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Review

A Survey on Sharp Oscillation Conditions of Differential Equations with Several Delays

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Abstract: This paper deals with the oscillation of the first-order differential equation with several delay arguments $x'(t)+\sum_{i=1}^{m} p_i(t)\,x(\tau_i(t))=0,\ t\geq t_0$, where the functions $p_i,\ \tau_i \in C([t_0,\infty),\mathbb{R}^+)$, for every $i=1,2,\ldots,m,\ \tau_i(t)\leq t$ for $t\geq t_0$ and $\lim_{t\to\infty}\tau_i(t)=\infty$. In this paper, the state-of-the-art on the sharp oscillation conditions are presented. In particular, several sufficient oscillation conditions are presented and it is shown that, under additional hypotheses dealing with slowly varying at infinity functions, some of the "liminf" oscillation conditions can be essentially improved replacing "liminf" by "limsup". The importance of the slowly varying hypothesis and the essential improvement of the sufficient oscillation conditions are illustrated by examples.

Keywords: oscillation; delay arguments; differential equations

1. Introduction

In this paper, we are concerned with the oscillatory behavior of all solutions to the first-order delay differential equation with several arguments of the form

$$x'(t)+\sum_{i=1}^{m} p_i(t)\,x(\tau_i(t))=0,\ t\geq t_0, \tag{1}$$

where the functions $p_i,\ \tau_i \in C([t_0,\infty),\mathbb{R}^+)$, for every $i=1,2,\ldots,m$, (here $\mathbb{R}^+=[0,\infty)$), $\tau_i(t)\leq t$ for $t\geq t_0$ and $\lim_{t\to\infty}\tau_i(t)=\infty$. Let $T_0\in[t_0,+\infty)$, $\tau(t)=\min\{\tau_i(t):i=1,\ldots,m\}$ and $\tau_{-1}(t)=\sup\{s:\tau(s)\leq t\}$.

By a solution of Equation (1) we understand the function $x\in C([T_0,+\infty),\mathbb{R})$, continuously differentiable on $[\tau_{-1}(T_0),+\infty)$ and which satisfies Equation (1) for $t\geq\tau_{-1}(T_0)$. Such a solution is called *oscillatory* if it has arbitrarily large zeros, otherwise, it is called *non-oscillatory*.

In the special case where $m=1$, Equation (1) reduces to the equation

$$x'(t)+p(t)\,x(\tau(t))=0,\ t\geq t_0, \tag{2}$$

where the functions $p,\ \tau\in C([t_0,\infty),\mathbb{R}^+)$, $\tau(t)\leq t$ for $t\geq t_0$ and $\lim_{t\to\infty}\tau(t)=\infty$. For the general theory of these equations, the reader is referred to [1–5]. The problem of setting sufficient conditions for the oscillation of all solutions of differential Equations (1) and (2) (and also to more general equations) was the subject of several investigations. See, for example, [1–35] and the references mentioned in it. In the case of monotonous arguments, several interesting sufficient oscillation conditions for

Equation (2) can be found in [6–10]. For equations with several arguments the following sufficient oscillation conditions have been established.

The objective of this paper is to point out that, under mild additional hypotheses dealing with slowly varying at infinity functions, several of these sufficient oscillation conditions can be essentially improved if "liminf" is replaced by "limsup".

2. Oscillation Criteria for Equation (1)

In 1982, several interesting sufficient conditions for the oscillation of all solutions to Equation (1) were established in an article by Ladas and Stavroulakis [11] (see also the paper in 1984 by Arino et al. [12]), where they studied the equation with several constant delay arguments of the form

$$x'(t) + \sum_{i=1}^{m} p_i(t)x(t-\tau_i) = 0\,,\ t \geq t_0, \tag{3}$$

under the assumption that $\liminf_{t\to\infty} \int_{t-\tau_i/2}^{t} p(s)ds > 0$, $i = 1, 2, \ldots, m$, and proved that each one of the following conditions

$$\liminf_{t\to\infty} \int_{t-\tau_i}^{t} p_i(s)ds > \frac{1}{\mathrm{e}} \ \text{ for some } \ i, \ \ i = 1, 2, \ldots, m, \tag{4}$$

$$\liminf_{t\to\infty} \int_{t-\tau}^{t} \sum_{i=1}^{m} p_i(s)ds > \frac{1}{\mathrm{e}}, \ \text{ where } \ \tau = \min\{\tau_{1,}\tau_{2,}, \ldots, \tau_m\}, \tag{5}$$

