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Octahedron Subgroups and Subrings

Jeong-Gon Lee [1,*], **Young Bae Jun** [2] **and Kul Hur** [1]

1 Division of Applied Mathematics, Wonkwang University, 460, Iksan-daero, Iksan-Si, Jeonbuk 54538, Korea; kulhur@wku.ac.kr
2 Department of Mathematics Education, Gyeongsang National University, Jinju 52828, Korea; skywine@gmail.com
* Correspondence: jukolee@wku.ac.kr

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Abstract: In this paper, we define the notions of i-octahedron groupoid and i-OLI [resp., i-ORI and i-OI], and study some of their properties and give some examples. Also we deal with some properties for the image and the preimage of i-octahedron groupoids [resp., i-OLI, i-ORI and i-OI] under a groupoid homomorphism. Next, we introduce the concepts of i-octahedron subgroup and normal subgroup of a group and investigate some of their properties. In particular, we obtain a characterization of an i-octahedron subgroup of a group. Finally, we define an i-octahedron subring [resp., i-OLI, i-ORI and i-OI] of a ring and find some of their properties. In particular, we obtain two characterizations of i-OLI [resp., i-ORI and i-OI] of a ring and a skew field, respectively.

Keywords: octahedron set; i-octahedron subgroupoid; i-octahedron ideal; i-sup-property, i-octahedron subgroup; i-octahedron subring

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1. Introduction

In 1965, Zadeh [1] proposed the concept of fuzzy sets as a generalization of crisp sets, in order express mathematically uncertainty problems. After then, Zadeh [2] and Atanassov [3] introduced the concept of interval-valued fuzzy sets and intuitionistic fuzzy sets, respectively. In traditional fuzzy logic, a number contained in the unit interval $[0, 1]$ is used as a measure of expert confidence in other statements. However, it is often difficult for experts to accurately quantify their certainty. In other words, the probability of quantified figures being accurate is low. Therefore, it is necessary to increase the accuracy of measurement using the sub-interval of $[0, 1]$, and there is an interval-value fuzzy set developed as a mathematical tool for this. Fuzzy sets or interval-valued fuzzy sets are a very useful tools for measuring against one factor, but it is not appropriate to measure two factors at the same time. Jun et al. [4] defined the notion of cubic sets, which is a kind of hybrid structure, by using a fuzzy set and an interval-valued fuzzy set. In addition, it is a good mathematical tool for evaluating both factors at the same time, and it is being applied in many places (see [5–9]). After the introduction of cubic set, various concepts related to it, i.e., cubic set, (generalized) cubic intuitionistic fuzzy set, cubic interval-valued intuitionistic fuzzy set, cubic picture fuzzy set, cubic hesitant fuzzy set, cubic bipolar fuzzy set, cubic Pythagorean fuzzy set, cubic soft set, etc., have emerged and are being applied in various ways. We can consider intuitionistic fuzzy set as a tool to measure both positive and negative factors for every outcome/assessment at the same time. We need the ability to handle three different tasks at the same time amid increasingly diverse social phenomena due to the development of science. In addition, mathematicians feel the need to develop mathematical tools to support this, and they have a desire to develop a wider hybrid structure. With a wider hybrid structure, Lee et al. [10] defined an octahedron set composed of an interval-valued fuzzy set, an intuitionistic fuzzy set and a fuzzy set that will provide more information about uncertainty. This structure allows point measurements, interval measurements, and positive and negative simultaneous measurements as event assessments at the same time. As mathematicians, the purpose of this paper is to carry out the study of applying octahedron sets to algebraic structures, in particular, groups and rings. From now on, we expect octahedron sets to be applied to several branches, including algebraic structures, topological structures, metric spaces, medical science, decision making systems, aggregation operators, expert systems, etc. The composition of this paper is as follows. In Section 2, we list some basic concepts needed in the next sections: for examples, an intuitionistic number, an intuitionistic fuzzy set, an interval number, an interval-valued fuzzy set, an octahedron number and an octahedron set. In Section 3, we define the i-product of two octahedron sets in a groupoid and introduce the concept of i-octahedron subgroupoids of a groupoid by using it. In particular, we obtain four characterizations of i-octahedron groupoids (See Theorems 2–4). Also, we define an i-OLI [resp., i-ORI and i-OI]] of a groupoid and study some of their properties. Moreover, we obtain some properties for the image and preimage of an i-octahedron subgroupoid [resp., i-OLI, i-ORI and i-OI] under groupoid homomorphism. In Section 4, we define an i-octahedron subgroup of a group and investigate some of its properties. In particular, we obtain two characterizations of i-octahedron subgroup and i-OLI [resp., i-ORI and i-OI] of a group (See Theorems 7 and 8). In Section 4, we introduce the concepts of i-octahedron subrings [resp., i-OLIs, i-ORIs and i-OIs] of a ring and obtain their characterizations (See Theorems 13 and 15). Furthermore, we find a sufficient condition for which a commutative ring with a unity e is a field (See Proposition 27).

2. Preliminaries

Let $I \oplus I = \{\bar{a} = (a^{\in}, a^{\notin}) \in I \times I : a^{\in} + a^{\notin} \leq 1\}$, where $I = [0, 1]$. Then each member $\bar{a}$ of $I \oplus I$ is called an intuitionistic point or intuitionistic number. In particular, we denote $(0, 1)$ and $(1, 0)$ as $\bar{0}$ and $\bar{1}$, respectively. Refer to [11] for the definitions of the order $(\leq)$ and the equality $(=)$ of two intuitionistic numbers, and the infimum and the supremum of any intuitionistic numbers.

Definition 1 ([3])**.** *For a nonempty set X, a mapping $A : X \to I \oplus I$ is called an intuitionistic fuzzy set (briefly, IF set) in X, where for each $x \in X$, $A(x) = (A^{\in}(x), A^{\notin}(x))$, and $A^{\in}(x)$ and $A^{\notin}(x)$ represent the degree of membership and the degree of nonmembership of an element x to A, respectively. Let $(I \oplus I)^X$ denote the set of all IF sets in X and for each $A \in (I \oplus I)^X$, we write $A = (A^{\in}, A^{\notin})$. In particular, $\bar{\mathbf{0}}$ and $\bar{\mathbf{1}}$ denote the IF empty set and the IF whole set in X defined by, respectively: for each $x \in X$,*

$$\bar{\mathbf{0}}(x) = \bar{0} \text{ and } \bar{\mathbf{1}}(x) = \bar{1}.$$

For the definitions of the inclusion, the equality, the union and the intersection of two IF sets, the complement of an IF set, two operations [ ] and $\diamond$ on $(I \oplus I)^X$, refer to [3].

The set of all closed subintervals of I is denoted by $[I]$, and members of $[I]$ are called interval numbers and are denoted by $\widetilde{a}$, $\widetilde{b}$, $\widetilde{c}$, etc., where $\widetilde{a} = [a^-, a^+]$ and $0 \leq a^- \leq a^+ \leq 1$. In particular, if $a^- = a^+$, then we write as $\widetilde{a} = \mathbf{a}$ (See [12]).

For the definitions of the order and the equality of two interval numbers, and the infimum and the supremum of any interval numbers, refer to [13,14].

Definition 2 ([2,15])**.** *For a nonempty set X, a mapping $A : X \to [I]$ is called an interval-valued fuzzy set (briefly, an IVF set) in X. Let $[I]^X$ denote the set of all IVF sets in X. For each $A \in [I]^X$ and $x \in X$, $A(x) = [A^-(x), A^+(x)]$ is called the degree of membership of an element x to A, where $A^-, A^+ \in I^X$ are called a lower fuzzy set and an upper fuzzy set in X, respectively. For each $A \in [I]^X$, we write $A = [A^-, A^+]$. In particular, $\widetilde{0}$ and $\widetilde{1}$ denote the interval-valued fuzzy empty set and the interval-valued fuzzy empty whole set in X defined by, respectively: for each $x \in X$,*

$$\widetilde{0}(x) = \mathbf{0} \text{ and } \widetilde{1}(x) = \mathbf{1}.$$

For the definitions of the inclusion, the equality, the union, the intersection of two IV sets and the complement of an IV set, refer to [2,15].

Now members of $[I] \times (I \oplus I) \times I$ are written as $\widetilde{\bar{a}} = \langle \widetilde{a}, \bar{a}, a \rangle = \langle [a^-, a^-], (a^{\in}, a^{\notin}), a \rangle$, $\widetilde{\bar{b}} = \langle \widetilde{b}, \bar{b}, b \rangle = \langle [b^-, b^-], (b^{\in}, b^{\notin}), b \rangle$, etc. and are called octahedron numbers. Furthermore, we will define the following order relations in $[I] \times (I \otimes I) \times I$ (see [10]):

(Oi) (Equality) $\widetilde{\bar{a}} = \widetilde{\bar{b}} \Leftrightarrow \widetilde{a} = \widetilde{b},\ \bar{a} = \bar{b},\ a = b,$

(Oii) (Type 1-order) $\widetilde{\bar{a}} \leq_1 \widetilde{\bar{b}} \Leftrightarrow a^- \leq b^-,\ a^+ \leq b^+,\ a^{\in} \leq b^{\in},\ a^{\notin} \geq b^{\notin},\ a \leq b,$

(Oiii) (Type 2-order) $\widetilde{\bar{a}} \leq_2 \widetilde{\bar{b}} \Leftrightarrow a^- \leq b^-,\ a^+ \leq b^+,\ a^{\in} \leq b^{\in},\ a^{\notin} \geq b^{\notin},\ a \geq b,$

(Oiv) (Type 3-order) $\widetilde{\bar{a}} \leq_3 \widetilde{\bar{b}} \Leftrightarrow a^- \leq b^-,\ a^+ \geq b^+,\ a^{\in} \geq b^{\in},\ a^{\notin} \leq b^{\notin},\ a \leq b,$

(Ov) (Type 4-order) $\widetilde{\bar{a}} \leq_4 \widetilde{\bar{b}} \Leftrightarrow a^- \leq b^-,\ a^+ \leq b^+,\ a^{\in} \geq b^{\in},\ a^{\notin} \leq b^{\notin},\ a \geq b.$

Definition 3 ([10]). *Let X be a nonempty set and let $\mathbf{A} = [A^-, A^+] \in [I]^X$, $A = (A^{\in}, A^{\notin}) \in (I \oplus I)^X$, $\lambda \in I^X$. Then the triple $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle$ is called an* octahedron *set in X. In fact, $\mathcal{A} : X \to [I] \times (I \oplus I) \times I$ is a mapping.*

We can consider following special octahedron sets in X:

$\left\langle \widetilde{0}, \bar{\mathbf{0}}, 0 \right\rangle = \ddot{0},$

$\left\langle \widetilde{0}, \bar{\mathbf{0}}, 1 \right\rangle,\ \left\langle \widetilde{0}, \bar{\mathbf{1}}, 0 \right\rangle,\ \left\langle \widetilde{1}, \bar{\mathbf{0}}, 0 \right\rangle,$

$\left\langle \widetilde{0}, \bar{\mathbf{1}}, 1 \right\rangle,\ \left\langle \widetilde{1}, \bar{\mathbf{0}}, 1 \right\rangle,\ \left\langle \widetilde{1}, \bar{\mathbf{1}}, 0 \right\rangle,$

$\left\langle \widetilde{1}, \bar{\mathbf{1}}, 1 \right\rangle = \ddot{1}.$

In this case, $\ddot{0}$ [resp., $\ddot{1}$] is called an octahedron empty set [resp., octahedron whole set] in X. We denote the set of all octahedron sets as $\mathcal{O}(X)$.

It is obvious that for each $A \in 2^X$, $\chi_{\mathcal{A}} = \langle [\chi_A, \chi_A], (\chi_A, \chi_{A^c}), \chi_A \rangle \in \mathcal{O}(X)$ and then $2^X \subset \mathcal{O}(X)$, where 2^X denotes the set of all subsets of X and χ_A denotes the characteristic function of A. Furthermore, we can easily see that for each $\mathbf{A} = \langle A, \lambda \rangle \in \mathcal{C}(X)$, $\mathbf{A} = \langle A, (A^-, A^+), \lambda \rangle$, $\mathbf{A} = \langle A, (\lambda, \lambda^c), \lambda \rangle \in \mathcal{O}(X)$ and then $\mathcal{C}(X) \subset \mathcal{O}(X)$. In this case, we denote $\langle A, (A^-, A^+), \lambda \rangle$ and $\langle A, (\lambda, \lambda^c), \lambda \rangle$ as $\mathcal{A}_A$ and $\mathcal{A}_\lambda$, respectively. In fact, we can consider octahedron sets as a generalization of cubic sets.

Definition 4 ([10]). *Let X be a nonempty set and let $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle$, $\mathcal{B} = \langle \mathbf{B}, B, \mu \rangle \in \mathcal{O}(X)$. Then we can define following order relations between $\mathcal{A}$ and $\mathcal{B}$:*

(i) (Equality) $\mathcal{A} = \mathcal{B} \Leftrightarrow \mathbf{A} = \mathbf{B},\ A = B,\ \lambda = \mu,$

(ii) (Type 1-order) $\mathcal{A} \subset_1 \mathcal{B} \Leftrightarrow \mathbf{A} \subset \mathbf{B},\ A \subset B, \lambda \leq \mu,$

(iii) (Type 2-order) $\mathcal{A} \subset_2 \mathcal{B} \Leftrightarrow \mathbf{A} \subset \mathbf{B},\ A \subset B, \lambda \geq \mu,$

(iv) (Type 3-order) $\mathcal{A} \subset_3 \mathcal{B} \Leftrightarrow \mathbf{A} \subset \mathbf{B},\ A \supset B, \lambda \leq \mu,$

(v) (Type 4-order) $\mathcal{A} \subset_4 \mathcal{B} \Leftrightarrow \mathbf{A} \subset \mathbf{B},\ A \supset B, \lambda \geq \mu.$

Definition 5 ([10]). *Let X be a nonempty set and let $(\mathcal{A}_j)_{j \in J} = (\langle \mathbf{A}_j, A_j, \lambda_j \rangle)_{j \in J}$ be a family of octahedron sets in X. Then the Type i-union $\cup^i$ and Type i-intersection $\cap^i$ of $(\mathcal{A}_j)_{j \in J}$, ($i = 1,\ 2,\ ,3, 4$), are defined as follows, respectively:*

(i) (Type i-union) $\bigcup^1_{j \in J} \mathcal{A}_j = \left\langle \bigcup_{j \in J} \mathbf{A}_j, \bigcup_{j \in J} A_j, \bigcup_{j \in J} \lambda_j \right\rangle,$

$\bigcup^2_{j \in J} \mathcal{A}_j = \left\langle \bigcup_{j \in J} \mathbf{A}_j, \bigcup_{j \in J} A_j, \bigcap_{j \in J} \lambda_j \right\rangle,$

$\bigcup^3_{j \in J} \mathcal{A}_j = \left\langle \bigcup_{j \in J} \mathbf{A}_j, \bigcap_{j \in J} A_j, \bigcup_{j \in J} \lambda_j \right\rangle,$

$\bigcup^4_{j \in J} \mathcal{A}_j = \left\langle \bigcup_{j \in J} \mathbf{A}_j, \bigcap_{j \in J} A_j, \bigcap_{j \in J} \lambda_j \right\rangle,$

(ii) (Type i-intersection) $\bigcap^1_{j \in J} \mathcal{A}_j = \left\langle \bigcap_{j \in J} \mathbf{A}_j, \bigcap_{j \in J} A_j, \bigcap_{j \in J} \lambda_j \right\rangle,$

$\bigcap^2_{j \in J} \mathcal{A}_j = \left\langle \bigcap_{j \in J} \mathbf{A}_j, \bigcap_{j \in J} A_j, \bigcup_{j \in J} \lambda_j \right\rangle,$

$$\bigcap^3_{j\in J}\mathcal{A}_j=\left\langle \bigcap_{j\in J}\mathbf{A}_j, \bigcup_{j\in J}A_j, \bigcap_{j\in J}\lambda_j\right\rangle,$$
$$\bigcap^4_{j\in J}\mathcal{A}_j=\left\langle \bigcap_{j\in J}\mathbf{A}_j, \bigcup_{j\in J}A_j, \bigcup_{j\in J}\lambda_j\right\rangle.$$

Definition 6 ([10])**.** *Let X be a nonempty set and let $\mathcal{A}=\langle \mathbf{A}, A, \lambda\rangle$ be an octahedron set in X. Then the complement $\mathcal{A}^c$, operators $[\,]$ and $\diamond$ of $\mathcal{A}$ are defined as follows, respectively: for each $x\in X$,*
(i) $\mathcal{A}^c=\langle \mathbf{A}^c, A^c, \lambda^c\rangle$,
(ii) $[\,]\mathcal{A}=\langle \mathbf{A}, [\,]A, \lambda\rangle$,
(iii) $\diamond\mathcal{A}=\langle \mathbf{A}, \diamond A, \lambda\rangle$.

Definition 7 ([10])**.** *Let $\mathcal{A}=\langle \mathbf{A}, A, \lambda\rangle\in O(X)$, let $\widetilde{\tilde{a}}=\langle \tilde{a}, \bar{a}, a\rangle\in[I]$ be an octahedron number such that $a^+>0$, $\bar{a}\in I\oplus I$ with $\bar{a}\neq\bar{0}$, $a\in I$ with $a\neq 0$. Then A is called an octahedron point with the support $x\in X$ and the value $\widetilde{\tilde{a}}$, denoted by $A=x_{\widetilde{\tilde{a}}}$, if for each $y\in X$,*

$$x_{\widetilde{\tilde{a}}}(y)=\begin{cases}\widetilde{\tilde{a}} & \text{if } y=x\\ \left\langle \tilde{0}, \bar{0}, 0\right\rangle & \text{otherwise.}\end{cases}$$

The set of all octahedron points in X is denoted by $\mathcal{O}_P(X)$.

Definition 8 ([16])**.** *Let $(X,\cdot)$ be a groupoid and let $\lambda,\ \mu\in I^X$. Then the product of λ and μ, denoted by $\lambda\circ_F\mu$, is a fuzzy set in X defined as follows: for each $x\in X$,*

$$(\lambda\circ_F\mu)(x)=\begin{cases}\bigvee_{yz=x,\ y,\ z\in X}[\lambda(y)\wedge\mu(z)] & \text{if } yz=x\\ 0 & \text{otherwise.}\end{cases}$$

Definition 9 ([17])**.** *Let $(X,\cdot)$ be a groupoid and let $A,\ B\in(I\oplus I)^X$. Then the product of A and B, denoted by $A\circ_{IF}B$, is an IF set in X defined as follows: for each $x\in X$,*

$$(A\circ_{IF}B)(x)$$
$$=\begin{cases}(\bigvee_{yz=x,\ y,\ z\in X}[A^{\in}(y)\wedge B^{\in}(z)], \bigwedge_{yz=x,\ y,\ z\in X}[A^{\notin}(y)\wedge B^{\notin}(z)] & \text{if } yz=x\\ (0,1) & \text{otherwise.}\end{cases}$$

Definition 10 ([18])**.** *Let $(X,\cdot)$ be a groupoid and let $\mathbf{A},\ \mathbf{B}\in[I]^X$. Then the product of $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A}\circ_{IV}\mathbf{B}$, is an IVF set in X defined as follows: for each $x\in X$,*

$$(\mathbf{A}\circ_{IV}\mathbf{B})(x)$$
$$=\begin{cases}[\ \bigvee_{yz=x,\ y,\ z\in X}[A^-(y)\wedge B^-(z)], \bigvee_{yz=x,\ y,\ z\in X}[A^+(y)\wedge B^+(z)]\ ] & \text{if } yz=x\\ [0,0] & \text{otherwise.}\end{cases}$$

3. Octahedron Subgroupoids

In this section, we list the product of fuzzy sets [resp., intuitionistic fuzzy sets and interval-valued fuzzy sets] and we define the product of octahedron sets by using each product. Next we introduce the concepts of octahedron subgroupoid and octahedron ideal in a groupoid X, and find some of their properties and give some examples.

