. \tag{12}$$

Adding (11) and (12), we get

$$d_s(g_n^\imath, g_{n+1}^\imath) \leq (a + b + s(c + e + \frac{h}{2}) + f + j) \cdot d_s(g_{n-1}^\imath, g_n^\imath) + \epsilon^n.$$

By Lemma 2, the sequence $\{g^\imath{}_n\}$ is a Cauchy sequence. Completeness of (X, d_s) gives $\lim_{n\to+\infty} d_s(g_n^\imath, g^{\imath*}) = 0$ for some $g^{\imath*} \in X$. We now show that $g^{\imath*} \in Tg^{\imath*}$. Suppose, on the contrary, that $g^{\imath*} \notin Tg^{\imath*}$. Then,

$$\begin{aligned} &\beta \cdot \delta_{d_s}(Tg_n^\imath, Tg^{\imath*}) + (1-\beta) \cdot \delta_{d_s}(Tg^{\imath*}, Tg_n^\imath) \leq H^\beta(Tg_n^\imath, Tg^{\imath*}) \\ &\leq a \cdot d_s(g_n^\imath, Tg_n^\imath) + b \cdot d_s(g^{\imath*}, Tg^{\imath*}) + c \cdot d_s(g_n^\imath, Tg^{\imath*}) + e \cdot d_s(g^{\imath*}, Tg_n^\imath) \\ &+h \cdot \frac{d_s(g_n^\imath, Tg^{\imath*}) + d_s(g^{\imath*}, Tg_n^\imath)}{2} + j \cdot \frac{d_s(g_n^\imath, Tg_n^\imath) d_s(g^{\imath*}, Tg^{\imath*})}{1 + d_s(g_n^\imath, g^{\imath*})} + f \cdot d_s(g_n^\imath, g^{\imath*}) \\ &\leq a \cdot d_s(g_n^\imath, g_{n+1}^\imath) + b \cdot d_s(g^{\imath*}, Tg^{\imath*}) + c \cdot d_s(g_n^\imath, Tg^{\imath*}) + e \cdot d_s(g^{\imath*}, g_{n+1}^\imath) \\ &+h \cdot \frac{d_s(g_n^\imath, Tg^{\imath*}) + d_s(g^{\imath*}, g_{n+1}^\imath)}{2} + \frac{d_s(g_n^\imath, g_{n+1}^\imath) d_s(g^{\imath*}, Tg^{\imath*})}{1 + d_s(g_n^\imath, g^{\imath*})} + f \cdot d_s(g_n^\imath, g^{\imath*}). \end{aligned}$$

and using the *-continuity of d_s, we get

$$\liminf_{n\to\infty} \beta \cdot \delta_{d_s}(Tg_n^\imath, Tg^{\imath*}) + (1-\beta) \cdot \delta_{d_s}(Tg^{\imath*}, Tg_n^\imath) \leq (b + c + \frac{h}{2}) \cdot d_s(g^{\imath*}, Tg^{\imath*}).$$

Similarly,

$$\liminf_{n\to\infty} \beta \cdot \delta_{d_s}(Tg^{\imath*}, Tg_n^\imath) + (1-\beta) \cdot \delta_{d_s}(Tg_n^\imath, Tg^{\imath*}) \leq (a + e + \frac{h}{2}) \cdot d_s(g^{\imath*}, Tg^{\imath*}).$$

It follows that

$$d_s(g^{\imath *}, Tg^{\imath *}) = \beta \cdot d_s(g^{\imath *}, Tg^{\imath *}) + (1-\beta) \cdot d_s(Tg^{\imath *}, g^{\imath *}) \leq s[\beta \cdot \delta_{d_s}(Tg^{\imath}_n, Tg^{\imath *}) \\ +(1-\beta) \cdot \delta_{d_s}(Tg^{\imath *}, Tg^{\imath}_n)] + s.d_s(g^{\imath}_{n+1}, g^{\imath *})$$

that is,

$$d_s(g^{\imath *}, Tg^{\imath *}) \leq s[\liminf_{n\to\infty}[\beta\, \delta_{d_s}(Tg^{\imath}_n, Tg^{\imath *}) + (1-\beta)\delta_{d_s}(Tg^{\imath *}, Tg^{\imath}_n)]] + s[\liminf_{n\to\infty} d_s(g^{\imath}_{n+1}, g^{\imath *})] \\ \leq s(b + c + \frac{h}{2}) d_s(x^*, Tg^{\imath *})$$

and

$$d_s(Tg^{\imath *}, g^{\imath *}) = \beta \cdot d_s(Tg^{\imath *}, g^{\imath *}) + (1-\beta) \cdot d_s(g^{\imath *}, Tg^{\imath *}) \leq s[\beta \cdot \delta_{d_s}(Tg^{\imath *}, Tg^{\imath}_n) \\ +(1-\beta) \cdot \delta_{d_s}(Tg^{\imath}_n, Tg^{\imath *})] + s \cdot d_s(g^{\imath *}, g^{\imath}_{n+1})$$

that is,

$$d_s(Tg^{\imath *}, g^{\imath *}) \leq s[\liminf_{n\to\infty}[\beta \cdot \delta_{d_s}(Tg^{\imath *}, Tg^{\imath}_n) + (1-\beta) \cdot \delta_{d_s}(Tg^{\imath}_n, Tg^{\imath *})]] + s[\liminf_{n\to\infty} d_s(g^{\imath *}, g^{\imath}_{n+1})] \\ \leq s(a + e + \frac{h}{2}) \cdot d_s(Tg^{\imath *}, x^*).$$

Since $\min\{s(a+e+\frac{h}{2}), s(c+e+\frac{h}{2}\} < 1$, we get $d_s(g^{\imath *}, Tg^{\imath *}) = 0$ which from Proposition 1 implies that $g^{\imath *} \in \overline{Tg^{\imath *}}$ and since $Tg^{\imath *}$ is closed it follows that $g^{\imath *} \in Tg^{\imath *}$. □