$$\left[\prod_{i=1}^{m} \left(\sum_{j=1}^{m} \liminf_{t\to\infty} \int_{t-\tau_j}^{t} p_i(s)ds \right)\right]^{1/m} > \frac{1}{\mathrm{e}}, \tag{6}$$

or

$$\frac{1}{m}\sum_{i=1}^{m}\left(\liminf_{t\to\infty}\int_{t-\tau_i}^{t} p_i(s)ds\right) + \frac{2}{m}\sum_{\substack{i<j\\ i,j=1}}^{m}\left[\left(\liminf_{t\to\infty}\int_{t-\tau_j}^{t} p_i(s)ds\right)\left(\liminf_{t\to\infty}\int_{t-\tau_i}^{t} p_i(s)ds\right)\right]^{\frac{1}{2}} > \frac{1}{\mathrm{e}}, \tag{7}$$

implies that all solutions of Equation (3) oscillate.

Later in 1996, Li [13] proved that the same conclusion holds if

$$\liminf_{t\to\infty} \sum_{i=1}^{m} \int_{t-\tau_i}^{t} p_i(s)ds > \frac{1}{e}. \tag{8}$$

In 1984, Hunt and Yorke [14] considered the equation with variable arguments of the form:

$$x'(t) + \sum_{i=1}^{m} p_i(t)x(t-\tau_i(t)) = 0, \ \ t \geq t_0, \tag{9}$$

under the assumption that there is a uniform upper bound τ_0 on the $\tau_i's$ and proved that if

$$\liminf_{t\to\infty} \sum_{i=1}^{m} \tau_i(t)p_i(t) > \frac{1}{\mathrm{e}}, \tag{10}$$

then all solutions of Equation (9) oscillate.

In 1984, Fukagai and Kusano [15], for Equation (1) established the following theorem.

Theorem 1. *([15], Theorem 1′(i)) Consider Equation (1) and assume that there is a continuous non-decreasing function $\tau^*(t)$ such that $\tau_i(t) \le \tau^*(t) \le t$ for $t \ge t_0$, $1 \le i \le m$. if*

$$\liminf_{t\to\infty} \int_{\tau^*(t)}^{t} \sum_{i=1}^{m} p_i(s)ds > \frac{1}{e}, \tag{11}$$

then all solutions of Equation (1) oscillate.

On the other hand, if there exists a continuous non-decreasing function $\tau_*(t)$ such that $\tau_*(t) \le \tau_i(t)$ for $t \ge t_0$, $1 \le i \le m$, $\lim_{t\to\infty} \tau_*(t) = \infty$ and

$$\int_{\tau_*(t)}^{t} \sum_{i=1}^{m} p_i(s)ds \le \frac{1}{e} \quad \textit{for all sufficiently large } t,$$

then Equation (1) has a non-oscillatory solution.

In 2000, Grammatikopoulos et al. [16] improved the above results as follows:

Theorem 2. *([16], Theorem 2.6) Assume that the functions τ_i are non-decreasing for all $i \in \{1, \dots, m\}$,*

$$\int_0^\infty \left| p_i(t) - p_j(t) \right| dt < +\infty, \quad i, j = 1, \dots, m$$

and

$$\liminf_{t\to\infty} \int_{\tau_i(t)}^{t} p_i(s)ds = \beta_i > 0, \quad i = 1, \dots, m.$$

if

$$\sum_{i=1}^{m} \left(\liminf_{t\to\infty} \int_{\tau_i(t)}^{t} p_i(s)ds \right) > \frac{1}{e}, \tag{12}$$

then all solutions of Equation (1) oscillate.

Note that all the conditions of oscillation mentioned above (4)–(12) involve lim inf only and in the case of the differential equation

$$x'(t) + p(t)x(t - \tau) = 0, \ \tau > 0, \ t \ge t_0, \tag{13}$$

reduce to the oscillation condition (cf. [8,17])

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > \frac{1}{e}. \tag{14}$$

At this point, we also mention that in the case of a differential equation with a constant coefficient and constant delay

$$x'(t) + px(t - \tau) = 0, \ p, \tau > 0, \ t \ge t_0, \tag{15}$$

the above condition (14) reduces to

$$p\tau > \frac{1}{e} \tag{16}$$

which is a *sufficient and necessary condition* [11,17] for all solutions of Equation (15) to oscillate.

It is also known [18] that if in addition τ is a non-decreasing function and

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > 1, \tag{17}$$

then all solutions of Equation (1) oscillate.