Throughout this section and next section, for an octahedron set $\mathcal{A}=\langle \mathbf{A}, A, \lambda\rangle$ in a set X, $\mathcal{A}\neq\ddot{0}$ [resp., $\left\langle \tilde{0}, \bar{\mathbf{0}}, 1\right\rangle$, $\left\langle \tilde{0}, \bar{\mathbf{1}}, 0\right\rangle$ and $\left\langle \tilde{0}, \bar{\mathbf{1}}, 1\right\rangle$] means that
$\mathbf{A}\neq\tilde{0},\ A\neq\bar{\mathbf{0}},\ \lambda\neq 0$ [resp., $\mathbf{A}\neq\tilde{0},\ A\neq\bar{\mathbf{0}},\ \lambda\neq 1$; $\mathbf{A}\neq\tilde{0},\ A\neq\bar{\mathbf{1}},\ \lambda\neq 0$ and $\mathbf{A}\neq\tilde{0},\ A\neq\bar{\mathbf{1}},\ \lambda\neq 1$].

Based on the order relations (Oi), (Oii), (Oiii), (Oiv) and (Ov), we can define the inf and the sup of octahedron numbers as follows:

Definition 11. *Let $\widetilde{\tilde{a}},\ \widetilde{\tilde{b}}\in[I]\times(I\oplus I)\times I$. Then*
(i) $\widetilde{\tilde{a}}\wedge^1\widetilde{\tilde{b}}=\left\langle [a^-\wedge b^-, a^+\wedge b^+], (a^{\in}\wedge b^{\in}, a^{\notin}\vee b^{\notin}), a\wedge b\right\rangle$,

$$\widetilde{\widetilde{a}} \wedge^2 \widetilde{\widetilde{b}} = \left\langle [a^- \wedge b^-, a^+ \wedge b^+], (a^{\in} \wedge b^{\in}, a^{\notin} \vee b^{\notin}), a \vee b \right\rangle,$$
$$\widetilde{\widetilde{a}} \wedge^3 \widetilde{\widetilde{b}} = \left\langle [a^- \wedge b^-, a^+ \wedge b^+], (a^{\in} \vee b^{\in}, a^{\notin} \wedge b^{\notin}), a \wedge b \right\rangle,$$
$$\widetilde{\widetilde{a}} \wedge^4 \widetilde{\widetilde{b}} = \left\langle [a^- \wedge b^-, a^+ \wedge b^+], (a^{\in} \vee b^{\in}, a^{\notin} \wedge b^{\notin}), a \vee b \right\rangle,$$
(ii) $$\widetilde{\widetilde{a}} \vee^1 \widetilde{\widetilde{b}} = \left\langle [a^- \vee b^-, a^+ \vee b^+], (a^{\in} \vee b^{\in}, a^{\notin} \wedge b^{\notin}), a \vee b \right\rangle,$$
$$\widetilde{\widetilde{a}} \vee^2 \widetilde{\widetilde{b}} = \left\langle [a^- \vee b^-, a^+ \vee b^+], (a^{\in} \vee b^{\in}, a^{\notin} \wedge b^{\notin}), a \wedge b \right\rangle,$$
$$\widetilde{\widetilde{a}} \vee^3 \widetilde{\widetilde{b}} = \left\langle [a^- \vee b^-, a^+ \vee b^+], (a^{\in} \wedge b^{\in}, a^{\notin} \vee b^{\notin}), a \vee b \right\rangle,$$
$$\widetilde{\widetilde{a}} \vee^4 \widetilde{\widetilde{b}} = \left\langle [a^- \vee b^-, a^+ \vee b^+], (a^{\in} \wedge b^{\in}, a^{\notin} \vee b^{\notin}), a \wedge b \right\rangle.$$

By using Definition 11, we can find the product of two octahedron sets as follows:

Definition 12. *Let $(X, \cdot)$ be a groupoid and let $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle$, $\mathcal{B} = \langle \mathbf{B}, B, \mu \rangle \in \mathcal{O}(X)$. Then the i-product of $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A} \circ_i \mathcal{B}$ ($i = 1,\ 2,\ ,3,\ 4$), is an octahedron set in X defined as follows: for each $x \in X$,*

$$(\mathcal{A} \circ_1 \mathcal{B})(x) = \begin{cases} \bigvee^1_{yz=x,\ y,\ z\in X}[\mathcal{A}(y) \wedge^1 \mathcal{B}(z)] & \text{if } yz = x \quad \text{for some} \quad y, z \in X \\ \ddot{0} & \text{otherwise,} \end{cases}$$

$$(\mathcal{A} \circ_2 \mathcal{B})(x) = \begin{cases} \bigvee^2_{yz=x,\ y,\ z\in X}[\mathcal{A}(y) \wedge^2 \mathcal{B}(z)] & \text{if } yz = x \quad \text{for some} \quad y, z \in X \\ \langle [0,0], (0,1), 1 \rangle & \text{otherwise,} \end{cases}$$

$$(\mathcal{A} \circ_3 \mathcal{B})(x) = \begin{cases} \bigvee^3_{yz=x,\ y,\ z\in X}[\mathcal{A}(y) \wedge^3 \mathcal{B}(z)] & \text{if } yz = x \quad \text{for some} \quad y, z \in X \\ \langle [0,0], (1,0), 0 \rangle & \text{otherwise,} \end{cases}$$

$$(\mathcal{A} \circ_4 \mathcal{B})(x) = \begin{cases} \bigvee^4_{yz=x,\ y,\ z\in X}[\mathcal{A}(y) \wedge^4 \mathcal{B}(z)] & \text{if } yz = x \quad \text{for some} \quad y, z \in X \\ \langle [0,0], (1,0), 1 \rangle & \text{otherwise.} \end{cases}$$

Remark 1. *From Definitions 8–12, we can easily see that followings hold:*
(1) $\mathcal{A} \circ_1 \mathcal{B} = \langle \mathbf{A} \circ_{IV} \mathbf{B}, A \circ_{IF} B, \lambda \circ_F \mu \rangle$,
(2) $\mathcal{A} \circ_2 \mathcal{B} = \langle \mathbf{A} \circ_{IV} \mathbf{B}, A \circ_{IF} B, \lambda \circ_2 \mu \rangle$, *where*

$$(\lambda \circ_2 \mu)(x) = \begin{cases} \bigwedge_{yz=x,\ y,\ z\in X}[\lambda(y) \vee \mu(z)] & \text{if } yz = x \quad \text{for some} \quad y, z \in X \\ 1 & \text{otherwise,} \end{cases}$$

(3) $\mathcal{A} \circ_3 \mathcal{B} = \langle \mathbf{A} \circ_{IV} \mathbf{B}, A \circ_3 B, \lambda \circ_F \mu \rangle$, *where*

$$(A \circ_3 B)(x) = \begin{cases} \left(\bigwedge_{yz=x,\ y,\ z\in X}[A^{\in}(y) \vee B^{\in}(z)], \bigvee_{yz=x,\ y,\ z\in X}[A^{\notin}(y) \wedge B^{\notin}(z)] \right) & \text{if } yz = x \\ & \text{for some} \quad y, z \in X \\ (1,0) & \text{otherwise,} \end{cases}$$

(4) $\mathcal{A} \circ_4 \mathcal{B} = \langle \mathbf{A} \circ_{IV} \mathbf{B}, A \circ_3 B, \lambda \circ_2 \mu \rangle$.

Example 1. *Let $X = \{a, b, c\}$ be the groupoid with the following Cayley Table 1:*

Table 1. Caley.

$\cdot$	a	b	c
a	a	a	a
b	b	a	b
c	c	c	a

Consider two octahedron sets $\mathcal{A}$ and $\mathcal{B}$ in X, respectively given by:
$\mathcal{A}(a) = \langle [0.3, 0.6], (0.7, 0.2), 0.5 \rangle$, $\mathcal{A}(b) = \langle [0.2, 0.4], (0.6, 0.3), 0.7 \rangle$,

$\mathcal{A}(c) = \langle [0.4, 0.7], (0.5, 0.4), 0.3 \rangle$, $\mathcal{B}(a) = \langle [0.2, 0.6], (0.6, 0.3), 0.7 \rangle$,
$\mathcal{B}(b) = \langle [0.3, 0.5], (0.5, 0.2), 0.6 \rangle$, $\mathcal{B}(c) = \langle [0.4, 0.7], (0.7, 0.2), 0.8 \rangle$.
Then we can easily calculate $\mathcal{A} \circ_i \mathcal{B}$ with Tables 2 and 3:

Table 2. $(\mathcal{A} \circ_1 \mathcal{B})(t)$ and $(\mathcal{A} \circ_2 \mathcal{B})(t)$.

	$(\mathcal{A} \circ_1 \mathcal{B})(t)$	$(\mathcal{A} \circ_2 \mathcal{B})(t)$
a	$\langle [0.4, 0.7], (0.5, 0.2), 0.6 \rangle$	$\langle [0.4, 0.7], (0.5, 0.2), 0.6 \rangle$
b	$\langle [0.2, 0.4], (0.6, 0.3), 0.7 \rangle$	$\langle [0.2, 0.4], (0.6, 0.3), 0.7 \rangle$
c	$\langle [0.3, 0.6], (0.5, 0.4), 0.3 \rangle$	$\langle [0.3, 0.6], (0.5, 0.4), 0.7 \rangle$

Table 3. $(\mathcal{A} \circ_3 \mathcal{B})(t)$ and $(\mathcal{A} \circ_4 \mathcal{B})(t)$.

	$(\mathcal{A} \circ_3 \mathcal{B})(t)$	$(\mathcal{A} \circ_4 \mathcal{B})(t)$
a	$\langle [0.4, 0.7], (0.6, 0.2), 0.6 \rangle$	$\langle [0.4, 0.7], (0.6, 0.2), 0.6 \rangle$
b	$\langle [0.2, 0.4], (0.6, 0.2), 0.7 \rangle$	$\langle [0.2, 0.4], (0.6, 0.2), 0.7 \rangle$
c	$\langle [0.3, 0.6], (0.6, 0.3), 0.3 \rangle$	$\langle [0.3, 0.6], (0.6, 0.3), 0.7 \rangle$

Proposition 1. *Let $(X, \cdot)$ be a groupoid, let $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle$, $\mathcal{B} = \langle \mathbf{B}, B, \mu \rangle \in \mathcal{O}(X)$ and let $x_{\tilde{\tilde{a}}} = x_{\langle \tilde{a}, \bar{a}, a \rangle}$, $y_{\tilde{\tilde{b}}} = y_{\langle \tilde{b}, \bar{b}, b \rangle} \in \mathcal{O}_P(X)$. Then we have*

(1) $x_{\tilde{\tilde{a}}} \circ_i y_{\tilde{\tilde{b}}} = (xy)_{\tilde{\tilde{a}} \wedge^i \tilde{\tilde{b}}}$, *for $i = 1,\ 2,\ 3,\ 4$, i.e.,*

$$x_{\tilde{\tilde{a}}} \circ_1 y_{\tilde{\tilde{b}}} = \left\langle (xy)_{\tilde{a} \wedge \tilde{b}}, (xy)_{\bar{a} \wedge \bar{b}}, (xy)_{a \wedge b} \right\rangle,\ x_{\tilde{\tilde{a}}} \circ_2 y_{\tilde{\tilde{b}}} = \left\langle (xy)_{\tilde{a} \wedge \tilde{b}}, (xy)_{\bar{a} \wedge \bar{b}}, (xy)_{a \vee b} \right\rangle,$$

$$x_{\tilde{\tilde{a}}} \circ_3 y_{\tilde{\tilde{b}}} = \left\langle (xy)_{\tilde{a} \wedge \tilde{b}}, (xy)_{\bar{a} \vee \bar{b}}, (xy)_{a \wedge b} \right\rangle,\ x_{\tilde{\tilde{a}}} \circ_4 y_{\tilde{\tilde{b}}} = \left\langle (xy)_{\tilde{a} \wedge \tilde{b}}, (xy)_{\bar{a} \vee \bar{b}}, (xy)_{a \vee b} \right\rangle,$$

(2) $\mathcal{A} \circ_i \mathcal{B} = \bigcup^i_{x_{\tilde{\tilde{a}}} \in_i \mathcal{A},\ y_{\tilde{\tilde{b}}} \in_i \mathcal{B}} x_{\tilde{\tilde{a}}} \circ_i y_{\tilde{\tilde{b}}}$, *for $i = 1,\ 2,\ 3,\ 4$.*

Proof. (1) The proofs are obvious from Definitions 7 and 12.

(2) Case 1: Let $i = 1$. Then the proof of the first part follows from Proposition 1.1 [16], Proposition 2.2 [17] and Proposition 3.2 [18].

Case 2: Let $i = 2$. From Remark 3.5 (2), it is sufficient to prove that $\lambda \circ_2 \mu = \bigcap_{x_a \in_2 \lambda,\ y_b \in_2 \mu} x_a \circ_2 y_b$. Let $C = \bigcap_{x_a \in_2 \lambda, y_b \in_2 \mu} x_a \circ_2 y_b$. For each $z \in X$, we may suppose that there are $u,\ v \in X$ such that $uv = z$, $x_a \neq 1$ and $y_b \neq 1$ without loss of generality. Then

$$\begin{aligned}(\lambda \circ_2 \mu)(z) &= \textstyle\bigwedge_{z=uv}[\lambda(u) \vee \mu(v)] \\ &\leq \textstyle\bigwedge_{z=uv}(\bigwedge_{x_a \in_2 \lambda,\ y_b \in_2 \mu}[x_a(u) \vee y_b(v)]) \\ &= \textstyle\bigwedge_{x_a \in_2 \lambda,\ y_b \in_2 \mu}(\bigwedge_{z=uv}[x_a(u) \vee y_b(v)]) \\ &= (\textstyle\bigcap_{x_a \in_2 \lambda,\ y_b \in_2 \mu} x_a \circ_2 y_b)(z) \\ &= C.\end{aligned}$$

Since $u_{\lambda(u)} \in_2 \lambda$ and $v_{\mu(v)} \in_2 \mu$,

$$\begin{aligned}(\textstyle\bigcap_{x_a \in_2 \lambda,\ y_b \in_2 \mu} x_a \circ_2 y_b)(z) &= \textstyle\bigwedge_{x_a \in_2 \lambda,\ y_b \in_2 \mu} \bigwedge_{z=uv}[x_a(u) \vee y_b(v)] \\ &\leq \textstyle\bigwedge_{z=uv}[u_{\lambda(u)}(u) \vee v_{\mu(v)}(v)] \\ &= \textstyle\bigwedge_{z=uv}[\lambda(u) \vee \mu(v)] \\ &= (\lambda \circ_2 \mu)(z).\end{aligned}$$

Thus, $(\lambda \circ_2 \mu)(z) = C(z)$. So $\mathcal{A} \circ_2 \mathcal{B} = \bigcup^2_{x_{\tilde{\tilde{a}}} \in_2 \mathcal{A},\ y_{\tilde{\tilde{b}}} \in_2 \mathcal{B}} x_{\tilde{\tilde{a}}} \circ_2 y_{\tilde{\tilde{b}}}$.

Case 3: Let $i = 3$. From Remark 1 (3), it is sufficient to prove that

$$A \circ_3 B = (\bigcap_{x_a \in_3 A,\ y_b \in_3 B} x_a \circ_3 y_b, \bigcup_{x_a \in_3 A,\ y_b \in_3 B} x_a \circ_3 y_b),$$

where $(A \circ_3 B)^{\in} = \bigcap_{x_a \in_3 A,\ y_b \in_3 B} x_a \circ_3 y_b$ and $(A \circ_3 B)^{\notin} = \bigcup_{x_a \in_3 A,\ y_b \in_3 B} x_a \circ_3 y_b$. Let $z \in X$. Then from the proof of Case 2 and Proposition 1.1 [16] (ii), we have

$$(A \circ_3 B)^{\in}(z) = (\bigcap_{x_a \in_3 A,\ y_b \in_3 B} x_a \circ_3 y_b)(z),\ (A \circ_3 B)^{\notin}(z) = (\bigcup_{x_a \in_3 A,\ y_b \in_3 B} x_a \circ_3 y_b)(z).$$

Thus, $\mathcal{A} \circ_3 \mathcal{B} = \bigcup^3_{x_{\tilde{a}} \in_3 \mathcal{A},\ y_{\tilde{b}} \in_3 \mathcal{B}} x_{\tilde{a}} \circ_3 y_{\tilde{b}}$.
For $i = 4$, from Cases 2 and 3, the proof is obvious. $\square$

The followings are immediate results of Definition 12.

Proposition 2. *Let $(X, \cdot)$ be a groupoid and let $i = 1,\ 2,\ 3,\ 4$.*
(1) *If "$\cdot$" is associative [resp., commutative] in X, then so is "$\circ_i$" in $\mathcal{O}(X)$.*
(2) *If "$\cdot$" has an identity $e \in X$, then we have*
(2_a) *$e_{\ddot{1}} \in \mathcal{O}_P(X)$ is an identity of "$\circ_1$" in $\mathcal{O}(X)$, i.e.,*
$\mathcal{A} \circ e_{\ddot{1}} = e_{\ddot{1}} \circ \mathcal{A} = \mathcal{A}$, for each $\mathcal{A} \in \mathcal{O}(X)$,
(2_b) *$e_{\langle \tilde{1}, \bar{\mathbf{1}}, 0\rangle} \in \mathcal{O}_P(X)$ is an identity of "$\circ_2$" in $\mathcal{O}(X)$, i.e.,*
$\mathcal{A} \circ e_{\langle \tilde{1}, \bar{\mathbf{1}}, 0\rangle} = e_{\langle \tilde{1}, \bar{\mathbf{1}}, 0\rangle} \circ \mathcal{A} = \mathcal{A}$, for each $\mathcal{A} \in \mathcal{O}(X)$,
(2_c) *$e_{\langle \tilde{1}, \bar{\mathbf{0}}, 1\rangle} \in \mathcal{O}_P(X)$ is an identity of "$\circ_3$" in $\mathcal{O}(X)$, i.e.,*
$\mathcal{A} \circ e_{\langle \tilde{1}, \bar{\mathbf{0}}, 1\rangle} = e_{\langle \tilde{1}, \bar{\mathbf{0}}, 1\rangle} \circ \mathcal{A} = \mathcal{A}$, for each $\mathcal{A} \in \mathcal{O}(X)$,
(2_d) *$e_{\langle \tilde{1}, \bar{\mathbf{0}}, 0\rangle} \in \mathcal{O}_P(X)$ is an identity of "$\circ_4$" in $\mathcal{O}(X)$, i.e.,*
$\mathcal{A} \circ e_{\langle \tilde{1}, \bar{\mathbf{0}}, 0\rangle} = e_{\langle \tilde{1}, \bar{\mathbf{0}}, 0\rangle} \circ \mathcal{A} = \mathcal{A}$, for each $\mathcal{A} \in \mathcal{O}(X)$.

Definition 13. *Let $(X, \cdot)$ be a groupoid and let $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle \in \mathcal{O}(X)$. Then*
(i) $\ddot{0} \neq \mathcal{A}$ is called a 1-octahedron subgroupoid in X, if $\mathcal{A} \circ_1 \mathcal{A} \subset^1 \mathcal{A}$, i.e.,

$$\mathbf{A} \circ_{IV} \mathbf{A} \subset \mathbf{A},\ A \circ_{IF} A \subset A,\ \lambda \circ_F \lambda \subset \lambda,$$

(ii) $\left\langle \tilde{0}, \bar{\mathbf{0}}, 1 \right\rangle \neq \mathcal{A}$ is called a 2-octahedron subgroupoid in X, if $\mathcal{A} \circ_2 \mathcal{A} \subset^2 \mathcal{A}$, i.e.,

$$\mathbf{A} \circ_{IV} \mathbf{A} \subset \mathbf{A},\ A \circ_{IF} A \subset A,\ \lambda \circ_2 \lambda \supset \lambda,$$

(iii) $\left\langle \tilde{0}, \bar{\mathbf{1}}, 0 \right\rangle \neq \mathcal{A}$ is called a 3-octahedron subgroupoid in X, if $\mathcal{A} \circ_3 \mathcal{A} \subset^3 \mathcal{A}$, i.e.,

$$\mathbf{A} \circ_{IV} \mathbf{A} \subset \mathbf{A},\ A \circ_3 A \supset A,\ \lambda \circ_F \lambda \subset \lambda,$$

(iv) $\left\langle \tilde{0}, \bar{\mathbf{1}}, 1 \right\rangle \neq \mathcal{A}$ is called a 4-octahedron subgroupoid in X, if $\mathcal{A} \circ_4 \mathcal{A} \subset^4 \mathcal{A}$, i.e.,

$$\mathbf{A} \circ_{IV} \mathbf{A} \subset \mathbf{A},\ A \circ_3 A \supset A,\ \lambda \circ_2 \lambda \supset \lambda.$$

We will denote the set of all i-octahedron subgroupoids in X as $OGP_i(X)$ ($i = 1,\ 2,\ 3,\ 4$).

