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Ore Extensions for the Sweedler's Hopf Algebra $\mathbb{H}_4$

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Abstract: The aim of this paper is to classify all Hopf algebra structures on the quotient of Ore extensions $\mathbb{H}_4[z;\sigma]$ of automorphism type for the Sweedler's 4-dimensional Hopf algebra $\mathbb{H}_4$. Firstly, we calculate all equivalent classes of twisted homomorphisms (σ, J) for $\mathbb{H}_4$. Then we give the classification of all bialgebra (Hopf algebra) structures on the quotients of $\mathbb{H}_4[z;\sigma]$ up to isomorphism.

Keywords: Ore extension; Drinfeld twist; twisted homomorphism; Hopf algebra

MSC: 16S36; 16T10; 16T05

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1. Introduction

Ore extensions play a key role in classifying pointed Hopf algebras, see for example [1–5]. It also can provide some neither pointed nor semisimple Hopf algebras, see for example ([6], Examples 3.1 and 3.3).

Panov [7] introduced the concept of Hopf-Ore extension $R[x;\sigma,\delta]$ of which the variable x is restricted to a skew primitive element and gave some equivalent descriptions. Later the Hopf-Ore extensions for some special Hopf algebras were obtained, such as quasitriangular Hopf algebras and multiplier Hopf algebras, see for example [8–11]. The authors [12] gave the realization of PBW-deformations of an quantum group via iterated Ore extensions. As is well-known, a Drinfeld's twist $\mathscr{J}$ for a Hopf algebra R gives rise to a new Hopf algebra $R^{\mathscr{J}}$ with the same underlying algebra and the coalgebra structure, which is twisted from Δ by $\mathscr{J}$. Moreover, if $(R,\mathscr{R})$ is a quasitriangular Hopf algebra, so is $(R^{\mathscr{J}},\mathscr{R}^{\mathscr{J}})$, where $\mathscr{R}^{\mathscr{J}} = \mathscr{J}_{21}\mathscr{R}\mathscr{J}^{-1}$ (see for example [13–17]). Now, let R be a bialgebra or Hopf algebra and σ an automorphism of R. Recently, Yang and Zhang [6] described Hopf algebra structures on the localization of skew polynomial ring $R[z;\sigma]$ and the quotients of $R[z;\sigma]/I$, for a certain Hopf ideal I of $R[z;\sigma]$, where $R[z;\sigma]$ is a Ore extension of automorphism type. Recall that the Sweedler's Hopf Algebra $\mathbb{H}_4$ is a noncommutative and noncocommutative quasitriangular Hopf algebra of the smallest dimension. It is one of the few examples discovered in the early stage of the exhibition, and now it still plays an important role in the theoretical development of Hopf algebra [18]. In this paper, we study Ore extensions of automorphism type for the Hopf algebra $\mathbb{H}_4$. Consequently, Ore extensions of $\mathbb{H}_4$ that are of bialgebras are classified. Some new examples of Hopf algebras of dimension $4n$, consisting some neither pointed nor semisimple Hopf algebras are given.

The paper is organized as follows.

In Section 1, some basic notions of Ore extensions, Hopf algebras, Drinfeld's twists and twisted homomorphisms are reviewed. One fundamental result (Theorem 1) about the Ore extensions of automorphism type for Hopf algebras is established. In Section 2, we firstly compute the equivalent classes of twisted homomorphisms for $\mathbb{H}_4$. Equivalently, the isomorphism classes of Ore extensions of automorphism type for $\mathbb{H}_4$ are described. In Section 3, up to bialgebra isomorphisms, the classes of the Ore extensions of automorphism type for $\mathbb{H}_4$ are determined completely (Theorem 3). All Hopf algebras structures on the quotients of Ore extensions of automorphism type for $\mathbb{H}_4$ are also classified (Theorem 4).

2. Preliminaries

Throughout the paper, we work over the fixed field $\Bbbk$ containing some primitive root ω of unity. All algebras, modules, homomorphisms and tensor products are defined over the field $\Bbbk$.

The group of automorphisms of an algebra R is denoted by $\mathrm{Aut}(R)$ and $\sigma \in \mathrm{Aut}(R)$ unless otherwise stated.

Let us recall some basic notions and results about Ore extensions and Hopf algebras. For more details, the readers can refer to [18,19].

Suppose that R is a ring, $\sigma : R \to R$ a ring homomorphism, and $\delta : R \to R$ a σ-derivation of R, which means that δ is a homomorphism of abelian groups satisfying

$$\delta(r_1 r_2) = \sigma(r_1)\delta(r_2) + \delta(r_1)r_2.$$

Then an Ore extension $R[x;\sigma,\delta]$, is defined by a noncommutative ring obtained by giving the ring of polynomials $R[x]$ a new multiplication, subject to the identity

$$xr = \sigma(r)x + \delta(r).$$

If $\delta = 0$, the Ore extension is denoted by $R[x;\sigma]$, and it is called an Ore extension of automorphism type for R.

A bialgebra over the the field $\Bbbk$ is a vector space which is both a unital associative algebra and a coalgebra. The algebraic and coalgebraic structures are compatible with a few more axioms: the comultiplication and the counit are both unital algebra homomorphisms, or equivalently, the multiplication and the unit of the algebra both are coalgebra morphisms.

Let R be a bialgebra with the comultiplication $\Delta : R \to R \otimes R$, the counit $\epsilon : R \to \Bbbk$. If there exist a $\Bbbk$-map $S : R \to R$ such that

$$\sum_{(h)} S(h_1)h_2 = \sum_{(h)} h_1 S(h_2) = \epsilon(h)$$

for all $h \in R$, then R is called a Hopf algebra, where we use the sigma notations

$$\Delta(h) = \sum_{(h)} h_1 \otimes h_2.$$

Example 1. *The Sweedler's 4-dimensional Hopf algebra $\mathbb{H}_4$ is defined by*

$$\begin{aligned} \textit{as an algebra:} \quad & \mathbb{H}_4 = \Bbbk\left\langle 1, g, x, gx \,|\, g^2 = 1, x^2 = 0, xg = -gx \right\rangle; \\ \textit{the coalgebra:} \quad & \Delta(g) = g \otimes g,\ \Delta(x) = x \otimes 1 + g \otimes x, \\ & \epsilon(g) = 1,\ \epsilon(x) = 0, \\ & S(g) = g,\ S(x) = -gx. \end{aligned}$$

$\mathbb{H}_4$ is a unique and non-commutative quasi-triangular Hopf algebra of dimension 4. It is one of the few examples discovered in the early stage of the exhibition, and now it still plays an important role in the theoretical development of Hopf algebra.

It is easy to see that the set of grouplikes of $\mathbb{H}_4$ is $G(\mathbb{H}_4) = \{1, g\}$ and $x \in P_{1,g}(\mathbb{H}_4)$, where

$$P_{g,h}(H) = \{a \in H | \Delta(a) = a \otimes g + h \otimes a\}.$$

Let

$$e_0 = \frac{1+g}{2},\ e_1 = \frac{1-g}{2}.$$

Then the $\{e_0, e_1, e_0x, e_1x\}$ forms another basis of $\mathbb{H}_4$ and

$$e_0e_1 = e_1e_0 = 0, \quad e_0^2 = e_1^2 = 1,$$

$$e_0g = ge_0 = e_0, \quad e_1g = ge_1 = -e_1,$$
$$e_0x = xe_1, \quad e_1x = xe_0.$$

It is straightforward to see that if $\sigma \in \mathrm{Aut}(\mathbb{H}_4)$, then

$$\sigma(x) = px + qgx = p(e_0 + e_1)x + q(e_0 - e_1)x = (p+q)e_0x + (p-q)e_1x$$

and

$$\sigma(g) = \nu g + mx + ngx = \nu g + (m+n)e_0x + (m-n)e_1x,$$

for some $p, q, m, n \in \Bbbk$ and $\nu = \pm 1$ with some relations.

We denote a copy of the Sweedler's Hopf algebra $\mathbb{H}_4$ by $\mathbb{H}_4'$, which generators are replaced by g', x' satisfying the same relations. The analogous notations are allowed for twisted homomorphisms for $\mathbb{H}_4'$.

Furthermore, if $\Phi : \mathbb{H}_4 \to \mathbb{H}_4'$ is the bialgebra isomorphism, then

$$\Phi(g) = g', \; \Phi(x) = \mu x', \text{ for some } \mu \neq 0$$

(see [20]).

Definition 1. ([21]) *Let R be a bialgebra or Hopf algebra, the invertible element $J \in R \otimes R$, and $\sigma \in \mathrm{Aut}(R)$. The pair (σ, J) is said to be a twisted homomorphism for R if*

$$(\Delta \otimes id)(J)(J \otimes 1) = (id \otimes \Delta)(J)(1 \otimes J), \tag{1}$$

$$(id \otimes \epsilon)(J) = (\epsilon \otimes id)(J) = 1, \tag{2}$$

$$J(\sigma \otimes \sigma)\Delta(h) = \Delta(\sigma(h))J, \tag{3}$$
$$\epsilon \circ \sigma = \epsilon \tag{4}$$

for all $h \in R$.

