


Soft Topological Transformation Groups

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Abstract: The aim of the present study is to introduce the concept of soft topological transformation groups by examining the topological transformation groups, which are the core subject of algebraic topology under the soft approach. Actions of soft topological groups on soft topological spaces are studied, and the category of soft topological transformation groups is constructed. Also, a translation and conjugation of the soft topological groups are described. Finally, the definitions of soft orbit spaces and soft homogeneous spaces are given, and some of the properties of these concepts are examined in detail.

Keywords: soft action; soft topological group; soft (continuous) action; soft topological transformation group

1. Introduction

Topology and algebra are two main areas of mathematics that play complementary roles. Topology deals with continuity and convergence, while algebra examines all kinds of operations. Although these two areas tend to develop independently, they are in a natural interaction with other areas of mathematics such as functional analysis, representation theory, dynamical systems. In particular, most of the important objects of mathematics contain a blend of topological and algebraic structures. Topological transformation groups are objects of this kind. Historically, the study of topological transformation groups were originated in the work of Montgomery and Zippin, which contains a solution of Hilbert's fifth problem [1]. This subject has reached a wide range of study potential and continues to be one of the most dominant topics in mathematics nowadays [2].

The other central theme in the work is the soft set theory. This theory proposed by Molodtsov is a recent mathematical approach for managing vagueness and uncertainty [3]. Researches based on the soft set theory continues to develop rapidly in many fields such as engineering, medicine, economics, environmental sciences. In recent years, this theory has been combined with some important mathematical theories such as algebra and topology. Many different algebraic properties of soft sets were presented in the literature by Maji, Aktaş, Jun and Feng [4–7]. Maji et al. defined new operations on soft sets [4]. In [5], Aktaş and Çağman improved the definitions of soft group and soft subgroup, and proved their new properties. Recently, Oguz et al. studied the actions of soft groups and obtained some important properties [8]. In addition, other algebraic properties of soft sets were studied in [9–12]. In the meantime, the pioneers of topological studies on soft sets are Shabir and Naz [13]. They investigated the notion of soft topological spaces. Later, Nazmul and Samanta presented the definition of a soft topological group [14]. Other than those, many topological studies have emerged [15–24].

In this paper, by using the concept of soft continuous we are going to define the action of soft topological groups and investigate some of their properties. Consequently, we are going to introduce the notion of soft topological transformation groups by studying the combination between the topological transformation groups and soft set theory. Finally, we are going to present some investigations about soft orbit spaces and soft homogeneous spaces.

2. Preliminaries

In this section, we review some definitions and results about the soft set theory used throughout the paper. For more details, we refer the reader to [3,4].

Throughout this paper, we assume that X is an initial universe set and E is a nonempty set of parameters. Let $P(X)$ denote the power set of X and $A \subset E$.

Definition 1 ([3]). A pair (F, A) is called a soft set over X , where F is a mapping defined by

$$F : A \longrightarrow P(X)$$

In general, one can see that a soft set over X is a parametrized family of subsets of the universe X . We will sometimes use the representation (X, F, A) instead of soft set (F, A) over X .

Definition 2 ([4]). A soft set (X, F, A) is called a null soft set indicated by Φ , if $F(\varepsilon) = \emptyset$ for all $\varepsilon \in A$.

Definition 3 ([4]). A soft set (X, F, A) is called an absolute soft set indicated by \tilde{A} , if $F(\varepsilon) = X$ for all $\varepsilon \in A$.

Definition 4 ([4]). Let (F, A) and (G, B) be two soft sets over the common universe X . Then (F, A) is said to be a soft subset of (G, B) , denoted by $(F, A) \widetilde{\subset} (G, B)$ if the following conditions are satisfied:

- i. $A \subset B$.
- ii. $\forall \varepsilon \in A$, $f(\varepsilon)$ and $G(\varepsilon)$ are identical approximations.

Definition 5 ([4]). Let (F, A) and (G, B) be two soft sets over the common universe X . Their intersection is a soft set (H, C) such that $C = A \cap B$ and $H(\varepsilon) = F(\varepsilon) \cap G(\varepsilon)$, $\forall \varepsilon \in C$.