Remark 5. *Theorem 5 is true even if we replace (9) by any of the following conditions:*
For some $0 \leq k < \frac{1}{s}$,

$$H^{\beta}(Tg^{\imath}, Tg^{\jmath}) \leq k \cdot \max\{d_s(g^{\imath}, g^{\jmath}), d_s(g^{\imath}, Tg^{\imath}), d_s(g^{\jmath}, Tg^{\jmath}), \frac{d_s(g^{\imath}, Tg^{\jmath}) + d_s(g^{\jmath}, Tg^{\imath})}{2s}, \\ \frac{d_s(g^{\imath}, Tg^{\imath}) d_s(g^{\jmath}, Tg^{\jmath})}{1 + d_s(g^{\imath}, g^{\jmath})}\}, \tag{13}$$

$$H^{\beta}(Tg^{\imath}, Tg^{\jmath}) \leq k \cdot \max\{d_s(g^{\imath}, g^{\jmath}), d_s(g^{\imath}, Tg^{\imath}), d_s(g^{\jmath}, Tg^{\jmath}), d_s(g^{\imath}, Tg^{\jmath}), \\ d_s(g^{\jmath}, Tg^{\imath}), \frac{d_s(g^{\imath}, Tg^{\imath}) d_s(g^{\jmath}, Tg^{\jmath})}{1 + d_s(g^{\imath}, g^{\jmath})}\}\} \tag{14}$$

The following result is a consequence of Theorem 5 and Remark 5:

Corollary 1. *Suppose d_s is $*$-continuous and $T : X \to CB^{d_s}(X)$ satisfy Condition (10) and any of the following conditions:*

(i) T is a multi-valued H^{β}-Ciric contraction.
(ii) T is a multi-valued H^{β}-Hardy–Roger's contraction.
(iii) T is a multi-valued H^{β}-quasi contraction.
(iv) T is a multi-valued H^{β}-weak quasi contraction.
(v) T is a multi-valued H^{β}-contraction.

Then $\mathcal{F}\{T\} \neq \phi$.

Taking $T : X \to X$ in Corollary 1 (ii) and using Theorem 2 (i), we have the following corollary.

Corollary 2. *Suppose d_s is $*$-continuous and $T: X \to X$. If there exists non-negative real numbers a, b, c, e, f such that $a + b + s(c + e) + f < 1$, $\min\{s(a + e), s(b + c)\} < 1$ and*

$$d_s(Tg^\imath, T^\jmath) \le a \cdot d_s(g^\imath, g^\jmath) + b \cdot d_s(g^\imath, Tg^\imath) + c \cdot d_s(g^\jmath, T^\jmath) + e \cdot d_s(g^\imath, T^\jmath) + f \cdot d_s(g^\jmath, Tg^\imath), \text{ for all } g^\imath, g^\jmath \in X, \tag{15}$$

then $\mathcal{F}(T) \neq \phi$.

Remark 6. *For $\beta = 1$, Condition (10) is obviously satisfied and hence, (Theorem 5 [3]), (Theorem 2.1 [8]), (Theorem 2.2 [9]), (Theorem 2.11 [10]), (Theorem 3.1 [12]) and (Theorem 3.1 [11]) are all particular cases of Corollary 1. However, the examples which follow illustrate that the converse is not necessarily true.*

We now furnish the following examples to validate our results.

Example 4. *Let X, d_s and T be as in Example 2. Then, as shown above, T belongs to the class of multi-valued H^β-contraction with $\beta \in (\frac{7}{16}, \frac{9}{16})$ and consequently T satisfies all the contraction conditions given in Definitions 11–14. We will show that T satisfies (10):*

For $g^\imath \in [0, \frac{7}{9}]$, $Tg^\imath$ is singleton and so the result is obvious. Now for $g^\imath = 1$, if $g^\jmath = 0 \in Tg^\imath$ then $g = 0 \in Tg^\jmath$ will satisfy (10). If $g^\jmath = \frac{1}{3} \in Tg^\imath$, then $g = \frac{1}{12} \in Tg^\jmath$ and if $g^\jmath = \frac{5}{12} \in Tg^\imath$ then $g = \frac{5}{48} \in T^\jmath$ will satisfy (10). Thus, T satisfies conditions of Theorem 5 and Corollary 1 and $0, 1 \in \mathcal{F}(T)$.

However, as shown in Example 2, T does not satisfy the contraction condition of Nadler [3] and Czervic [8].

Example 5. *Let X, d_s and T be as in Example 3. Then as shown above, T belongs to the class of multi-valued H^β-contraction with $\beta \in (\frac{7}{16}, \frac{9}{16})$ and consequently T satisfies all the contraction conditions given in Definitions 11–14.*

We will show that T satisfies (10):

For $g^\imath \in \{0, \frac{1}{4}\}$, $Tg^\imath$ is singleton and so the result is obvious. Now for $g^\imath = 1$, if $g^\jmath = 0 \in Tg^\imath$ then $g = 0 \in Tg^\jmath$ will satisfy (10). If $g^\jmath = 1 \in Tg^\imath$ then $g = 1 \in Tg^\jmath$ will satisfy (10). Thus, Theorem 5 and Corollary 1 are applicable and $0, 1 \in \mathcal{F}(T)$. However, we see that T does not satisfy the conditions of (Theorem 2.2 [9]), (Theorem 2.11 [10]) and (Theorem 3.1 [12]).

Example 6. *Let $X = \{0, \frac{1}{12}, \frac{1}{3}, \frac{5}{12}, \frac{34}{48}, 1\}$, $d_s(g^\imath, g^\jmath) = |g^\imath - g^\jmath|$ for all $g^\imath, g^\jmath \in X$ and $T: X \to CB^{d_s}(X)$ be as follows:*

$$T(0) = T(\frac{1}{12}) = \{0\}, \;\; T(\frac{1}{3}) = T(\frac{5}{12}) = T(\frac{34}{48}) = \left\{\frac{1}{12}\right\}, \;\; T(1) = \{0, \frac{1}{3}, \frac{34}{48}, 1\}.$$

Then, T is a multi-valued H^β-quasi contraction for $\beta = \frac{3}{4}$ with $\frac{34}{44} \le k < 1$ as shown below:

(1) If $g^\imath = \frac{34}{48}$ and $g^\jmath = 1$, then $\delta_{d_s}(T(\frac{34}{48}), T1) = \delta_{d_s}(\{\frac{1}{12}\}, \{0, \frac{1}{3}, \frac{34}{48}, 1\}) = \frac{1}{12}$ and $\delta_{d_s}(T1, T(\frac{34}{48})) = \delta_{d_s}(\{0, \frac{1}{3}, \frac{34}{48}, 1\}, \{\frac{1}{12}\}) = \frac{11}{12}$.