Let us denote the set of all fuzzy [resp., intuitionistic fuzzy, interval-valued fuzzy] subgroupoids in a groupoid X in the sense of Liu [16] [resp., Hur et al. [17], Kang and Hur [18]] as $FGP(X)$ [resp., $IFGP(X)$, $IVGP(X)$].

Remark 2. *Let $(X, \cdot)$ be a groupoid and let $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle \in \mathcal{O}(X)$. Then*
(1) $\mathcal{A} \in OGP_1(X)$ if and only if $\mathbf{A} \in IVGP(X),\ A \in IFGP(X),\ \lambda \in FGP(X)$,
(2) $\mathcal{A} \in OGP_2(X)$ if and only if $\mathbf{A} \in IVGP(X),\ A \in IFGP(X),\ \lambda \circ_2 \lambda \supset \lambda$,
(3) $\mathcal{A} \in OGP_3(X)$ if and only if $\mathbf{A} \in IVGP(X),\ A \circ_3 A \supset A,\ \lambda \in FGP(X)$,
(4) $\mathcal{A} \in OGP_3(X)$ if and only if $\mathbf{A} \in IVGP(X),\ A \circ_3 A \supset A,\ \lambda \circ_2 \lambda \supset \lambda$.

Example 2. *(1) Let* $(X, \cdot)$ *be the subgroupoid and let* $\mathcal{A}$ *be the octahedron set in* X *given in Example 1. Then we can easily calculate that*

$$(\mathbf{A} \circ_{IV} \mathbf{A})(a) = [0.4, 0.7] \not\leq [0.3, 0.6] = \mathbf{A}(a),$$

$$(\lambda \circ_2 \lambda)(a) = 0.3 \not\geq 0.5 = \lambda(a),$$

$$(A \circ_3 A)(a) = (0.5, 0.4) \not\geq (0.7, 0.2) = A(a).$$

Thus, $\mathcal{A} \notin OGP_i(X)$*, for* $i = 1,\ 2,\ 3,\ 4$.

(2) Let $X = \{a, b, c\}$ *be the groupoid with the following Cayley Table 4:*

Table 4. Caley.

$\cdot$	a	b	c
a	a	a	a
b	b	a	b
c	c	c	c

Consider the octahedron set $\mathcal{A}$ *in* X *given in Example 1. Then we can easily see that* $\mathcal{A} \in OGP_i(X)$ *for* $i = 1,\ 2$ *but* $\mathcal{A} \notin OGP_i(X)$ *for* $i = 3,\ 4$.

(3) Let $(X, \cdot)$ *be a groupoid and let* $\mathbf{A} \in IVGP(X)$*. Then clearly,* $\mathcal{O}_{\mathbf{A}} \in OGP_1(X)$*, where* $\mathcal{O}_{\mathbf{A}}$ *is the octahedron set in* X *induced by* $\mathbf{A}$ *(See Example 3.2 (3) in [10]).*

(4) Let $(X, \cdot)$ *be a groupoid and let* $A \in IFGP(X)$*. Then clearly,* $\mathcal{O}_A \in OGP_1(X)$*, where* $\mathcal{O}_A$ *is the octahedron set in* X *induced by* A *(See Example 3.2 (4) in [10]).*

(5) Let $(X, \cdot)$ *be a groupoid and let* $\mathcal{A} \in OGP_i(X)$*. Then clearly,* $[\,]\mathcal{A},\ \diamond\mathcal{A} \in OGP_i(X)$ *(*$i = 1,\ 2,\ 3,\ 4$*).*

The followings are immediate results of Definitions 11–13, Proposition 2 (2) and Remark 2 (1).

Theorem 1. *Let* $(X, \cdot)$ *be a groupoid and let* $\ddot{0} \neq \mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle \in \mathcal{O}(X)$*. Then the followings are equivalent:*

(1) $\mathcal{A} \in OGP_1(X)$,

(2) *for every* $x_{\tilde{\tilde{a}}},\ y_{\tilde{\tilde{b}}} \in \mathcal{A}$, $x_{\tilde{\tilde{a}}} \circ_1 y_{\tilde{\tilde{b}}} \in \mathcal{A}$*, i.e.,* $(\mathcal{A}, \circ_1)$ *is a groupoid,*

(3) *for every* $x,\ y \in X$, $\mathcal{A}(xy) \geq \mathcal{A}(x) \wedge^1 \mathcal{A}(y)$*, i.e.,*

(i) $A^-(xy) \geq A^-(x) \wedge A^-(y),\ A^+(xy) \geq A^+(x) \wedge A^+(y)$,

(ii) $A^{\in}(xy) \geq A^{\in}(x) \wedge A^{\in}(y),\ A^{\notin}(xy) \leq A^{\notin}(x) \vee A^{\notin}(y)$,

(iii) $\lambda(xy) \geq \lambda(x) \wedge \lambda(y)$.

From Definitions 8–10, Remark 1 (1) and the above proposition, it is obvious that $(\mathcal{A}, \circ_1)$ is a groupoid if and only if $(\mathbf{A}, \circ_{IV})$, $(A, \circ_{IF})$ and $(\lambda, \circ_F)$ are groupoids.

Proposition 3. *Let* $(X, \cdot)$ *be a groupoid and let* $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle \in OGP_1(X)$.

(1) *If "*$\cdot$*" is associative in* X*, then so is "*$\circ_1$*" in* $\mathcal{A}$*, i.e., for every* $x_{\tilde{\tilde{a}}},\ y_{\tilde{\tilde{b}}},\ z_{\tilde{\tilde{c}}} \in \mathcal{A}$,

$$(x_{\tilde{\tilde{a}}} \circ_1 y_{\tilde{\tilde{b}}}) \circ_1 z_{\tilde{\tilde{c}}} = x_{\tilde{\tilde{a}}} \circ_1 (y_{\tilde{\tilde{b}}} \circ_1 z_{\tilde{\tilde{c}}}),$$

(2) *If "*$\cdot$*" is commutative in* X*, then so is "*$\circ_1$*" in* $\mathcal{A}$*, i.e., for every* $x_{\tilde{\tilde{a}}},\ y_{\tilde{\tilde{b}}} \in \mathcal{A}$,

$$x_{\tilde{\tilde{a}}} \circ_1 y_{\tilde{\tilde{b}}} = y_{\tilde{\tilde{b}}} \circ_1 x_{\tilde{\tilde{a}}},$$

(3) *If "*$\cdot$*" has an identity* $e \in X$*, then for each* $x_{\tilde{\tilde{a}}} \in \mathcal{A}$,

$$e_{\ddot{1}} \circ_1 x_{\tilde{\tilde{a}}} = x_{\tilde{\tilde{a}}} = x_{\tilde{\tilde{a}}} \circ_1 e_{\ddot{1}}.$$

The followings are immediate consequences of Definitions 11–13, Proposition 2 (2) and Remark 2 (2).

Theorem 2. *Let $(X, \cdot)$ be a groupoid and let $\left\langle \widetilde{0}, \bar{\mathbf{0}}, 1 \right\rangle \neq \mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle \in \mathcal{O}(X)$. Then the followings are equivalent:*

(1) $\mathcal{A} \in OGP_2(X)$,
(2) *for every* $x_{\widetilde{a}}$, $y_{\widetilde{b}} \in \mathcal{A}$, $x_{\widetilde{a}} \circ_2 y_{\widetilde{b}} \in \mathcal{A}$, *i.e.,* $(\mathcal{A}, \circ_2)$ *is a groupoid,*
(3) *for every* $x,\ y \in X$, $\mathcal{A}(xy) \geq \mathcal{A}(x) \wedge^2 \mathcal{A}(y)$, *i.e.,*
(i) $A^-(xy) \geq A^-(x) \wedge A^-(y),\ A^+(xy) \geq A^+(x) \wedge A^+(y)$,
(ii) $A^{\in}(xy) \geq A^{\in}(x) \wedge A^{\in}(y),\ A^{\notin}(xy) \leq A^{\notin}(x) \vee A^{\notin}(y)$,
(iii) $\lambda(xy) \leq \lambda(x) \vee \lambda(y)$.

Proposition 4. *Let $(X, \cdot)$ be a groupoid and let $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle \in OGP_2(X)$.*

(1) *If "$\cdot$" is associative in X, then so is "$\circ_2$" in $\mathcal{A}$, i.e., for every* $x_{\widetilde{a}}$, $y_{\widetilde{b}}$, $z_{\widetilde{c}} \in \mathcal{A}$,

$$(x_{\widetilde{a}} \circ_2 y_{\widetilde{b}}) \circ_2 z_{\widetilde{c}} = x_{\widetilde{a}} \circ_2 (y_{\widetilde{b}} \circ_2 z_{\widetilde{c}}),$$

(2) *If "$\cdot$" is commutative in X, then so is "$\circ_2$" in $\mathcal{A}$, i.e., for every* $x_{\widetilde{a}}$, $y_{\widetilde{b}} \in \mathcal{A}$,

$$x_{\widetilde{a}} \circ_2 y_{\widetilde{b}} = y_{\widetilde{b}} \circ_2 x_{\widetilde{a}},$$

(3) *If "$\cdot$" has an identity $e \in X$, then for each* $x_{\widetilde{a}} \in \mathcal{A}$,

$$e_{\langle \widetilde{1}, \bar{\mathbf{1}}, 0 \rangle} \circ_2 x_{\widetilde{a}} = x_{\widetilde{a}} = x_{\widetilde{a}} \circ_2 e_{\langle \widetilde{1}, \bar{\mathbf{1}}, 0 \rangle}.$$

The followings are immediate consequences of Definitions 11–13, Proposition 2 (3) and Remark 2 (3).

Theorem 3. *Let $(X, \cdot)$ be a groupoid and let $\left\langle \widetilde{0}, \bar{\mathbf{1}}, 0 \right\rangle \neq \mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle \in \mathcal{O}(X)$. Then the followings are equivalent:*

(1) $\mathcal{A} \in OGP_3(X)$,
(2) *for every* $x_{\widetilde{a}}$, $y_{\widetilde{b}} \in \mathcal{A}$, $x_{\widetilde{a}} \circ_3 y_{\widetilde{b}} \in \mathcal{A}$, *i.e.,* $(\mathcal{A}, \circ_3)$ *is a groupoid,*
(3) *for every* $x,\ y \in X$, $\mathcal{A}(xy) \geq \mathcal{A}(x) \wedge^3 \mathcal{A}(y)$, *i.e.,*
(i) $A^-(xy) \geq A^-(x) \wedge A^-(y),\ A^+(xy) \geq A^+(x) \wedge A^+(y)$,
(ii) $A^{\in}(xy) \leq A^{\in}(x) \vee A^{\in}(y),\ A^{\notin}(xy) \geq A^{\notin}(x) \wedge A^{\notin}(y)$,
(iii) $\lambda(xy) \geq \lambda(x) \wedge \lambda(y)$.

Proposition 5. *Let $(X, \cdot)$ be a groupoid and let $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle \in OGP_3(X)$.*

(1) *If "$\cdot$" is associative in X, then so is "$\circ_3$" in $\mathcal{A}$, i.e., for every* $x_{\widetilde{a}}$, $y_{\widetilde{b}}$, $z_{\widetilde{c}} \in \mathcal{A}$,

$$(x_{\widetilde{a}} \circ_3 y_{\widetilde{b}}) \circ_3 z_{\widetilde{c}} = x_{\widetilde{a}} \circ_3 (y_{\widetilde{b}} \circ_3 z_{\widetilde{c}}),$$

(2) *If "$\cdot$" is commutative in X, then so is "$\circ_3$" in $\mathcal{A}$, i.e., for every* $x_{\widetilde{a}}$, $y_{\widetilde{b}} \in \mathcal{A}$,

$$x_{\widetilde{a}} \circ_3 y_{\widetilde{b}} = y_{\widetilde{b}} \circ_3 x_{\widetilde{a}},$$

(3) *If "$\cdot$" has an identity $e \in X$, then for each* $x_{\widetilde{a}} \in \mathcal{A}$,

$$e_{\langle \widetilde{1}, \bar{\mathbf{0}}, 1 \rangle} \circ_3 x_{\widetilde{a}} = x_{\widetilde{a}} = x_{\widetilde{a}} \circ_3 e_{\langle \widetilde{1}, \bar{\mathbf{0}}, 1 \rangle}.$$

The followings are immediate consequences of Definitions 11–13, Proposition 2 (4) and Remark 2 (4).

Theorem 4. *Let $(X, \cdot)$ be a groupoid and let $\left\langle \widetilde{0}, \bar{\mathbf{1}}, 1 \right\rangle \neq \mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle \in \mathcal{O}(X)$. Then the followings are equivalent:*

(1) $\mathcal{A} \in OGP_4(X)$,
(2) *for every* $x_{\tilde{a}},\ y_{\tilde{b}} \in \mathcal{A}$, $x_{\tilde{a}} \circ_3 y_{\tilde{b}} \in \mathcal{A}$, *i.e.,* $(\mathcal{A}, \circ_4)$ *is a groupoid,*
(3) *for every* $x,\ y \in X$, $\mathcal{A}(xy) \geq \mathcal{A}(x) \wedge^4 \mathcal{A}(y)$, *i.e.,*
(i) $A^-(xy) \geq A^-(x) \wedge A^-(y),\ A^+(xy) \geq A^+(x) \wedge A^+(y)$,
(ii) $A^{\in}(xy) \leq A^{\in}(x) \vee A^{\in}(y),\ A^{\notin}(xy) \geq A^{\notin}(x) \wedge A^{\notin}(y)$,
(iii) $\lambda(xy) \leq \lambda(x) \vee \lambda(y)$.

Proposition 6. *Let* $(X, \cdot)$ *be a groupoid and let* $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle \in OGP_4(X)$.
(1) *If "$\cdot$" is associative in X, then so is "$\circ_4$" in $\mathcal{A}$, i.e., for every* $x_{\tilde{a}},\ y_{\tilde{b}},\ z_{\tilde{c}} \in \mathcal{A}$,

$$(x_{\tilde{a}} \circ_4 y_{\tilde{b}}) \circ_4 z_{\tilde{c}} = x_{\tilde{a}} \circ_4 (y_{\tilde{b}} \circ_4 z_{\tilde{c}}),$$

(2) *If "$\cdot$" is commutative in X, then so is "$\circ_4$" in $\mathcal{A}$, i.e., for every* $x_{\tilde{a}},\ y_{\tilde{b}} \in \mathcal{A}$,

$$x_{\tilde{a}} \circ_4 y_{\tilde{b}} = y_{\tilde{b}} \circ_4 x_{\tilde{a}},$$

(3) *If "$\cdot$" has an identity $e \in X$, then for each* $x_{\tilde{a}} \in \mathcal{A}$,

$$e_{\langle \tilde{1}, \bar{0}, 0 \rangle} \circ_3 x_{\tilde{a}} = x_{\tilde{a}} = x_{\tilde{a}} \circ_4 e_{\langle \tilde{1}, \bar{0}, 0 \rangle}.$$

Remark 3. *Let* $(X, \cdot)$ *be a groupoid and let* $A \in 2^X$. *Then we have*

$$\chi_{\mathcal{A}} \in OGP_1(X) \iff A \text{ is a subgroupoid of } X.$$

Definition 14. *Let* $(X, \cdot)$ *be a groupoid,* $\mathcal{A} \in \mathcal{O}(X)$ *and let* $i = 1,\ 2,\ 3,\ 4$. *Then* $\mathcal{A}$ *is called a:*
(i) i-octahedron left ideal (briefly, i-OLI) of X, if for every $x,\ y \in X$,

$$\mathcal{A}(xy) \geq_i \mathcal{A}(y), \text{ i.e.,}$$

(ii) i-octahedron right ideal (briefly, i-ORI) of X, if for every $x,\ y \in X$,

$$\mathcal{A}(xy) \geq_i \mathcal{A}(x), \text{ i.e.,}$$

(iii) i-octahedron ideal (simply, i-OI) of X, if it is both an i-OLI and an i-ORI of X.
In this case, we will denote the set of all i-OIs [resp., i-OLIs and i-ORIs] of X as $OI_i(X)$ [resp., $OLI_i(X)$ and $ORI_i(X)$].

Remark 4. *From the above Definition, we have the followings.*
(1) $\mathcal{A} \in OLI_1(X)$
$\iff A^-(xy) \geq A^-(y),\ A^+(xy) \geq A^+(y),\ A^{\in}(xy) \geq A^{\in}(y)$,
$A^{\notin}(xy) \leq A^{\notin}(y),\ \lambda(xy) \geq \lambda(y)$,
$\mathcal{A} \in OLI_2(X)$
$\iff A^-(xy) \geq A^-(y),\ A^+(xy) \geq A^+(y),\ A^{\in}(xy) \geq A^{\in}(y)$,
$A^{\notin}(xy) \leq A^{\notin}(y),\ \lambda(xy) \leq \lambda(y)$,
$\mathcal{A} \in OLI_3(X)$
$\iff A^-(xy) \geq A^-(y),\ A^+(xy) \geq A^+(y),\ A^{\in}(xy) \leq A^{\in}(y)$,
$A^{\notin}(xy) \geq A^{\notin}(y),\ \lambda(xy) \geq \lambda(y)$,
$\mathcal{A} \in OLI_4(X)$
$\iff A^-(xy) \geq A^-(y),\ A^+(xy) \geq A^+(y),\ A^{\in}(xy) \leq A^{\in}(y)$,
$A^{\notin}(xy) \geq A^{\notin}(y),\ \lambda(xy) \leq \lambda(y)$,
(2) $\mathcal{A} \in ORI_1(X)$
$\iff A^-(xy) \geq A^-(x),\ A^+(xy) \geq A^+(x),\ A^{\in}(xy) \geq A^{\in}(x)$,

$$A^{\notin}(xy) \leq A^{\notin}(x),\ \lambda(xy) \geq \lambda(x),$$
$$\mathcal{A} \in ORI_2(X)$$
$$\Longleftrightarrow A^-(xy) \geq A^-(x),\ A^+(xy) \geq A^+(x),\ A^{\in}(xy) \geq A^{\in}(x),$$
$$A^{\notin}(xy) \leq A^{\notin}(x),\ \lambda(xy) \leq \lambda(x),$$
$$\mathcal{A} \in ORI_3(X)$$
$$\Longleftrightarrow A^-(xy) \geq A^-(x),\ A^+(xy) \geq A^+(x),\ A^{\in}(xy) \leq A^{\in}(x),$$
$$A^{\notin}(xy) \geq A^{\notin}(x),\ \lambda(xy) \geq \lambda(x),$$
$$\mathcal{A} \in ORI_4(X)$$
$$\Longleftrightarrow A^-(xy) \geq A^-(x),\ A^+(xy) \geq A^+(x),\ A^{\in}(xy) \leq A^{\in}(x),$$
$$A^{\notin}(xy) \geq A^{\notin}(x),\ \lambda(xy) \leq \lambda(x),$$

(3) $\mathcal{A} \in OI_1(X)$
$$\Longleftrightarrow A^-(xy) \geq A^-(x) \vee A^-(y),\ A^+(xy) \geq A^+(x) \vee A^+(y),$$
$$A^{\in}(xy) \geq A^{\in}(x) \vee A^{\in}(y),\ A^{\notin}(xy) \leq A^{\notin}(x) \wedge A^{\notin}(x),\ \lambda(xy) \geq \lambda(x) \vee \lambda(y),$$
$$\mathcal{A} \in ORI_2(X)$$
$$\Longleftrightarrow A^-(xy) \geq A^-(x) \vee A^-(y),\ A^+(xy) \geq A^+(x) \vee A^+(y),$$
$$A^{\in}(xy) \geq A^{\in}(x) \vee A^{\in}(y),\ A^{\notin}(xy) \leq A^{\notin}(x) \wedge A^{\notin}(x),\ \lambda(xy) \leq \lambda(x) \wedge \lambda(y),$$
$$\mathcal{A} \in OI_3(X)$$
$$\Longleftrightarrow A^-(xy) \geq A^-(x) \vee A^-(y),\ A^+(xy) \geq A^+(x) \vee A^+(y),$$
$$A^{\in}(xy) \leq A^{\in}(x) \wedge A^{\in}(y),\ A^{\notin}(xy) \geq A^{\notin}(x) \vee A^{\notin}(x),\ \lambda(xy) \geq \lambda(x) \vee \lambda(y),$$
$$\mathcal{A} \in OI_4(X)$$
$$\Longleftrightarrow A^-(xy) \geq A^-(x) \vee A^-(y),\ A^+(xy) \geq A^+(x) \vee A^+(y),$$
$$A^{\in}(xy) \leq A^{\in}(x) \wedge A^{\in}(y),\ A^{\notin}(xy) \geq A^{\notin}(x) \vee A^{\notin}(x),\ \lambda(xy) \leq \lambda(x) \wedge \lambda(y).$$

Remark 5. *An i-octahedron left ideal [resp., right ideal and ideal] in a semigroup S, a group G and a ring G is defined as Definition 14.*

For a groupoid $(X, \cdot)$, let us denote the set of all fuzzy ideals [resp., left ideals and right ideals] (See [19]), the set of all IVIs [resp., IVLIs and IVRIs] (See [18]) and the set of all IFIs [resp., IFLIs, IFRIs] (See [17]) of X as $FI(X)$ [resp., $FLI(X)$ and $FRI(X)$], $IVI(X)$ [resp., $IVLI(X)$ and $IVRI(X)$] and $IFI(X)$ [resp., $IFLI(X)$ and $IFRI(X)$]. Then we can easily see that $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle \in OI_1(X)$ [resp., $OLI_1(X)$ and $ORI_1(X)$] if and only if $\mathbf{A} \in IVI(X)$, $A \in IFI(X)$, $\lambda \in FI(X)$ [resp., $\mathbf{A} \in IVLI(X)$, $A \in IFLI(X)$, $\lambda \in FLI(X)$ and $\mathbf{A} \in IVRI(X)$, $A \in IFRI(X)$, $\lambda \in FRI(X)$]. Furthermore, it is obvious that $\mathcal{A} \in OGP_i(X)$, for each $\mathcal{A} \in OI_i(X)$ [resp., $OLI_i(X)$ and $ORI_i(X)$] ($i = 1, 2, 3, 4$) but the converse is not true in general (See Example 3 (1)).