It is denoted by $(F, A) \widetilde{\cap} (G, B) = (H, C)$.

Definition 6 ([4]). Let (F, A) and (G, B) be two soft sets over the common universe X . Their union is a soft set (H, C) , where $C = A \cup B$, and $\forall \varepsilon \in C$,

$$H(\varepsilon) = \begin{cases} F(\varepsilon), & \text{if } \varepsilon \in A - B \\ G(\varepsilon), & \text{if } \varepsilon \in B - A \\ F(\varepsilon) \cup G(\varepsilon), & \text{if } \varepsilon \in A \cap B \end{cases}$$

This situation is denoted by $(F, A) \widetilde{\cup} (G, B) = (H, C)$.

Below, we give soft group and related concepts.

Definition 7 ([5]). Let G be a group and A be a nonempty set. For the soft set (F, A) over G , it is said that (F, A) is a soft group over G if and only if $F(\varepsilon) < G$ for all $\varepsilon \in A$.

As it can be seen from the definition, the soft group (F, A) is actually a parameterized family of subgroups of the group G . From now on, the triplet (G, F, A) stands for the soft group (F, A) over G .

Example 1 ([5]). Assume that $G = A = S_3 = \{e, (12), (13), (23), (123), (132)\}$, and $F(e) = \{e\}$, $F(12) = \{e, (12)\}$, $F(13) = \{e, (13)\}$, $F(23) = \{e, (23)\}$, $F(123) = F(132) = \{e, (123), (132)\}$. Then (F, A) is a soft group over G , since $F(\varepsilon)$ is a subgroup of G for all $\varepsilon \in A$.

Definition 8 ([5]). Let (F, A) be a soft set over G . Then, for all $\varepsilon \in A$

- i. (F, A) is said to be an identity soft group over G if $F(\varepsilon) = \{e\}$, where e is the identity element of G .
- ii. (F, A) is said to be an absolute soft group over G if $F(\varepsilon) = G$.

Here we state the action of a soft group, introduced by Oguz et al.

Definition 9 ([8]). Let $G = (G, F, A)$ be a soft group and let $X = (X, F', A)$ be a soft set. Then, a left soft (group) action of G on X is defined by the binary operation

$$\Pi_\varepsilon : F(\varepsilon) \times F'(\varepsilon) \longrightarrow F'(\varepsilon)$$

satisfying the following properties for all $\varepsilon \in A$:

- i. $\Pi_\varepsilon(e, x) = x$ for all $x \in F'(\varepsilon)$.
- ii. $\Pi_\varepsilon(g, \Pi_\varepsilon(h, x)) = \Pi_\varepsilon(gh, x)$ for all $x \in F'(\varepsilon)$ and $g, h \in F(\varepsilon)$.

The following example of soft (group) action seems to be illustrative.

Example 2 ([8]). Suppose $A = G = S_3$ is the group of permutations on the $S = \{1, 2, 3\}$. Consider the soft group $G = (G, F, A)$ as follows

$$\begin{aligned} F(e) &= \{e\}, \\ F(12) &= \{e, (12)\}, \\ F(13) &= \{e, (13)\}, \\ F(23) &= \{e, (23)\}, \\ F(123) &= F(132) = \{e, (123), (132)\}. \end{aligned}$$

Also, the soft set $X = (X, F', A)$ with $X = \{1, 2, 3\}$ is given by

$$\begin{aligned} F'(e) &= \{1\}, \\ F'(12) &= \{1, 2\}, \\ F'(13) &= \{1, 3\}, \\ F'(23) &= \{2, 3\}, \\ F(123) &= F(132) = \{1, 2, 3\}. \end{aligned}$$

For all $a \in A$, the mapping

$$\begin{aligned} \Pi_\varepsilon : F(\varepsilon) \times F'(\varepsilon) &\longrightarrow F'(\varepsilon) \\ (g, x) &\mapsto \Pi_a(g, x) = \sigma(x) \end{aligned}$$

is a left (group) action such that $e \in G$ is the identity element and $\Pi_\varepsilon(e, x) = x$ as desired. Moreover, it is obvious that $\Pi_\varepsilon(g, \Pi_\varepsilon(h, x)) = \Pi_\varepsilon(gh, x)$ for $g, h \in F(\varepsilon)$.