$$\begin{aligned}
H^{\frac{3}{4}}(T(\frac{34}{48}), T1) &= \max\{\frac{3}{4}\delta_{d_s}(T(\frac{34}{48}), T1) + \frac{1}{4}\delta_{d_s}(T1, T(\frac{34}{48}), \frac{3}{4}\delta_{d_s}(T1, T(\frac{34}{48})) + \frac{1}{4}\delta_{d_s}(T(\frac{34}{48}), T1)\} \\
&= \max\{\frac{3}{4}.\frac{1}{12} + \frac{1}{4}.\frac{11}{12}, \frac{3}{4}.\frac{11}{12} + \frac{1}{4}.\frac{1}{12}\} = \frac{34}{48} \\
&\leq k\frac{44}{48}, \quad \text{for any } k \geq \frac{34}{44} \\
&= kd_s(1, T(\frac{34}{48})) \\
&\leq k\max\{d_s(\frac{34}{48}, 1), d_s(\frac{34}{48}, T(\frac{34}{48}), d_s(1, T1), d_s(\frac{34}{48}, T1), d_s(1, T(\frac{34}{48}))\}.
\end{aligned}$$

(2) If $g^\imath = \frac{1}{12}$ and $g^\jmath = 1$. $\delta_{d_s}(T(\frac{1}{12}), T1) = \delta_{d_s}(\{0, \{0, \frac{1}{3}, \frac{34}{48}, 1\}) = 0$. $\delta_{d_s}(T1, T(\frac{1}{12})) = \delta_{d_s}(\{0, \frac{1}{3}, \frac{34}{48}, 1\}, 0\}) = 1$.

$$\begin{aligned}
H^{\frac{3}{4}}(T(\frac{1}{12}), T1) &= \max\{\frac{3}{4}\delta_{d_s}(T(\frac{1}{12}), T1) + \frac{1}{4}\delta_{d_s}(T1, T(\frac{1}{12}), \frac{3}{4}\delta_{d_s}(T1, T(\frac{1}{12})) + \frac{1}{4}\delta_{d_s}(T(\frac{1}{12}), T1)\} = \frac{3}{4} \\
&\leq k.1, \quad \text{for any } k \geq \frac{3}{4} \\
&= k \cdot d_s(1, T(\frac{1}{12})) \\
&\leq k \cdot \max\{d_s(\frac{1}{12}, 1), d_s(\frac{1}{12}, T(\frac{1}{12}), d_s(1, T1), d_s(\frac{1}{12}, T1), d_s(1, T(\frac{1}{12}))\}.
\end{aligned}$$

(3) If $g^\imath = \frac{1}{12}$ and $g^\jmath = \frac{1}{3}$, then $\delta_{d_s}(T(\frac{1}{12}), T(\frac{1}{3})) = \delta_{d_s}(\{0, \{\frac{1}{12}\}) = \frac{1}{12}$ and $\delta_{d_s}(\frac{1}{3}, T(\frac{1}{12})) = \delta_{d_s}(\{\frac{1}{12}\}, 0\}) = \frac{1}{12}$.

$$\begin{aligned}
H^{\frac{3}{4}}(T(\frac{1}{12}), T(\frac{1}{3})) &= \max\{\frac{3}{4}\delta_{d_s}(T(\frac{1}{12}), T(\frac{1}{3})) + \frac{1}{4}\delta_{d_s}(T(\frac{1}{3}), T(\frac{1}{12}), \frac{3}{4}\delta_{d_s}(T(\frac{1}{3}), T(\frac{1}{12}) + \frac{1}{4}\delta_{d_s}(T(\frac{1}{12}), T(\frac{1}{3}))\} \\
&= \frac{1}{12} \leq k.\frac{4}{12}, \quad \text{for any } k \geq \frac{1}{4} \\
&= k \cdot d_s(\frac{1}{3}, T(\frac{1}{12}) \\
&\leq k \cdot \max\{d_s(\frac{1}{12}, \frac{1}{3}), d_s(\frac{1}{12}, T(\frac{1}{12}), d_s(\frac{1}{3}, T(\frac{1}{3})), d_s(\frac{1}{12}, T(\frac{1}{3})), d_s(\frac{1}{3}, T(\frac{1}{12}))\}.
\end{aligned}$$

For all other values of $g^\imath$ and $g^\jmath$, a similar argument as above follows. Thus, T is a multi-valued H^β-quasi contraction. We will show that T satisfies (10): For $g^\imath \in \{0, \frac{1}{12}, \frac{1}{3}, \frac{5}{12}, \frac{34}{48}\}$, $Tg^\imath$ is singleton and so the result is obvious. Now, for $g^\imath = 1$, if $g^\jmath = 0 \in Tg^\imath$ then $g = 0 \in Tg^\jmath$ will satisfy (10). If $g^\jmath = \frac{1}{3}$ or $\frac{34}{48} \in Tg^\imath$ then, $g = \frac{1}{12} \in Tg^\jmath$ will satisfy (10). Thus, Theorem 5 and Corollary 1 are applicable and $0, 1 \in \mathcal{F}(T)$. However, we see that $H(T(\frac{34}{48}), T(1)) = \frac{11}{12}$, where $d(\frac{34}{48}, 1) = \frac{14}{48}$, $d(\frac{34}{48}, T(\frac{34}{48})) = \frac{30}{48}$, $d(1, T(1)) = 0$, $d(\frac{34}{48}, T(1) = 0$ and $d(1, T(\frac{34}{48}))\} = \frac{11}{12}$ and so T does not satisfy the conditions of (Theorem 2.2 [9]), (Theorem 2.11 [10]), (Theorem 3.1 [12]) and (Theorem 3.1 [11]).