Note that for every $\mathcal{A} \in OGP_i(X)$ ($i = 1, 2, 3, 4$), we have: for each $x \in X$,

$$\mathcal{A}(x^n) \geq_i \mathcal{A}(x), \text{ i.e.,}$$

where x^n is any composite of x's.

Example 3. *(1) Let $(X, \cdot)$ be the groupoid and $\mathcal{A} \in OGP_1(X)$ given in Example 2 (2). Then clearly, $\lambda(ab) = 0.5 \ngeq 0.7 = \lambda(b)$. Thus, $\lambda \notin FLI(X)$. So $\mathcal{A} \notin OLI_1(X)$.*

(2) Let $X = \{a, b, c\}$ be the groupoid with the following Cayley Table 5:

Table 5. Caley.

$\cdot$	a	b	c
a	a	a	a
b	a	a	c
c	a	b	c

Consider the octahedron set $\mathcal{A}$ in X given by:

$$\mathcal{A}(a) = \langle [0.4, 0.8], (0.7, 0.2), 0.8 \rangle ,$$

$$\mathcal{A}(b) = \langle [0.3, 0.7], (0.6, 0.3), 0.7 \rangle ,$$

$$\mathcal{A}(c) = \langle [0.2, 0.6], (0.5, 0.4), 0.6 \rangle .$$

Then we can easily calculate that $\mathcal{A} \in OLI_1(X)$. But $A^-(bc) = 0.2 \not\geq 0.3 = A^-(b)$. Thus, $\mathbf{A} \notin IVRI(X)$. So $\mathcal{A} \notin ORI_1(X)$.

(3) Let $X = \{a, b, c\}$ be the groupoid with the following Cayley Table 6:

Table 6. Caley.

$\cdot$	***a***	***b***	***c***
a	a	a	a
b	b	b	a
c	c	a	c

Consider the octahedron set $\mathcal{A}$ in X given by:

$$\mathcal{A}(a) = \langle [0.4, 0.8], (0.7, 0.2), 0.9 \rangle ,$$

$$\mathcal{A}(b) = \langle [0.3, 0.7], (0.6, 0.3), 0.7 \rangle ,$$

$$\mathcal{A}(c) = \langle [0.2, 0.6], (0.5, 0.4), 0.8 \rangle .$$

Then we can easily calculate that $\mathcal{A} \in ORI_1(X)$. But $A^-(ba) = 0.3 \not\geq 0.4 = A^-(a)$. Thus, $\mathbf{A} \notin IVLI(X)$. So $\mathcal{A} \notin OLI_1(X)$.

From Proposition 3.2 in [19], we have the following result.

Theorem 5. *Let $(X, \cdot)$ be a groupoid and let $A \in 2^X$. Then $\chi_{\mathcal{A}} \in OLI_1(X)$ [resp., $ORI_1(X)$ and $OI_1(X)$] if and only if A is a left ideal [resp., a right ideal and an ideal] of X.*

Definition 15 (**[10]**)**.** *Let X be a nonempty set, let $\tilde{\tilde{a}} = \langle \tilde{a}, \bar{a}, a \rangle \in [I] \times (I \oplus I) \times I$ and let $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle \in \mathcal{O}(X)$. Then two subsets $[\mathcal{A}]_{\tilde{\tilde{a}}}$ and $[\mathcal{A}]^*_{\tilde{\tilde{a}}}$ of X are defined as follows:*

$$[\mathcal{A}]_{\tilde{\tilde{a}}} = \{x \in X : \mathbf{A}(x) \geq \tilde{a},\ A(x) \geq \bar{a},\ \lambda(x) \geq a\},$$

$$[\mathcal{A}]^*_{\tilde{\tilde{a}}} = \{x \in X : \mathbf{A}(x) > \tilde{a},\ A(x) > \bar{a},\ \lambda(x) > a\}.$$

In this case, $[\mathcal{A}]_{\tilde{\tilde{a}}}$ is called an $\tilde{\tilde{a}}$-level set of $\mathcal{A}$ and $[\mathcal{A}]^*_{\tilde{\tilde{a}}}$ is called a strong $\tilde{\tilde{a}}$-level set of $\mathcal{A}$.

The following is an immediate consequence of Theorem 1, Definitions 13 and 14.

Proposition 7. *Let $(X, \cdot)$ be a groupoid and let $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle \in \mathcal{O}(X)$. If $\mathcal{A} \in OGP_1(X)$ or $\mathcal{A} \in OLI_1(X)$ [resp., $ORI_1(X)$ and $OI_1(X)$], then $[\mathcal{A}]_{\tilde{\tilde{a}}}$ is a subgroupoid or a left ideal [resp., a right ideal and an ideal] of X, for each $\tilde{\tilde{a}} \in [I] \times (I \oplus I) \times I$.*

Proposition 8. *Let $(X, \cdot)$ be a groupoid and let $i = 1,\ 2,\ 3,\ 4$. If $(\mathcal{A}_j)_{j \in J} = (\langle \mathbf{A}_j, A_j, \lambda_j \rangle)_{j \in J} \subset OGP_i(X)$, then $\bigcap^i_{j \in J} \mathcal{A}_j \in OGP_i(X)$, where J denotes an index set.*

Proof. Case 1: Suppose $(\mathcal{A}_j)_{j\in J} = (\langle \mathbf{A}_j, A_j, \lambda_j\rangle)_{j\in J} \subset OGP_1(X)$. Then from Propositions 3.8 in [18], 3.9 in [17] and 3.1 in [19], we have

$$\bigcap_{j\in J} \mathbf{A}_j \in IVGP(X),\ \bigcap_{j\in J} A_j \in IFGP(X),\ \bigcap_{j\in J} \lambda_j \in FGP(X).$$

Thus, $\bigcap^1_{j\in J} \mathcal{A}_j \in OGP_1(X)$.

Case 2: Suppose $(\mathcal{A}_j)_{j\in J} = (\langle \mathbf{A}_j, A_j, \lambda_j\rangle)_{j\in J} \subset OGP_2(X)$ and let $x,\ y \in X$. Then by Definition 5 and Theorem 2,

$$\begin{aligned}(\bigcup_{j\in J}\lambda_j)(xy) &= \bigvee_{j\in J}\lambda_j(xy) \leq \bigvee_{j\in J}(\lambda_j(x)\vee\lambda_j(y))\\ &= (\bigvee_{j\in J}\lambda_j(x))\vee(\bigvee_{j\in J}\lambda_j(y))\\ &= (\bigcup_{j\in J}\lambda_j)(x)\vee(\bigcup_{j\in J}\lambda_j)(y).\end{aligned}$$

Thus, $(\bigcup_{j\in J}\lambda_j)$ satisfies the the condition (iii) of Theorem 2 (3). By the hypothesis and Case 1, $\bigcap_{j\in J}\mathbf{A}_j$ and $\bigcap_{j\in J} A_j$ satisfy the conditions (i) and (ii) of Theorem 2 (3). So $\bigcap^2_{j\in J}\mathcal{A}_j \in OGP_2(X)$.

Case 3: Suppose $(\mathcal{A}_j)_{j\in J} = (\langle \mathbf{A}_j, A_j, \lambda_j\rangle)_{j\in J} \subset OGP_3(X)$ and let $x,\ y \in X$. Then by Definition 5 and Theorem 3 (ii),

$$\begin{aligned}(\bigcup_{j\in J}A_j^{\in})(xy) &= \bigvee_{j\in J}A_j^{\in}(xy) \leq \bigvee_{j\in J}(A_j^{\in}(x)\vee A_j^{\in}(y))\\ &= (\bigvee_{j\in J}A_j^{\in}(x))\vee(\bigvee_{j\in J}A_j^{\in}(y))\\ &= (\bigcup_{j\in J}A_j^{\in})(x)\vee(\bigcup_{j\in J}A_j^{\in})(y).\end{aligned}$$

Similarly, we have $(\bigcap_{j\in J}A_j^{\notin})(xy) \geq (\bigcap_{j\in J}A_j^{\notin})(x)\wedge(\bigcap_{j\in J}A_j^{\notin})(y)$. Thus, $(\bigcup_{j\in J}A_j)$ satisfies the the condition Theorem 3 (ii). By the hypothesis and Case 1, $\bigcap_{j\in J}\mathbf{A}_j$ and $\bigcap_{j\in J}\lambda_j$ satisfy the conditions (i) and (iii) of Theorem 2. So $\bigcap^3_{j\in J}\mathcal{A}_j \in OGP_3(X)$.

Case 4: Suppose $(\mathcal{A}_j)_{j\in J} = (\langle \mathbf{A}_j, A_j, \lambda_j\rangle)_{j\in J} \subset OGP_3(X)$. Then by Case 2 and 3, we can easily see that $\bigcap^4_{j\in J}\mathcal{A}_j \in OGP_4(X)$. □

Remark 6. *for every $\mathcal{A},\ \mathcal{B} \in OGP_i(X)$, $\mathcal{A}\cup^i\mathcal{B} \notin OGP_i(X)$ in general ($i = 1,\ 2,\ 3,\ 4$).*

Example 4. *Let $(X,\cdot)$ be the groupoid and $\mathcal{A} \in OGP_1(X)$ given in Example 2 (2). Consider the octahedron subgroupoid in X given by:*

$$\mathcal{B}(a) = \mathcal{B}(b) = \mathcal{B}(c) = \langle[0.1, 0.7], (0.5, 0.4), 0.6\rangle.$$

Then $(A\cup B)(ab) = (0.7, 0.4) \ngeq (0.6, 0.3) = (A\cup B)(a)\wedge(A\cup B)(b)$. Thus, $A\cup B \notin IFGP(X)$. So $\mathcal{A}\cup^1\mathcal{B} \notin OGP_1(X)$.

Remark 7. *Let $(X,\cdot)$ be a groupoid and let $(\mathcal{A}_j)_{j\in J} = (\langle \mathbf{A}_j, A_j, \lambda_j\rangle)_{j\in J} \subset OGP_i(X)$ ($i = 1,\ 2,\ 3,\ 4$). Then from Proposition 8, we can easily see that*

$$\bigcap^i\{\mathcal{A} \in OGP_i(X) : \bigcup^i_{j\in J}\mathcal{A}_j \subset_i \mathcal{A}\} \in OGP_i(X).$$

In this case, we will denote $\bigcap^i\{\mathcal{A} \in OGP_i(X) : \bigcup^i_{j\in J}\mathcal{A}_j \subset_i \mathcal{A}\}$ as $\bigvee^i_{j\in J}\mathcal{A}_j$.

In particular, it is obvious that $(OGP_1(X), \subset_1)$ is a complete lattice with the least element $\ddot{0}$ and the greatest element $\ddot{1}$, where for each $(\mathcal{A}_j)_{j\in J} \subset_1 OGP_1(X)$, the inf and the sup of $(\mathcal{A}_j)_{j\in J}$ are $inf_{j\in J}\mathcal{A}_j = \bigcap^1_{j\in J}\mathcal{A}_j$ and $sup_{j\in J}\mathcal{A}_j = \bigvee^1_{j\in J}\mathcal{A}_j$.

The following is an immediate result of Proposition 8.

Corollary 1. *Let* $(X, \cdot)$ *be the groupoid,* $\mathcal{A} \in \mathcal{O}(X)$ *and let*

$$(\mathcal{A}) = \bigcap^{1} \{\mathcal{B} \in OGP_1(X) : \mathcal{A} \subset_1 \mathcal{B}\}.$$

Then $(\mathcal{A}) \in OGP_1(X)$.

In this case, $(\mathcal{A})$ is called the octahedron subgroupoid in X generated by $\mathcal{A}$.

Proposition 9. *Let* $(X, \cdot)$ *be a groupoid, and let* (A) *be the subgroupoid generated by* A *and* $\chi_{(A)} = \left\langle [\chi_{(A)}, \chi_{(A)}], (\chi_{(A)}, \chi_{(A^c)}), \chi_{(A)} \right\rangle$ *for each* $A \in 2^X$. *Then*

$$(\chi_{\mathcal{A}}) = \chi_{(A)}.$$

Proof. From Remark 3 and Corollary 1, it is obvious that $\chi_{(A)} \in OGP_1(X)$. Let $\mathcal{B} \in OGP_1(X)$ such that $\mathcal{B} \supset_1 \chi_{(A)}$. Then clearly,

$$\mathcal{B}(x) = \langle [1,1], (1,0), 1 \rangle \text{, for each } x \in A.$$

Since $\mathcal{B} \in OGP_1(X)$, $\mathcal{B}(xy) = \langle [1,1], (1,0), 1 \rangle$ for every $x,\ y \in A$. Thus, $\mathcal{B} \supset_1 \chi_{(\mathcal{A})}$. So

$$\chi_{(A)} \subset_1 \bigcap^{1} \{\mathcal{B} \in OGP_1(X) : \mathcal{B} \supset_1 \chi_{\mathcal{A}}\} = (\chi_{\mathcal{A}}).$$

We can easily prove that $(\chi_{\mathcal{A}}) \subset_1 \chi_{(A)}$. Hence $(\chi_{\mathcal{A}}) = \chi_{(A)}$. □

From the above Proposition, the subgoupoid lattice of X can be regarded as a sublattice of the octahedron subgroupoid lattice of X.

Proposition 10. *Let* $(X, \cdot)$ *be a groupoid and let* $i = 1,\ 2,\ 3,\ 4$. *Then the i-intersection or the i-union of any i-octahedron (left, right) ideals is an i-octahedron (left, right) ideal.*

Proof. Let $(\mathcal{A}_j)_{j \in J} \subset OLI_i(X)$ [resp., $ORI_i(X)$ and $OI_i(X)$], where $\mathcal{A}_j = \langle \mathbf{A}_j, A_j, \lambda_j \rangle$. We only prove that $\bigcup^{i}_{j \in J} \mathcal{A}_j \in OLI_i(X)$ and the remainder's proofs are omitted.

Case 1: $(\mathcal{A}_j)_{j \in J} \subset OLI_1(X)$ and let $x,\ y \in X$. Then by Definition 5 and Remark 4 (1), we have

$$(\bigcup_{j \in J} \mathbf{A}^-)(xy) = \bigvee_{j \in J} \mathbf{A}^-(xy) \geq \bigvee_{j \in J} \mathbf{A}^-(y) = (\bigcup_{j \in J} \mathbf{A}^-)(y).$$

Similarly, $(\bigcap_{j \in J} \mathbf{A}^+)(xy) \geq (\bigcap_{j \in J} \mathbf{A}^+)(y)$. From Proposition 3.3 in [19] and 3.10 in [17], we have

$$(\bigcup_{j \in J} \lambda)(xy) \geq (\bigcup_{j \in J} \lambda)(y),\ (\bigcup_{j \in J} A)(xy) \geq (\bigcup_{j \in J} A)(y).$$

Thus, $(\mathcal{A}_j)_{j \in J} \in OLI_1(X)$.

Case 2: $(\mathcal{A}_j)_{j \in J} \subset OLI_2(X)$ and let $x,\ y \in X$. Then by Definition 5 and Remark 4 (1), we have

$$(\bigcap_{j \in J} \lambda)(xy) = \bigwedge_{j \in J} \lambda(xy) \leq \bigwedge_{j \in J} \lambda(y) = (\bigcap_{j \in J} \lambda)(y).$$

Thus, by Case 1, $(\mathcal{A}_j)_{j \in J} \in OLI_2(X)$.

Case 3: $(\mathcal{A}_j)_{j \in J} \subset OLI_3(X)$ and let $x,\ y \in X$. Then by Definition 5 and Remark 4 (1), we have

$$(\bigcap_{j \in J} A^{\in})(xy) = \bigwedge_{j \in J} A^{\in}(xy) \leq \bigwedge_{j \in J} A^{\in}(y) = (\bigcap_{j \in J} A^{\in})(y).$$

Similarly, $(\bigcup_{j\in J} A^{\notin})(xy) \geq (\bigcup_{j\in J} A^{\notin})(y)$. Thus, by Case 1, $(\mathcal{A}_j)_{j\in J} \in OLI_3(X)$.

Case 4: $(\mathcal{A}_j)_{j\in J} \subset OLI_4(X)$. Then by Cases 2 and 3, $(\mathcal{A}_j)_{j\in J} \in OLI_4(X)$. This completes the proof. $\square$

Definition 16 ([10]). *Let X, Y be two sets, let $f : X \to Y$ be a mapping and let $\mathcal{A} = \langle \mathbf{A}, A, \lambda\rangle \in \mathcal{O}(X)$, $\mathcal{B} = \langle \mathbf{B}, B, \mu\rangle \in \mathcal{O}(Y)$.*

(i) The preimage of $\mathcal{B}$ under f, denoted by $f^{-1}(\mathcal{B}) = \langle f^{-1}(\mathbf{B}), f^{-1}(B), f^{-1}(\mu)\rangle$, is the octahedron set in X defined as follows: for each $x \in X$,

$$f^{-1}(\mathcal{B})(x) = \left\langle [(B^{-} \circ f)(x), (B^{+} \circ f)(x))], ((B^{\in} \circ f)(x), (B^{\notin} \circ f)(x)), (\mu \circ f)(x) \right\rangle.$$

(ii) The image of $\mathcal{A}$ under f, denoted by $f(\mathcal{A}) = \langle f(\mathbf{A}), f(A), f(\lambda)\rangle$, is the octahedron set in Y defined as follows: for each $y \in Y$,

$$f(\mathbf{A})(y) = \begin{cases} [\bigvee_{x\in f^{-1}(y)} A^{-}(x), \bigvee_{x\in f^{-1}(y)} A^{+}(x)] & \text{if } f^{-1}(y) \neq \phi \\ \mathbf{0} & \text{otherwise,} \end{cases}$$

$$f(A)(y) = \begin{cases} (\bigvee_{x\in f^{-1}(y)} A^{\in}(x), \bigwedge_{x\in f^{-1}(y)} A^{\notin}(x)) & \text{if } f^{-1}(y) \neq \phi \\ \bar{0} & \text{otherwise,} \end{cases}$$

$$f(\lambda)(y) = \begin{cases} \bigvee_{x\in f^{-1}(y)} \lambda(x) & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{otherwise.} \end{cases}$$

It is obvious that $f(x_{\langle \tilde{a}, \bar{b}, \alpha\rangle}) = [f(x)]_{\langle \tilde{a}, \bar{b}, \alpha\rangle}$, for each $x_{\langle \tilde{a}, \bar{b}, \alpha\rangle} \in \mathcal{O}_P(X)$.