It is clear that there is a gap between conditions (14) and (17) when the $\lim_{t\to\infty}\int_{\tau(t)}^{t} p(s)ds$ does not exist. Moreover, it is an interesting problem to investigate Equation (1) with non-monotone arguments and derive sufficient oscillation conditions that include lim sup (as the condition (17) for the Equation (2) with one argument). Concerning the differential Equation (1) with several non-monotone arguments the following oscillation results have been recently published. Assume that there exist non-decreasing functions $\sigma_i \in C\left([t_0,\infty),\mathbb{R}^+\right)$ such that

$$\tau_i(t) \le \sigma_i(t) \le t, i = 1,2,\ldots,m. \tag{18}$$

In 2015 Infante et al. [19] proved that if

$$\limsup_{t\to+\infty}\prod_{j=1}^{m}\left[\prod_{i=1}^{m}\int_{\sigma_j(t)}^{t} p_i(s)\exp\left(\int_{\tau_i(s)}^{\sigma_i(t)}\sum_{i=1}^{m} p_i(\xi)\exp\left(\int_{\tau_i(\xi)}^{\xi}\sum_{i=1}^{m} p_i(u)du\right)d\xi\right)ds\right]^{\frac{1}{m}} > \frac{1}{m^m}, \tag{19}$$

then all solutions of Equation (1) oscillate.

Also in 2015 Kopladatze [20] improved the above condition as follows: Let there exist some $k \in \mathbb{N}$ such that

$$\limsup_{t\to\infty}\prod_{j=1}^{m}\left[\prod_{i=1}^{m}\int\limits_{\sigma_j(t)}^{t} p_i(s)\exp\left(m\int\limits_{\tau_i(s)}^{\sigma_i(t)}\left(\prod_{\ell=1}^{m} p_\ell(\xi)\right)^{\frac{1}{m}}\psi_k(\xi)\,d\xi\right)ds\right]^{\frac{1}{m}} > \frac{1}{m^m}\left[1-\prod_{i=1}^{m} c_i(\alpha_i)\right], \tag{20}$$

where

$$\psi_1(t) = 0,\ \psi_i(t) = \exp\left(\sum_{j=1}^{m}\int\limits_{\tau_j(t)}^{t}\left(\prod_{\ell=1}^{m} p_\ell(s)\right)^{\frac{1}{m}}\psi_{i-1}(s)\,ds\right),\ i = 2,3,\ldots, \tag{21}$$

$$0 < \alpha_i := \liminf_{t\to\infty}\int\limits_{\sigma_i(t)}^{t} p_i(s)\,ds < \frac{1}{e},\ i = 1,2,\ldots,m, \tag{22}$$

and

$$c_i(\alpha_i) = \frac{1-\alpha_i-\sqrt{1-2\alpha_i-\alpha_i^2}}{2},\ i = 1,2,\ldots,m, \tag{23}$$

then all solutions of Equation (1) oscillate.

In 2016 Braverman et al. [21] obtained the following iterative sufficient oscillation conditions:

$$\limsup_{t\to\infty}\int\limits_{h(t)}^{t}\sum_{i=1}^{m} p_i(u)\,a_r(h(t),\tau_i(u))\,du > 1, \tag{24}$$

$$\limsup_{t\to\infty}\int\limits_{h(t)}^{t}\sum_{i=1}^{m} p_i(u)\,a_r(h(t),\tau_i(u))\,du > 1-\frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}, \tag{25}$$

$$\liminf_{t\to\infty}\int\limits_{h(t)}^{t}\sum_{i=1}^{m} p_i(u)\,a_r(h(t),\tau_i(u))\,du > \frac{1}{e}, \tag{26}$$

where

$$h(t) = \max_{1\le i\le m} h_i(t) \text{ and } h_i(t) = \sup_{t_0\le s\le t} \tau_i(s),\ i = 1,2,\ldots,m,$$

$$0 < \alpha := \liminf_{t\to\infty} \int\limits_{h(t)}^{t} \sum_{i=1}^{m} p_i\,(s)\,ds \le \frac{1}{e} \tag{27}$$

and

$$a_1\,(t,s) = \exp\left(\int\limits_{s}^{t} \sum_{i=1}^{m} p_i\,(u)\,du\right),$$

$$a_{r+1}\,(t,s) = \exp\left(\int\limits_{s}^{t} \sum_{i=1}^{m} p_i\,(u)\,a_r\,(u, \tau_i\,(u))\,du\right),\ r \in \mathbb{N}.$$