Proposition 3. *Let $T_1, T_2 : X \to CB^{d_s}(X)$, satisfy the following:*
(3.1) For all $q, r \in \{1, 2\}$, every $g^\imath$ in X, $g^\jmath$ in $T_q(g^\imath)$ and $\epsilon > 0$, there exists g in $T_r(g^\jmath)$ satisfying

$$d_s(g^\jmath, g) \leq H^\beta(T_q g^\imath, T_r g^\jmath) + \epsilon.$$

(3.2) Any of the following conditions holds:

(i) T_1 *and* T_2 *is a multi-valued* H^β*-Ciric contraction;*

(ii) T_1 *and* T_2 *is a multi-valued* H^β*-quasi contraction;*

(iii) T_1 *and* T_2 *is a multi-valued* H^β*-weak quasi contraction;*

Then, for any $u \in \mathcal{F}\{T_q\}$*, there exist* $w \in \mathcal{F}\{T_r\}$ *(*$q \neq r$*) such that*

$$d_s(u, w) \leq \frac{s}{1-k} \sup_{x \in X} H^\beta(T_q x, T_r x),$$

where k is the Lipschitz's constant.

Proof. Let $g_0^\iota \in \mathcal{F}\{T_1\}$. By (3.1) we can find $g_1^\iota \in T_2 g_0^\iota$ such that

$$d_s(g_0^\iota, g_1^\iota) \leq H^\beta(T_1 g_0^\iota, T_2 g_1^\iota) + \epsilon.$$

By (3.1), choose $g_2^\iota \in T_2 g_1^\iota$ such that

$$d_s(g_1^\iota, g_2^\iota) \leq H^\beta(T_2 g_0^\iota, T_2 g_1^\iota).$$

Inductively, we define sequence $\{g_n^\iota\}$ such that $g_{n+1}^\iota \in T_2(g_n^\iota)$ and

$$d_s(g_n^\iota, g_{n+1}^\iota) \leq H^\beta(T_2 g_{n-1}^\iota, T_2 g_n^\iota) + \epsilon. \tag{16}$$

Now, following the same technique as in the proof of Theorem 5, we see that the sequence $\{g_n^\iota\}$ converges to some g_*^ι in X and $g_*^\iota \in \mathcal{F}\{T_2\}$. Since ϵ is arbitrary, taking $\epsilon \to 0$ in (16) we get

$$d_s(g_n^\iota, g_{n+1}^\iota) \leq H^\beta(T_2 g_{n-1}^\iota, T_2 g_n^\iota).$$

Then, using (Section 3.2), we get

$$d_s(g_n^\iota, g_{n+1}^\iota) \leq k^n d_s(g_0^\iota, g_1^\iota).$$

Then, we have $d(g_0^\iota, g_*^\iota) \leq \sum_{n=0}^{\infty} s^{n+1} d_s(g_{n+1}^\iota, g_n^\iota) \leq s(1 + sk + (sk)^2 + \cdots) d_s(g_1^\iota, g_0^\iota) \leq \frac{s}{1-sk}(H^\beta(T_2 g_0^\iota, T_1 g_0^\iota) + \epsilon)$. Interchanging the roles of T_1 and T_2 and proceeding as above, it gives that for each $g_0^J \in \mathcal{F}\{T_2\}$ there exist $g_1^J \in T_1 g_0^J$ and $g^\ell \in F(T_1)$ such that

$$d(g_0^J, g^\ell) \leq \frac{s}{1-sk}(H^\beta(T_1 g_0^J, T_2 g_0^J) + \epsilon).$$

Now the result follows as $\epsilon > 0$ is arbitrary. □

3.3. Application to Multi-Valued Fractals

Inspiring from some recent works in [18,22,23], we provide an application of our result to multi-valued fractals. Let $P_i : X \to CB^{d_s}(X)$, $i = 1, 2, \cdots n$ be upper semi continuous mappings. Then, $P = (P_1, P_2, \cdots P_n)$ is an iterated multifunction system (in short IMS) defined on the b-metric space (X, d_s). The operator $T_P : CB(X) \to CB(X)$ defined by $T_P(Y) = \bigcup_{i=1}^n P_i(Y)$ is called the extended multifractal operator generated by the IMS $P = (P_1, P_2, \cdots P_n)$. Any non empty compact subset of X which is a fixed point of T_P is called a multi-valued fractal of the iterated multifunction system $P = (P_1, P_2, \cdots P_n)$.

Theorem 6. *Let* $P_i : X \to CB(X)$*,* $i = 1, 2, \cdots n$ *be upper semi continuous mappings such that for each* $i = 1, 2, \cdots n$ *the following conditions hold:*
We can find $\beta \in [0, 1]$ *and* $a, e \in (0, 1)$*,* $a + 2se < 1$*, such that for all* $x, y \in X$*,* $i = 1, 2 \cdots n$

$$H^\beta(P_i x, P_i y) \leq a\, d_s(x, y) + e[d_s(x, P_i y) + d_s(y, P_i x)]. \tag{17}$$

Then,

(i) *For all $U_1, U_2 \in CB(X)$, $H^\beta(T_P(U_1), T_P(U_2)) \leq a\, H^\beta(U_1, U_2) + e[H^\beta(U_1, T_P(U_2)) + H^\beta(U_2, T_P(U_1))]$.*

(ii) *A unique multi-valued fractal U^* exists for the iterated multifunction system $P = (P_1, P_2, \cdots P_n)$.*

Proof. Suppose condition (17) holds. Then, for $U_1, U_2 \in CB(X)$, we have

$$\begin{aligned} R^\beta(P_i(U_1), P_i(U_2)) &= \beta\delta(P_i(U_1), P_i(U_2)) + (1-\beta)\delta(P_i(U_2), P_i(U_1)) \\ &= \beta \sup_{x\in U_1}(\inf_{y\in U_2} H^\beta(P_i(x), P_i(y)) + \\ &\quad (1-\beta)\sup_{y\in U_2}(\inf_{x\in U_1} H^\beta(P_i(x), P_i(y)) \\ &\leq \beta \sup_{x\in U_1}(\inf_{y\in U_2} \Big\{ a\, d_s(x,y) + e[d_s(x, P_i y) + d_s(y, P_i x)] \Big\} \\ &\quad +(1-\beta)\sup_{y\in U_2}(\inf_{x\in U_1} \Big\{ a\, d_s(x,y) + e[d_s(x, P_i y) + d_s(y, P_i x)] \Big\} \\ &= a\, H^\beta(U_1, U_2) + e[H^\beta(U_1, P_i(U_2) + H^\beta(U_2, P_i(U_1))]. \end{aligned}$$