Similar to Molodtsov's perspective, the definition of a soft topology proposed by Kandemir is as follows:

Definition 10 ([16]). Let the initial universe set X be a topological space with a topology τ and let (F, A) be a soft set defined over X . Then, (F, A) is called a soft topology over X if $F(\varepsilon)$ is a subspace of X with respect to the topology $\tau_{F(\varepsilon)}$ induced by τ for all $\varepsilon \in A$. Also, (F, A, τ) is called a soft topological space over X .

Now, we shall recall the definition of a soft topological group.

Definition 11 ([18]). Let G be a group that is furnished with a topology τ and let (F, A) be a non-null soft set described over G . Then, the structure (F, A, τ) is defined as a soft topological group over G if the following statements are satisfied for all $\varepsilon \in A$:

- i. $F(\varepsilon)$ is a subgroup of G .
- ii. The mapping $(x, y) \mapsto x - y$ of the topological space $F(\varepsilon) \times F(\varepsilon)$ onto $F(\varepsilon)$ is continuous.

Example 3 ([18]). Consider $G = S_3 = \{e, (12), (13), (23), (123), (132)\}$ as a group, $A = \{e_1, e_2, e_3\}$ and $F(e_1) = \{e\}$, $F(e_2) = \{e, (12)\}$, $F(e_3) = \{e, (123), (132)\}$. Let $\mathcal{B} = \{\{e\}, \{(12)\}, \{(123)\}, \{(132)\}\}$ be a base for the topology τ . It is easy to check that $F(a)$ is a subgroup of G for all $\varepsilon \in A$. In addition, the second condition of above definition is provided. This means that (F, A, τ) is a soft topological group.

Definition 12 ([18]). Let (F, A, τ) and (F', B, τ') be two soft topological groups over G and G' , respectively. Let $f : G \rightarrow G'$ and $g : A \rightarrow B$ be two mappings. Then, the pair (f, g) is called a soft topological group homomorphism if

- i. f is a group epimorphism and g is a surjection.
- ii. $f(F(\varepsilon)) = F'(g(\varepsilon))$ for all $\varepsilon \in A$.
- iii. $f_\varepsilon : (F(\varepsilon), \tau_{F(\varepsilon)}) \rightarrow (F'(g(\varepsilon)), \tau'_{F'(g(\varepsilon))})$ is continuous for all $\varepsilon \in A$.

It is worth noting that (F, A, τ) is called soft topologically homomorphic to (F', B, τ') and written as $(F, A, \tau) \sim (F', B, \tau')$.

3. Actions of Soft Topological Groups

In this section, after introducing the actions of soft topological groups we define the concept of a soft topological transformation group. We first begin by giving the following definitions.

Definition 13. Let (F, A, τ) be a soft topological group over G and (F', A, τ') a soft topological space over X . Then, a left soft (continuous) action of (F, A, τ) on (F', A, τ') is a continuous map

$$\theta_\varepsilon : F(\varepsilon) \times F'(\varepsilon) \rightarrow F'(\varepsilon)$$

such that it has the following properties for all $\varepsilon \in A$:

- i. $\theta_\varepsilon(e, x) = x$ for all $x \in F'(\varepsilon)$.
- ii. $\theta_\varepsilon(g, \theta_\varepsilon(h, x)) = \theta_\varepsilon(gh, x)$ for all $x \in F'(\varepsilon)$ and $g, h \in F(\varepsilon)$.

Also, the soft topological space (F', A, τ') is called a soft G -space. It is clear that each Π_ε is a left (continuous) action mapping for all $\varepsilon \in A$. In a similar way, a right soft (continuous) action is defined.

Definition 14. Let the soft topological space (F', A, τ') be a soft G -space. Then, (F, A, τ) is called a soft topological transformation group on (F', A, τ') .