Proposition 11. *Let $f : X \to Y$ be a groupoid homomorphism, let $\mathcal{B} = \langle \mathbf{B}, B, \mu\rangle \in \mathcal{O}(Y)$ and let $i = 1, 2, 3, 4$.*

(1) If $\mathcal{B} \in OGP_i(Y)$, then $f^{-1}(\mathcal{B}) \in OGP_i(X)$.

(2) If $\mathcal{B} \in OLI_i(Y)$ [resp., $ORI_i(Y)$ and $OI_i(Y)$], then $f^{-1}(\mathcal{B}) \in OLI_i(X)$ [resp., $ORI_i(X)$ and $OI_i(X)$].

Proof. (1) Case 1: Suppose $\mathcal{B} = \langle \mathbf{B}, B, \mu\rangle \in OGP_1(Y)$. Then clearly, Propositions 3.9 in [18], 4.1 in [19] and 4.1 (1) in [17], $f^{-1}(\mu) \in FGP(X)$, $f^{-1}(\mathbf{B}) \in IVGP(X)$ and $f^{-1}(B) \in IFGP(X)$. Thus, $f^{-1}(\mathcal{B}) \in OGP_1(X)$.

Case 2: Suppose $\mathcal{B} \in OGP_2(Y)$ and let $x, y \in X$. Then by Theorem 2 (iii),

$$\begin{aligned} f^{-1}(\mu)(xy) &= (\mu \circ f)(xy) = \mu(f(xy)) \\ &= \mu(f(x)f(y)) \text{ [Since } f \text{ is a groupoid homomorphism]} \\ &\leq \mu(f(x)) \vee \mu(f(y)) \\ &= f^{-1}(\mu)(x) \vee f^{-1}(\mu)(y). \end{aligned}$$

Thus, $f^{-1}(\mu)$ satisfies the condition (iii) of Theorem 2 (3). So by Case 1, $f^{-1}(\mathcal{B}) \in OGP_2(X)$.

Case 3: Suppose $\mathcal{B} \in OGP_3(Y)$ and let $x, y \in X$. Then by Theorem 3 (ii),

$$\begin{aligned} f^{-1}(B^{\in})(xy) &= (B^{\in} \circ f)(xy) = B^{\in}(f(xy)) \\ &= B^{\in}(f(x)f(y)) \\ &\leq B^{\in}(f(x)) \vee B^{\in}(f(y)) \\ &= f^{-1}(B^{\in})(x) \vee f^{-1}(B^{\in})(y). \end{aligned}$$

Similarly, we have $f^{-1}(B^{\notin})(xy) \geq f^{-1}(B^{\notin})(x) \wedge f^{-1}(B^{\notin})(y)$. Thus, $f^{-1}(B)$ satisfies the condition (ii) of Theorem 3 (3). So by Case 1, $f^{-1}(\mathcal{B}) \in OGP_3(X)$.

Case 4: Suppose $\mathcal{B} \in OGP_4(Y)$. Then by Cases 2 and 3, $f^{-1}(\mathcal{B}) \in OGP_4(X)$.

(2) We only prove that $f^{-1}(\mathcal{B}) \in OLI_i(X)$ and the other proofs are omitted.

Case 1: Suppose $\mathcal{B} \in OLI_1(Y)$ and let $x, y \in X$. Then

$$f^{-1}(\mathbf{B})(xy) = [B^{-}(f(x)f(y)), B^{+}(f(x)f(y))]$$

$\geq [B^{-}(f(y)), B^{+}(f(y))]$ [Since $\mathbf{B} \in IVLI(Y)$]
$= f^{-1}(\mathbf{B})(y)$.

Thus, $f^{-1}(\mathbf{B}) \in IVLI(X)$. Moreover, from Propositions 4.1 in [19] and 4.1 in [17], $f^{-1}(\mu) \in FLL(X)$ and $f^{-1}(B) \in IFLI(X)$. So $f^{-1}(\mathcal{B}) \in OLI_1(X)$.

Case 2: Suppose $\mathcal{B} \in OLI_2(Y)$ and let $x,\ y \in X$. Then by Remark 4 (1),

$$f^{-1}(\mu)(xy) = \lambda(f(xy)) = \mu(f(x)f(y)) \leq \mu(f(y)) = f^{-1}(\mu)(y).$$

Thus, $f^{-1}(\mu)(xy) \leq f^{-1}(\mu)(y)$. So by Case 1, $f^{-1}(\mathcal{B}) \in OLI_2(X)$.

Case 3: Suppose $\mathcal{B} \in OLI_3(Y)$ and let $x,\ y \in X$. Then by Remark 4 (1),

$$f^{-1}(B^{\in})(xy) = B^{\in}(f(xy)) = B^{\in}(f(x)f(y)) \leq B^{\in}(f(y)) = f^{-1}(B^{\in})(y).$$

Similarly, we have $f^{-1}(B^{\notin})(xy) \geq f^{-1}(B^{\notin})(y)$. Thus, by Case 1, $f^{-1}(\mathcal{B}) \in OLI_3(X)$.

Case 4: Suppose $\mathcal{B} \in OLI_3(Y)$. Then by Cases 2 and 3, $f^{-1}(\mathcal{B}) \in OLI_4(X)$. This completes the proof. □

Definition 17. *Let X be a nonempty set and let $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle \in \mathcal{O}(X)$. Then we say that $\mathcal{A}$ has the i-sup-property ($i = 1,\ 2,\ 3,\ 4$), if for each $T \in 2^X$, there is $t_0 \in T$ such that*

$$\mathcal{A}(t_0) = \bigvee_{t \in T}^{i} \mathcal{A}(t).$$

It is obvious that if $\mathcal{A}$ takes on only finitely many values, then it has the i-sup-property. In particular, $\mathcal{A} \in \mathcal{O}(X)$ has the 1-sup-property if and only if $\mathbf{A}$, A and λ have the sup-property.

Proposition 12. *Let $f : X \to Y$ be a groupoid homomorphism, let $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle \in \mathcal{O}(X)$ has the i-sup-property and let $i = 1,\ 2,\ 3,\ 4$.*

(1) If $\mathcal{A} \in OGP_i(X)$, then $f(\mathcal{A}) \in OGP_i(Y)$.

(2) If $\mathcal{A} \in OLI_i(X)$ [resp., $ORI_i(X)$ and $OI_i(X)$], then $f(\mathcal{A}) \in OLI_i(Y)$ [resp., $ORI_i(Y)$ and $OI_i(Y)$].

Proof. (1) Case 1: Suppose $\mathcal{A} \in OGP_1(X)$. Then from Propositions 3.11 in [18], 4.2 in [19] and 4.4 (1) in [17],

$$f(\mathbf{A}) \in IVGP(X),\ f(\lambda) \in FGP(X),\ f(A) \in IFGP(X).$$

Thus, by Remark 2 (1), $f(\mathcal{A}) \in OGP_1(Y)$.

Case 2: Suppose $\mathcal{A} \in OGP_2(X)$. Since $f(\mathbf{A}) \in IVGP(X)$, $f(A) \in IFGP(X)$ by Case 1, it is sufficient to show that $f(\lambda)$ satisfies the condition (iii) of Theorem 2. Let $y,\ y' \in Y$. Then we can consider four cases:

(i) $f^{-1}(y) \neq \emptyset,\ f^{-1}(y') \neq \emptyset$, (ii) $f^{-1}(y) \neq \emptyset,\ f^{-1}(y') = \emptyset$,
(iii) $f^{-1}(y) = \emptyset,\ f^{-1}(y') \neq \emptyset$, (iv) $f^{-1}(y) = \emptyset,\ f^{-1}(y') = \emptyset$.

We prove only case (i) and omit the other ones. Since $\mathcal{A}$ has the 2-sup-property, there are $x_0,\ x_0' \in X$ such that $\lambda(x_0) = \bigwedge_{t \in f^{-1}(y)} \lambda(t)$, $\lambda(x_0') = \bigwedge_{t' \in f^{-1}(y')} \lambda(t')$. Then

$f(\lambda)(yy') = \bigwedge_{z \in f^{-1}(yy')} \lambda(z)$
$\leq \lambda(x_0 x_0')$ [Since $f(x_0 x_0') = f(x_0)f(x_0') = yy'$.]
$\leq \lambda(x_0) \vee \lambda(x_0')$ [Since by the hypothesis and Theorem 2 (iii)]
$= (\bigwedge_{t \in f^{-1}(y)} \lambda(t)) \vee (\bigwedge_{t' \in f^{-1}(y')} \lambda(t'))$
$= f(\lambda)(y) \vee f(\lambda)(y')$.

Thus, by Theorem 2, $f(\mathcal{A}) \in OGP_2(Y)$.

Case 3: Suppose $\mathcal{A} \in OGP_3(X)$. Since $f(\mathbf{A}) \in IVGP(X)$, $f(\lambda) \in FGP(X)$ by Case 1, it is sufficient to show that $f(A)$ satisfies the condition (ii) of Theorem 3. Let y, $y^{'} \in Y$ and we show only case (i) of Case 2. Since $\mathcal{A}$ has the 3-sup-property, there are x_0, $x_0^{'} \in X$ such that

$$A^{\in}(x_0) = \bigwedge_{t \in f^{-1}(y)} A^{\in}(t),\ A^{\in}(x_0^{'}) = \bigwedge_{t^{'} \in f^{-1}(y^{'})} A^{\in}(t^{'})$$

and

$$A^{\notin}(x_0) = \bigvee_{t \in f^{-1}(y)} A^{\notin}(t),\ A^{\notin}(x_0^{'}) = \bigvee_{t^{'} \in f^{-1}(y^{'})} A^{\notin}(t^{'}).$$

Then

$$\begin{aligned}
[f(A)]^{\in}(yy^{'}) &= \textstyle\bigwedge_{z \in f^{-1}(yy^{'})} A^{\in}(z) \\
&\leq A^{\in}(x_0 x_0^{'}) \\
&\leq A^{\in}(x_0) \vee A^{\in}(x_0^{'}) \\
&\quad \text{[Since by the hypothesis and Theorem 3 (ii)]} \\
&= (\textstyle\bigwedge_{t \in f^{-1}(y)} A^{\in}(t)) \vee (\bigwedge_{t^{'} \in f^{-1}(y^{'})} A^{\in}(t^{'})) \\
&= [f(A)]^{\in}(y) \vee [f(A)]^{\in}(y^{'}).
\end{aligned}$$

Similarly, we have $[f(A)]^{\notin}(yy^{'}) \geq [f(A)]^{\notin}(y) \wedge [f(A)]^{\notin}(y^{'})$. Thus, by Theorem 3, $f(\mathcal{A}) \in OGP_3(Y)$.

Case 4: Suppose $\mathcal{A} \in OGP_4(X)$. Then by Cases 2, 3 and Theorem 4, we can easily prove that $f(\mathcal{A}) \in OGP_4(Y)$.

(2) We prove only that $f(\mathcal{A}) \in OLI_i(Y)$.

Case 1: Suppose $\mathcal{A} \in OLI_1(X)$. From Propositions 4.2 in [19] and 4.4 (2) in [17], $f(\lambda) \in FLI(Y)$ and $f(B) \in IFLI(Y)$. Then it is sufficient to show that $f(\mathbf{A}) \in IVLI(Y)$. Let y, $y^{'} \in X$ and we prove only case (i) of Case 2 in (1). Since $\mathcal{A}$ has the 1-sup-property, there are $x_0 \in f^{-1}(y)$ and $x_0^{'} \in f^{-1}(y^{'})$ such that

$$\mathcal{A}(x_0) = \bigvee_{t \in f^{-1}(y)} \mathcal{A}(x) \text{ and } \mathcal{A}(x_0^{'}) = \bigvee_{t^{'} \in f^{-1}(y^{'})} \mathcal{A}(t^{'}).$$

Then

$$\begin{aligned}
f(\mathbf{A})^{-}(yy^{'}) &= \textstyle\bigvee_{z \in f^{-1}(yy^{'})} A^{-}(z) \\
&\geq A^{-}(x_0 x_0^{'}) \text{ [Since } f(x_0 x_0^{'}) = f(x_0) f(x_0^{'}) = yy^{'}] \\
&\geq A^{-}(x_0^{'}) \text{ [Since } \mathbf{A} \in IVLI(X)] \\
&= \textstyle\bigvee_{t^{'} \in f^{-1}(y^{'})} A^{-}(t^{'}) \\
&= f(\mathbf{A})^{-}(y^{'}).
\end{aligned}$$

Similarly, we have $f(\mathbf{A})^{+}(yy^{'}) \geq f(\mathbf{A})^{+}(y^{'})$. Thus, $f(\mathbf{A})(yy^{'}) \geq f(\mathbf{A})(y^{'})$. So $f(\mathbf{A}) \in IVLI(Y)$. Hence by Remark 4 (1), $f(\mathcal{A}) \in OLI_1(Y)$.

Case 2: Suppose $\mathcal{A} \in OLI_2(X)$ and let y, $y^{'} \in Y$. Since $f(\mathbf{A}) \in IVLI(Y)$, $f(A) \in IFLI(Y)$ by Case (1), it is sufficient to prove that $f(\lambda)(yy^{'}) \leq f(\lambda)(y^{'})$. Since $\mathcal{A}$ has the 2-sup-property, there are x_0, $x_0^{'} \in X$ such that

$$\lambda(x_0) = \bigwedge_{t \in f^{-1}(y)} \lambda(t),\ \lambda(x_0^{'}) = \bigwedge_{t^{'} \in f^{-1}(y^{'})} \lambda(t^{'}).$$

Then

$$\begin{aligned}
f(\lambda)(yy^{'}) &= \textstyle\bigwedge_{z \in f^{-1}(yy^{'})} \lambda(z) \\
&\leq \lambda(x_0 x_0^{'}) \\
&\leq \lambda(x_0^{'}) \text{ [Since by the hypothesis and Remark 4 (1)]} \\
&= \textstyle\bigwedge_{t^{'} \in f^{-1}(y^{'})} \lambda(t^{'}) \\
&= f(\lambda)(y^{'}).
\end{aligned}$$

Thus, by Remark 4 (1), $f(\mathcal{A}) \in OLI_2(Y)$.

Case 3: Suppose $\mathcal{A} \in OLI_3(X)$ and let $y,\ y^{'} \in Y$. Since $f(\mathbf{A}) \in IVLI(Y),\ f(\lambda) \in FLI(Y)$ by Case (1), it is sufficient to prove that $[f(A)]^{\in}(yy^{'}) \leq [f(A)]^{\in}(y^{'})$ and $[f(A)]^{\notin}(yy^{'}) \geq [f(A)]^{\notin}(y^{'})$. Since $\mathcal{A}$ has the 3-sup-property, there are $x_0,\ x_0^{'} \in X$ such that

$$A^{\in}(x_0) = \bigwedge_{t \in f^{-1}(y)} A^{\in}(t),\ A^{\in}(x_0^{'}) = \bigwedge_{t^{'} \in f^{-1}(y^{'})} A^{\in}(t^{'})$$

and

$$A^{\notin}(x_0) = \bigvee_{t \in f^{-1}(y)} A^{\notin}(t),\ A^{\notin}(x_0^{'}) = \bigvee_{t^{'} \in f^{-1}(y^{'})} A^{\notin}(t^{'}).$$

Then

$[f(A)]^{\in}(yy^{'}) = \bigwedge_{z \in f^{-1}(yy^{'})} A^{\in}(z)$
$\leq A^{\in}(x_0x_0^{'})$
$\leq A^{\in}(x_0^{'})$
[Since by the hypothesis and Remark 4 (1)]
$= \bigwedge_{t^{'} \in f^{-1}(y^{'})} A^{\in}(t^{'})$
$= [f(A)]^{\in}(y^{'})$.

Similarly, we have $[f(A)]^{\notin}(yy^{'}) \geq [f(A)]^{\notin}(y^{'})$. Thus, by Remark 4 (1), $f(\mathcal{A}) \in OLI_3(Y)$.

Case 4: Suppose $\mathcal{A} \in OLI_4(X)$. Then by Cases 2 and 3, we can easily show that $f(\mathcal{A}) \in OLI_4(Y)$. This completes the proof. □

Definition 18. *Let $X,\ Y$ be sets, $f : X \to Y$ be a mapping and let $\mathcal{A} \in \mathcal{O}(X)$. Then $\mathcal{A}$ is said to be f-invariant, if for every $x,\ y \in X$, $f(x) = f(y)$ implies $\mathcal{A}(x) = \mathcal{A}(y)$.*

It is obvious that $\mathcal{A}$ is f-invariant if and only if $\mathbf{A}$, A and λ are f-invariant. Moreover, we can easily see that if $\mathcal{A}$ is f-invariant, then $f^{-1}(f(\mathcal{A})) = \mathcal{A}$.

The following is the immediate result of Definition 18.

Proposition 13. *Let $X,\ Y$ be sets, let $f : X \to Y$ be a mapping and let*

$$\Omega = \{\mathcal{A} \in \mathcal{O}(X) : \mathcal{A}\ is\ f - invariant\}.$$

Then there is a one-to-one correspondence between Ω and $\mathcal{O}(Imf)$, where Imf denotes the image of f.

The following is the immediate result of Propositions 12 (1) and 13.

Proposition 14. *Let $f : X \to Y$ be a groupoid homomorphism and let*

$$\Phi = \{\mathcal{A} \in OGP_i(X) : \mathcal{A}\ is\ f - invariant\ and\ has\ the\ i - sup\ property\},$$

where $i = 1,\ 2,\ 3,\ 4$. Then there is a one-to-one correspondence between Φ and $OGP_i(Imf)$.

4. Octahedron Subgroups

Unless stated otherwise in this section, G denotes a group and e is the identity of G.

Definition 19 ([19]). *Let $\lambda \in FGP(G)$. Then λ is called a fuzzy subgroup of G, if it satisfies the following condition:*

$$\lambda(x^{-1}) \geq \lambda(x),\ \text{for each}\ x \in G.$$

We will denote the set of fuzzy subgroups of G as $FG(G)$.

Definition 20 ([20]). *Let $A \in IFGP(G)$. Then A is called an intuitionistic fuzzy subgroup (briefly, IFG) of G, if it satisfies the following condition: for each $x \in G$,*

$$A(x^{-1}) \geq A(x), \text{ i.e., } A^{\in}(x^{-1}) \geq A^{\in}(x),\ A^{\notin}(x^{-1}) \leq A^{\notin}(x).$$

We will denote the set of IFGs of G as $IFG(G)$.

Definition 21 ([18,21]). *Let $\mathbf{A} \in IVFGP(X)$. Then $\mathbf{A}$ is called an interval-valued fuzzy subgroup (briefly, IVG) of G, if it satisfies the following condition: for each $x \in G$,*

$$\mathbf{A}(x^{-1}) \geq A(x), \text{ i.e., } A^{-}(x^{-1}) \geq A^{-}(x),\ A^{+}(x^{-1}) \geq A^{+}(x).$$

We will denote the set of IVGs of G as $IVG(G)$.

Definition 22. *Let $\mathcal{A} \in OGP_i(G)$ $(i = 1,\ 2,\ 3,\ 4)$. Then $\mathcal{A}$ is called a i-octahedron subgroup of G, if it satisfies the following condition: for each $x \in G$,*

$$\mathcal{A}(x^{-1}) \geq_i \mathcal{A}(x).$$

We will denote the set of all i-octahedron subgroups of G as $OG_i(G)$.
In particular, if $\mathcal{A} \in OGP_1(G)$, then $\mathcal{A}$ will simply called an octahedron subgroup of G.