Also, in 2016 Akca et al. [22] improved the above condition (24) replacing it by the condition

$$\limsup_{t\to\infty} \int\limits_{h(t)}^{t} \sum_{i=1}^{m} p_i\,(u)\,a_r\,(h\,(u)\,, \tau_i\,(u))\,du > \frac{1+\ln\lambda_0}{\lambda_0}, \tag{28}$$

where λ_0 is the smaller root of the equation $\lambda = e^{\alpha\lambda}$,

$$0 < \alpha := \liminf_{t\to\infty} \int\limits_{\tau(t)}^{t} \sum_{i=1}^{m} p_i\,(s)\,ds \le \frac{1}{e},$$

and $\tau\,(t) = \max\limits_{1\le i\le m} \{\tau_i\,(t)\}$.

In 2017 Chatzarakis [23] derived the following results: Assume that for some $k \in \mathbb{N}$

$$\limsup_{t\to\infty} \int\limits_{h(t)}^{t} P(s)\exp\left(\int\limits_{\tau(s)}^{h(t)} P_k\,(u)\,du\right) ds > 1, \tag{29}$$

or

$$\limsup_{t\to\infty} \int\limits_{h(t)}^{t} P(s)\exp\left(\int\limits_{\tau(s)}^{h(t)} P_k\,(u)\,du\right) ds > 1 - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}, \tag{30}$$

or

$$\limsup_{t\to\infty} \int_{h(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{t} P_k(u)du\right) ds > \frac{2}{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}, \tag{31}$$

or

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{\sigma(s)} P_k(u)du\right) ds > \frac{1+\ln\lambda_0}{\lambda_0} - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}, \tag{32}$$

or

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} p(s)\exp\left(\int_{\tau(s)}^{\sigma(s)} P_k(u)du\right) ds > \frac{1}{e}, \tag{33}$$

where $h(t), \tau\,(t)\,, \alpha$ and λ_0 are defined as above, and

$$P_k\,(t) = P(t)\left[1 + \int\limits_{\tau(t)}^{t} P\,(s)\exp\left(\int\limits_{\tau(s)}^{t} P\,(u)\exp\left(\int\limits_{\tau(u)}^{u} P_{k-1}\,(\xi)\,d\xi\right) du\right) ds\right]$$

with $P_0(t) = P(t) = \sum_{i=1}^{m} p_i\,(t)$. Then all solutions of Equation (1) oscillate.

In 2018 Attia et al. [24] established the following oscillation conditions under the assumption that there exists a family of nondecreasing continuous functions $g_i(t)$, $i = 1, 2, ..., m$ and a nondecreasing continuous functions $g(t)$ such that for some $t_1 \geq t_0$

$$\tau_i(t) \leq g_i(t) \leq g(t) \leq t, \; i = 1, 2, ..., m$$

Assume that

$$0 < \rho := \liminf_{t\to\infty} \int_{g(t)}^{t} \sum_{i=1}^{m} p_i(s)ds \leq \frac{1}{e},$$

and

$$\limsup_{t\to\infty} \left(\int_{g(t)}^{t} Q(v)dv + c(\rho) exp \left[\int_{g(t)}^{t} \sum_{i=1}^{m} p_i(s)ds \right] \right) > 1,$$

where

$$Q(t) = \sum_{k=1}^{m} \sum_{i=1}^{m} p_i(t) \int_{\tau_i(t)}^{t} p_k(s) exp \left(\int_{g_k(t)}^{t} \sum_{i=1}^{m} p_i(s)ds + (\lambda(\rho) - \epsilon) \int_{\tau_k(s)}^{g_k(t)} \sum_{\ell=1}^{m} p_\ell(u)du \right) ds, \epsilon \in (0, \lambda(\rho)),$$

or

$$\limsup_{t\to\infty} \left(\int_{g(t)}^{t} Q_1(v)dv + c(\rho) exp \left(\int_{g(t)}^{t} \sum_{i=1}^{m} p_i(s)ds \right) \right) > 1,$$

where

$$Q_1(t) = \sum_{k=1}^{m} \sum_{i=1}^{m} p_i(t) \int_{\tau_i(t)}^{t} p_k(s) exp \left(\int_{g_k(t)}^{t} \sum_{i=1}^{m} p_i(s)ds + \int_{\tau_k(s)}^{g_k(t)} \sum_{\ell=1}^{m} (\lambda(q_\ell) - \epsilon_\ell) p_\ell(u)du \right) ds, \epsilon_\ell \in (0, \lambda(q_\ell)),$$

