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An Alternating Sum of Fibonacci and Lucas Numbers of Order k

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Abstract: During the last decade, many researchers have focused on proving identities that reveal the relation between Fibonacci and Lucas numbers. Very recently, one of these identities has been generalized to the case of Fibonacci and Lucas numbers of order k. In the present work, we state and prove a new identity regarding an alternating sum of Fibonacci and Lucas numbers of order k. Our result generalizes recent works in this direction.

Keywords: Fibonacci numbers; Lucas numbers; order k; relation

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1. Introduction

Let m be a fixed positive integer greater than or equal to two, and let n be a nonnegative integer, unless otherwise specified. Denote by F_n and L_n the Fibonacci and Lucas numbers, respectively, i.e., $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ $(n \geq 2)$, and $L_0 = 2$, $L_1 = 1$, $L_n = L_{n-1} + L_{n-2}$ $(n \geq 2)$.

Fibonacci numbers, Lucas numbers, and their generalizations appear in many areas of mathematics, like graph theory [1], Optimization Theory [2], probability theory [3], and combinatorics [4]. They also appear in computer science [5], mathematical biology [6], reliability [7], etc.

Among the various properties of the sequences, a class of special interest has to do with the demonstration of identities that present the relation between the Fibonacci and the Lucas numbers. Edgar [8] stated and proved the following identity:

$$\sum_{i=0}^{n} m^i (L_i + (m-2)F_{i+1}) = m^{n+1} F_{n+1}. \tag{1}$$

Special cases of (1), for $m = 2$ and $m = 3$, respectively, have been proven by Benjamin and Quinn [4] and Marques [9].

Let k be a fixed positive integer greater than or equal to two. Dafnis, Philippou, and Livieris [10] generalized (1) to the Fibonacci and Lucas numbers of order k (for the definitions of the Fibonacci and Lucas numbers of order k, we refer to [3,11], respectively; see, also, [7,12,13]), deriving the following identity by means of color tiling.

$$\sum_{i=0}^{n} m^i \left(\left(L_i^{(k)} + (m-2)F_{i+1}^{(k)} - \sum_{j=3}^{k} (j-2) F_{i-j+1}^{(k)} \right) \right) = m^{n+1} F_{n+1}^{(k)} + k - 2. \tag{2}$$

A simple proof of (2) has been given by Philippou and Dafnis [14].

In the present note, we state and prove a new identity regarding an alternating sum of Fibonacci and Lucas numbers of order k, analogous to (2). As a special case of this identity for $k = 2$, a new identity follows that further reduces to a Fibonacci–Lucas relation derived recently by Martinjak [15].

2. A New Identity

We presently state and prove the following:

Theorem 1. *Let $(F_n^{(k)})_{n\geq 0}$ be the sequence of Fibonacci numbers of order k, and set $F_{-1}^{(k)} = \cdots F_{-k+1}^{(k)} = 0$, i.e., $F_n^{(k)} = 0$ for $-k+1 \leq n \leq 0, F_1^{(k)} = 1$, and $F_n^{(k)} = \sum_{j=1}^{k} F_{n-j}^{(k)}$ for $n \geq 2$. Furthermore, let $(L_n^{(k)})_{n\geq 0}$ be the sequence of Lucas numbers or order k, i.e., $L_0^{(k)} = k, L_1^{(k)} = 1, L_n^{(k)} = n + \sum_{j=1}^{n-1} L_{n-j}^{(k)}$ for $2 \leq n \leq k$, and $L_n^{(k)} = \sum_{j=1}^{k} L_{n-j}^{(k)}$ for $n \geq k+1$. Then:*

$$\sum_{i=0}^{n} (-1)^i m^{n-i} \left(L_{i+1}^{(k)} + (m-2)\, F_i^{(k)} - \sum_{j=3}^{k} jF_{i-j+2}^{(k)} \right) = (-1)^n F_{n+1}^{(k)}. \tag{3}$$

Proof. We employ the following relation (see Charalambides [11], (2.18), and our definition of $F_n^{(k)}$):

$$L_n^{(k)} = \sum_{j=1}^{\min\{n,k\}} jF_{n-j+1}^{(k)} = \sum_{j=1}^{k} jF_{n-j+1}^{(k)},\ n \geq 1,$$

which implies:

$$L_{n+1}^{(k)} = \sum_{j=1}^{k} jF_{n-j+2}^{(k)} = F_{n+1}^{(k)} + 2F_n^{(k)} + \sum_{j=3}^{k} jF_{n-j+2}^{(k)},\ n \geq 0.$$

Then,

$$\begin{aligned}
&\sum_{i=0}^{n} (-1)^i m^{n-i} \left(L_{i+1}^{(k)} + (m-2)\, F_i^{(k)} - \sum_{j=3}^{k} jF_{i-j+2}^{(k)} \right) \\
&= \sum_{i=0}^{n} (-1)^i\, m^{n-i} \left(mF_i^{(k)} + F_{i+1}^{(k)} \right) \\
&= m^n F_1^{(k)} - m^{n-1} \left(mF_1^{(k)} + F_2^{(k)} \right) + m^{n-2} \left(mF_2^{(k)} + F_3^{(k)} \right) - \ldots + (-1)^n \left(mF_n^{(k)} + F_{n+1}^{(k)} \right) \\
&= (-1)^n\, F_{n+1}^{(k)},
\end{aligned}$$

which is to be shown. □

For $k = 2$, Identity (3) reduces to a new Fibonacci–Lucas relation, analogous to (1). We present it in the following corollary.