From Theorems 1–4 and Definition 22, we obtain easily the characterizations of i-octahedron subgroups of G.

Theorem 6. *Let $\mathcal{A} \in \mathcal{O}(G)$.*
(1) $\mathcal{A} \in OG_1(G) \iff \mathbf{A} \in IVG(G),\ A \in IFG(G),\ \lambda \in FG(G)$.
(2) $\mathcal{A} \in OG_2(G) \iff \mathbf{A} \in IVG(G),\ A \in IFG(G)$ *and λ satisfies the following conditions: for every $x,\ y \in G$,*

$$\lambda(xy) \leq \lambda(x) \vee \lambda(y),\ \lambda(x^{-1}) \leq \lambda(x).$$

(3) $\mathcal{A} \in OG_3(G) \iff \mathbf{A} \in IVG(G),\ \lambda \in FG(G)$ *and A satisfies the following conditions: for every $x,\ y \in G$,*
(i) $A^{\in}(xy) \leq A^{\in}(x) \vee A^{\in}(y),\ A^{\notin}(xy) \geq A^{\notin}(x) \wedge A^{\notin}(y)$,
(ii) $A^{\in}(x^{-1}) \leq A^{\in}(x),\ A^{\notin}(x^{-1}) \geq A^{\notin}(x)$.
(4) $\mathcal{A} \in OG_4(G) \iff \mathbf{A} \in IVG(G)$, *$A$ and λ satisfies the following conditions: for every $x,\ y \in G$,*
(i) $\lambda(xy) \leq \lambda(x) \vee \lambda(y),\ \lambda(x^{-1}) \leq \lambda(x)$,
(ii) $A^{\in}(xy) \leq A^{\in}(x) \vee A^{\in}(y),\ A^{\notin}(xy) \geq A^{\notin}(x) \wedge A^{\notin}(y)$,
(iii) $A^{\in}(x^{-1}) \leq A^{\in}(x),\ A^{\notin}(x^{-1}) \geq A^{\notin}(x)$.

Example 5. *(1) Consider the additive group $(\mathbb{Z}, +)$. We define five mappings $\mathbf{A} = [A^{-}, A^{+}] : \mathbb{Z} \to [I]$, $A = (A^{\in}, A^{\notin}),\ B = (B^{\in}, B^{\notin}) : \mathbb{Z} \to I \oplus I$ and $\lambda,\ \mu : \mathbb{Z} \to I$, respectively as follows: for each $0 \neq n \in \mathbb{Z}$,*

$$\mathbf{A}(0) = [1,1],\ A(0) = (1,0),\ B(0) = (0,1),\ \lambda(0) = 1,\ \mu(0) = \frac{1}{6}, \tag{1}$$

$$\mathbf{A}(n) = \begin{cases} \left[\frac{1}{2}, \frac{2}{3}\right] & \textit{if } n \text{ is odd} \\ \left[\frac{1}{3}, \frac{4}{5}\right] & \textit{if } n \text{ is even,} \end{cases} \tag{2}$$

$$A(n) = \begin{cases} \left(\frac{1}{2}, \frac{1}{3}\right) & \text{if } n \text{ is odd} \\ \\ \left(\frac{2}{3}, \frac{1}{5}\right) & \text{if } n \text{ is even,} \end{cases} \tag{3}$$

$$B(n) = \begin{cases} \left(\frac{2}{3}, \frac{1}{5}\right) & \text{if } n \text{ is odd} \\ \\ \left(\frac{1}{2}, \frac{1}{3}\right) & \text{if } n \text{ is even,} \end{cases} \tag{4}$$

$$\lambda(n) = \begin{cases} \frac{1}{2} & \text{if } n \text{ is odd} \\ \\ \frac{3}{5} & \text{if } n \text{ is even,} \end{cases} \tag{5}$$

$$\mu(n) = \begin{cases} \frac{3}{5} & \text{if } n \text{ is odd} \\ \\ \frac{1}{2} & \text{if } n \text{ is even.} \end{cases} \tag{6}$$

Then we can easily check that $\lambda \in FG(\mathbb{Z})$. Moreover, $\mathbf{A} \in IVG(\mathbb{Z})$ and $A \in IFG(\mathbb{Z})$ from Example 4.1 in [18] and Example 2.1 in [20]. Thus, $\langle \mathbf{A}, A, \lambda \rangle$ is an octahedron subgroup of $\mathbb{Z}$.

On the other hand, we can easily check that μ and B satisfy the conditions of Theorem 6 (2) and (i) and (ii) of Theorem 6 (3), respectively. Hence by Theorem 6 (2) and (3), $\langle \mathbf{A}, A, \mu \rangle \in OG_2(X)$ and $\langle \mathbf{A}, B, \lambda \rangle \in OG_3(X)$. Thus, by Theorem 6 (2), $\mathcal{A} = \langle \mathbf{A}, A, \mu \rangle \in OG_2(X)$. Furthermore, from Theorem 6 (4), we can easily see that $\langle \mathbf{A}, B, \mu \rangle \in OG_4(X)$.

(2) If $\mathbf{A} \in IVGP(G)$, then $\mathcal{O}_{\mathbf{A}} \in OG_1(G)$. Also if $A \in IFGP(G)$, then $\mathcal{O}_A \in OG_1(G)$.

(3) If $\mathcal{A} \in OG_i(X)$, then clearly, $[\,]\mathcal{A}$, $\diamond \mathcal{A} \in OG_i(X)$ ($i = 1, 2, 3, 4$).

Remark 8. *(1) If $\lambda \in FG(G)$, then we have*

$$\langle [\lambda, \lambda], (\lambda, \lambda^c), \lambda \rangle \in OG_1(G), \ \langle [\lambda, \lambda], (\lambda, \lambda^c), \lambda^c \rangle \in OG_2(G),$$

$$\left\langle [\lambda, \lambda], (\lambda^C, \lambda), \lambda \right\rangle \in OG_3(G), \ \left\langle [\lambda, \lambda], (\lambda^C, \lambda), \lambda^c \right\rangle \in OG_4(G).$$

(2) If $\mathbf{A} \in IVG(G)$, then we have

$$\left\langle \mathbf{A}, (A^-, (A^+)^c), A^- \right\rangle, \ \left\langle \mathbf{A}, (A^-, (A^+)^c), A^+ \right\rangle \in OG_1(G),$$

$$\left\langle \mathbf{A}, (A^+, (A^-)^c), A^+ \right\rangle, \ \left\langle \mathbf{A}, (A^+, (A^-)^c), A^- \right\rangle \in OG_1(G),$$

$$\left\langle \mathbf{A}, (A^-, (A^+)^c), (A^-)^c \right\rangle, \ \left\langle \mathbf{A}, (A^-, (A^+)^c), (A^+)^c \right\rangle \in OG_2(G),$$

$$\left\langle \mathbf{A}, (A^+, (A^-)^c), (A^+)^C \right\rangle, \ \left\langle \mathbf{A}, (A^+, (A^-)^c), (A^-)^c \right\rangle \in OG_2(G),$$

$$\left\langle \mathbf{A}, ((A^-)^c, A^+), A^- \right\rangle, \ \left\langle \mathbf{A}, ((A^-)^c, A^+), A^+ \right\rangle \in OG_3(G),$$

$$\left\langle \mathbf{A}, ((A^+)^c, A^-), A^+ \right\rangle, \ \left\langle \mathbf{A}, ((A^+)^c, A^-), A^- \right\rangle \in OG_3(G),$$

$$\left\langle \mathbf{A}, ((A^-)^c, A^+), (A^-)^c \right\rangle, \ \left\langle \mathbf{A}, ((A^-)^c, A^+), (A^+)^c \right\rangle \in OG_4(G),$$

$$\left\langle \mathbf{A}, ((A^+)^c, A^-), (A^+)^c \right\rangle, \ \left\langle \mathbf{A}, ((A^+)^c, A^-), (A^-)^c \right\rangle \in OG_4(G).$$

(3) If $A \in IFG(G)$, then we have

$$\left\langle [A^{\in}, (A^{\notin})^c], A, A^{\in} \right\rangle \in OG_1(G), \ \left\langle [A^{\in}, (A^{\notin})^c], A, A^{\notin} \right\rangle \in OG_2(G),$$

$$\left\langle [A^{\in}, (A^{\notin})^c], A^c, A^{\in} \right\rangle \in OG_3(G), \ \left\langle [A^{\in}, (A^{\notin})^c], A^c, A^{\notin} \right\rangle \in OG_4(G).$$

The following is an immediate result of Theorem 6 (1) and Remark 3.

Proposition 15. *For every $H \subset G$, H is a subgroup of G if and only if $\chi_{\mathcal{H}} \in OG_1(G)$.*

The following is an immediate result of Theorem 6 (1) and Proposition 8.

Proposition 16. *If $(\mathcal{A}_j)_{j\in J} = (\langle \mathbf{A}_j, A_j, \lambda_j \rangle)_{j\in J} \subset OG_1(G)$, then $\bigcap^1_{j\in J} \mathcal{A}_j \in OG_1(G)$, where J denotes an index set.*

The following is an immediate result of Proposition 16 and Corollary 1.

Corollary 2. *Let $\mathcal{A} \in \mathcal{O}(G)$ and let*

$$(\mathcal{A}) = \bigcap^1 \{\mathcal{B} \in OG_1(G) : \mathcal{A} \subset_1 \mathcal{B}\}.$$

Then $(\mathcal{A}) \in OG_1(G)$.
In this case, $(\mathcal{A})$ is called the octahedron subgroup of G generated by $\mathcal{A}$.

The following is an immediate result of Proposition 9 and Corollary 2.

Corollary 3. *For each $A \in 2^X$, let (A) be the subgroup generated by A and let $\chi_{(A)} = \left\langle [\chi_{(A)}, \chi_{(A)}], (\chi_{(A)}, \chi_{(A^c)}), \chi_{(A)} \right\rangle$. Then*

$$(\chi_{\mathcal{A}}) = \chi_{(A)}.$$

Proposition 17. *Let $\mathcal{A} \in OG_i(G)$ $(i = 1,\ 2,\ 3,\ 4)$. Then for each $x \in G$,*

$$\mathcal{A}(x^{-1}) = \mathcal{A}(x),\ \mathcal{A}(e) \geq_i \mathcal{A}(x).$$

Proof. Case 1: Suppose $\mathcal{A} \in OG_1(G)$. Then by Theorem 6 (1), we have

$$\mathbf{A} \in IVG(G),\ A \in IFG(G),\ \lambda \in FG(G).$$

Thus, by Propositions 3.1 in [21], 2.6 in [20] and 5.4 in [19], we have

$$\mathbf{A}(x^{-1}) = \mathbf{A}(x),\ A(x^{-1}) = A(x),\ \lambda(x^{-1}) = \lambda(x),$$

$$\mathbf{A}(e) \geq \mathbf{A}(x),\ A(e) \geq A(x),\ \lambda(e) \geq \lambda(x).$$

So $\mathcal{A}(x^{-1}) = \mathcal{A}(x)$ and $\mathcal{A}(e) \geq_1 \mathcal{A}(x)$.
Case 2: Suppose $\mathcal{A} \in OG_2(G)$. Then by Theorem 6 (2), we have

$$\mathbf{A} \in IVG(G),\ A \in IFG(G)$$

and

$$\lambda(xy) \leq \lambda(x) \vee \lambda(y),\ \lambda(x^{-1}) \leq \lambda(x),\ \text{for every } x,\ y \in G.$$

Thus, $\lambda(x) = \lambda((x^{-1})^{-1}) \leq \lambda(x^{-1}) \leq \lambda(x)$, i.e., $\lambda(x^{-1}) = \lambda(x)$.
On the other hand, $\lambda(e) = \lambda(xx^{-1}) \leq \lambda(x) \vee \lambda(x^{-1}) = \lambda(x)$. By Case 1, we have

$$\mathbf{A}(x^{-1}) = \mathbf{A}(x),\ A(x^{-1}) = A(x) \text{ and } \mathbf{A}(e) \geq \mathbf{A}(x),\ A(e) \geq A(x).$$

So $\mathcal{A}(x^{-1}) = \mathcal{A}(x)$ and $\mathcal{A}(e) \geq_2 \mathcal{A}(x)$.

Case 3: Suppose $\mathcal{A} \in OG_2(G)$. Then by Case 1 and Theorem 6 (3), we have

$$\mathbf{A}(x^{-1}) = \mathbf{A}(x),\ \lambda(x^{-1}) = \lambda(x),\ \mathbf{A}(e) \geq \mathbf{A}(x),\ \lambda(e) \geq \lambda(x)$$

and A satisfies the conditions (i) and (ii). By (ii),

$$A^{\in}(x) = A^{\in}((x^{-1})^{-1}) \leq A^{\in}(x^{-1}) \leq A^{\in}(x), \text{ i.e., } A^{\in}(x) = A^{\in}(x^{-1}).$$

Similarly, we have $A^{\notin}(x) = A^{\notin}(x^{-1})$. Thus, $A(x) = A(x^{-1})$. By (i), we can easily prove that $A(e) \leq A(x)$.. So $\mathcal{A}(x^{-1}) = \mathcal{A}(x)$ and $\mathcal{A}(e) \geq_3 \mathcal{A}(x)$.

Case 4: Suppose $\mathcal{A} \in OG_2(G)$. Then by Cases 1 and 2, we have $\mathcal{A}(x^{-1}) = \mathcal{A}(x)$ and $\mathcal{A}(e) \geq_4 \mathcal{A}(x)$. This completes the proof. □

Theorem 7. *Let $\mathcal{A} \in \mathcal{O}(G)$ and let $i = 1, 2, 3, 4$. Then $\mathcal{A} \in OG_i(G)$ if and only if $\mathcal{A}(xy^{-1}) \geq_i \mathcal{A}(x) \wedge_i \mathcal{A}(x)(y)$,* for every $x, y \in G$.

Proof. We prove only the necessity of the condition.

Case 1: Suppose $\mathcal{A} \in OG_1(G)$ and let $x, y \in G$. Then by Theorem 6 (1), we have

$$\mathbf{A} \in IVG(G),\ A \in IFG(G),\ \lambda \in FG(G).$$

Thus, by Propositions 3.2 in [21] and 5.6 in [19], we have

$$\mathbf{A}(xy^{-1}) = \mathbf{A}(x) \wedge \mathbf{A}(y),\ \lambda(xy^{-1}) = \lambda(x) \wedge \lambda(y).$$

On the other hand, by Proposition 17,

$$A^{\in}(xy^{-1}) \geq A^{\in}(x) \wedge A^{\in}(y^{-1}) = A^{\in}(x) \wedge A^{\in}(y),$$

$$A^{\notin}(xy^{-1}) \leq A^{\notin}(x) \vee A^{\notin}(y^{-1}) = A^{\notin}(x) \vee A^{\notin}(y).$$

So $\mathcal{A}(xy^{-1}) \geq_1 \mathcal{A}(x) \wedge^1 \mathcal{A}(y)$.

Case 2: Suppose $\mathcal{A} \in OG_2(G)$ and let $x, y \in G$. Then by Theorem 6 (2) and Case 1, we have

$$\mathbf{A}(xy^{-1}) = \mathbf{A}(x) \wedge \mathbf{A}(y),\ A(xy^{-1}) = A(x) \wedge A(y)$$

and

$$\lambda(xy) \leq \lambda(x) \vee \lambda(y),\ \lambda(x^{-1}) \leq \lambda(x).$$

Thus, by Proposition 17,

$$\lambda(xy^{-1}) \leq \lambda(x) \vee \lambda(y^{-1}) = \lambda(x) \vee \lambda(y).$$

So $\mathcal{A}(xy^{-1}) \geq_2 \mathcal{A}(x) \wedge^2 \mathcal{A}(y)$.

Case 3: Suppose $\mathcal{A} \in OG_2(G)$ and let $x, y \in G$. Then by Case 1 and Theorem 6 (3), we have

$$\mathbf{A}(xy^{-1}) \geq \mathbf{A}(x) \wedge \mathbf{A}(x),\ \lambda(xy^{-1}) \geq \lambda(x) \wedge \lambda(y)$$

and A satisfies the conditions (i) and (ii). Thus, by (i) and Proposition 17,

$$A^{\in}(xy^{-1}) \leq A^{\in}(x) \vee A^{\in}(y^{-1}) = A^{\in}(x) \vee A^{\in}(y),$$

$$A^{\notin}(xy^{-1}) \geq A^{\notin}(x) \wedge A^{\in}(y^{-1}) = A^{\in}(x) \vee A^{\in}(y).$$

So $\mathcal{A}(xy^{-1}) \geq_3 \mathcal{A}(x) \wedge^3 \mathcal{A}(y)$.

Case 4: Suppose $\mathcal{A} \in OG_2(G)$. Then by Cases 1 and 2, we can easily prove that $\mathcal{A}(xy^{-1}) \geq_4 \mathcal{A}(x) \wedge^4 \mathcal{A}(y)$. This completes the proof. $\square$

The following is an immediate consequence of Corollary in [19], Propositions 4.6 in [18] and 2.7 in [20].

Proposition 18. *If $\mathcal{A} \in OG_1(G)$. Then $G_{\mathcal{A}} = \{x \in G : \mathcal{A}(x) = \mathcal{A}(e)\}$ is a subgroup of G.*

Proposition 19. *Let $\mathcal{A} \in OG_1(G)$ $(i = 1,\ 2,\ 3,\ 4)$ and let $x,\ y \in G$. If $\mathcal{A}(xy^{-1}) = \mathcal{A}(e)$, then $\mathcal{A}(x) = \mathcal{A}(y)$.*

Proof. (1) Case 1: Suppose $\mathcal{A} \in OG_1(G)$ and let $x,\ y \in G$. Then by Theorem 6 (1), we have

$$\mathbf{A} \in IVG(G),\ A \in IFG(G),\ \lambda \in FG(G).$$

Thus, by Propositions 4.7 in [18], 2.8 in [20] and 5.5 in [19], we have

$$\mathbf{A}(x) = \mathbf{A}(y),\ A(x) = A(y)\ \lambda(x) = \lambda(y).$$

So $\mathcal{A}(x) = \mathcal{A}(y)$.

Case 2: Suppose $\mathcal{A} \in OG_2(G)$ and let $x,\ y \in G$. Then by Theorem 6 (2) and Case 1, we have

$$\mathbf{A}(x) = \mathbf{A}(y),\ A(x) = A(y)$$

and

$$\lambda(xy) \leq \lambda(x) \vee \lambda(y),\ \lambda(x^{-1}) \leq \lambda(x).$$

By Proposition 17,

$$\lambda(x) = \lambda((xy^{-1})y) \leq \lambda(xy^{-1}) \vee \lambda(y) = \lambda(e) \vee \lambda(y) = \lambda(y)$$
$$= \lambda((yx^{-1})x) \leq \lambda(e) \vee \lambda(x) = \lambda(x).$$

Thus, $\lambda(x) = \lambda(y)$. So $\mathcal{A}(x) = \mathcal{A}(y)$.

Case 3: Suppose $\mathcal{A} \in OG_2(G)$ and let $x,\ y \in G$. Then by Case 1 and Theorem 6 (3), we have $\mathbf{A}(x) = \mathbf{A}(y)$, $\lambda(x) = \lambda(y)$ and A satisfies the conditions (i) and (ii). By (i) and Proposition 17,

$$A^{\in}(x) = A^{\in}((xy^{-1})y) \leq A^{\in}(e) \vee A^{\in}(y) = A^{\in}(y)$$
$$= A^{\in}((yx^{-1})x) \leq A^{\in}(e) \vee A^{\in}(x) = A^{\in}(x).$$

Thus, $A^{\in}(x) = A^{\in}(y)$. Similarly, we have $A^{\notin}(x) = A^{\notin}(y)$.

$$A^{\notin}(xy^{-1}) \geq A^{\notin}(x) \wedge A^{\in}(y^{-1}) = A^{\in}(x) \vee A^{\in}(y).$$

So $\mathcal{A}(x) = \mathcal{A}(y)$.