and

$$q_\ell = \liminf_{t\to\infty} \int_{\tau_\ell(t)}^{t} p_\ell(s)ds, \;\; \ell = 1, 2, ..., m$$

or

$$\limsup_{t\to\infty} \left(\left(\prod_{j=1}^{m} \left(\prod_{k=1}^{m} \int_{g_j(t)}^{t} R_k(s)ds \right) \right)^{\frac{1}{n}} + \frac{\prod_{k=1}^{m} c(\beta_k)}{n^n} exp \left(\sum_{k=1}^{m} \int_{g_k(t)}^{t} \sum_{\ell=1}^{m} p_\ell(s)ds \right) \right) > \frac{1}{m^m},$$

where

$$R_k(s) = exp \left(\int_{g_k(s)}^{s} \sum_{i=1}^{m} p_i(u)du \right) \sum_{i=1}^{m} p_i(s) \int_{\tau_i(s)}^{s} p_k(u) exp \left((\lambda(\rho) - \epsilon) \int_{\tau_k(u)}^{g_k(s)} \sum_{\ell=1}^{m} p_\ell(v)dv \right) du, \epsilon \in (0, \lambda(\rho)),$$

and

$$0 < \beta_k := \liminf_{t\to\infty} \int_{\sigma_i(t)}^{t} p_i(s)ds \leq \frac{1}{e},$$

then Equation (1) is oscillatory.

In 2019 Bereketoglu et al. [25] derived the following oscillation conditions: Assume that there exist non-decreasing functions $\sigma_i \in C([t_0, \infty), \mathbb{R}^+)$ such that (18) is satisfied and for some $k \in \mathbb{N}$

$$\limsup_{t\to\infty} \prod_{j=1}^{m} \left[\prod_{i=1}^{m} \left(\int_{\sigma_j(t)}^{t} p_i(s) \exp \left(\int_{\tau_i(s)}^{\sigma_i(t)} P_k(u)\, du \right) ds \right) \right]^{1/m} > \frac{1}{m^m}, \tag{34}$$

or

$$\limsup_{t\to\infty} \prod_{j=1}^{m} \left[\prod_{i=1}^{m} \left(\int_{\sigma_j(t)}^{t} p_i(s) \exp \left(\int_{\tau_i(s)}^{\sigma_i(t)} P_k(u)\, du \right) ds \right) \right]^{1/m} > \frac{1}{m^m} \left[1 - \prod_{i=1}^{m} c_i(\alpha_i) \right], \tag{35}$$

where

$$P_k(t) = \sum_{j=1}^{m} p_j(t) \left\{ 1 + m \left[\prod_{i=1}^{m} \int_{\sigma_j(t)}^{t} p_i(s) \exp\left(\int_{\tau_i(s)}^{t} P_{k-1}(u)\, du \right) ds \right]^{1/m} \right\},$$

with

$$P_0(t) = m \left[\prod_{\ell=1}^{m} p_\ell(t) \right]^{1/m},$$

α_i is given by (22) and $c_i(\alpha_i)$ by (23). Then all solutions of Equation (1) oscillate.

In 2019, Moremedi et al. [26] improved further the above result as follows: Assume that there exist non-decreasing functions $\sigma_i \in C([t_0, \infty), \mathbb{R}^+)$ such that (18) is satisfied and for some $k \in \mathbb{N}$

$$\limsup_{t\to\infty} \prod_{j=1}^{m} \left[\prod_{i=1}^{m} \int_{\sigma_j(t)}^{t} p_i(s) \exp\left(\int_{\tau_i(s)}^{\sigma_i(t)} P_k(u)\, du \right) ds \right]^{1/m} > \frac{1}{m^m}, \tag{36}$$

or

$$\limsup_{t\to\infty} \prod_{j=1}^{m} \left[\prod_{i=1}^{m} \left(\int_{\sigma_j(t)}^{t} p_i(s) \exp\left(\int_{\tau_i(s)}^{\sigma_i(t)} P_k(u)\, du \right) ds \right) \right]^{1/m} > \frac{1}{m^m} \left[1 - \prod_{i=1}^{m} c_i(\alpha_i) \right], \tag{37}$$

where

$$P_k(t) = P(t) \left[1 + \int_{\sigma_i(t)}^{t} P(s) \exp\left(\int_{\tau_i(s)}^{t} P(u) \exp\left(\int_{\tau_i(u)}^{u} P_{k-1}(\xi)\, d\xi \right) du \right) ds \right]$$

with

$$P_0(t) = P(t) = \sum_{i=1}^{m} p_i(t)$$

and α_i, $c_i(\alpha_i)$ are given by (22) and (23) respectively. Then all solutions of Equation (1) oscillate.