Assume that R is a Hopf algebra. By ([22], Theorem 2.4), $R[z;\sigma]$ is a bialgebra with $R \subset R[z;\sigma]$ defined by

$$\Delta(z) = J(z \otimes z), \quad \epsilon(z) = 1_{\Bbbk} \tag{5}$$

if and only if (σ, J) is a twisted homomorphism for R. In this case, $R[z;\sigma]$ is called a bialgebra Ore extension of automorphism type (simply, BOEA) for Hopf algebra R.

Let R be a Hopf algebra with the antipode S, $\sigma \in \mathrm{Aut}(R)$, and (σ, J) a twisted homomorphism for R. Let

$$\vartheta = \prod_{i=0}^{n-1} (\sigma^i \otimes \sigma^i)(J) = \prod_{i=0}^{n-1} \left(\sum_J \sigma^i(J^1) \otimes \sigma^i(J^2) \right) = \sum_\vartheta \vartheta^1 \otimes \vartheta^2.$$

We denote $S^\sigma = \sigma \circ S \circ \sigma^{-1}$ and

$$\theta := \theta_l = \sum_J S(J^1)J^2, \; \theta_r = \sum_J J^1 S^\sigma(J^2),$$

and

$$\varrho := \varrho_l = \sum_{\vartheta} S(\vartheta^1)\vartheta^2, \quad \varrho_r = \sum_{\vartheta} \vartheta^1 S(\vartheta^2),$$

where $J = \sum J^1 \otimes J^2 \in R \otimes R$.

Theorem 1. ([6], Theorem 2.6) *Let R be a Hopf algebra with the antipode S, (σ, J) a twisted homomorphism for R, and $R[z;\sigma]$ a BOEA for R. Suppose that there exists a nonzero $t \in R$, with*

$$\Delta(t) = \prod_{i=0}^{n-1} (\sigma^i \otimes \sigma^i)(J)(t \otimes t), \quad th = \sigma^n(h)t$$

for all $h \in R$. Then $H = R[z;\sigma]/\langle z^n - t\rangle$ is a Hopf algebra with the antipode S such that $S(z) = z^{-1}\theta^{-1}$ if and only if

1. $\theta_r = \theta$ *and* $\sigma^n(\varrho_r) = \varrho$;
2. $S^\sigma(h) = \theta^{-1}S(h)\theta$ *for all* $h \in R$;
3. $\prod_{i=0}^{n-1} \sigma^i(\theta) = \varrho$.

The authors in [6] gave some nontrivial examples on Theorem 1. Here we give more example as follows.

Example 2. *For the Hopf algebra $\mathbb{H}_4$, $\omega \in \Bbbk$ a primitive $2n$-th root of unity. Let*

$$\sigma(g) = g,\ \sigma(x) = \omega x, t = g$$

$$J = 1 \otimes 1 + dxg \otimes x.$$

for any $d \in \Bbbk$. Then (σ, J) is a twisted homomorphism for $\mathbb{H}_4$ and satisfies the condtions of Theorem 1. Thus up to isomorphism, we get a Hopf algebra $\mathbb{H}_{4n}$, generated by x, z with the relations

$$z^{2n} = 1,\ x^2 = 0,\ zx = \omega xz.$$

The coalgebra is

$$\begin{aligned} &\Delta(x) = x \otimes 1 + z^n \otimes x,\ \Delta(z) = z \otimes z + dxz^{n+1} \otimes xz, \\ &\varepsilon(z) = 1,\ \varepsilon(x) = 0, \\ &S(z) = z^{-1}, S(x) = -z^n x. \end{aligned}$$

$\mathbb{H}_{4n}$ is a neither pointed nor semisimple quasitrangular Hopf algebra of dimension $4n$ extended by $\mathbb{H}_4$. In the present paper, we shall investigate the bialgebra (Hopf algebra) structures on the quotients $\mathbb{H}_4[z;\sigma]/\langle z^n - t\rangle$ in general.

3. Classification of Twisted Homomorphisms for $\mathbb{H}_4$

In this section, we give the classification of twisted homomorphisms for $\mathbb{H}_4$.

Let (σ, J) and (σ', J') be twisted homomorphisms for $\mathbb{H}_4$ and $\mathbb{H}'_4$ respectively, and $\mathbb{H}_4[z,\sigma]$ and $\mathbb{H}'_4[z',\sigma']$ the corresponding Ore extensions of automorphism type. The datum (σ, J) is said to be equivalent to (σ', J'), denoted by $(\sigma, J) \approx (\sigma', J')$, if there is a bi-algebraic isomorphism $\Phi : \mathbb{H}_4[z,\sigma] \to \mathbb{H}'_4[z',\sigma']$ such that $\Phi(\mathbb{H}_4) = \mathbb{H}'_4$ as bialgebras.

Therefore, if $\Phi : \mathbb{H}_4[z;\sigma] \cong \mathbb{H}'_4[z';\sigma']$ as bialgebras, then

$$(\Phi \otimes \Phi)(J)(\Phi(z) \otimes \Phi(z)) = \Delta'(\Phi(z)) \tag{6}$$

$$\Phi(z)\Phi(h) = \Phi(\sigma(h))\Phi(z), \tag{7}$$

$$\epsilon(\Phi(z)) = 1 \tag{8}$$

for all $h \in R$.

We have the following main result.

Theorem 2. *Any twisted homomorphism (σ, J) for $\mathbb{H}_4$ is equivalent to one of the following lists.*

(a) *the pair $(\sigma_{1,s}, J_{1,d})$:*

$$\sigma_{1,s}(g) = g, \quad \sigma_{1,s}(x) = sx,$$
$$J_{1,d} = 1 \otimes 1 + d\,(gx \otimes x)\,.$$

for any $d \in \Bbbk$ and $0 \neq s \in \Bbbk$.

(b) *the pair $(\sigma_{2,s}, J_{2,d})$:*

$$\sigma_{2,s}(g) = g + 2gx,\ \sigma_{2,s}(x) = sx,$$
$$J_{2,d} = 1 \otimes 1 + 2e_1 \otimes x + d(gx \otimes x).$$

for any $d \in \Bbbk$ and $0 \neq s \in \Bbbk$.

(c) *the pair $(\sigma_{3,s}, J_3)$:*

$$\sigma_{3,s}(g) = g + 2x,\ \sigma_{3,s}(x) = s\,gx,$$
$$J_3 = 1 \otimes 1 - 2e_1 \otimes e_1 + 2e_1 \otimes x.$$

for any $0 \neq s \in \Bbbk$.

Proof. Let (σ, J) be a twisted homomorphism for $\mathbb{H}_4$. The proof is given in three steps as follows.

Step 1: Firstly, we assume that

$$J = \sum_{i,j=0}^{1} (a_{ij}g^i \otimes g^j + b_{ij}g^i \otimes g^j x + c_{ij}g^i x \otimes g^j + d_{ij}g^i x \otimes g^j x) \in \mathbb{H}_4 \otimes \mathbb{H}_4$$

satisfies Equations (1) and (2). Tedious computations and comparing the coefficients of terms by Equations (1) and (2) show that the Drinfeld twist J for $\mathbb{H}_4$ must be one of the following

(1)
$$J_1 = 1 \otimes 1 + 4(a-1)e_1 \otimes e_1 + 2cgx \otimes e_1 + d\,(gx \otimes x + 4(a-1)e_0 x \otimes e_0 x)$$
for any a, c, d.

(2)
$$J_2 = 1 \otimes 1 + 2be_1 \otimes x + d(gx \otimes x),$$
for any $b \neq 0$ and d.

(3)
$$J_3 = 1 \otimes 1 + 4(a-1)e_1 \otimes e_1 + 2be_1 \otimes x + 2cgx \otimes e_1 + d'(gx \otimes x)$$
for any $a \neq 1$, $b \neq 0$ and c, where $d' = \frac{bc}{a-1}$.

Step 2: Secondly, we note that

$$J_1(\sigma \otimes \sigma)\Delta(h) = \Delta(\sigma(h))J_1, \tag{9}$$
$$\epsilon \circ \sigma = \epsilon. \tag{10}$$

The Equation (10) and $\sigma \in \mathrm{Aut}(\mathbb{H}_4)$ imply that

$$\sigma(g) = g + mx + ngx, \quad \sigma(x) = px + qgx.$$

for some $p, q, m, n \in \Bbbk$. Recall that J_1 can be written as

$$\begin{aligned} J_1 = \quad & a(1 \otimes 1) + (1-a)(g \otimes 1) + (1-a)(1 \otimes g) + (a-1)(g \otimes g) + c(gx \otimes 1 - gx \otimes g) \\ & + d((a-1)1 \otimes 1 + ag \otimes 1 + (a-1)1 \otimes g + (a-1)g \otimes g)(x \otimes x) \end{aligned}$$

for any a, c, d. Equation (9) imply that

$$J_1(\sigma \otimes \sigma)\Delta(g) = \Delta(\sigma(g))J_1, \quad J_1(\sigma \otimes \sigma)\Delta(x) = \Delta(\sigma(x))J_1.$$

The above two equations show that

$$2(a-1)p = (2a-1)q, \tag{11}$$
$$2(a-1)m = (2a-1)n, \tag{12}$$
$$2(a-1)n + 2c = (2a-1)m, \tag{13}$$
$$am^2 + 2(1-a)mn + (a-1)n^2 = 2cm - cn, \tag{14}$$
$$amn + (1-a)n^2 + (1-a)m^2 + (a-1)mn = cn, \tag{15}$$
$$an^2 + 2(1-a)mn + (a-1)m^2 = cn, \tag{16}$$
$$amp + (1-a)np + (1-a)mq + (a-1)nq = 2cp - cq, \tag{17}$$
$$amq + (1-a)nq + (1-a)mp + (a-1)np = cq, \tag{18}$$
$$anp + (1-a)mp + (1-a)nq + (a-1)mq = cq, \tag{19}$$
$$anq + (1-a)mq + (1-a)np + (a-1)mp = cq. \tag{20}$$

If $a = \frac{3}{4}$, then $c = 0$, $m + n = 0$, $p + q = 0$ by Equations (11)–(13) and Equations (14)–(20) hold automatically. However, $\sigma \notin \mathrm{Aut}(\mathbb{H}_4)$. Therefore, $a \neq \frac{3}{4}$. Now we set $\kappa = \frac{1}{4a-3}$, then

$$m = (1+\kappa)c, \quad n = (1-\kappa)c$$

by Equations (12) and (13) and we can rewrite

$$p = \frac{1}{2}(1+\kappa)s, \quad q = \frac{1}{2}(1-\kappa)s$$

by Equation (11). Also, such m, n and p, q enjoy Equations (14)–(20) automatically and $\sigma_1 \in \mathrm{Aut}(\mathbb{H}_4)$.