Case 4: Suppose $\mathcal{A} \in OG_2(G)$. Then by Cases 1 and 2, we can easily prove that $\mathcal{A}(x) = \mathcal{A}(y)$. This completes the proof. $\square$

The following is an immediate consequence of Corollary in [19], Corollaries 4.7-1 in [18] and 2.8-1 in [20].

Corollary 4. *Let $\mathcal{A} \in OG_1(G)$. If $G_{\mathcal{A}}$ is a normal subgroup of G, then $\mathcal{A}$ is constant on each coset of $G_{\mathcal{A}}$.*

The following is an immediate result of Corollary in [19], Corollaries 4.7-2 in [18] and 2.8-2 in [20].

Corollary 5. *Let $\mathcal{A} \in OG_1(G)$ and let $G_{\mathcal{A}}$ be a normal subgroup of G. If $G_{\mathcal{A}}$ has a finite index, then $\mathcal{A}$ has the sup property.*

The following is an immediate result of Propositions 5.7 in [19], 4.8 in [18] and 2.10 in [20].

Proposition 20. *A group G cannot be the* 1*-union of two proper* 1*-octahedron subgroups of G.*

Theorem 8. *$\mathcal{A} \in OLI_i(G)$ [resp., $\mathcal{A} \in ORI_i(G)$ and $\mathcal{A} \in OI_i(G)$] if and only if $\mathcal{A}$ is a constant mapping ($i = 1, 2, 3, 4$).*

Proof. We prove only that $\mathcal{A} \in OLI_i(G)$ if and only if $\mathcal{A}$ is a constant mapping. Suppose $\mathcal{A}$ is a constant mapping. Then we can easily show that $\mathcal{A} \in OLI_i(G)$. Thus, it is sufficient to prove only that the necessary condition holds.

Case 1: Suppose $\mathcal{A} \in OLI_1(G)$. Then by Propositions 4.14 in [18], 2.16 in [20] and 5.9 in [19], $\mathbf{A}$, A and λ are constant mappings. Thus, $\mathcal{A}$ is a constant mapping.

Case 2: Suppose $\mathcal{A} \in OLI_2(G)$. Since $\mathbf{A}$ and A are constant mappings by Case 1, it is enough to show that λ is a constant mapping. Let $x, y \in G$. Since $\mathcal{A} \in OLI_2(G)$, by Remark 4 (1), we have $\lambda(xy) \leq \lambda(x) \vee \lambda(y)$. Let $y = e$. Then by Proposition 17, we have

$$\lambda(x) \leq \lambda(x) \vee \lambda(e) = \lambda(e), \text{ for each } x \in G.$$

Now let $x = y^{-1}$. Then by Proposition 17, we have

$$\lambda(e) \leq \lambda(y^{-1}) \vee \lambda(y) = \lambda(y), \text{ for each } y \in G.$$

Thus, $\lambda(x) = \lambda(y) = \lambda(e)$. So λ is a constant mapping. Hence $\mathcal{A}$ is a constant mapping.

Case 3: Suppose $\mathcal{A} \in OLI_3(G)$. Then by Remark 4 (1) and Proposition 17, we can easily see that A is a constant mapping. Thus, $\mathcal{A}$ is a constant mapping.

Case 4: Suppose $\mathcal{A} \in OLI_4(G)$. Then by Cases (2) and (3), we can easily prove that $\mathcal{A}$ is a constant mapping. This completes the proof. □

Proposition 21. *Let $f : G \to G'$ be a group homomorphism, let $\mathcal{A} \in OG_i(G)$ and let $\mathcal{B} \in OG_i(G')$ ($i = 1, 2, 3, 4$).*

(1) *If $\mathcal{A}$ has the i-sup-property, $f(\mathcal{A}) \in OG_i(G')$.*

(2) *$f^{-1}(\mathcal{B}) \in OG_i(G)$.*

Proof. (1) Since $f(\mathcal{A}) \in OGP_i(G')$ by Proposition 12 (1), It is sufficient to show that $f(\mathcal{A})(y^{-1}) \geq_i f(\mathcal{A})(y)$ for each $y \in f(G)$.

Case 1: Suppose $\mathcal{A} \in OG_1(G)$. Then by Propositions 4.11 in [18], 2.13 in [20] and 5.8 in [19], $f(\mathbf{A}) \in IVG(G')$, $f(A) \in IFG(G')$, $f(\lambda) \in FG(G')$. Thus, by Theorem 6 (1), $f(\mathcal{A}) \in OG_1(G')$.

Case 2: Suppose $\mathcal{A} \in OG_2(G)$ and let $y \in f(G)$. Since $f(\mathbf{A}) \in IVG(G')$ and $f(A) \in IFG(G')$ by Case 1, it is enough to prove that $f(\lambda)(y^{-1}) \leq f(\lambda)(y)$. Since $\mathcal{A}$ has the 2-sup-property, there is $x_0 \in f^{-1}(y)$ such that

$$\lambda(x_0) = \bigwedge_{t \in f^{-1}(y)} \lambda(t).$$

Then $f(\lambda)(y^{-1}) = \bigwedge_{t \in f^{-1}(y^{-1})} \lambda(t) \leq \lambda(x_0^{-1}) \leq \lambda(x_0) = f(\lambda)(y)$. Thus, by Theorem 6 (2), $f(\mathcal{A}) \in OG_2(G')$.

Case 3: Suppose $\mathcal{A} \in OG_3(G)$ and let $y \in f(G)$. Since $f(\mathbf{A}) \in IVG(G')$ and $f(\lambda) \in FG(G')$ by Case 1, it is sufficient to show that $[f(A)]^{\in}(y^{-1}) \leq [f(A)]^{\in}(y)$ and $[f(A)]^{\notin}(y^{-1}) \geq [f(A)]^{\in}(y)$. Since $\mathcal{A}$ has the 3-sup-property, there is $x_0 \in f^{-1}(y)$ such that

$$A^{\in}(x_0) = \bigwedge_{t \in f^{-1}(y)} A^{\in}(t), \; A^{\notin}(x_0) = \bigvee_{t \in f^{-1}(y)} A^{\notin}(t).$$

Then $[f(A)]^{\in}(y^{-1}) = \bigwedge_{t \in f^{-1}(y^{-1})} A^{\in}(t) \leq A^{\in}(x_0^{-1}) \leq A^{\in}(x_0) = [f(A)]^{\in}(y)$. Similarly, we have $[f(A)]^{\notin}(y^{-1}) \geq [f(A)]^{\notin}(y^{-1})$. Thus, by Theorem 6 (3), $f(\mathcal{A}) \in OG_3(G')$.

Case 4: Suppose $\mathcal{A} \in OG_4(G)$. Then from Cases (2), (3) and Theorem 6 (4), we can easily prove that $f(\mathcal{A}) \in OG_4(G^{'})$.

(2) Case 1: Suppose $\mathcal{B} \in OG_1(G^{'})$. Then Propositions 4.11 in [18], 2.13 in [20] and 5.8 in [19], $f^{-1}(\mathbf{B}) \in IVG(G)$, $f^{-1}(B) \in IFG(G)$, $f^{-1}(\mu) \in FG(G)$. Thus, $f^{-1}(\mathcal{B}) \in OG_1(G)$.

Case 2: Suppose $\mathcal{B} \in OG_2(G^{'})$ and let $x \in G$. Since $f^{-1}(\mathbf{B}) \in IVG(G)$, $f^{-1}(B) \in IFG(G)$ by Case 1, it is sufficient to prove that $f^{-1}(\mu)(x^{-1}) \leq f^{-1}(\mu)(x)$. Then

$$f^{-1}(\mu)(x^{-1}) = \mu(f(x^{-1})) = \mu(f(x)^{-1}) \leq \mu(f(x)) = f^{-1}(\mu)(x).$$

Thus, by Theorem 6 (2), $f(\mathcal{A}) \in OG_2(G^{'})$.

Case 3: Suppose $\mathcal{B} \in OG_3(G^{'})$ and let $x \in G$. Since $f^{-1}(\mathbf{B}) \in IVG(G)$, $f^{-1}(\mu) \in FG(G)$ by Case 1, it is enough to show that $[f^{-1}(A)]^{\in}(x^{-1}) \leq [f^{-1}(A)]^{\in}(x)$ and $[f^{-1}(A)]^{\notin}(x^{-1}) \geq [f^{-1}(A)]^{\notin}(x)$. Then

$$[f^{-1}(A)]^{\in}(x^{-1}) = A^{\in}(f(x^{-1})) = A^{\in}(f(x)^{-1}) \leq A^{\in}(f(x)) = [f^{-1}(A)]^{\in}(x).$$

Similarly, we have $[f^{-1}(A)]^{\notin}(x^{-1}) \geq [f^{-1}(A)]^{\notin}(x)$. Thus, by Theorem 6 (3), $f(\mathcal{A}) \in OG_3(G^{'})$.

Case 4: Suppose $\mathcal{B} \in OG_3(G^{'})$. Then from Cases (2), (3) and Theorem 6 (4), we can easily prove that $f(\mathcal{A}) \in OG_4(G^{'})$. This completes the proof. □

From Propositions 4.16 and 4.17 in [18], 2.18 and 2.19 in [20], and Theorems 2.1 and 2.2 in [22], we have the following result.

Theorem 9. *If $\mathcal{A} \in OG_1(G)$, then $[\mathcal{A}]_{\widetilde{\tilde{a}}}$ is a subgroup of G, for each $\widetilde{\tilde{a}} \in [I] \times (I \oplus I) \times I$ such that $\widetilde{\tilde{a}} \leq_1 \mathcal{A}(e)$. Conversely, if $\mathcal{A} \in \mathcal{O}(G)$ such that $[\mathcal{A}]_{\widetilde{\tilde{a}}}$ is a subgroup of G, for each $\widetilde{\tilde{a}} \in [I] \times (I \oplus I) \times I$ such that $\widetilde{\tilde{a}} \leq_1 \mathcal{A}(e)$, then $\mathcal{A} \in OG_1(G)$.*

Theorem 10. *Let G_p be the cyclic group of prime order p. Then $\mathcal{A} \in OG_i(G_p)$ if and only if $\mathcal{A}(x) = \mathcal{A}(1) \leq_i \mathcal{A}(0)$, for $i = 1,\ 2,\ 3,\ 4$.*

Proof. We prove only for $i = 1,\ 2$ and the proofs are omitted for $i = 3,\ 4$.

For $i = 1$, from Propositions 4.12 in [18], 2.14 in [20] and 5.10 in [19], we have for each $0 \neq x \in G_p$,

$$\mathbf{A} \in IVG(G_p) \text{ iff } \mathbf{A}^{-}(x) = \mathbf{A}^{-}(1) \leq \mathbf{A}^{-}(0) \text{ and } \mathbf{A}^{+}(x) = \mathbf{A}^{+}(1) \leq \mathbf{A}^{+}(0), \tag{7}$$

$$A \in IFG(G_p) \text{ iff } A^{\in}(x) = A^{\in}(1) \leq A^{\in}(0) \text{and } A^{\notin}(x) = A^{\notin}(1) \geq A^{\notin}(0), \tag{8}$$

$$\lambda \in FG(G_p) \text{ iff } \lambda(x) = \lambda(1) \leq \lambda(0). \tag{9}$$

Thus, by Theorem 6 (1), $\mathcal{A} \in OG_1(G_p)$ iff $\mathcal{A}(x) = \mathcal{A}(1) \leq_i \mathcal{A}(0)$.

For $i = 2$, suppose $\mathcal{A} \in OG_2(G_p)$ and let $y \in G_p$. Then by Theorem 6 (2), $\lambda(xy) \leq \lambda(x) \vee \lambda(y)$. Since G_p is the cyclic group of prime order p, $G_p = \{0, 1, 2, \ldots, p-1\}$. Since x is the sum of i's and i is the sum of x's, $\lambda(x) \leq \lambda(1) \leq \lambda(x)$. Thus, $\lambda(x) = \lambda(1)$. Since 0 is the identity of G_p, $\lambda(0) \leq \lambda(x)$. Thus, $\lambda(x) = \lambda(1) \geq \lambda(0)$. So $\mathcal{A}(x) = \mathcal{A}(1) \leq_2 \mathcal{A}(0)$.

Conversely, suppose $\mathcal{A}(x) = \mathcal{A}(1) \leq_2 \mathcal{A}(0)$. Then by Theorem 7, $\mathcal{A} \in OG_2(G_p)$. This completes the proof. □

Definition 23. *Let $\mathcal{A} \in \mathcal{O}(G)$ and let $i = 1,\ 2,\ 3,\ 4$. Then $\mathcal{A}$ is called an i-octahedron normal subgroup (briefly, an i-ONG) of G, if it satisfies the following conditions:*

$$\mathcal{A} \in OG_i(G) \text{ and } \mathcal{A}(xy) = \mathcal{A}(yx), \text{ for every} x,\ y \in G.$$

We will denote the set of all i-ONGs of G as $ONG_i(G)$. It is obvious that if G is abelian, then $\mathcal{A} \in ONG_i(G)$, for each $\mathcal{A} \in OG_i(G)$. Furthermore,

$$\mathcal{A} \in OG_1(G) \iff \mathbf{A} \in IVNG(G),\ A \in IFNG(G),\ \lambda \in FNG(G),$$

where $IVNG(G)$ [resp., $IFNG(G)$ and $FNG(G)$] is denoted by the set of all interval-valued fuzzy normal subgroups [resp., intuitionistic fuzzy normal subgroups and fuzzy normal subgroups or fuzzy invariant subgroups] of G (See [18] [resp., See [16,20]]).

Example 6. *Let $GL(n,R)$ be the general linear group of degree n and let I_n be the unit matrix of $GL(n,R)$. Then clearly, $GL(n,R)$ is a non abelian group. We define the interval-valued fuzzy set $\mathbf{A}$, two intuitionistic fuzzy sets $A,\ B$ and two fuzzy sets $\lambda,\ \mu$ in $GL(n,R)$ as follows: for each $I_n \neq M \in GL(n,R)$,*

$$\mathbf{A}(I_n) = [1,1],\ A(I_n) = (1,0),\ B(I_n) = (0,1),\ \lambda) = 1,\ \mu(I_n) = 0,$$

$$\mathbf{A}(M) = \begin{cases} \left[\frac{1}{5}, \frac{2}{3}\right] & \text{if } M \text{ is not a triangular matrix} \\ \left[\frac{1}{3}, \frac{1}{2}\right] & \text{if } M \text{ is a triangular matrix,} \end{cases}$$

$$A(M) = \begin{cases} \left[\frac{2}{3}, \frac{1}{5}\right] & \text{if } M \text{ is not a triangular matrix} \\ \left[\frac{1}{2}, \frac{1}{3}\right] & \text{if } M \text{ is a triangular matrix,} \end{cases}$$

$$A(M) = \begin{cases} \left[\frac{1}{5}, \frac{2}{3}\right] & \text{if } M \text{ is not a triangular matrix} \\ \left[\frac{1}{3}, \frac{1}{2}\right] & \text{if } M \text{ is a triangular matrix,} \end{cases}$$

$$\lambda(M) = \begin{cases} \frac{2}{3} & \text{if } M \text{ is not a triangular matrix} \\ \frac{1}{3} & \text{if } M \text{ is a triangular matrix,} \end{cases}$$

$$\mu(M) = \begin{cases} \frac{1}{3} & \text{if } M \text{ is not a triangular matrix} \\ \frac{2}{3} & \text{if } M \text{ is a triangular matrix.} \end{cases}$$

Then we can easily check that that the followings hold:

$$\langle \mathbf{A}, A, \lambda \rangle \in ONG_1(GL(n,R)),\ \langle \mathbf{A}, A, \mu \rangle \in ONG_2(GL(n,R)),$$

$$\langle \mathbf{A}, B, \lambda \rangle \in ONG_3(GL(n,R)),\ \langle \mathbf{A}, B, \mu \rangle \in ONG_4(GL(n,R)).$$

From Propositions 5.2 in [18], 3.2 in [20] and 2.1 (i) in [16], and Remark 1 (1), we have the following.

Proposition 22. *Let $\mathcal{A} \in \mathcal{O}(G)$ and let $\mathcal{B} \in ONG_1(G)$. Then $\mathcal{A} \circ_1 \mathcal{B} = \mathcal{B} \circ_1 \mathcal{A}$.*

Also from Propositions 5.3 in [18], 3.3 in [20] and 2.1 (ii) in [16], and Remark 1 (1), we have the following.

Proposition 23. *Let $\mathcal{A} \in ONG_1(G)$. if $\mathcal{B} \in OG_1(G)$, then $\mathcal{B} \circ_1 \mathcal{A} \in OG_1(G)$.*

Proposition 24. *If $\mathcal{A} \in ONG_1(G)$, then $G_{\mathcal{A}}$ is a normal subgroup of G.*

Proof. From Propositions 5.4 in [18], 3.5 in [20], 2.2 (ii) in [16] and Proposition 18, the proof is clear. □

It is obvious that A is a normal subgroup of G, then $\chi_{\mathcal{A}} \in ONG_1(G)$ and $G_{\chi_{\mathcal{A}}} = A$.

Definition 24. *Let $\mathcal{A} \in ONG_1(G)$. Then the quotient group $G/G_{\mathcal{A}}$ is called the octahedron quotient group (briefly, OQG) of G with respect to $\mathcal{A}$.*

Now let $\pi : G \to G/G_{\mathcal{A}}$ be the natural projection.

Proposition 25. *If $\mathcal{A} \in ONG_1(G)$ and $\mathcal{B} \in \mathcal{O}(G)$, then $\pi^{-1}(\pi(\mathcal{B})) = G/G_{\mathcal{A}} \circ_1 \mathcal{B}$.*

Proof. From Propositions 5.6 in [18], 3.7 in [20] and 2.3 (ii) in [16], the proof is obvious. □

5. Octahedron Ideals

Definition 25. *Let $(R, +, \cdot)$ be a ring and let $\mathcal{A} \in \mathcal{O}(R)$. Then*

(i) $\ddot{0} \neq \mathcal{A}$ is called a 1-octahedron subring of R, if it satisfies the following conditions: for every $x,\ y \in R$,

(a) $\mathcal{A}(x+y) \geq_1 \mathcal{A}(x) \wedge^1 \mathcal{A}(y)$,

(b) $\mathcal{A}(x^{-1}) \geq_1 \mathcal{A}(x)$,

(c) $\mathcal{A}(xy) \geq_1 \mathcal{A}(x) \wedge^1 \mathcal{A}(y)$,

(ii) $\left\langle \widetilde{0}, \mathbf{\bar{0}}, 1 \right\rangle \neq \mathcal{A}$ is called a 2-octahedron subring of R, if it satisfies the following conditions: for every $x,\ y \in R$,

(a) $\mathcal{A}(x+y) \geq_2 \mathcal{A}(x) \wedge^2 \mathcal{A}(y)$,

(b) $\mathcal{A}(x^{-1}) \geq_2 \mathcal{A}(x)$,

(c) $\mathcal{A}(xy) \geq_2 \mathcal{A}(x) \wedge^2 \mathcal{A}(y)$,

(iii) $\left\langle \widetilde{0}, \mathbf{\bar{1}}, 0 \right\rangle \neq \mathcal{A}$ is called a 3-octahedron subring of R, if it satisfies the following conditions: for every $x,\ y \in R$,

(a) $\mathcal{A}(x+y) \geq_3 \mathcal{A}(x) \wedge^3 \mathcal{A}(y)$,

(b) $\mathcal{A}(x^{-1}) \geq_3 \mathcal{A}(x)$,

(c) $\mathcal{A}(xy) \geq_3 \mathcal{A}(x) \wedge^3 \mathcal{A}(y)$,

(iv) $\left\langle \widetilde{0}, \mathbf{\bar{1}}, 1 \right\rangle \neq \mathcal{A}$ is called a 4-octahedron subring of R, if it satisfies the following conditions: for every $x,\ y \in R$,

(a) $\mathcal{A}(x+y) \geq_4 \mathcal{A}(x) \wedge^4 \mathcal{A}(y)$,

(b) $\mathcal{A}(x^{-1}) \geq_4 \mathcal{A}(x)$,

(c) $\mathcal{A}(xy) \geq_4 \mathcal{A}(x) \wedge^4 \mathcal{A}(y)$.