Remark 1. *It is clear that the left-hand side of both conditions (34) and (35) and also of (36) and (37) are identically the same and also the right-hand side of (35) and (37) reduce to (34) and (36) respectively, when $c_i(\alpha_i) = 0$. Thus, it seems that conditions (35) and (37) are exactly the same as (34) and (36) when $c_i(\alpha_i) = 0$. One may notice, however, that the condition (22) is required in (35) and (37) but not in (34) and (36).*

In 2017, Pituk [27] and in 2019, Garab et al. [28] studied the delay differential equation with constant delay

$$x'(t) + p(t)x(t-\tau) = 0, \;\; \tau > 0, \; t \ge t_0,$$

under additional assumptions dealing with *slowly varying at infinity* functions. Recall that a function $g\colon [t_0, \infty) \to \mathbb{R}$ is called *slowly varying at infinity* (or simply *slowly varying*) if for every $\xi \ge 0$,

$$g(t+\xi) - g(t) \to 0, \quad \text{as } t \to \infty.$$

Also Pituk [27] gave the following characterization of continuous slowly varying functions: A continuous function $g\colon [t_0, \infty) \to \mathbb{R}$ is slowly varying if and only if there exists $t_1 \ge t_0$, such that g can be written in the form

$$g(t) = a(t) + b(t), \text{ for all } t \ge t_1, \tag{38}$$

where $a\colon [t_1, \infty) \to \mathbb{R}$ is a continuous function which tends to some finite limit as $t \to \infty$, and $b\colon [t_1, \infty) \to \mathbb{R}$ is a continuously differentiable function for which $\lim_{t\to\infty} b'(t) = 0$ holds. For more information about slowly varying functions and their characterization the reader is referred to the papers [27–30] and the references cited therein.

In a subsequent paper, Garab [29] studied the case of the differential equation with variable delay

$$x'(t) + p(t)\,x(t - \tau(t)) = 0,\; t \geq t_0.$$

Very recently Garab and Stavroulakis [30] considered the linear differential equation with several variable delays:

$$x'(t) + \sum_{i=1}^{m} p_i(t)x(t - \tau_i(t)) = 0\,,\; t \geq t_0,$$

where $p_i\colon [t_0, \infty) \to [0, \infty)$ and $\tau_i\colon [t_0, \infty) \to (0, \infty)$ are continuous functions, such that $t - \tau_i(t) \to \infty$ (as $t \to \infty$) for all $1 \leq i \leq m$. Note that functions $t \mapsto t - \tau_i(t)$ are not necessarily nondecreasing. Let $t_{-1} \inf\{s - \tau_i(s) : s \in [t_0, \infty)$ and $1 \leq i \leq m\}$ and observe that $t_{-1} \in (-\infty, t_0)$ holds. Then a continuous function $x\colon [t_{-1}, \infty) \to \mathbb{R}$ is called a *solution* of Equation (9), if it is continuously differentiable on $[t_0, \infty)$ and satisfies (9) there.

In the sequel, we will assume the following hypotheses:

(H_1) there exists $K > 0$ such that $0 < \tau_i(t) \leq K$ for all $t \geq t_0$ and $1 \leq i \leq m$;

(H_2) there exists $L > 0$ such that $0 \leq p_i(t) \leq L$ for all $t \geq t_0$ and $1 \leq i \leq m$.

The conditions in the next theorem, established in [30], essentially improve related conditions in the literature.

Theorem 3. *([30]) Suppose that*

$$\liminf_{t\to\infty} \int_{t-\tau_i(t)}^{t} p_i(s)ds > 0 \text{ for all } i,\;\; i = 1, 2, \ldots, m, \tag{39}$$

and hypotheses (H_1) and (H_2) are fulfilled. Furthermore, suppose that the functions p_i and τ_i are uniformly continuous. Then each one of the following conditions implies that all of solutions of Equation (9) oscillate.