As an illustration, let us consider the following examples.

Example 4. Let G be a soft topological group and $X = G$. Then, define the mapping

$$\begin{aligned} \theta_\varepsilon : F(\varepsilon) \times F(\varepsilon) &\longrightarrow F(\varepsilon) \\ (g, h) &\mapsto \theta_\varepsilon(g, h) = h. \end{aligned}$$

It is easy to verify that θ_ε is continuous action for all $\varepsilon \in A$. Thus, every soft topological group G acts on itself by translation as above.

Example 5. Consider a soft topological group G and let $X = G$. In this case, the soft (continuous) action of G on itself by multiplication is as follows. For all $\varepsilon \in A$,

$$\begin{aligned}\theta_\varepsilon : F(\varepsilon) \times F(\varepsilon) &\longrightarrow F(\varepsilon) \\ (g, h) &\mapsto \theta_\varepsilon(g, h) = gh.\end{aligned}$$

We have already seen that each mapping Π_ε is continuous action for all $\varepsilon \in A$.

Example 6. Every soft topological group G acts on itself by conjugation as follows. Let $X = G$. For all $\varepsilon \in A$, define the mapping

$$\begin{aligned}\theta_\varepsilon : F(\varepsilon) \times F(\varepsilon) &\longrightarrow F(\varepsilon) \\ (g, h) &\mapsto \theta_\varepsilon(g, h) = ghg^{-1}.\end{aligned}$$

It is easy to notice that each Π_ε is a continuous action.

Apart from these examples, a different construction is as follows.

Example 7. Let (F', A, τ') be a soft topological space over X and (F, A, τ) be a soft topological group acting on the soft topological space (F', A, τ') as given below

$$\begin{aligned}\theta_\varepsilon : F(\varepsilon) \times F'(\varepsilon) &\longrightarrow F'(\varepsilon) \\ (g, x) &\mapsto \theta_\varepsilon(g, x).\end{aligned}$$

Suppose that $f : (X, \tau') \longrightarrow (Y, \tau'')$ is an injective continuous function. Now defining $F'' = f(F')$, we establish

$$\begin{aligned}\theta'_\varepsilon : F(\varepsilon) \times f(F'(\varepsilon)) &\longrightarrow f(F'(\varepsilon)) \\ (g, f(x)) &\mapsto \theta'_\varepsilon(g, f(x)) = f(\theta_\varepsilon(g, x))\end{aligned}$$

for all $\varepsilon \in A$. So using Definition 13, one can easily see that Π'_ε is a soft (continuous) action of (F, A, τ) on (F'', A, τ'') .

Proposition 1. Let (F', A, τ') be a soft topological space over X and (F'', A, τ'') be a soft topological space over Y . If (F', A, τ') and (F'', A, τ'') are two soft G -space, then $(F' \times F'', A, \tau' \times \tau'')$ is a soft G -space.

Proof. Let (F, A, τ) be a soft topological group over G . For all $\varepsilon \in A$, consider a soft G -space (F', A, τ') with the soft (continuous) action

$$\begin{aligned}\theta_\varepsilon : F(\varepsilon) \times F'(\varepsilon) &\longrightarrow F'(\varepsilon) \\ (g, x) &\mapsto \theta_\varepsilon(g, x)\end{aligned}$$

and a soft G -space (F'', A, τ'') with the soft (continuous) action

$$\begin{aligned}\phi_\varepsilon : F(\varepsilon) \times F''(\varepsilon) &\longrightarrow F''(\varepsilon) \\ (g, y) &\mapsto \phi_\varepsilon(g, y).\end{aligned}$$

Then, we can establish a soft (continuous) action denoted as

$$\varphi_\varepsilon : F(\varepsilon) \times (F'(\varepsilon) \times F''(\varepsilon)) \longrightarrow (F'(\varepsilon) \times F''(\varepsilon))$$

$$(g, (x, y)) \mapsto \varphi_\epsilon(g, (x, y)) = (\theta_\epsilon(g, x), \phi_\epsilon(g, y)).$$

It is easy to verify that φ_ϵ is a soft (continuous) action of (F, A, τ) on $(F' \times F'', A, \tau' \times \tau'')$. This gives us a soft G -space of $(F' \times F'', A, \tau' \times \tau'')$. \square

Let us present some types of soft (continuous) action here.