Similarly, we get

$$R^\beta(P_i(U_2), P_i(U_1)) \leq a\, H^\beta(U_2, U_1) + e[H^\beta(U_2, P_i(U_1) + H^\beta(U_1, P_i(U_2))].$$

Thus, we have, for $i = 1, 2, \cdots n$,

$$H^\beta(P_i(U_1), P_i(U_2)) \leq a\, H^\beta(U_1, U_2) + e[H^\beta(U_2, P_i(U_1) + H^\beta(U_1, P_i(U_2))].$$

Note that

$$H^\beta(\bigcup_{i=1}^n P_i(U_1), \bigcup_{i=1}^n P_i(U_2)) \leq \max\{H^\beta(P_1(U_1), P_1(U_2)), H^\beta(P_2(U_1), P_2(U_2)), \cdots H^\beta(P_n(U_1), P_n(U_2))\}$$

and so

$$H^\beta(T_P(U_1), T_P(U_2)) \leq a\, H^\beta(U_1, U_2) + e[H^\beta(U_1, T_P(U_2)) + H^\beta(U_2, T_P(U_1))].$$

Thus, $T_P : CB(X) \to CB(X)$ satisfies the conditions of Corollary 2 in the metric space $\{CB(X), H^\beta\}$, with $b = c = 0$ and $e = f$ and hence has a fixed point U^* in $CB(X)$, which in turn is the unique multi-valued fractal of the iterated multifunction system $P = (P_1, P_2, \cdots P_n)$. □

Remark 7. *Since $H^\beta(A, B) \leq H(A, B)$, Theorem 6 is a proper improvement and generalization of (Theorem 3.4 [18]), (Theorem3.1 [22]) and (Theorem 3.8 [23]).*

3.4. Application to Nonconvex Integral Inclusions

We will begin this section by introducing the following generalized norm on a vector space:

Definition 15. *Let V be a vector space over the field K. For some $\rho > 0$ and $\gamma \geq 1$, a real valued function $\|.\|_\gamma^\rho : V \to R$ is a generalized (ρ, γ)-norm if for all $x, y \in V$ and $\lambda \in K$*

(1) $\|x\|_\gamma^\rho \geq 0$ *and* $\|x\|_\gamma^\rho = 0$ *if and only if* $x = 0$.

(2) $\|\lambda\, x\|_\gamma^\rho \leq |\lambda|^\rho \|x\|_\gamma^\rho$.

(3) $\|x + y\|_\gamma^\rho \leq \gamma[\|x\|_\gamma^\rho + \|y\|_\gamma^\rho]$.

We say that $(V, \|.\|_\gamma^\rho$ is a generalized (ρ,γ)-normed linear space.

Remark 8. *The following are immediate consequences of the above definition:*

(i) *Every norm is a generalized (ρ,γ)-norm with $\rho = 1$ and $\gamma = 1$.*
(ii) *Every generalized (ρ,γ)-norm induces a b-metric with coefficient γ, given by $d_\gamma(x,y) = \|x-y\|_\gamma^\rho$.*

Example 7. *Every norm defined on a vector space is a generalized (ρ,γ)-norm.*

Example 8. *Let $V = R$. Define $\|x\|_\gamma^\rho = |x|^2$. Then $\|.\|_\gamma^\rho$ is a generalized $(2,2)$-norm.*

Example 9. *Let $V = R^n$. Define $\|x\|_\gamma^\rho = \sum_k |x_k|^p$, $1 \le p < \infty$. Then $\|.\|_\gamma^\rho$ is a generalized $(p, 2^{p-1})$-norm.*

The convergence, Cauchy sequence and completeness in a generalized (ρ,γ)-normed linear space is defined in the same way as that in a normed linear space.

Throughout this section we will use the following notations and functions:

(i) $A = [0,\tau], \quad \tau > 0$.
(ii) $\mathcal{L}(A)$: is the σ-algebra of all Lebesgue measurable subsets of A.
(iii) Z: is a real separable Banach space with the generalized (ρ,γ)-norm $\|.\|_\gamma^\rho$, for some $\rho > 0$ and $\gamma \ge 1$.
(iv) $\mathcal{P}(Z)$: is the family of all nonempty closed subsets of Z.
(v) d_γ is the b-metric induced by the generalized (ρ,γ)-norm $\|.\|_\gamma^\rho$ and H^β is the H^β-Hausdorff–Pompeiu b-metric on $\mathcal{P}(Z)$, induced by the b-metriv d_γ.
(vi) $\mathcal{B}(Z)$: is the collection of all Borel subsets of Z.
(vii) $\mathcal{C}(A,Z)$: is the Banach space of all continuous functions $g(.) : \ A \to Z$ with norm $\|g(.)\|_* = \sup_{t\in A} \|g(t)\|_\gamma^\rho$.
(viii) $\lambda^\ell(.) : \ A \to Z$.
(ix) $p(.,.) : A \times Z \to Z$.
(x) $Q(.,.) : \ A \times Z \to \mathcal{P}(Z)$.
(xi) $q(.,.,.) : \ A \times A \times Z \to Z$.
(xii) $V : \ \mathcal{C}(A,Z) \to \mathcal{C}(A,Z)$.
(xiii) $\alpha_1, \alpha_2 : \ A \times A \to (-\infty, +\infty)$.
(xiv) $L_{\lambda^\ell,\sigma}(t) = Q(t, V(x_{\sigma,\lambda^\ell})(t)), x \in Z, \lambda^\ell \in \mathcal{C}(A,Z), \sigma \in \mathcal{L}^1(A,Z)$.
(xv) $S_{\lambda^\ell}(\sigma) = \{\psi(.) \in \mathcal{L}^1(A,Z) : \ \psi(t) \in L_{\lambda^\ell,\sigma}(t)\}$.
(xvi) $\mathcal{L}^1(A,Z)$: is the Banach space of all integrable functions u: $A \to Z$, endowed with the norm

$$\|u(.)\|_1 = \int_0^T e^{-\alpha(M_4M_2+M_5M_1)M_3m(t)} \|u(t)\|_\gamma^\rho \, dt,$$

where $m(t) = \int_0^t k(s)\,ds, t \in A, \ \ M_1, M_2, M_3, M_4, M_5$ are positive real constants.