Corollary 1. *Let F_n be the Fibonacci numbers and L_n be the Lucas numbers. Then, for any $m \geq 2$,*

$$\sum_{i=0}^{n} (-1)^i m^{n-i}\ (L_{i+1} + (m-2)\, F_i) = (-1)^n F_{n+1}. \tag{4}$$

For $m = 2$, Corollary 1 further reduces to:

$$\sum_{i=0}^{n} (-1)^i 2^{n-i}\, L_{i+1} = (-1)^n F_{n+1}, \tag{5}$$

which is Theorem 1 of Martinjak [15].

Next, in Corollary 2, we present a new relation between the Lucas numbers of order three (see [11]) and the tribonacci numbers (see [3,16,17]), as another special case of Theorem 1 for $k = 3$.

Corollary 2. *Let $(T_n)_{n\geq 0}$ be the sequence of tribonacci numbers, i.e., $T_0 = 0$, $T_1 = 1$, $T_2 = 1$, and $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for $n \geq 3$. Furthermore, let $(V_n)_{n\geq 0}$ be the sequence of Lucas numbers of order three, i.e., $V_0 = 3$, $V_1 = 1$, $V_2 = 3$, and $V_n = V_{n-1} + V_{n-2} + V_{n-3}$ for $n \geq 3$. Set $T_{-1} = 0$. Then:*

$$\sum_{i=0}^{n} (-1)^i m^{n-i} \left(V_{i+1} + (m-2)\,T_i - 3T_{i-1}\right) = (-1)^n T_{n+1}. \tag{6}$$

Finally, in Table 1, we give the first 10 terms of the Fibonacci ($k = 2$), tribonacci ($k = 3$), tetranacci or quadranacci ($k = 4$), pentanacci or pentacci ($k = 5$), hexanacci or esanacci ($k = 6$), heptanacci ($k = 7$), and octanacci ($k = 8$) numbers. In Table 2, we give the first 10 terms of the Lucas numbers of order k for $k = 2, 3, \ldots, 8$. Each table is completed with the last column containing the A-number, according to the On-Line Encyclopedia of Integer Sequences (OEIS) [18], for each sequence.

Table 1. Fibonacci sequences of order k.

k \ n	1	2	3	4	5	6	7	8	9	10	A-Number
2	1	1	2	3	5	8	13	21	34	55	A000045
3	1	1	2	4	7	13	24	44	81	149	A000073
4	1	1	2	4	8	15	29	56	108	208	A000078
5	1	1	2	4	8	16	31	61	120	236	A001591
6	1	1	2	4	8	16	32	63	125	248	A001592
7	1	1	2	4	8	16	32	64	127	253	A066178
8	1	1	2	4	8	16	32	64	128	255	A079262

Table 2. Lucas sequences of order k.

k \ n	1	2	3	4	5	6	7	8	9	A-Number
2	1	3	4	7	11	18	29	47	76	A000032
3	1	3	7	11	21	39	71	131	241	A001644
4	1	3	7	15	26	51	99	191	367	A073817
5	1	3	7	15	31	57	113	223	439	A074048
6	1	3	7	15	31	63	120	239	475	A074584
7	1	3	7	15	31	63	127	247	493	A104621
8	1	3	7	15	31	63	127	255	502	A105754

Tables 1 and 2 may provide illustrations of (3), (4) and (6) for small values of n. We shall presently give three examples. Fixing $k = 4$, $n = 6$ and exploiting the appropriate entries of the third row of each table, we may easily calculate the LHS of (3) to be equal to -29, as was expected, since $F_7^{(4)} = 29$. Fixing $k = 2$, $n = 7$ and exploiting the appropriate entries of the first row of each table, we may easily calculate the LHS of (4) to be equal to -21, as was expected, since $F_8 = 21$. Finally, fixing $k = 3$, $n = 5$ and exploiting the appropriate entries of the second row of each table, we may easily calculate the LHS of (6) to be equal to -13, as was expected, since $T_6 = 13$.

3. Discussion and Conclusions

In the present work, we explore the relation between Fibonacci and Lucas numbers of order k. We establish a new identity, which generalizes or extends results appearing in the literature during the last decade. The proof of our new result is simple, since our aim is to further reveal the aforementioned relation. Regarding future research, the new identity may be proven employing different techniques, such as combinatorial methods or methods based on the generating functions of the numbers. It is the authors' strong belief that the new identity may prove quite profitable to applied research, since Fibonacci and Lucas numbers are met in a wide range of applications in various fields.

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