Therefore, we get a twisted homomorphism $(\sigma_1^{\kappa,s}, J_1^{c,d})$, for $\mathbb{H}_4$, where

$$\begin{aligned} & \sigma_1^{\kappa,s}(g) = g + 2c\,(e_0 + \kappa e_1)\,x, \quad \sigma_1^{\kappa,s}(x) = s(e_0 + \kappa e_1)x(s \neq 0), \\ & J_1^{c,d} = 1 \otimes 1 + \left(\frac{1}{\kappa} - 1\right) 1 \otimes e_1 + 2cgx \otimes e_1 + d\left(gx \otimes x + \left(\frac{1}{\kappa} - 1\right) e_0 x \otimes e_0 x\right) \end{aligned}$$

for any $\kappa \neq 0, c, d$.

Similarly, we get the twisted homomorphism $(\sigma_2^s, J_2^{b,d})$, where

$$\sigma_2^s(g) = g + 2bgx,\ \sigma_2^s(x) = sx (s \neq 0),$$
$$J_2^{b,d} = 1 \otimes 1 + 2b(e_1 \otimes x) + d(gx \otimes x)$$

for any $b \neq 0$ and d, and the twisted homomorphism $(\sigma_3^{\kappa,s}, J_3^{b,c})$, where

$$\sigma_3^{\kappa,s}(g) = g + 2\left((b+c)e_0 + \kappa(c-b)e_1)\right) x,\ \sigma_3^{\kappa,s}(x) = s(e_0 + \kappa e_1)x (s \neq 0),$$
$$J_3^{b,c} = 1 \otimes 1 + \left(\frac{1}{\kappa} - 1\right) e_1 \otimes e_1 + 2be_1 \otimes x + 2cgx \otimes e_1 + \frac{4\kappa bc}{1-\kappa}(gx \otimes x),$$

for any $\kappa \neq 0, 1, b \neq 0, c$.

Step 3: Now, assume that $\Phi : H = \mathbb{H}_4[z, \sigma] \to H' = \mathbb{H}'_4[z', \sigma']$ is a bialgebra isomorphism. Define $\deg r = 0$ for all $r \in \mathbb{H}_4$ and $\deg z = 1$. Then we can extend this to define the lexicographic order on $\mathbb{H}_4[z; \sigma]$. Since Φ is bialgebra isomorphism from H to H', we have $\Phi(z) = A'z' + B'$ by considering the expression for $\Phi(z)$ as a polynomial z with coefficients in $\mathbb{H}'_4$, where $A', B' \in \mathbb{H}'_4$. It is easy to see that A' is invertible in $\mathbb{H}'_4$ since Φ is an isomorphism.

By Equation (6), we have

$$\begin{aligned} & \Delta'(A')J'(z' \otimes z') + \Delta'(B') \\ = \ & (\Phi \otimes \Phi)(J)((A' \otimes A')(z' \otimes z') + (A' \otimes B')(z' \otimes 1) + (B' \otimes A')(1 \otimes z') + B' \otimes B'). \end{aligned}$$

Comparing the coefficients of $z' \otimes 1$ and $1 \otimes z'$, we get $B' = 0$. Hence we have $\Phi(z) = A'z'$. Equation (6) holds if and only if

$$\Delta'(A')J' = (\Phi \otimes \Phi)(J)(A' \otimes A'). \tag{21}$$

Now, by Equation (8) , it follows that

$$A' = e'_0 + b_0 e'_1 + c_0 e'_0 x' + d_0 e'_1 x'.$$

Let us investigate them case by case.

(1) For the twisted homomorphism $(\sigma_1^{\kappa,s}, J_1^{c,d})$, where

$$\sigma_1^{\kappa,s}(g) = g + 2c(e_0 + \kappa e_1)x, \quad \sigma_1^{\kappa,s}(x) = s(e_0 + \kappa e_1)x \ (s \neq 0),$$
$$J_1^{c,d} = 1 \otimes 1 + (\kappa^{-1} - 1)e_1 \otimes e_1 + 2cgx \otimes e_1 + d((\kappa^{-1} - 1)e_0 \otimes e_0 + g \otimes 1)(x \otimes x).$$

Assume that

$$(\sigma_1^{\kappa,s}, J_1^{c,d}) \approx (\sigma_1^{\kappa',s'}, J_1^{c',d'}),$$

where

$$\sigma_1^{\kappa',s'}(g') = g' + 2c'(e'_0 + \kappa' e'_1)x', \quad \sigma_1^{\kappa',s'}(x') = s'(e'_0 + \kappa' e'_1)x' \ (s' \neq 0),$$
$$J_1^{c',d'} = 1 \otimes 1 + (\kappa'^{-1} - 1)e'_1 \otimes e'_1 + 2c'g'x' \otimes e'_1 + d'((\kappa'^{-1} - 1)e'_0 \otimes e'_0 + g' \otimes 1)(x' \otimes x')$$

in $\mathbb{H}'_4$. By Equation (7), we have

$$\Phi(z)\Phi(x) = \Phi(\sigma(x))\Phi(z), \quad \Phi(z)\Phi(g) = \Phi(\sigma(g))\Phi(z).$$

Therefore, we have

$$sb_0 = s',\ s\kappa = b_0 s'\kappa',\ c_0 = c' - c\mu b_0,\ d_0 = c\mu\kappa - c' b_0 \kappa'. \tag{22}$$

It is noted that $b_0 = \frac{s'}{s}$, $b_0^2 = \frac{\kappa}{\kappa'}$.
On the other hand, it is straightforward to see that by Equation (21),

$$(\Phi \otimes \Phi)(J_1^{c,d})\left(A' \otimes A'\right) = \Delta'(A') J_1^{c',d'}.$$

Comparing coefficients of all terms and noting that Equation (22), one get

$$d' = d\mu^2 + 2c\mu d_0 - \kappa^{-1} d_0^2.$$

Hence

$$\begin{aligned}
&a_0 = 1, \quad b_0 = \frac{s'}{s} = \varepsilon\sqrt{\frac{\kappa}{\kappa'}},\\
&c_0 = c' - c\mu\sqrt{\frac{\kappa}{\kappa'}}\varepsilon, \quad d_0 = -\sqrt{\kappa\kappa'}(c' - c\mu\sqrt{\frac{\kappa}{\kappa'}}\varepsilon),\\
&d' = d\mu^2 + \kappa c^2 \mu^2 - \kappa' c'^2
\end{aligned}$$

for any c, d, c', d' and $s\kappa \neq 0$, $s'\kappa'\mu \neq 0$.
In particular, for any c, d and $s\kappa \neq 0$, we can choose suitable triples (b_0, c_0, d_0), such that

$$c' = 0,\ \kappa' = 1,\ s' = \varepsilon\sqrt{\kappa}s,\ d' = d\mu^2 + \kappa c^2\mu^2.$$

Therefore, we get

$$(\sigma_1^{\kappa,s}, J_1^{c,d}) \approx (\sigma_{1,s'}, J_{1,d'}),$$

where

$$\sigma_{1,s'}(g) = g, \quad \sigma_{1,s'}(x) = s'x(s' \neq 0), \quad J_{1,d'} = 1 \otimes 1 + d'\left(gx \otimes x\right),$$

for any $d' \in \Bbbk$ and $0 \neq s' \in \Bbbk$.

(2) For the twisted homomorphism $(\sigma_2^s, J_2^{b,d})$, where

$$\sigma_2^s(g) = g + 2bgx, \quad \sigma_2^s(x) = sx\ (s \neq 0),\ J_2^{b,d} = 1 \otimes 1 + 2b(e_1 \otimes x) + d(gx \otimes x)$$

for any d and $b \neq 0$.
Assume that

$$(\sigma_2^s, J_2^{b,d}) \approx (\sigma_2^{s'}, J_2{}^{b',d'}),$$

where

$$\sigma_2^{s'}(g') = g' + 2b'g'x', \quad \sigma_2^{s'}(x') = s'x'\ (s' \neq 0),\ J_2^{b',d'} = 1 \otimes 1 + 2b'(e_1' \otimes x') + d'(g'x' \otimes x')$$

in $\mathbb{H}_4'$. By Equation (7), we have

$$\Phi(z)\Phi(x) = \Phi(\sigma(x))\Phi(z), \quad \Phi(z)\Phi(g) = \Phi(\sigma(g))\Phi(z).$$

Therefore, we have

$$sb_0 = s',\ s = b_0 s',\ c_0 = b' - b\mu b_0,\ d_0 = b_0 b' - b\mu\kappa. \tag{23}$$

It is noted that $b_0 = \frac{s'}{s}$ and get that $b_0^2 = 1$.