It is well known (see [24]) that $L_{\lambda^\ell,\sigma}(t)$ is measurable and $S_\lambda^\ell(\sigma)$ is nonempty with closed values.

We consider the following integral inclusion

$$x^\ell(t) = \lambda^\ell(t) + \int_0^t [\alpha_1(t,s)\, p(t,u(s)) + \alpha_2(t,s)\, q(t,s,u(s))], ds \tag{18}$$

$$u(t) \in Q(t, V(x^\ell)(t)) \quad a.e.\ t \in A. \tag{19}$$

We will analyze the above problem (18) and (19) under the following assumptions:
$(\mathbf{AS_1})$ $Q(\cdot,\cdot)$ is $\mathcal{L}(I) \otimes \mathcal{B}(X)$ measurable.

$(\mathbf{AS_2(i)})$ There exists $k(\cdot) \in L^1(A, \mathbf{R}_+)$ such that, for almost all $t \in A, Q(t, \cdot)$ satisfies

$$H^\beta(Q(t,x), Q(t,y)) \le k(t)\, \|x - y\|_\gamma^\rho$$

for all x, y in Z.

$(\mathbf{AS_2(ii)})$ For all $x, y \in Z, \epsilon > 0$, if $w_1 \in Q(t,x)$ then there exists $w_2 \in Q(t,y)$ such that

$$\|w_1(t) - w_2(t)\|_\gamma^\rho \le H^\beta(Q(t,x), Q(t,y)) + \epsilon.$$

$(\mathbf{AS_2(iii)})$ For any $\sigma \in \mathcal{L}^1(A, Z), \epsilon > 0$ and $\sigma_1 \in S_{\lambda^\ell}(\sigma)$, there exists $\sigma_2 \in S_{\lambda^\ell}(\sigma_1)$ such that

$$\|\sigma_1 - \sigma_2\|_1 \le H^\beta(S_{\lambda^\ell}(\sigma), S_{\lambda^\ell}(\sigma_1)) + \epsilon.$$

$(\mathbf{AS_3})$ The mappings $f: A \times A \times Z \to Z, g: A \times Z \to Z$ are continuous, $V: C(A,Z) \to C(A,Z)$
and there exist the constants $M_1, M_2, M_3, M_4 > 0$ such that $(AS_3(i))$ and either $(AS_3(ii)(a))$
or $(AS_3(ii)(b))$ holds $\forall t, s \in A, u_1, u_2 \in \mathcal{L}^1(A,Z), x_1, x_2 \in \mathcal{C}(A,Z)$.

$(\mathbf{AS_3(i)})\|V(x_1)(t) - V(x_2)(t)\|_\gamma^\rho \le M_3\|x_1(t) - x_2(t)\|_\gamma^\rho$.

$(\mathbf{AS_3(ii)(a)})$ $\|q(t,s,u_1(s)) - q(t,s,u_2(s))\|_\gamma^\rho \le M_1\, N(u_1,u_2),$

$$\|p(s,u_1(s)) - p(s,u_2(s))\|_\gamma^\rho \le M_2\, N(u_1,u_2).$$

$(\mathbf{AS_3(ii)(b)})$ $\|q(t,s,u_1(s)) - q(t,s,u_2(s))\|_\gamma^\rho \le M_1\, n(u_1,u_2),$

$$\|p(s,u_1(s)) - p(s,u_2(s))\|_\gamma^\rho \le M_2\, n(u_1,u_2),$$

where

$$N(u_1,u_2) = max\,\{\|u_1(s) - u_2(s)\|_\gamma^\rho, \|u_1(s) - S_{\lambda^\ell}(u_1)\|_\gamma^\rho, \|u_2(s) - S_{\lambda^\ell}(u_2)\|_\gamma^\rho, \|u_1(s) - S_{\lambda^\ell}(u_2)\|_\gamma^\rho, \|u_2(s) - S_{\lambda^\ell}(u_1)\|_\gamma^\rho\},$$

$$n(u_1,u_2) = max\,\{\|u_1(s) - u_2(s)\|_\gamma^\rho, \|u_1(s) - S_{\lambda^\ell}(u_1)\|_\gamma^\rho, \|u_2(s) - S_{\lambda^\ell}(u_2)\|_\gamma^\rho\} + K\,\|u_1(s) - S_{\lambda^\ell}(u_2)\|_\gamma^\rho$$

and

$$\|u(s) - S_\lambda^\ell(v)\|_\gamma^\rho = \inf_{w \in S_{\lambda^\ell}(v)} \|u(s) - w(s)\|_\gamma^\rho.$$

(AS_4) α_1, α_2 are continuous, $|\alpha_1(t,s)|^\rho \le M_4$ and $|\alpha_2(t,s)|^\rho \le M_5$.

Theorem 7. *Suppose assumptions (AS_1) to (AS_4) hold and let $\lambda^\ell(\cdot), \mu^\ell(\cdot) \in \mathcal{C}(A,Z), v(\cdot) \in \mathcal{L}^1(A,Z)$ be such that $d(v(t), Q(t, V(y^\ell)(t)) \le l(t)$ a.e. $t \in A$, where $l(\cdot) \in \mathcal{L}^1(A, \mathbf{R}_+)$ and $y^\ell(t) = \mu^\ell(t,u(t)) + \Phi(u)(t),\ \forall\, t \in A$ with $\Phi(u)(t) = \int_0^t [\alpha_1(t,\tau)p(\tau,u(\tau)) + \alpha_2 q(t,\tau,u(\tau))]$ $d\tau,\ t \in A$. Then, for every $\eta > \gamma$ and $\epsilon > 0$, we can find a solution $x^\ell(\cdot)$ of the problem (18) and (19) such that for every $t \in A$*

$$\|x^\ell(t) - y^\ell(t)\| \le \|\lambda^\ell - \mu^\ell\|_*\Big[1 + \frac{\gamma\, e^{\eta(M_4M_2 + M_5M_1)M_3m(T)}}{\eta - \gamma}\Big]$$

$$+\frac{\gamma\eta}{\eta - \gamma}(M_4M_2 + M_5M_1)e^{\eta(M_4M_2 + M_1)M_3m(T)} \int_0^T e^{-\eta(M_4M_2 + M_5M_1)M_3m(t)}l(t)dt.$$