Definition 15. A soft (continuous) action of the soft topological group (F, A, τ) on the soft topological space (F', A, τ') is said to be

- i. transitive if for each pair x, y in $F'(\epsilon)$ there exists an element g in $F(\epsilon)$ such that $\theta_\epsilon(g, x) = y$.
- ii. effective (or faithful) if for every two distinct elements g, h in $F(\epsilon)$ there is an element x in $F'(\epsilon)$ such that $\theta_\epsilon(g, x) \neq \theta_\epsilon(h, x)$.
- iii. free if given g, h in $F(\epsilon)$, the existence of an element x in $F'(\epsilon)$ with $\theta_\epsilon(g, x) = \theta_\epsilon(h, x)$ implies $g = h$.

By this definition, we get the following desired result.

Proposition 2. Every free soft (continuous) action is faithful.

Definition 16. A soft (continuous) action of the soft topological group (F, A, τ) on the soft topological space (F', A, τ') is said to be regular if it is both transitive and free or equivalently, if for every element x, y in $F'(\epsilon)$ there exists only one element g in $F(\epsilon)$ such that $\theta_\epsilon(g, x) = y$.

Example 8. The soft (continuous) action of any soft topological group (F, A, τ) on itself by multiplication is both regular and faithful.

Let us continue with soft homogeneous spaces.

Definition 17. A soft G -space is called soft homogeneous if its soft (continuous) action is transitive.

Example 9. Choose (F', B, τ) over G' as a soft topological subgroup group of (F, A, τ) . Then (F, A, τ) is a soft homogeneous G' - space with the soft (continuous) action given by

$$\begin{aligned} \theta_\epsilon : F'(\epsilon) \times F(\epsilon) &\longrightarrow F(\epsilon) \\ (x, g) &\mapsto \theta_\epsilon(x, g) = xg \end{aligned}$$

for all $\epsilon \in B$.

On the frame of this example, let (F', B, τ) be a soft topological space over the quotient space G/G' . We can define a soft (continuous) action

$$\begin{aligned} \psi_\epsilon : F(\epsilon) \times F(\epsilon)/F'(\epsilon) &\longrightarrow F(\epsilon)/F'(\epsilon) \\ (g, g_1x) &\mapsto \psi_\epsilon(g, g_1x) = (gg_1)x \end{aligned}$$

for all $\epsilon \in B$. Thus, we say that (F', B, τ) is a soft homogeneous G -space.

Below, we present the category of soft G -spaces as a new category.

Definition 18. Let (F', A, τ') and (F'', A, τ'') be two soft topological spaces over X and Y , respectively. Also, let (F', A, τ') and (F'', A, τ'') are soft G -spaces with the soft actions θ_ϵ and θ'_ϵ . Then a map $f_\epsilon : F'(\epsilon) \longrightarrow F''(\epsilon)$, defined by $f_\epsilon(\theta_\epsilon(g, x)) = \theta'_\epsilon(g, f_\epsilon(x))$ for every $\epsilon \in A$, is said to be a soft G -equivariant.

Notice that soft G -spaces together with soft G -equivariants between them form a category.

We shall now obtain some important characterizations giving concepts such as the fixed point set, centralizer and normalizer related to the soft (continuous) action.

Definition 19. Let (F', A, τ') be a G -soft space. Then, for any soft subset $(H, F'', B) \widetilde{\subseteq} (X, F', A)$, we define the fixed point set as follows

$$\text{Fix}_{F''}(\varepsilon) = \{g \in F(\varepsilon) : \theta_\varepsilon(g, x) = x, x \in F''(\varepsilon)\}.$$

In particular, for all $\varepsilon \in A$ and $x \in F'(\varepsilon)$, the fixed point set x is defined by

$$\text{Stab}_{F'}(\varepsilon) = \{g \in F(\varepsilon) : \theta_\varepsilon(g, x) = x\}.$$

At this point, we can give a result.