Computing

$$(\Phi \otimes \Phi)(J_2^{b,d})\,(A' \otimes A') \quad \text{ and } \quad \Delta'(A')J_2^{b',d'}$$

as in the case (1), and comparing coefficients of all terms and noticing that Equation (23), one get that

$$d' = d\mu^2 - 2b\mu d_0 - d_0^2.$$

Hence

$$\begin{aligned} &a_0 = 1, \quad b_0 = \frac{s'}{s} = \varepsilon, \\ &c_0 = b' - b\mu\varepsilon, \quad d_0 = \varepsilon b' - b\mu, \\ &d' = d\mu^2 - b'^2 + b^2\mu^2 \end{aligned}$$

for any $d, d', b \neq 0, b' \neq 0$ and $s \neq 0,\ s'\mu \neq 0$.

In particular, for any $d, b \neq 0$ and $s \neq 0$, we can choose suitable triples (b_0, c_0, d_0), such that

$$b' = 1,\ s' = \varepsilon s,\ d' = d\mu^2 + b^2\mu^2 - 1.$$

This means that

$$(\sigma_2^s, J_2^{b,d}) \approx (\sigma_{2,s'}, J_{2,d'}),$$

where

$$\begin{aligned} &\sigma_{2,s'}(g) = g + 2gx,\ \sigma_{2,s'}(x) = s'x(s' \neq 0), \\ &J_{2,d'} = 1 \otimes 1 + 2(e_1 \otimes x) + d'\,(gx \otimes x) \end{aligned}$$

for any $d' \in \Bbbk$ and $0 \neq s' \in \Bbbk$.

(3) For the twisted homomorphism pair $(\sigma_3^{\kappa,s}, J_3^{b,c})$, where

$$\begin{aligned} &\sigma_3^{\kappa,s}(g) = 2\left((b+c)e_0 + \kappa(c-b)e_1)\right)x, \quad \sigma_3^{\kappa,s}(x) = s(e_0 + \kappa e_1)x\ (s \neq 0), \\ &J_3^{b,c} = 1 \otimes 1 + (\kappa^{-1} - 1)e_1 \otimes e_1 + 2be_1 \otimes x + 2cgx \otimes e_1 + d'(gx \otimes x) \end{aligned}$$

for any $b \neq 0$ and c, where $d' = \frac{4\kappa\, bc}{1-\kappa}$ with $\kappa \neq 0, 1$.

Now, we assume that

$$(\sigma_3^{\kappa,s}, J_3^{b,c}) \approx (\sigma_3^{\kappa',s'}, {J_3}^{b',c'}),$$

where

$$\begin{aligned} &\sigma_3^{\kappa',s'}(g') = g' + 2\left((b'+c')e_0' + \kappa'(c'-b')e_1')\right)x', \quad \sigma_3^{\kappa',s'}(x') = s'(e_0' + \kappa' e_1')x'\ (s' \neq 0), \\ &J_3^{b',c'} = 1 \otimes 1 + (\kappa'^{-1} - 1)e_1' \otimes e_1' + 2b'e_1' \otimes x' + 2c'g'x' \otimes e_1' + d''(g'x' \otimes x') \end{aligned}$$

in $\mathbb{H}_4'$. By Equation (7), we have

$$\Phi(z)\Phi(x) = \Phi(\sigma(x))\Phi(z), \quad \Phi(z)\Phi(g) = \Phi(\sigma(g))\Phi(z).$$

From the above equations, we easily get that

$$sb_0 = s',\ s\kappa = b_0 s'\kappa',\ c_0 = b' + c' - \mu b_0(b+c),\ d_0 = \mu\kappa(c-b) - b_0\kappa'(c'-b'). \tag{24}$$

It is noted that $b_0 = \frac{s'}{s}$ and $b_0^2 = \frac{\kappa}{\kappa'}$.

Furthermore, tediously computing

$$(\Phi \otimes \Phi)(J_3^{b,c})\left(A' \otimes A'\right) \quad \text{and} \quad \Delta'(A')J_3^{b',c'}$$

and comparing coefficients of all terms of them, one get that

$$d'' = d'\mu^2 + 2c\mu d_0 - 2b\mu d_0 - \kappa^{-1}d_0^2.$$

Hence

$$\begin{aligned} &a_0 = 1, \quad b_0 = \frac{s'}{s} = \varepsilon\sqrt{\frac{\kappa}{\kappa'}}, \\ &c_0 = (b'+c') - (b+c)\mu\sqrt{\frac{\kappa}{\kappa'}}\varepsilon, \quad d_0 = \mu\kappa(c-b) - \sqrt{\kappa\kappa'}(c'-b'), \\ &d'' = d'\mu^2 + \kappa(c-b)^2\mu^2 - \kappa'(c'-b')^2 \end{aligned}$$

for any $c,\ c',\ b,\ b' \neq 0$ and $s\kappa \neq 0,\ s'\kappa'\mu \neq 0$.

In particular, for any c and $s\kappa \neq 0$, we can choose suitable triples (b_0, c_0, d_0), such that

$$b' = 1,\ c' = 0,\ \kappa' = -1,\ s' = \varepsilon\sqrt{-\kappa}s,\ d'' = 0$$

and we get

$$(\sigma_3^{\kappa,s}, J_3^{b,c}) \approx (\sigma_{3,s'}, J_3),$$

where

$$\sigma_{3,s'}(g) = g + 2x, \quad \sigma_{3,s'}(x) = s'gx(s' \neq 0), \quad J_3 = 1 \otimes 1 - 2e_1 \otimes e_1 + 2e_1 \otimes x$$

for any $0 \neq s' \in \Bbbk$.

The proof is completed. □

Using Theorem 2 and ([22], Theorem 2.4), we deduce the following result.

Corollary 1. *Assume that $H = \mathbb{H}_4[z;\sigma]$, a BOEA for $\mathbb{H}_4$. Then H is one of the following lists up to isomorphism.*

(1) $H_1^{s,d}$: *$H_1^{s,d}$ is generated by g, x, z subjecting to the relations*

$$g^2 = 1,\ x^2 = 0,\ xg = -gx,\ zg = gz,\ zx = sxz\ (s \neq 0).$$

The coalgebra is defined by

$$\begin{aligned} &\Delta(g) = g \otimes g,\ \ \Delta(x) = x \otimes 1 + g \otimes x, \\ &\Delta(z) = z \otimes z + d\, gxz \otimes xz, \\ &\epsilon(g) = 1,\ \epsilon(z) = 1,\ \epsilon(x) = 0 \end{aligned}$$

for any d.

(2) $H_2^{s,d}$: *$H_2^{s,d}$ is generated by g, x, z subjecting to the relations*

$$g^2 = 1,\ x^2 = 0,\ xg = -gx,\ zx = sxz,\ zg = gz + 2gxz\ (s \neq 0).$$

The coalgebra is defined by

$$\Delta(g) = g \otimes g, \Delta(x) = x \otimes 1 + g \otimes x,$$
$$\Delta(z) = z \otimes z + 2e_1 z \otimes xz + d\, gxz \otimes \ xz$$
$$\epsilon(g) = 1, \epsilon(z) = 1, \epsilon(x) = 0$$

for any d.

(3) H_3^s: *H_3^s is generated by g, x, z subjecting to the relations*

$$g^2 = 1,\ x^2 = 0,\ xg = -gx,\ zx = sgxz\ (s \neq 0),\ zg = gz + 2xz.$$

The coalgebra is defined by

$$\Delta(g) = g \otimes g, \Delta(x) = x \otimes 1 + g \otimes x,$$
$$\Delta(z) = z \otimes z - 2e_1 z \otimes e_1 z + 2e_1 z \otimes xz,$$
$$\epsilon(g) = 1, \epsilon(z) = 1, \epsilon(x) = 0.$$

In the sequel, we always suppose that $H = \mathbb{H}_4[z; \sigma]$, a BOEA for $\mathbb{H}_4$.

4. The Quotients of the BOEA for $\mathbb{H}_4$

Let R be a Hopf algebra, and $R[z; \sigma]$ a BOEA for R. Suppose that there exists $0 \neq t \in R$ such that

$$\Delta(t) = \prod_{i=0}^{n-1} (\sigma^i \otimes \sigma^i)(J)(t \otimes t) \text{ and } th = \sigma^n(h)t \qquad (*)$$

for all $h \in R$. we get by ([6], Lemma 2.5) that $R[z; \sigma]/\langle z^n - t\rangle$ is a bialgebra.

The aim of this section is to investigate all bialgebra structures on the quotients $\mathbb{H}_4[z; \sigma]/\langle z^n - t\rangle$, where $t \in \mathbb{H}_4$ satisfying ($*$). Firstly, up to equivalence, (σ, J) should be one of the twisted homomorphisms given in Theorem 2.

Let us determine all $t \in \mathbb{H}_4$ satisfying ($*$).