Proof. For $\lambda^\ell \in \mathcal{C}(A,Z)$ and $u \in \mathcal{L}^1(A,Z)$, define

$$x^\ell{}_{u,\lambda^\ell}(t) = \lambda^\ell(t) + \int_0^t [\alpha_1(t,s)\, p(t,u(s)) + \alpha_2(t,s)q(t,s,u(s))]\, ds.$$

Let $\sigma_1, \sigma_2 \in \mathcal{L}^1(A,Z)$, $w_1 \in S_{\lambda^\ell}(\sigma_1)$ and

$$\mathcal{H}(t) := L_{\lambda^\ell,\sigma_2(t)} \cap \Big\{ z \in Z : \ \|w_1(t) - z\| \leq (M_4M_2 + M_5M_1)M_3k(t) \int_0^t N(\sigma_1,\sigma_2)\, ds + \delta \Big\}.$$

By assumption $(AS_2(ii))$, we have

$$\begin{aligned} d_\gamma(w_1(t), L_{\lambda^\ell,\sigma_2}) &\leq H^\beta\Big(Q(t, V(x_{\sigma_1,\lambda^\ell})(t)), Q(t, V(x_{\sigma_2,\lambda^\ell})(t))\Big) + \epsilon \\ &\leq k(t)\|V(x_{\sigma_1,\lambda^\ell})(t)) - V(x_{\sigma_2,\lambda^\ell})(t))\|_\gamma^\rho + \epsilon \\ &\leq M_3k(t)\|x_{\sigma_1,\lambda^\ell}(t) - x_{\sigma_2,\lambda^\ell}(t)\|_\gamma^\rho + \epsilon \\ &\leq M_3k(t)\Big[\int_0^t |\alpha_1(t,s)|^\rho \|p(t,\sigma_1(s)) - p(t,\sigma_2(s))\|_\gamma^\rho ds \\ &\quad + \int_0^t |\alpha_2(t,s)|^\rho \|q(t,s,\sigma_1(s)) - q(t,s,\sigma_2(s))\|_\gamma^\rho ds\Big] + \epsilon \\ &\leq M_3k(t)\Big[(M_4M_2 + M_5M_1)\int_0^t N(\sigma_1,\sigma_2)ds\Big] + \epsilon. \end{aligned}$$

Since ϵ is arbitrary, we conclude that $\mathcal{H}(\cdot)$ is nonempty, closed, bounded and measurable.

Let $w_2(\cdot)$ be a measurable selector of $\mathcal{H}(\cdot)$. Then, $w_2 \in S_{\lambda^\ell}(\sigma_2)$. If assumption $AS_3(ii)(a)$ is assumed, then we have

$$\begin{aligned} \|w_1 - w_2\|_1 &= \int_0^T e^{-\eta(M_4M_2+M_5M_1)M_3m(t)}\|w_1(t) - w_2(t)\|_\gamma^\rho dt \\ &\leq \int_0^T e^{-\eta(M_4M_2+M_5M_1)M_3m(t)} M_3k(t)\Big[(M_4M_2 + M_5M_1)\int_0^t N(\sigma_1,\sigma_2)ds\Big]dt \\ &\quad + \delta\int_0^T e^{-\eta(M_4M_2+M_5M_1)M_3m(t)}dt \\ &\leq \frac{1}{\eta}N^1(\sigma_1,\sigma_2) + \delta\int_0^T e^{-\eta(M_4M_2+M_5M_1)M_3m(t)}dt, \end{aligned}$$

where $N^1(\sigma_1,\sigma_2) = max\,\{\|\sigma_1 - \sigma_2\|_1, \|\sigma_1 - S_{\lambda^\ell}(\sigma_1)\|_1, \|\sigma_2 - S^\ell_\lambda(\sigma_2)\|_1, \|\sigma_1 - S_{\lambda^\ell}(\sigma_2)\|_1, \|\sigma_2 - S_{\lambda^\ell}(\sigma_1)\|_1\}$. Since δ is arbitrary, we have

$$d_\gamma(w_1, S_{\lambda^\ell}(\sigma_2) = \inf_{w_2 \in S_{\lambda^\ell}(\sigma_2)} \|w_1 - w_2\|_1 \leq \frac{1}{\eta}N^1(\sigma_1,\sigma_2).$$

Therefore,

$$\delta_\gamma(S_{\lambda^\ell}(\sigma_1), S_{\lambda^\ell}(\sigma_2) = \sup_{w_1 \in S_{\lambda^\ell}(\sigma_1)} d_\gamma(w_1, S_{\lambda^\ell}(\sigma_2) \leq \frac{1}{\eta}N^1(\sigma_1,\sigma_2). \tag{20}$$

Similarly, we also get

$$\delta_\gamma(S_{\lambda^\ell}(\sigma_2), S_{\lambda^\ell}(\sigma_1) = \sup_{w_1 \in S_{\lambda^\ell}(\sigma_1)} d_\gamma(w_1, S_{\lambda^\ell}(\sigma_2) \leq \frac{1}{\eta}N^1(\sigma_1,\sigma_2). \tag{21}$$

Multiplying (20) by β and (21) by $1-\beta$ and adding, we get

$$H^{\beta}(S_{\lambda^{\ell}}(\sigma_1), S_{\lambda^{\ell}}(\sigma_2)) \le \frac{1}{\eta} N^1(\sigma_1, \sigma_2).$$

Thus, $S_{\lambda^{\ell}}(\cdot)$ is a H^{β}-quasi contraction on $\mathcal{L}^1(A, Z)$.

Now let

$$\tilde{Q}(t, x) := Q(t, x) + l(t),$$
$$\tilde{M}_{\lambda^{\ell},\sigma}(t) := \tilde{Q}(t, V(x_{\sigma,\lambda^{\ell}})(t)), \qquad t \in I,$$
$$\tilde{S}_{\mu^{\ell}}(\sigma) := \{\psi(\cdot) \in \mathcal{L}^1(A, Z); \psi(t) \in \tilde{L}_{\mu^{\ell},\sigma}(t).$$

It is obvious that $\tilde{Q}(\cdot, \cdot)$ satisfies Hypothesis 5.1.