We will denote the set of all i-octahedron subrings of R as $OR_i(R)$ ($i = 1,\ 2,\ 3,\ 4$).

It is clear that if A is a subring of R, then $\chi_{\mathcal{A}} \in OR_1(R)$.

Example 7. *Consider the ring $(\mathbb{Z}_2, +, \cdot)$, where $\mathbb{Z}_2 = \{0, 1\}$. Let us define the interval-valued fuzzy set $\mathbf{A}$, two intuitionistic fuzzy sets $A,\ B$ and two fuzzy sets $\lambda,\ \mu$ in $\mathbb{Z}_2$ as follows:*

$$\mathbf{A}(0) = [0.5, 0.8],\ \mathbf{A}(1) = [0.4, 0.6],$$

$$A(0) = (0.7, 0.2),\ A(1) = (0.5, 0.3),\ B(0) = (0.6, 0.3),\ B(1) = (0.8, 0.2),$$

$$\lambda(0) = 0.8,\ \lambda(1) = 0.5,\ \mu(0) = 0.6,\ \mu(1) = 0.7.$$

Then we can easily check that the followings hold:

$$\langle \mathbf{A}, A, \lambda \rangle \in OR_1(\mathbb{Z}_2),\ \langle \mathbf{A}, A, \mu \rangle \in OR_2(\mathbb{Z}_2),$$

$$\langle \mathbf{A}, B, \lambda \rangle \in OR_3(\mathbb{Z}_2),\ \langle \mathbf{A}, B, \mu \rangle \in OR_4(\mathbb{Z}_2).$$

From the definitions of orders of two octahedron numbers and Definition 11, and Theorems 1–4 and 6, we have the following.

Theorem 11. *Let $(R,+,\cdot)$ be a ring and let $\mathcal{A} \in \mathcal{O}(R)$. Then*
(1) $\ddot{0} \neq \mathcal{A} \in OR_1(R) \iff \mathcal{A} \in OG_1((R,+)), \mathcal{A} \in OGP_1((R,\cdot))$,
(2) $\left\langle \widetilde{0}, \bar{\mathbf{0}}, 1 \right\rangle \neq \mathcal{A} \in OR_2(R) \iff \mathcal{A} \in OG_2((R,+)),\ \mathcal{A} \in OGP_2((R,\cdot))$,
(3) $\left\langle \widetilde{0}, \bar{\mathbf{0}}, 1 \right\rangle \neq \mathcal{A} \in OR_3(R) \iff \mathcal{A} \in OG_3((R,+)),\ \mathcal{A} \in OGP_3((R,\cdot))$,
(4) $\left\langle \widetilde{0}, \bar{\mathbf{0}}, 1 \right\rangle \neq \mathcal{A} \in OR_4(R) \iff \mathcal{A} \in OG_4((R,+)),\ \mathcal{A} \in OGP_4((R,\cdot))$.
The following is an immediate result of Theorems 7 and 11.

Corollary 6. *Let R be a ring and let $\mathcal{A} \in \mathcal{O}(R)$. Then $\mathcal{A} \in OR_i(R)$ if and only if it satisfies the following conditions: for every $x,\ y \in R$ and for every $i = 1,\ 2,\ 3,\ 4$,*
(i) $\mathcal{A}(x-y) \geq_i \mathcal{A}(x) \wedge^i \mathcal{A}(y)$,
(ii) $\mathcal{A}(xy) \geq_i \mathcal{A}(x) \wedge^i \mathcal{A}(y)$.
The following is an immediate result of Remark 3 and Proposition 15.

Theorem 12. *Let R be a ring. Then A be a subring of R if and only if $\chi_{\mathcal{A}} \in OR_1(R)$.*

Definition 26. *Let R be a ring and let $\mathcal{A} \in OR_i(R)$ $(I = 1,\ 2,\ 3,\ 4)$. Then*
(i) $\mathcal{A}$ is called an i-octahedron left ideal (briefly, i-OLI) if for every $x,\ y \in R$,

$$\mathcal{A}(xy) \geq_i \mathcal{A}(y),$$

(ii) $\mathcal{A}$ is called an i-octahedron right ideal (briefly, i-ORI), if for every $x,\ y \in R$,

$$\mathcal{A}(xy) \geq_i \mathcal{A}(x),$$

(iii) $\mathcal{A}$ is called an i-octahedron ideal (briefly, i-OI), if it is an i-octahedron left ideal and an i-octahedron right ideal of R.
we will denote the set of all i-OLIs [resp., i-ORIs and i-OIs] in R as $OLI_i(R)$ [resp., $ORI_i(R)$ and $OI_i(R)$].

Example 8. *Consider the ring $(\mathbb{Z}_4,+,\cdot)$, where $\mathbb{Z}_4 = \{0,1,2,3\}$. Let us define an interval-valued fuzzy set $\mathbf{A},\ \mathbf{B}$, two intuitionistic fuzzy sets $A,\ B$ and two fuzzy sets $\lambda,\ \mu,\ \eta,\ \delta$ in $\mathbb{Z}_2$ as follows:*

$$\mathbf{A}(0) = [0.6, 0.8],\ \mathbf{A}(1) = \mathbf{A}(3) = [0.4, 0.6],\ \mathbf{A}(2) = [0.5, 0.7],$$

$$A(0) = (0.7, 0.2),\ A(1) = A(3) = (0.5, 0.4),\ A(2) = (0.6, 0.3),$$

$$B(0) = (0.5, 0.4),\ B(1) = B(3) = (0.7, 0.2),\ B(2) = (0.6, 0.3),$$

$$\lambda(0) = 0.9,\ \lambda(1) = \lambda(3) = 0.6,\ \lambda(2) = 0.7,$$

$$\mu(0) = 0.5,\ \mu(1) = \mu(3) = 0.8,\ \mu(2) = 0.6.$$

Then we can easily check that the followings hold:

$$\langle \mathbf{A}, A, \lambda \rangle \in OLI_1(\mathbb{Z}_4),\ \langle \mathbf{A}, A, \mu \rangle \in OLI_2(\mathbb{Z}_4),$$

$$\langle \mathbf{A}, B, \lambda \rangle \in OLI_3(\mathbb{Z}_4),\ \langle \mathbf{A}, B, \mu \rangle \in OLI_4(\mathbb{Z}_4).$$

Remark 9. *(1) Let R be a ring. If $\lambda \in FLI(R)$ [resp., $FRI(R)$ and $FI(R)$], then*
$\langle [\lambda,\lambda], (\lambda,\lambda^c), \lambda \rangle \in OLI_1(R)$ *[resp., $ORI_1(R)$ and $OI_1(R)$],*
$\langle [\lambda,\lambda], (\lambda,\lambda^c), \lambda^c \rangle \in OLI_2(R)$ *[resp., $ORI_2(R)$ and $OI_2(R)$],*

$\langle [\lambda, \lambda], (\lambda^c, \lambda), \lambda \rangle \in OLI_3(R)$ *[resp., $ORI_3(R)$ and $OI_3(R)$],*
$\langle [\lambda, \lambda], (\lambda^c, \lambda), \lambda^c \rangle \in OLI_4(R)$ *[resp., $ORI_4(R)$ and $OI_4(R)$].*
(2) Let R be a ring. If $A \in IFLI(R)$ [resp., $IFRI(R)$ and $IFI(R)$], then
$\langle [A^{\in}, (A^{\notin})^c], A, A^{\in} \rangle \in OLI_1(R)$ *[resp., $ORI_1(R)$ and $OI_1(R)$],*
$\langle [A^{\in}, (A^{\notin})^c], A, (A^{\notin})^c \rangle \in OLI_2(R)$ *[resp., $ORI_2(R)$ and $OI_2(R)$],*
$\langle [A^{\in}, (A^{\notin})^c], A^c, A^{\in} \rangle \in OLI_3(R)$ *[resp., $ORI_3(R)$ and $OI_3(R)$],*
$\langle [A^{\in}, (A^{\notin})^c], A^c, (A^{\notin})^c \rangle \in OLI_4(R)$ *[resp., $ORI_4(R)$ and $OI_4(R)$].*

The following is an immediate result of Propositions 11, 12 and 21.

Proposition 26. *Let $f : R \to R'$ be a ring homomorphism and let $i = 1,\ 2,\ 3,\ 4$.*
(1) *If $\mathcal{A} \in OR_i(R)$ or $\mathcal{A} \in OLI_i(R)$ [resp., $ORI_i(R)$ and $OI_i(R)$], then so is $f(\mathcal{A})$.*
(2) *If $\mathcal{B} \in OR_i(R')$ or $\mathcal{B} \in OLI_i(R')$ [resp., $ORI_i(R')$ and $OI_i(R')$], then so is $f^{-1}(\mathcal{B})$.*

The following is an immediate result of Corollary 6 and Definition 26.

Theorem 13. *Let R be a ring, $\mathcal{A} \in \mathcal{O}(R)$ and let $i = 1,\ 2,\ 3,\ 4$. Then $\mathcal{A} \in OI_i(R)$ [resp., $\mathcal{A} \in OLI_i(R)$ and $\mathcal{A} \in ORI_i(R)$] if and only if it satisfies the following conditions: for every $x,\ y \in R$,*
(i) *$\mathcal{A}(x - y) \geq_i \mathcal{A}(x) \wedge^i \mathcal{A}(y)$,*
(ii) *$\mathcal{A}(xy) \geq_i \mathcal{A}(x) \vee^i \mathcal{A}(y)$ [resp., $\mathcal{A}(xy) \geq_i \mathcal{A}(y)$ and $\mathcal{A}(xy) \geq_i \mathcal{A}(x)$].*

The following is an immediate result of Theorems 12 and 13.

Theorem 14. *Let R be a ring. Then A an ideal [resp., a left ideal and a right ideal] of R if and only $\chi_{\mathcal{A}} \in OI_1(R)$ [resp., $OLI_1(R)$ and $ORI_1(R)$].*

Theorem 15. *Let R be a skew field (also division ring) and let $\mathcal{A} \in \mathcal{O}(R)$, where 0 and e denote the identity for "$+$" and "$\cdot$". Then $\mathcal{A} \in OI_i(R)$ [resp., $OLI_i(R)$ and $ORI_i(R)$] if and only if for each $0 \neq x \in R$, $\mathcal{A}(x) = \mathcal{A}(e) \leq_i \mathcal{A}(0)$ for $i = 1,\ 2,\ 3,\ 4$.*

Proof. We show only that $\mathcal{A} \in OLI_i(R)$ iff for each $0 \neq x \in R$, $\mathcal{A}(x) = \mathcal{A}(e) \leq_i \mathcal{A}(0)$ for $i = 1,\ 2$. The remainder's proofs are omitted.

Case 1: Let $i = 1$. Then from Propositions 6.6 in [18], 4.7 in [20] and 3.3 in [16],
$\mathcal{A} \in OLI_1(R)$ iff for each $0 \neq x \in R$, $\mathcal{A}(x) = \mathcal{A}(e) \leq_1 \mathcal{A}(0)$, i.e.,

$$\mathbf{A}(x) = \mathbf{A}(e) \leq \mathbf{A}(0),\ A(x) = A(e) \leq A(0),\ \lambda(x) = \lambda(e) \leq \lambda(0). \tag{10}$$

Case 2: Let $i = 2$. Suppose $\mathcal{A} \in OLI_2(R)$ and let $0 \neq x \in R$. Then by Definition 26, $\lambda(x) = \lambda(xe) \leq \lambda(e)$ and $\lambda(e) = \lambda(x^{-1}x) \leq \lambda(x)$. Thus, $\lambda(x) = \lambda(e)$. Since $\mathcal{A} \in OLI_2(R)$, $\mathcal{A} \in OR_2(R)$. By Corollary 6 (i) and Definition 2,

$$\lambda(0) = \lambda(e - e) \leq \lambda(e) \vee \lambda(e) = \lambda(e).$$

So $\lambda(x) = \lambda(e) \geq \lambda(0)$. Hence by the first and the second parts of (5.1),

$$\mathcal{A}(x) = \mathcal{A}(e) \leq_2 \mathcal{A}(0).$$

Conversely, suppose the necessary condition holds and let $x,\ y \in R$. Then we have four cases:
(i) $x \neq 0,\ y \neq 0,\ x \neq y$, (ii) $x \neq 0,\ y \neq 0,\ x = y$,
(iii) $x \neq 0,\ y = 0$, (iv) $x = 0,\ y \neq 0$.

Case (i). Suppose $x \neq 0,\ y \neq 0,\ x \neq y$. Then by the hypothesis, we have

$$\mathbf{A}(x) = \mathbf{A}(y) = \mathbf{A}(e) = \mathbf{A}(x-y) = \mathbf{A}(xy),$$

$$A(x) = A(y) = A(e) = A(x-y) = A(xy),$$
$$\lambda(x) = \lambda(y) = \lambda(e) = \lambda(x-y) = \lambda(xy).$$

Thus,

$$\mathbf{A}(x-y) \geq \mathbf{A}(x) \wedge \mathbf{A}(y),\ \mathbf{A}(xy) \geq \mathbf{A}(y),$$
$$A(x-y) \geq A(x) \wedge A(y),\ A(xy) \geq A(y),$$
$$\lambda(x-y) \leq \lambda(x) \vee \lambda(y),\ \lambda(xy) \leq \lambda(y).$$

So $\mathcal{A}(x-y) \geq_2 \mathcal{A}(x) \wedge^2 \mathcal{A}(y)$ and $\mathcal{A}(xy) \geq_2 \mathcal{A}(y)$.

Case (ii). Suppose $x \neq 0,\ y \neq 0,\ x = y$. Then by the hypothesis, we have

$$\mathbf{A}(x) = \mathbf{A}(y) = \mathbf{A}(e) = \mathbf{A}(xy) = \mathbf{A}(x-y) = \mathbf{A}(0),$$

$$A(x) = A(y) = A(e) = A(xy) = A(x-y) = A(0),$$
$$\lambda(x) = \lambda(y) = \lambda(e)) = \lambda(x-y) = \lambda(xy) = \lambda(0).$$

Thus, we have the same result in Case (i):

$$\mathcal{A}(x-y) \geq_2 \mathcal{A}(x) \wedge^2 \mathcal{A}(y) \text{ and } \mathcal{A}(xy) \geq_2 \mathcal{A}(y).$$

Case (iii). Suppose $x \neq 0,\ y = 0$. Then by the hypothesis, we have

$$\mathbf{A}(x-y) = \mathbf{A}(x) = \mathbf{A}(e) \geq \mathbf{A}(x) \wedge \mathbf{A}(y),\ \mathbf{A}(xy) = \mathbf{A}(0) \geq \mathbf{A}(y)$$

$$A(x-y) = A(x) = A(e) \geq A(x) \wedge A(y),\ A(xy) = A(0) \geq A(y),$$
$$\lambda(x-y) = \lambda(x) = \lambda(e)) \leq \lambda(x) \vee \lambda(y),\ \lambda(xy) = \lambda(0) \leq \lambda(y).$$

Thus, we have the same result in Case (i):

$$\mathcal{A}(x-y) \geq_2 \mathcal{A}(x) \wedge^2 \mathcal{A}(y) \text{ and } \mathcal{A}(xy) \geq_2 \mathcal{A}(y).$$

Case (iv). Suppose $x = 0,\ y \neq 0$. Then by the similar proof to Case (iii), we have

$$\mathcal{A}(x-y) \geq_2 \mathcal{A}(x) \wedge^2 \mathcal{A}(y) \text{ and } \mathcal{A}(xy) \geq_2 \mathcal{A}(y).$$

Hence in either cases, by Theorem 13, $\mathcal{A} \in OLI_2(R)$. □

Remark 10. *Theorem 15 shows that an i-OLI (ORI) is an i-OI in a skew field.*

The following gives a characteristic of a (usual) field by a 1-OI.

Proposition 27. *Let R be a commutative ring with a unity e. Suppose for each $\mathcal{A} \in OI_1(R)$,*

$$\mathcal{A}(x) = \mathcal{A}(e) \leq_1 \mathcal{A}(0) \text{ for each } 0 \neq x \in R.$$

Then R is a field.

Proof. Let A be an ideal of R such that $A \neq R$. Then clearly by Theorem 14, $\chi_{\mathcal{A}} \in OI_1(R)$ such that $A \neq \ddot{1}$. Thus, there is $y \in R$ such that $y \notin A$. So $\chi_{\mathcal{A}}(y) = \ddot{0}$. By the hypothesis, $\chi_{\mathcal{A}}(x) = \chi_{\mathcal{A}}(e) \leq_1 \chi_{\mathcal{A}}(0)$. Hence $\chi_{\mathcal{A}}(0) = \ddot{1}$, i.e., $A = \{0\}$. Therefore R is a field. □

6. Conclusions

By using the i-product of two octahedron sets, we introduce the concept of i-octahedron subgroupoids of a groupoid. In particular, we obtain four characterizations of i-octahedron groupoids. Also, we defined an i-OLI [resp., i-ORI and i-OI]] of a groupoid and investigated some of their properties. Moreover, we obtain some properties for the image and preimage of an i-octahedron subgroupoid [resp., i-OLI, i-ORI and i-OI] under groupoid homomorphism. Next, we define i-octahedron subgroups of a group and study some of their properties. In particular, we obtain two characterizations of i-octahedron subgroups and i-OLI [resp., i-ORI and i-OI] of a group. We introduce the concepts of i-octahedron subrings [resp., i-OLIs, i-ORIs and i-OIs] of a ring and obtain their characterizations. Furthermore, we found a sufficient condition for which a commutative ring with a unity e is a field.

In the future, we expect that one applies octahedron sets to BCI/BCK-algebras, topologies, decision-making, measures and entropy measures, etc.

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Abbreviations

The following abbreviations are used in this manuscript:

$\mathcal{O}(X)$	the set of all octahedron sets in X
$\mathcal{O}_P(X)$	the set of all octahedron points in X
$FGP(X)$	the set of all fuzzy subgroupoids in a groupoid X
$IFGP(X)$	the set of all intuitionistic fuzzy subgroupoids in a groupoid X
$IVGP(X)$	the set of all interval-valued fuzzy subgroupoids in a groupoid X
$OGP_i(X)$	the set of all i-octahedron subgroupoids in X
i-OLI	i-octahedron left ideal
i-ORI	i-octahedron right ideal
i-OI	i-octahedron ideal
$OLI_i(X)$	the set of all i-OLIs of X
$ORI_i(X)$	the set of all i-ORIs of X
$OI_i(X)$	the set of all i-OIs of X
$FI(X)$	the set of all fuzzy ideals of X
$FLI(X)$	the set of all fuzzy left ideals of X
$FRI(X)$	the set of all fuzzy right ideals of X
IVI	interval-valued fuzzy ideals
$IVLI$	interval-valued fuzzy left ideal
$IVRI$	interval-valued fuzzy right ideal
$IVI(X)$	the set of all IVIs of X
$IVLI(X)$	the set of all IVLIs of X
$IVRI(X)$	the set of all IVRIs of X
IFI	intuitionistic fuzzy ideal
$IFLI$	intuitionistic fuzzy left ideal

$IFRI$	intuitionistic fuzzy right ideal
$IFI(X)$	the set of all IFIs of X
$IFLI(X)$	the set of all IFLIs of X
$IFRI(X)$	the set of all IFRIs of X
$FGP(G)$	the set of all fuzzy subgroupoids in a group G
$IFGP(G)$	the set of all intuitionistic fuzzy subgroupoids in a group G
$OGP_i(X)$	the set of all i-octahedron groupoids in a groupoid X
$FG(G)$	the set of fuzzy subgroups of G
IFG	intuitionistic fuzzy subgroup
$IFG(G)$	the set of IFGs of G
IVG	interval-valued fuzzy subgroup
$IVG(G)$	the set of IVGs of G
$OG_i(G)$	the set of all i-octahedron subgroups of G
i-ONG	i-octahedron normal subgroup
$ONG_i(G)$	the set of all i-ONGs of G
$IVNG(G)$	the set of all interval-valued fuzzy normal subgroups
$IFNG(G)$	the set of all intuitionistic fuzzy normal subgroups
$FNG(G)$	the set of all fuzzy normal subgroups
$OR_i(R)$	the set of all i-octahedron subrings of R

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