(a) For the twisted homomorphism $(\sigma_{1,s}, J_{1,d})$, where

$$\sigma_{1,s}(g) = g, \quad \sigma_{1,s}(x) = sx,$$
$$J_{1,d} = 1 \otimes 1 + d\,(gx \otimes x)$$

for any $d \in \Bbbk$ and $s \neq 0$.

Now we assume that

$$t = a_1 e_0 + b_1 e_1 + c_1 e_0 x + d_1 e_1 x.$$

It is easy to see that $a_1 = 1$ since $\epsilon(t) = 1$. On the other hand, since

$$tx = \sigma_{1,s}^n(x)t, \quad tg = \sigma_{1,s}^n(g)t,$$

It follows that

$$b_1 = s^n, \quad b_1 s^n = 1,$$
$$c_1 = d_1 = 0.$$

Hence $t = e_0 + \varepsilon e_1$, where $b_1 = s^n = \varepsilon = \pm 1$. For simplicity of discussion, we denote t by t_ε and $\sigma_{1,s}$ by σ. It is easy to see that

$$\Delta(t_\varepsilon) = t_\varepsilon \otimes t_\varepsilon.$$

We have the following lemma for the case (a).

Lemma 1. *The element t_ε satisfies the following.*

(1) If $s^2 = 1$, then $\Delta(t_\varepsilon) = \prod_{i=0}^{n-1} (\sigma^i \otimes \sigma^i)(J_{1,d})(t_\varepsilon \otimes t_\varepsilon)$ if and only if $d = 0$.

(2) If $s^2 \neq 1$, then $\Delta(t_\varepsilon) = \prod_{i=0}^{n-1} (\sigma^i \otimes \sigma^i)(J_{1,d})(t_\varepsilon \otimes t_\varepsilon)$.

Proof. Noting that

$$(\sigma \otimes \sigma)(J_{1,d}) = (\sigma \otimes \sigma)(1 \otimes 1 + dgx \otimes x) = 1 \otimes 1 + ds^2 gx \otimes x,$$

we have

$$(\sigma^i \otimes \sigma^i)(J_{1,d}) = 1 \otimes 1 + ds^{2i} gx \otimes x$$

and

$$\prod_{i=0}^{\ell-1} \left(\sigma^i \otimes \sigma^i\right)(J_{1,d}) = \prod_{i=0}^{\ell-1} \left(1 \otimes 1 + d\, s^{2i} gx \otimes x\right) = 1 \otimes 1 + d \left(\sum_{i=0}^{\ell-1} s^{2i}\right) gx \otimes x.$$

(i) If $s^2 = 1$, then we have

$$\sum_{i=0}^{n-1} s^{2i} = \underbrace{1 + 1 + \cdots + 1}_{n \text{ times}} = n,$$

$$\prod_{i=0}^{n-1} \left(\sigma^i \otimes \sigma^i\right)(J_{1,d}) = 1 \otimes 1 + nd\, gx \otimes x.$$

One sees that

$$\prod_{i=0}^{n-1} \left(\sigma^i \otimes \sigma^i\right)(J_{1,d})(t_\varepsilon \otimes t_\varepsilon) = t_\varepsilon \otimes t_\varepsilon + nd\, t_{-\varepsilon} x \otimes t_\varepsilon x.$$

Hence if $s^2 = 1$, then $\Delta(t_\varepsilon) = \prod_{i=0}^{n-1} (\sigma^i \otimes \sigma^i)(J_{1,d})(t_\varepsilon \otimes t_\varepsilon)$ if and only if $d = 0$.

(ii) If $s^2 \neq 1$, then

$$\prod_{i=0}^{\ell-1} \left(\sigma^i \otimes \sigma^i\right)(J_{1,d}) = 1 \otimes 1 + d \frac{s^{2\ell} - 1}{s^2 - 1} gx \otimes x.$$

Noting that

$$\sum_{i=0}^{n-1} s^{2i} = \frac{s^{2n} - 1}{s^2 - 1} = 0,$$

we have

$$\prod_{i=0}^{n-1} \left(\sigma^i \otimes \sigma^i\right)(J_{1,d}) = 1 \otimes 1$$

and

$$\Delta(t_\varepsilon) = \prod_{i=0}^{n-1} \left(\sigma^i \otimes \sigma^i\right)(J_{1,d})(t_\varepsilon \otimes t_\varepsilon)$$

for any d.

The proof is completed. □

For suitable elements $d \in \Bbbk$ and $s = \omega \in \Bbbk$, if t_ε satisfies the hypothesis of Lemma 1, then

$$\overline{H} = \mathbb{H}_4[z;\sigma]/\langle z^n - t_\varepsilon \rangle$$

is a bialgebra.

In this case, we have $\omega^{2n} = 1$ and the bialgebra $\overline{H}$ is one of the following lists.

Case 1: If ω is 2-th primitive root of unity: $\omega^2 = 1$.

(i) if $n \geq 2$ is even, for example $n = 2m(m \geq 1)$, then $\omega^{2m} = 1$ and $t_\varepsilon = e_0 + \omega^{2m} e_1 = 1$. Thus we get the bialgebra $H^1_{8m}(m \geq 1)$ generated by g, x, z with the relations

$$g^2 = 1, xg = -gx, x^2 = 0, zg = gz, zx = -xz,\ z^{2m} = 1.$$

The coalgebra is

$$\Delta(g) = g \otimes g, \Delta(x) = x \otimes 1 + g \otimes x, \Delta(z) = z \otimes z;$$
$$\epsilon(g) = 1,\ \epsilon(z) = 1,\ \epsilon(x) = 0.$$

(ii) if $n > 2$ is odd, for example $n = 2m+1(m \geq 1)$, then $\omega^{2m+1} = -1$ and $t_\varepsilon = e_0 + \omega^{2m+1} e_1 = g$. Then we get the bialgebra $H^2_{4(2m+1)}(m \geq 1)$ generated by g, x, z with the relations

$$g^2 = 1, xg = -gx, x^2 = 0, zg = gz, zx = -xz,\ z^{2m+1} = g.$$

The coalgebra is

$$\Delta(g) = g \otimes g,\ \Delta(x) = x \otimes 1 + g \otimes x,\ \Delta(z) = z \otimes z;$$
$$\epsilon(g) = 1,\ \epsilon(z) = 1,\ \epsilon(x) = 0.$$

In fact, up to isomorphism, $H^2_{4(2m+1)}(m \geq 1)$ is generated by z, x with the relations

$$z^{4m+2} = 1,\ x^2 = 0,\ zx = -xz.$$

The coalgebra is

$$\Delta(x) = x \otimes 1 + z^{2m+1} \otimes x, \quad \Delta(z) = z \otimes z;$$
$$\epsilon(z) = 1,\ \epsilon(x) = 0.$$

Case 2: Assume that $n \geq 2$ and let ω be the $2r$-th primitive root of unity with $r > 1$. Since $\omega^{2n} = 1$, we have $r|n$. Let $n = \ell r$. Then $\omega^n = (-1)^\ell$.

(i) If ℓ is even, for example $\ell = 2m(m \geq 1)$, then $\omega^n = 1$ and $t_\varepsilon = e_0 + \omega^n e_1 = 1$. Thus we get the bialgebra $H^1_{8mr}(2m, r, d)(m \geq 1)$ generated by g, x, z with the relations

$$g^2 = 1, xg = -gx, x^2 = 0, zg = gz, zx = \omega xz,\ z^{2mr} = 1.$$

The coalgebra is

$$\Delta(g) = g \otimes g,\ \Delta(x) = x \otimes 1 + g \otimes x,\ \Delta(z) = z \otimes z - d\, xgz \otimes xz;$$

$$\epsilon(g) = 1,\ \epsilon(z) = 1,\ \epsilon(x) = 0.$$

(ii) If ℓ is odd, for example $\ell = 2m + 1 (m \geq 0)$, then $\omega^n = -1$ and $t_\varepsilon = e_0 + \omega^n e_1 = g$. Thus we get the bialgebra $H^2_{4(2m+1)r}(2m + 1, r, d)(m \geq 0)$ generated by g, x, z with the relations

$$g^2 = 1, xg = -gx, x^2 = 0, zg = gz, zx = \omega xz, \ z^{(2m+1)r} = g.$$

The coalgebra is

$$\Delta(g) = g \otimes g, \ \Delta(x) = x \otimes 1 + g \otimes x, \ \Delta(z) = z \otimes z - dx\, gz \otimes xz;$$

$$\epsilon(g) = 1, \ \epsilon(z) = 1, \ \epsilon(x) = 0.$$

In fact, up to isomorphism, $H^2_{4(2m+1)}(2m + 1, r, d)(m \geq 0)$ is generated by z, x with the relations

$$z^{2(2m+1)r} = 1, \ x^2 = 0, \ zx = \omega xz.$$

The coalgebra is

$$\Delta(x) = x \otimes 1 + z^{(2m+1)r} \otimes x, \quad \Delta(z) = z \otimes z - dx\, z^{(2m+1)r+1} \otimes xz;$$

$$\epsilon(z) = 1, \ \epsilon(x) = 0.$$

In particular, if $r = n$, then $\ell = 1$ and the bialgebra $H_{4n}(1, n, d)$ is generated by x, z with the relations

$$z^{2n} = 1, x^2 = 0, zx = \omega xz.$$

The coalgebra is

$$\Delta(z) = z \otimes z - dxz^{n+1} \otimes xz, \quad \Delta(x) = x \otimes 1 + z^n \otimes x;$$

$$\epsilon(g) = 1, \ \epsilon(z) = 1, \ \epsilon(x) = 0.$$

Case 3: Let ℓ be odd and ω be ℓ-th primitive root of unity with $\ell > 1$. Setting $\omega' = -\omega$, then ω' is the 2ℓ-th primitive root of unity with order 2ℓ and $(\omega')^{2n} = \omega^{2n} = 1$. This case turns into **Case 2**.