Let $\phi \in S_{\lambda^{\ell}}(\sigma), \delta > 0$ and define

$$\tilde{\mathcal{H}}(t) := \tilde{L}_{\lambda^{\ell},\sigma(t)} \cap \Big\{z \in Z : \ \|\phi(t) - z\| \le M_3 k(t) \|\lambda^{\ell} - \mu^{\ell}\|_* + l(t) + \delta\Big\}.$$

Proceeding in the same way as in the case of $\mathcal{H}(\cdot)$ above, we see that $\tilde{\mathcal{H}}(\cdot)$ is measurable, nonempty and has closed values.

Let $\omega(\cdot) \in S_{\mu^{\ell}}(\sigma)$. Then

$$\begin{aligned}
\|\phi - \omega\|_1 &\le \int_0^T e^{-\eta(M_4M_2+M_5M_1)M_3m(t)} \|\phi(t) - \omega(t)\|_{\gamma}^{\rho} dt \\
&\le \int_0^T e^{-\eta(M_4M_2+M_5M_1)M_3m(t)} [M_3 k(t) \|\lambda^{\ell} - \mu^{\ell}\|_* + l(t) + \delta] dt \\
&= \|\lambda^{\ell} - \mu^{\ell}\|_* \int_0^T e^{-\eta(M_4M_2+M_5M_1)M_3m(t)} M_3 k(t) dt \\
&+ \int_0^T e^{-\eta(M_4M_2+M_5M_1)M_3m(t)} l(t) dt + \delta \int_0^T e^{-\eta(M_4M_2+M_5M_1)M_3m(t)} dt \\
&\le \frac{1}{\eta(M_4M_2 + M_5M_1)} \|\lambda^{\ell} - \mu^{\ell}\|_* \\
&+ \int_0^T e^{-\eta(M_4M_2+M_5M_1)M_3m(t)} l(t) dt + \delta \int_0^T e^{-\eta(M_4M_2+M_5M_1)M_3m(t)} dt.
\end{aligned}$$

As $\delta \to 0$ we get

$$\begin{aligned}
H^{\beta}(S_{\lambda^{\ell}}(\sigma), \tilde{S}_{\mu^{\ell}}(\sigma)) \le &\frac{1}{\eta(M_4M_2 + M_5M_1)} \|\lambda^{\ell} - \mu^{\ell}\|_* \\
&+ \int_0^T e^{-\eta(M_4M_2+M_5M_1)M_3m(t)} l(t) dt.
\end{aligned} \tag{22}$$

Since $S_{\lambda^{\ell}}(.,.)$ and $\tilde{S}^{\ell}_{\mu}(.,.)$ are H^{β}-quasi contractions with Lipschitz constant $\dfrac{1}{\eta}$ and since $v(\cdot) \in \mathcal{F}\{\tilde{S}_{\mu^{\ell}}\}$ by Proposition 3 there exists $u(\cdot) \in \mathcal{F}\{S_{\lambda^{\ell}}\}$ such that

$$\|v - u\|_1 \le \frac{\gamma\eta}{\eta - \gamma} \sup_{x \in X} H^{\beta}(\tilde{S}_{\mu^{\ell}} x, S_{\lambda^{\ell}} x).$$

Using (22), we have

$$\begin{aligned}
\|v - u\|_1 \le &\frac{\gamma}{(\eta - \gamma)(M_4M_2 + M_5M_1)} \|\lambda^{\ell} - \mu^{\ell}\|_* \\
&+ \frac{\gamma\eta}{\eta - \gamma} \int_0^T e^{-\eta(M_4M_2+M_5M_1)M_3m(t)} l(t) dt.
\end{aligned} \tag{23}$$

Now let

$$x^\ell(t) = \lambda^\ell(t) + \int_0^t [\alpha_1(t,s)\, p(t,u(s)) + \alpha_2(t,s)q(t,s,u(s))]\, ds.$$

Then, we have

$$\begin{aligned}\|x^\ell(t) - y^(t)\| &\le \|\lambda^\ell(t) - \mu^\ell(t)\| + (M_4M_2 + M_5M_1)\int_0^t \|u(s) - v(s)\| ds \\ &\le \|\lambda^\ell - \mu^\ell\|_* + (M_4M_2 + M_5M_1)e^{\eta(M_4M_2+M_5M_1)M_3m(T)}\|u - v\|_1.\end{aligned}$$

Using (23) we get

$$\|x^\ell(t) - y^\ell(t)\| \le \|\lambda^\ell - \mu^\ell\|_*\Big[1 + \frac{\gamma\, e^{\eta(M_4M_2+M_5M_1)M_3m(T)}}{\eta - \gamma}\Big]$$

$$+\frac{\gamma\eta}{\eta-\gamma}(M_4M_2 + M_5M_1)e^{\eta(M_4M_2+M_1)M_3m(T)}\int_0^T e^{-\eta(M_4M_2+M_5M_1)M_3m(t)}l(t)dt.$$

This completes the proof. □

Remark 9. *Since $H^\beta(A,B) \le H(A,B)$ and the class of generalized (ρ,γ)-norms includes the usual norm $\|.\|$, we note that the hypothesis conditions $AS_2(i)$ and $AS_3(i),(ii)$ are much weaker than the corresponding hypothesis conditions (Hypothesis 2.1 (ii) and (iii)) of [24]).*

3.5. Conclusions

The H^β-Hausdorff–Pompeiu b-metric is introduced as a new tool in metric fixed point theory and new variants of Nadler, Ciric, Hardy–Rogers contraction principles for multi-valued mappings are established in a b-metric space. The examples and applications provided illustrates the advantages of using H^β-Hausdorff–Pompeiu b-metric in fixed point theory and its applications. The new tool of H^β-Hausdorff–Pompeiu b-metric can be utilized by young researchers in extending and generalizing many of the fixed point results for multi-valued mappings existing in literature and investigate how the new tool would enhance, extend and generalize the applications of the fixed-point theory to linear differential and integro-differential equations, nonlinear phenomena, algebraic geometry, game theory, non-zero-sum game theory and the Nash equilibrium in economics.

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