Proposition 3. The sets $\text{Fix}_{F''}(\varepsilon)$ and $\text{Stab}_{F'}(\varepsilon)$ defined above are soft topological groups over G .

Proof. We have the map

$$\begin{aligned} \text{Fix}_{F''} : A &\longrightarrow P(G) \\ \varepsilon &\longmapsto \text{Fix}_{F''}(\varepsilon). \end{aligned}$$

It is clear that $\text{Fix}_{F''}(\varepsilon)$ is a topological subgroup with respect to the topologies induced by G for all $\varepsilon \in A$. Then, the pair $(\text{Fix}_{F''}, A)$ is a soft topological group over G . Analogously, we consider the map

$$\begin{aligned} \text{Stab}_{F'} : A &\longrightarrow P(G) \\ \varepsilon &\longmapsto \text{Stab}_{F'}(\varepsilon) \end{aligned}$$

which implies that $\text{Stab}_{F'}(\varepsilon)$ is a topological subgroup with respect to the topologies induced by G for all $\varepsilon \in A$. Therefore, the pair $(\text{Stab}_{F'}, A)$ is a soft topological group over G . \square

Remark 1. It is well-known that the fixed point sets are subgroups of the topological group G in the classical theory. Also, the soft analog of this statement is true. In other words, $\text{Stab}_G(x)$ and $\text{Fix}_G Y$ are soft topological subgroups of the soft topological group G .

Proposition 4. For $\text{Fix}_{F''}(\varepsilon)$ and $\text{Stab}_{F'}(\varepsilon)$ as given above, the following equality holds:

$$\text{Fix}_{F''}(\varepsilon) = \cap_{x \in F''(\varepsilon)} \text{Stab}_{F'}(\varepsilon).$$

Proof. For proof, consider $g \in \text{Fix}_{F''}(\varepsilon)$. Then $\theta_\varepsilon(g, x) = x$ for all $x \in F''(\varepsilon)$, which implies that $g \in \text{Stab}_{F'}(\varepsilon)$. That is, $g \in \cap_{x \in F''(\varepsilon)} \text{Stab}_{F'}(\varepsilon)$ and so $\text{Fix}_{F''}(\varepsilon) \subseteq \cap_{x \in F''(\varepsilon)} \text{Stab}_{F'}(\varepsilon)$ is obtained.

Conversely, take $g \in \cap_{x \in F''(\varepsilon)} \text{Stab}_{F'}(\varepsilon)$. Then $g \in \text{Stab}_{F'}(\varepsilon)$ for all $x \in F''(\varepsilon)$ and so $\theta_\varepsilon(g, x) = x$. Thus, it is readily verified that $g \in \text{Fix}_{F''}(\varepsilon)$. By this newly obtained fact, the proof is completed. \square

Definition 20. Let (F', A, τ') be a G -soft space. Then, for all $x \in F'(\varepsilon)$ the soft orbit of $F'(\varepsilon)$ under $F(\varepsilon)$ is defined as

$$\text{Orb}_{F'}(\varepsilon) = \{\theta_\varepsilon(g, x) : g \in F(\varepsilon)\}.$$

It is not difficult to show that the pair $(\text{Orb}_{F'}, A)$ is a soft set over X with the mapping

$$\text{Orb}_{F'} : A \longrightarrow P(X).$$

Also, it is clear that $F'(\varepsilon)$ and $F'(\epsilon)$ belong to the same soft orbit if $\text{Orb}_{F'}(\varepsilon) = \text{Orb}_{F'}(\epsilon)$.

Proposition 5. For the G -soft space (F', A, τ') , if $x \in F'(\varepsilon)$, $g \in F(\varepsilon)$, and $y = \theta_\varepsilon(g, x)$, then $x = \theta_\varepsilon(g^{-1}, y)$. Moreover, if $x \neq x'$ then $\theta_\varepsilon(g, x) \neq \theta_\varepsilon(g, x')$.