(b) For the twisted homomorphism $(\sigma_{2,s}, J_{2,d})$, where

$$\sigma_{2,s}(g) = g + 2gx, \quad \sigma_{2,s}(x) = sx,$$
$$J_{2,d} = 1 \otimes 1 + 2e_1 \otimes x + d\,(gx \otimes x)$$

for any $d \in \Bbbk$ and $s \neq 0$.

It is easy to see that if $s = 1$, then

$$\sigma^n_{2,s}(x) = x, \quad \sigma^n_{2,s}(g) = g + 2ngx \tag{25}$$

and if $s \neq 1$, then

$$\sigma^n_{2,s}(x) = s^n x, \quad \sigma^n_{2,s}(g) = g + \frac{2(1 - s^n)}{1 - s} gx. \tag{26}$$

Now we assume that

$$t = a_2 e_0 + b_2 e_1 + c_2 e_0 x + d_2 e_1 x.$$

One see that $a_2 = 1$ since $\epsilon(t) = 1$. On the other hand,

$$tx = \sigma^n_{2,s}(x)t, \quad tg = \sigma^n_{2,s}(g)t.$$

(i) If $s = 1$, then

$$e_0x + b_2e_1x = e_1x + b_2e_0x,$$
$$g - c_2e_0x + d_2e_1x = g + (c_1 + 2n)e_0x - (d_2 + 2n)e_1x.$$

Therefore, we have $b_2 = 1, c_2 = d_2 = -n$, and we get $t_0 = 1 - nx \in \mathbb{H}_4$ satisfying $(*)$ and

$$\Delta(t_0) = 1 \otimes 1 - nx \otimes 1 - ng \otimes x.$$

(ii) If $s \neq 1$, then

$$e_0x + b_1e_1x = s^ne_1x + s^nb_2e_0x,$$
$$-c_2e_0x + d_2e_1x = \left(c_2 + b_2\frac{2(1-s^n)}{1-s}\right)e_0x - \left(d_2 + \frac{2(1-s^n)}{1-s}\right)e_1x.$$

It follows that

$$b_2 = s^n, \quad b_2s^n = 1,$$
$$c_2 = b_2\frac{s^n-1}{1-s}, \quad d_2 = \frac{s^n-1}{1-s}.$$

Hence
$$t_\varepsilon = e_0 + \varepsilon e_1 + \frac{1-\varepsilon}{1-s}e_0x + \frac{\varepsilon-1}{1-s}e_1x = e_0 + \varepsilon e_1 + \frac{1-\varepsilon}{1-s}gx,$$
where $b_1 = s^n = \varepsilon = \pm 1$. We denote t by t_ε and $\sigma_{2,s}$ by σ in discussion.

We have the following in the case (b).

Lemma 2. *The element t_ε satisfies the following condition:*

(1) If $s^2 = 1$, then $\Delta(t_\varepsilon) = \prod_{i=0}^{n-1}(\sigma^i \otimes \sigma^i)(J_{2,d})(t_\varepsilon \otimes t_\varepsilon)$ if and only if $d = -1$.

(2) If $s^2 \neq 1$, then $\Delta(t_\varepsilon) = \prod_{i=0}^{n-1}(\sigma^i \otimes \sigma^i)(J_{2,d})(t_\varepsilon \otimes t_\varepsilon)$.

Proof. Analogous argument to the proof of Lemma 1. □

For suitable elements $d \in \Bbbk$ and $s = \omega \in \Bbbk$, if t_ε satisfies the hypothesis of Lemma 2, then $\overline{B} = \mathbb{H}_4[z;\sigma]/\langle z^n - t_\varepsilon\rangle$ is a bialgebra.

In this case, $\omega^{2n} = 1$ and $\varepsilon = \omega^n$. The bialgebra $\overline{B}$ is one of the following lists.

Case 1: If $\omega = 1$, of course $\omega^2 = \omega^n = \omega^{2n} = 1$. Thus we get the bialgebra B^1_{4n} generated by g, x, z with the relations

$$g^2 = 1, xg = -gx, x^2 = 0, zg = gz + 2gxz, zx = xz,\ z^n = 1 - nx.$$

The coalgebra is

$$\Delta(g) = g \otimes g, \Delta(x) = x \otimes 1 + g \otimes x, \Delta(z) = z \otimes z + 2e_1z \otimes xz - gxz \otimes xz;$$

$$\epsilon(g) = 1,\ \epsilon(z) = 1,\ \epsilon(x) = 0.$$

Case 2: If $\omega = -1$, then $\omega^2 = 1$.

(i) Assume that $n \geq 2$ is even, for example $n = 2m(m \geq 1)$, then $\omega^{2m} = 1$. Thus we get the bialgebra $B^2_{8m}(m \geq 1)$ generated by g, x, z with the relations

$$g^2 = 1, xg = -gx, x^2 = 0, zg = gz + 2gxz, zx = -xz,\ z^{2m} = 1.$$

The coalgebra is

$$\Delta(g) = g \otimes g, \Delta(x) = x \otimes 1 + g \otimes x, \Delta(z) = z \otimes z + 2e_1 z \otimes xz - gxz \otimes xz;$$

$$\epsilon(g) = 1,\ \epsilon(z) = 1,\ \epsilon(x) = 0.$$

(ii) if $n > 2$ is odd, for example $n = 2m + 1(m \geq 1)$, then $\omega^n = \omega^{2m+1} = -1$. Thus we get the bialgebra $B^3_{4(2m+1)}(m \geq 1)$ generated by g, x, z with the relations

$$g^2 = 1, xg = -gx, x^2 = 0, zg = gz + 2gxz, zx = -xz,\ z^{2m+1} = g + gx.$$

The coalgebra is

$$\Delta(g) = g \otimes g,\ \Delta(x) = x \otimes 1 + g \otimes x,\ \Delta(z) = z \otimes z + 2e_1 z \otimes xz - gxz \otimes xz;$$

$$\epsilon(g) = 1,\ \epsilon(z) = 1,\ \epsilon(x) = 0.$$

Case 3: Let ω be $2r$-th primitive root of unity with $r > 1$. Since $\omega^{2n} = 1$, we have $r|n$. Let $n = \ell r$. Therefore, $\omega^n = (-1)^\ell$.

(i) If ℓ is even, for example $\ell = 2m$, we get the bialgebra $B^1_{8mr}(2m, r, d)$ generated by g, x, z with the relations

$$g^2 = 1, xg = -gx, x^2 = 0, zg = gz + 2gxz, zx = \omega xz,\ z^{2mr} = 1.$$

The coalgebra is

$$\Delta(g) = g \otimes g,\ \Delta(x) = x \otimes 1 + g \otimes x,\ \Delta(z) = z \otimes z + 2e_1 z \otimes xz + dg\, xz \otimes xz;$$

$$\epsilon(g) = 1,\ \epsilon(z) = 1,\ \epsilon(x) = 0.$$

(ii) If ℓ is odd with $\ell = 2m + 1$, we get a bialgebra $B^2_{4(2m+1)r}(2m + 1, r, d)$ generated by g, x, z with the relations

$$g^2 = 1, xg = -gx, x^2 = 0, zg = gz + 2gxz, zx = \omega xz,\ z^{(2m+1)r} = g + \frac{2}{1-\omega} gx.$$

The coalgebra is

$$\Delta(g) = g \otimes g,\ \Delta(x) = x \otimes 1 + g \otimes x,\ \Delta(z) = z \otimes z + 2e_1 z \otimes xz + dg\, xz \otimes xz;$$

$$\epsilon(g) = 1,\ \epsilon(z) = 1,\ \epsilon(x) = 0.$$

Case 4: Let ℓ be odd and ω be ℓ-th primitive root of unity with $\ell > 1$. Setting $\omega' = -\omega$, then ω' is the 2ℓ-th primitive root of unity with order 2ℓ and $(\omega')^{2n} = \omega^{2n} = 1$. This case turns into **Case 3**.