(a) The delay functions τ_i $(1 \leq i \leq m)$ are all constant, the function $A\colon [t_0 + K, \infty) \to [0, \infty)$,

$$A(t) = \sum_{i=1}^{m} \int_{t-\tau_i}^{t} p_i(s)\,ds$$

is slowly varying at infinity, and

$$\limsup_{t\to\infty} \sum_{i=1}^{m} \int_{t-\tau_i}^{t} p_i(s)\,ds > \frac{1}{e}. \tag{40}$$

(b) The function $A\colon [t_0 + K, \infty) \to [0, \infty)$,

$$A(t) = \sum_{i=1}^{m} p_i(t)\,\tau_i(t)$$

is slowly varying at infinity, and

$$\limsup_{t\to\infty} \sum_{i=1}^{m} p_i(t)\,\tau_i(t) > \frac{1}{e}. \tag{41}$$

(c) There exists a uniformly continuous function $\delta\colon [t_0, \infty) \to [0, \infty)$ such that $0 \leq \delta(t) \leq \tau_i(t)$ for all $t \geq t_0$ and $i = 1, 2, ..., m$, and that the function $A\colon [t_0 + K, \infty) \to [0, \infty)$,

$$A(t) = \int_{t-\delta(t)}^{t} \sum_{i=1}^{m} p_i(s)ds$$

is slowly varying at infinity and

$$\limsup_{t\to\infty} \int_{t-\delta(t)}^{t} \sum_{i=1}^{m} p_i(s)\, ds > \frac{1}{e}. \tag{42}$$

3. Examples

In the following examples, it is shown that the conditions of Theorem 3 are independent (cf. [11]) and also improve related results in the literature.

3.1. Example

([30]) Consider the delay equation

$$x'(t) + \left(c_1 + \varepsilon \cos\sqrt{t}\right) x(t-1) + \left(c_2 + \varepsilon \cos\sqrt{t}\right) x(t-2) = 0,\ t \geq 1, \tag{43}$$

where $0 < \varepsilon < c_1 \leq c_2$.

The coefficient functions are uniformly positive (i.e., bounded from below by a positive number), uniformly continuous, and bounded. Thus Equation (43) is of the form (9) with $m = 2$, $t_0 = 1$, $p_j(t) = \left(c_j + \varepsilon\cos\sqrt{t}\right)$ $(j = 1, 2)$ and constant delays $\tau_1 = 1$ and $\tau_2 = 2$, and also condition (39) is satisfied. Note that the derivative of the function $\cos\sqrt{t}$ vanishes at infinity and therefore characterization (38) implies that p_1 and p_2 are slowly varying, and also the constant functions $\tau_1 = 1$ and $\tau_2 = 2$ are slowly varying by definition. It is a matter of elementary calculations to see that the equations

$$\liminf_{t\to\infty} \sum_{i=1}^{2} \int_{t-\tau_i}^{t} p_i(s) ds = \liminf_{t\to\infty} [\tau_1 p_1(t) + \tau_2 p_2(t)] = c_1 + 2c_2 - 3\varepsilon,$$

$$\limsup_{t\to\infty} \sum_{i=1}^{2} \int_{t-\tau_i}^{t} p_i(s)\, ds = \limsup_{t\to\infty} [\tau_1 p_1(t) + \tau_2 p_2(t)] = c_1 + 2c_2 + 3\varepsilon$$

hold (consider i.e., the sequences $t_n = (2n+1)^2\pi^2$ and $t_n' = (2n)^2\pi^2$).

Therefore, if $c_1 + 2c_2 + 3\varepsilon > \frac{1}{e}$ both (a) and (b) of Theorem 3 imply that all solutions of Equation (43) oscillate. Observe, however, that conditions (8) and (10) lead to this conclusion if the stronger condition $c_1 + 2c_2 - 3\varepsilon > \frac{1}{e}$ is satisfied.

Concerning part (c) of Theorem 3, note that $\delta(t) := \min\{\tau_1(t), \tau_2(t)\} = 1$ and as a constant is slowly varying. By simple calculations, we get

$$\liminf_{t\to\infty} \int_{t-1}^{t} [p_1(s) + p_2(s)] ds = c_1 + c_2 - 2\varepsilon,$$

and

$$\limsup_{t\to\infty} \int_{t-1}^{t} [p_1(s) + p_2(s)] ds = c_1 + c_2 + 2\varepsilon.$$

Thus, if $c_1 + c_2 + 2\varepsilon > \frac{1}{e}$ part (c) of Theorem 3 implies that all solutions of Equation (43) oscillate, while the condition (5) requires the stronger condition $c_1 + c_2 - 2\varepsilon > \frac{1}{e}$.