Proof. Let $y = \theta_\varepsilon(g, x)$ such that

$$\theta_\varepsilon(g^{-1}, y) = \theta_\varepsilon(g^{-1}, \theta_\varepsilon(g, x)) = \theta_\varepsilon(g^{-1}g, x) = \theta_\varepsilon(e, x) = x.$$

Assume $x \neq x'$, so $\theta_\varepsilon(g, x) = \theta_\varepsilon(g, x')$. When g^{-1} is applied to both sides, it follows that

$$\theta_\varepsilon(g^{-1}, \theta_\varepsilon(g, x)) = \theta_\varepsilon(g^{-1}, \theta_\varepsilon(g, x'))$$

$$\theta_\varepsilon(g^{-1}g, x) = \theta_\varepsilon(g^{-1}g, x')$$

$$\theta_\varepsilon(e, x) = \theta_\varepsilon(e, x')$$

$$x = x'$$

This contradicts by the hypothesis $x \neq x'$. Hence, $\theta_\varepsilon(g, x) \neq \theta_\varepsilon(g, x')$. \square

Definition 21. Let the soft topological group (F, A, τ) acts on itself by conjugation. For all $\varepsilon \in A$ and $h \in F(\varepsilon)$, the soft centralizer of $F(\varepsilon)$ is a set defined by

$$C_F(\varepsilon) = \{g \in F(\varepsilon) : \theta_\varepsilon(g, h) = \theta_\varepsilon(h, g)\}.$$

Then, we have the followings:

Proposition 6. The soft centralizer $C_F(\varepsilon)$ given above is a soft topological group over G .

Proof. For all $\varepsilon \in A$ consider

$$\begin{aligned} C_F : A &\longrightarrow P(G) \\ \varepsilon &\longmapsto C_F(\varepsilon). \end{aligned}$$

Then one can easily check that $C_F(\varepsilon)$ is a topological subgroup of G . Thus, the pair (C_F, A) is a soft topological group over G . \square

Definition 22. Let the soft topological group (F, A, τ) act on itself by conjugation and let (H, F'', B) be a soft subset of (G, F, A) . The soft stability subgroup is defined

$$N_{F''}(\varepsilon) = \{g \in F(\varepsilon) : \theta_\varepsilon(g, h) = h, h \in F''(\varepsilon)\}$$

and is said to be a soft normalizer of (H, F'', B) .

From this definition, we can get the following:

Proposition 7. The soft normalizer $N_{F''}(\varepsilon)$ given above is a soft topological group over G .

Proof. For all $\varepsilon \in A$ we define

$$\begin{aligned} N_{F''} : A &\longrightarrow P(G) \\ \varepsilon &\longmapsto N_{F''}(\varepsilon). \end{aligned}$$

Then it is easy to verify that $N_{F''}(\varepsilon)$ is a topological subgroup of G . Hence, the pair $(N_{F''}, A)$ is a soft topological group over G . \square

Lastly, we can directly obtain the following result, which completes this study.

Corollary 1. *If (H, F'', B) is a soft topological subgroup of (G, F, A) , then (H, F'', B) is a soft topological subgroup of $(G, N_{F''}, A)$. Furthermore, if (H, F'', B) is a normal soft topological subgroup of (G, F, A) , $(G, N_{F''}, A)$ is the largest soft topological subgroup of (G, F, A) .*

4. Conclusions

In summary, it can be said that this study is the beginning of a new structure between topological transformation groups and soft set theory, which are one of the important topics of algebraic topology. Several fundamental properties of soft topological transformation groups are also investigated. At the same time, some notions such as stabilizer, centralizer and normalizer related to the soft (continuous) action are described, and the characterizations between them are presented. Furthermore, by defining soft G -equivariants, the category of soft G -spaces is established. To extend our work in this paper, by using the concept of soft (continuous) action defined on soft topological groups, soft topological crossed modules can be defined so that by constructing the category of soft topological crossed modules, some novel categorical equivalents between category theory and soft set theory can be studied.

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