(c) For the twisted homomorphism $(\sigma_{3,s}, J_3)$, we also denote $\sigma_{3,s}$ by σ in the case. It is easy to see that

$$\sigma^i(e_0) = e_0 + f_i(s)e_0 x + f_i(-s)e_1 x, \quad \sigma^i(e_1) = e_1 - f_i(s)e_0 x - f_i(-s)e_1 x.$$

$$\sigma^i(e_0x) = s^i e_0 x, \quad \sigma^i(e_1x) = (-s)^i e_1 x,$$

where $f_0(x) = 0$ and for $i > 0$

$$f_i(x) = 1 + x + \cdots + x^{i-1} = \begin{cases} i, \text{ for } x = 1, \\ \frac{x^i - 1}{x - 1}, \text{ for } x \neq 1. \end{cases}$$

We see that

$$f_i(x) + x^i = f_{i+1}(x)$$

and the following equations hold by induction.

$$2\sum_{i=0}^{k}(-1)^{k-i} f_i(x) - f_{k+1}(x) = (-1)^{k+1} f_{k+1}(-x), \tag{27}$$

$$2\sum_{i=0}^{k}(-1)^{k-i} f_{i+1}(x) - f_{k+2}(x) = (-1)^{k+2} f_{k+2}(-x). \tag{28}$$

We also see that

$$\begin{aligned} & \left(\sigma^i \otimes \sigma^i\right)(J_3) \\ = & \left(\sigma^i \otimes \sigma^i\right)(1 \otimes 1 - 2e_1 \otimes e_1 + 2e_1 \otimes e_0 x + 2e_1 \otimes e_1 x) \\ = & \mathcal{A} - 2f_i(s) f_{i+1}(s) e_0 x \otimes e_0 x - 2f_i(-s) f_{i+1}(-s) e_1 x \otimes e_1 x \\ & +2f_{i+1}(s) e_1 \otimes e_0 x + 2f_{i+1}(-s) e_1 \otimes e_1 x + 2f_i(s) e_0 x \otimes e_1 \\ & +2f_i(-s) e_1 x \otimes e_1 - 2f_i(s) f_{i+1}(-s) e_0 x \otimes e_1 x - 2f_i(-s) f_{i+1}(s) e_1 x \otimes e_0 x \end{aligned}$$

where $\mathcal{A} = 1 \otimes 1 - 2e_1 \otimes e_1$. It is noted that

$$\mathcal{A}^l = \begin{cases} 1 \otimes 1, \text{ if } l \text{ is even;} \\ \mathcal{A}, \text{ if } l \text{ is odd.} \end{cases}$$

Lemma 3. *We have*

$$\begin{aligned} & \prod_{i=0}^{l}\left(\sigma^i \otimes \sigma^i\right)(J_3) = \mathcal{A}^{l+1} \\ & +2\left(\sum_{i=0}^{l}(-1)^{l-i} f_{i+1}(s)\right) e_1 \otimes e_0 x + 2\left(\sum_{i=0}^{l}(-1)^{i} f_{i+1}(-s)\right) e_1 \otimes e_1 x \\ & +2\left(\sum_{i=0}^{l}(-1)^{l-i} f_{i}(s)\right) e_0 x \otimes e_1 + 2\left(\sum_{i=0}^{l}(-1)^{i} f_{i}(-s)\right) e_1 x \otimes e_1 \\ & +2\left(\sum_{i=0}^{l}(-1)^{l-i+1} f_i(s) f_{i+1}(s)\right) e_0 x \otimes e_0 x + 2\left(\sum_{i=0}^{l}(-1)^{i+1} f_i(-s) f_{i+1}(-s)\right) e_1 x \otimes e_1 x \\ & +2\left(\sum_{i=0}^{l}(-1)^{i} f_i(-s) f_{i+1}(-s)\right) e_0 x \otimes e_1 x + 2\left(\sum_{i=0}^{l}(-1)^{i+1} f_i(-s) f_{i+1}(-s)\right) e_1 x \otimes e_0 x. \end{aligned}$$

Proof. The equation is trivial if $l = 0$. Applying Equations (27) and (28), one can get the result by induction.

The proof is finished. □

Assume that

$$tx = \sigma_{3,s}^n(x)t, \quad tg = \sigma_{3,s}^n(g)t,$$

and

$$t = a_3 e_0 + b_3 e_1 + c_3 e_0 x + d_3 e_1 x.$$

It is easy to see that $a_3 = 1$ since $\epsilon(t) = 1$ and

$$\sigma^i(e_0 x) = s^i e_0 x, \quad \sigma^i(e_1 x) = (-s)^i e_1 x,$$

$$\begin{aligned} &e_0 x + b_3 e_1 x = b_3 s^n e_0 x + (-s)^n e_1 x, \\ &e_0 - b_1 e_1 - c_1 e_0 x + d_1 e_1 x = e_0 - b_1 e_1 + (c_1 + 2b_1 f_n(s)) e_0 x + (2 f_n(-s) - d_1) e_1 x. \end{aligned}$$

It follows that

$$b_3 = s^{-n} = (-s)^n, \ \ c_3 = -b_3 f_n(s), \quad d_3 = f_n(-s).$$

Therefore, we have

$$t = e_0 + b_3 e_1 - b_3 f_n(s) e_0 x + f_n(-s) e_1 x,$$

where $(-s^2)^n = 1$ and $b_3 = (-1)^n s^n$. We also have $b_3^2 = s^{2n} = (-1)^n$.

We denote t by t_ν where $\nu = (-1)^n s^n$ with $(-s^2)^n = 1$. In this case

$$t_\nu = e_0 + \nu e_1 - \nu f_n(s) e_0 x + f_n(-s) e_1 x.$$

It is easy to see that

$$\begin{aligned} \Delta(t_\nu) \quad = \quad & e_0 \otimes e_0 + e_1 \otimes e_1 + b_3 e_1 \otimes e_0 + b_3 e_0 \otimes e_1 + c_3 e_0 x \otimes e_0 + c_3 e_0 \otimes \ e_0 x \\ & + c_3 e_1 x \otimes e_1 - c_3 e_1 \otimes e_1 x + d_3 e_1 x \otimes e_0 - d_3 e_1 \otimes e_0 x + d_3 e_0 x \otimes e_1 + d_3 e_0 \otimes e_1 x. \end{aligned}$$

Lemma 4. *The following condition holds*

$$\Delta(t_\nu) = \prod_{i=0}^{n-1} \left(\sigma^i \otimes \sigma^i\right)(J_3)(t_\nu \otimes t_\nu)$$

if and only if

$$\sum_{i=0}^{n-1} (-s^2)^i = 0.$$

Proof. (Sketch) By Lemma 3, it is straightforward to see

$$\begin{aligned} & \prod_{i=0}^{n-1} \left(\sigma^i \otimes \sigma^i\right)(J_3)(t_\nu \otimes t_\nu) \\ = \quad & \Delta(t_\nu) + \left(\sum_{i=0}^{n-1} (-1)^{i+1} s^{2i}\right) e_0 x \otimes e_0 x + \nu \left(\sum_{i=0}^{n-1} (-1)^{i+1} s^{2i}\right) e_0 x \otimes \ e_1 x \\ & + \nu \left(\sum_{i=0}^{n-1} (-1)^i s^{2i}\right) e_1 x \otimes e_0 x + \left(\sum_{i=0}^{n-1} (-1)^{n-i} s^{2i}\right) e_1 x \otimes e_1 x. \end{aligned}$$

Comparing the coefficients of each term of two-hand side of $\Delta(t_\nu)$, we have

$$\Delta(t_\nu) = \prod_{i=0}^{n-1} \left(\sigma^i \otimes \sigma^i\right)(J_3)(t_\nu \otimes t_\nu)$$

if and only if

$$\sum_{i=0}^{n-1}(-s^2)^i = 0.$$

The proof is completed. $\square$

Now we assume that $n \geq 2$, $s = \omega \in \Bbbk$, and

$$t_\nu = e_0 + \nu e_1 - \nu f_n(s)e_0 x + f_n(-s)e_1 x,$$

where

$$\nu = (-1)^n \omega^n, \quad (-\omega^2)^n = 1, \quad \sum_{i=0}^{n-1}(-\omega^2)^i = 0.$$

Thus, we get that

$$\overline{C} = \mathbb{H}_4[z;\sigma]/\langle z^n - t_\nu \rangle$$

is a bialgebra.

In this case, $(-\omega^2)^n = 1 \ (n \geq 2)$, $\omega^2 \neq -1$, and $\nu = (-1)^n\omega^n$. The bialgebra $\overline{C}$ is one of the following lists.

Case 1: If $-\omega^2 = -1$, then $\omega = \pm 1$. Hence n must be an even and set $n = 2m$. Then

$$\sum_{i=0}^{n-1}(-\omega^2)^i = 0.$$

(i) If $\omega = 1$, then $\nu = (-1)^n\omega^n = 1$ and

$$t_1 = 1 - 2m\, e_0 x.$$

We get the bialgebra $C^1_{8m} (m \geq 1)$ generated by g, x, z with the relations

$$g^2 = 1, xg = -gx, x^2 = 0, zg = gz + 2xz, zx = gxz,\ z^{2m} = 1 - 2m\, e_0 x.$$

The coalgebra is

$$\Delta(g) = g \otimes g,\ \Delta(x) = x \otimes 1 + g \otimes x,\ \Delta(z) = z \otimes z - 2e_1 z \otimes e_1 z + 2e_1 z \otimes xz;$$

$$\epsilon(g) = 1,\ \epsilon(z) = 1,\ \epsilon(x) = 0.$$

(ii) If $\omega = -1$, then $\nu = (-1)^n\omega^n = 1$ and

$$t_1 = 1 + 2m\, e_1 x.$$

We get the bialgebra $C^2_{8m} (m \geq 1)$ generated by g, x, z with the relations

$$g^2 = 1, xg = -gx, x^2 = 0, zg = gz + 2xz, zx = -gxz,\ z^{2m} = 1 + 2m\, e_1 x.$$

The coalgebra is

$$\Delta(g) = g \otimes g,\ \Delta(x) = x \otimes 1 + g \otimes x,\ \Delta(z) = z \otimes z - 2e_1 z \otimes e_1 z + 2e_1 z \otimes xz;$$

$$\epsilon(g) = 1,\ \epsilon(z) = 1,\ \epsilon(x) = 0.$$

Case 2: Assume that $\omega^2 \neq \pm 1$ and $n \geq 4$ is even, we have $\omega^{2n} = 1$ and

$$\sum_{i=0}^{n-1} (-\omega^2)^i = 0$$

always holds.