In the particular case that $c_1 = \frac{1}{9}, c_2 = \frac{1}{8}$ and $\varepsilon = \frac{1}{14}$, that is, in the case of the delay equation

$$x'(t) + \left(\frac{1}{9} + \frac{1}{14}\cos\sqrt{t}\right) x(t-1) + \left(\frac{1}{8} + \frac{1}{14}\cos\sqrt{t}\right) x(t-2) = 0,\ t \geq 1, \tag{44}$$

we have

$$c_1 + 2c_2 + 3\varepsilon \approx 0.57539 > \frac{1}{e} \ \text{ and } \ c_1 + c_2 + 2\varepsilon \approx 0.37896 > \frac{1}{e},$$

that is, the conditions in parts (a), (b) and (c) of Theorem 3 are satisfied, and therefore, all solutions of Equation (44) oscillate. Observe, however, that

$$c_1 + 2c_2 - 3\varepsilon \approx 0.14682 < \frac{1}{e} \text{ and } c_1 + c_2 - 2\varepsilon \approx 0.09325 < \frac{1}{e},$$

and therefore none of the conditions (8), (10) and (5) are satisfied.

Remark 2. *([30]) As we have seen in this example, both (a) and (b) of Theorem 3 outperform part (c). However, in the next example we show that part (c) of Theorem 3 can be applied and gives more efficient criteria than the conditions (10) and (5), while none of the conditions (8), (40) and (41) of parts (a) and (b) of Theorem 3 applies.*

3.2. Example

([30]) Consider the equation with variable delays

$$x'(t) + c_1 x(t - 2 - \sin\sqrt{t}) + c_2 x(t - 4 - \cos t) = 0, \ t \geq 1, \tag{45}$$

where c_1 and c_2 are positive constants. Equation (45) is of the form (9) with $m = 2, t_0 = 1$, constant coefficient functions $p_1 = c_1$ and $p_2 = c_2$, and uniformly continuous delay functions $\tau_1(t) = 2 + \sin\sqrt{t}$ and $\tau_2(t) = 4 + \cos t$. Observe that $\tau_1(t) \leq \tau_2(t)$ holds for all $t \geq t_0$, and that, in view of characterization (38), the map $t \to \sin\sqrt{t}$ is slowly varying since its derivative vanishes at infinity. Thus the map

$$A(t) := \int_{t-\tau_1(t)}^{t} [p_1(s) + p_2(s)]ds = (c_1 + c_2)(2 + \sin\sqrt{t}),$$

is slowly varying and also condition (39) is satisfied.

It is easy to see that

$$\liminf_{t\to\infty} A(t) = c_1 + c_2$$

and

$$\limsup_{t\to\infty} A(t) = 3(c_1 + c_2).$$

Thus, if $3(c_1 + c_2) > \frac{1}{e}$ Theorem 3(c) implies that all solutions of Equation (45) oscillate. Observe, however, that the condition of Theorem 2.7.1 in [5]

$$\liminf_{t\to\infty} \int_{t-\tau_{\min}(t)}^{t} \sum_{i=1}^{m} p_i(s)ds > \frac{1}{\mathrm{e}}, \tag{46}$$

where $\tau_{\min}(t) := \min_{1\leq i\leq m} \tau_i(t)$, and (10) require the stronger conditions $c_1 + c_2 > \frac{1}{e}$ and $c_1 + 3c_2 > \frac{1}{e}$ respectively. Moreover, condition (8) and part (a) of Theorem 3 cannot be applied, as we have variable delays.

Finally, we show that part (b) of Theorem 3 cannot be applied in this case. The function

$$\bar{A}(t) := \sum_{i=1}^{2} \int_{t-\tau_i(t)}^{t} p_i(s)\,ds = \sum_{i=1}^{2} p_i(s)\tau_i(t) = c_1(2 + \sin\sqrt{t}) + c_2(4 + \cos t), \text{ for all } t \geq 1,$$

is *not slowly varying* because of the function *cost* which is nonconstant and 2π-periodic. Therefore part (b) of Theorem 3 does not apply.

4. Conclusions

Several sufficient conditions for the oscillation of all solutions to differential equations with several delays were presented. Also, under mild additional assumptions dealing with slowly varying at infinity functions, some of these sufficient oscillation conditions involving "liminf" were essentially improved replacing "liminf" by "limsup". The importance of the slowly varying hypothesis and the essential improvement of the sufficient oscillation conditions was demonstrated by suitable examples.

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