(a) Let ω be a $2r$-th primitive root of unity with $r > 2$. Then we also have $r|n$. Let $n = \ell r$. Then $\nu = (-\omega)^n = \omega^n = (-1)^\ell$.

(i) If ℓ is even with $\ell = 2m$, then $\nu = 1$ and

$$f_n(\omega) = \frac{\omega^n - 1}{\omega - 1} = 0,\ f_n(-\omega) = \frac{(-\omega)^n - 1}{-\omega - 1} = 0.$$

Hence $t_1 = e_0 + e_1 = 1$ and we get a bialgebra $C^1_{8mr}(2m, r)(m \geq 1)$ generated by g, x, z with the relations

$$g^2 = 1, xg = -gx, x^2 = 0, zg = gz + 2xz, zx = \omega gxz,\ z^{2mr} = 1.$$

The coalgebra is

$$\Delta(g) = g \otimes g,\ \Delta(x) = x \otimes 1 + g \otimes x,\ \Delta(z) = z \otimes z - 2e_1 z \otimes e_1 z + 2e_1 z \otimes xz;$$

$$\epsilon(g) = 1,\ \epsilon(z) = 1,\ \epsilon(x) = 0.$$

(ii) If ℓ is odd with $\ell = 2m + 1(m \geq 1)$, then $\nu = -1$ and

$$f_n(\omega) = \frac{\omega^n - 1}{\omega - 1} = \frac{2}{1 - \omega},\ f_n(-\omega) = \frac{(-\omega)^n - 1}{-\omega - 1} = \frac{2}{1 + \omega}.$$

Hence

$$t_{-1} = g + \frac{2}{1 - \omega} e_0 x + \frac{2}{1 + \omega} e_1 x.$$

We get a bialgebra $C^2_{4(2m+1)r}(2m + 1, r)(m \geq 1, r$ is even) generated by g, x, z with the relations

$$g^2 = 1, xg = -gx, x^2 = 0, zg = gz + 2xz, zx = \omega gxz,$$
$$z^{(2m+1)r} = g + \frac{2}{1 - \omega} e_0 x + \frac{2}{1 + \omega} e_1 x.$$

The coalgebra is

$$\Delta(g) = g \otimes g,\ \Delta(x) = x \otimes 1 + g \otimes x,\ \Delta(z) = z \otimes z - 2e_1 z \otimes e_1 z + 2e_1 z \otimes xz;$$

$$\epsilon(g) = 1,\ \epsilon(z) = 1,\ \epsilon(x) = 0.$$

(b) Let ℓ be odd and ω be ℓ-th primitive root of unity with $\ell > 1$. We can replace ω by $-\omega$. This case turns into the case (a) above.

Case 3: Assume that $\omega^2 \neq \pm 1$ and $n = 2m + 1(m \geq 1)$, we have $\omega^n = \pm \mathbf{i}$ and

$$\sum_{i=0}^{n-1} (-\omega^2)^i = 0.$$

It is easy to see that

$$\nu = (-\omega)^n = -\omega^n, \quad f_n(\omega) = \frac{\omega^n - 1}{\omega - 1}, \quad f_n(-\omega) = \frac{\omega^n + 1}{\omega + 1}.$$

Hence

$$t_\nu = e_0 - \omega^n e_1 + \frac{\omega^n + 1}{1 - \omega} e_0 x + \frac{\omega^n + 1}{1 + \omega} e_1 x.$$

We get the bialgebra $C^{\pm}_{4(2m+1)}$ $(m \geq 1)$ generated by g, x, z with the relations

$$g^2 = 1, xg = -gx, x^2 = 0, zg = gz + 2xz, zx = \omega gxz,$$

$$z^{2m+1} = e_0 \mp \mathbf{i} e_1 + \frac{\pm\mathbf{i} + 1}{1 - \omega} e_0 x + \frac{\pm\mathbf{i} + 1}{\omega + 1} e_1 x,$$

where $\omega^{2m+1} = \pm\mathbf{i}$.

The coalgebra is

$$\Delta(g) = g \otimes g,\ \Delta(x) = x \otimes 1 + g \otimes x,\ \Delta(z) = z \otimes z - 2e_1 z \otimes e_1 z + 2e_1 z \otimes xz;$$

$$\epsilon(g) = 1,\ \epsilon(z) = 1,\ \epsilon(x) = 0.$$

In summary we get the following result.

Theorem 3. *Let $H = \mathbb{H}_4[z; \sigma]$ be the BOEA for $\mathbb{H}_4$ and $\overline{H} := H / \langle z^n - t \rangle$, where $\sigma \in \mathrm{Aut}(\mathbb{H}_4)$ and $t \in \mathbb{H}_4$ satisfy*

$$\Delta(t) = \prod_{i=0}^{n-1} (\sigma^i \otimes \sigma^i)(J)(t \otimes t)$$

and

$$th = \sigma^n(h)t, \text{ for all } h \in \mathbb{H}_4.$$

Then $\overline{H}$ is a bialgebra and up to isomorphism, it is one of the following lists.

(a) $H^1_{8m}(m \geq 1),\ H^2_{4(2m+1)}(m \geq 1),$

$H^1_{8mr}(2m, r, d)(m \geq 1),\ H^2_{4(2m+1)r}(2m+1, r, d)(m \geq 0)$, *where ω is $2r$-th root of unity with $r > 1$;*

(b) $B^1_{4m}(m \geq 1), B^2_{8m}(m \geq 1),$

$B^3_{4(2m+1)}(m \geq 1), B^1_{8mr}(2m, r, d)(m \geq 1), B^2_{4(2m+1)r}(2m+1, r, d)(m \geq 0)$, *where ω is $2r$-th root of unity with $r > 1$;*

(c) $C^1_{8m}(m \geq 1), C^2_{8m}(m \geq 1),$

$C^3_{8mr}(2m, r)(m \geq 1), C^4_{4(2m+1)r}(2m+1, r)(m \geq 1, r$ *is even*$)$, *where ω is an $2r$-th root of unity with $r > 1$;*

(d) $C^{\pm}_{4(2m+1)}(m \geq 1)$, *where $\omega^{2m+1} = \pm\mathbf{i}$.*

5. Hopf Algebra Structures for $\mathbb{H}_4$-Ore Extension of Automorphism Type

Let $H = \mathbb{H}_4[z; \sigma]$ be the BOEA for $\mathbb{H}_4$, and H_z an algebra obtained from H by adding a new generator z^{-1} such that

$$zz^{-1} = z^{-1}z = 1.$$

Theorem 4. *Keeping notations as above. Then up to isomorphism, H_z is a Hopf algebra if and only if H_z is generated by g, x, z subjecting to relations*

$$g^2 = 1,\ x^2 = 0,\ zz^{-1} = z^{-1}z = 1,$$
$$xg = -gx,\ zg = gz,\ zx = sxz\ (s \neq 0).$$

The coalgebra is defined by

$$\Delta(g) = g \otimes g,\ \ \Delta(x) = x \otimes 1 + g \otimes x,$$
$$\Delta(z) = z \otimes z + d\, gxz \otimes xz,$$
$$\epsilon(g) = 1,\ \epsilon(z) = 1,\ \epsilon(x) = 0,$$
$$S(g) = g,\ S(z) = z^{-1},\ S(x) = -gx$$

for any d.

Proof. Up to equivalence we have yielded the twisted homomorphism (σ, J) for $\mathbb{H}_4$ listed in Theroem 2.

(a) For the twisted homomorphism $(\sigma_{1,s}, J_{1,d})$, where

$$\sigma_{1,s}(g) = g,\quad \sigma_{1,s}(x) = sx,$$
$$J_{1,d} = 1 \otimes 1 + d\,(gx \otimes x)$$

for any $d \in \mathbb{k}$ and $s \neq 0$. One see that

$$S^{\sigma_{1,s}}(g) = g = S(g),\quad S^{\sigma_{1,s}}(x) = \sigma_{1,s}S(s^{-1}x) = -s^{-1}\sigma(gx) = -gx = S(x).$$

It follows that $\theta_l = \theta_r = 1$ and

$$S^{\sigma_{1,s}}(h) = S(h)$$

for all $h \in \mathbb{H}_4$. Hence H_z is a Hopf algebra with $S(z) = z^{-1}$.

(b) For the twisted homomorphism $(\sigma_{2,s}, J_{2,d})$, where

$$\sigma_{2,s}(g) = g + 2gx,\ \sigma_{2,s}(x) = sx,$$
$$J_{2,d} = 1 \otimes 1 + 2e_1 \otimes x + d(gx \otimes x)$$

for any $d \in k$ and $s \neq 0$. one see that

$$\sigma_{2,s}^{-1}(g) = g - 2s^{-1}gx,\quad \sigma_{2,s}^{-1}(x) = s^{-1}x.$$

Hence we have

$$S^{\sigma_{2,s}}(g) = g + 4e_0x,\quad S^{\sigma_{2,s}}(x) = gx.$$

In this case, we have $\theta := \theta_l = 1 + 2e_1x = \theta_r$. But

$$S^{\sigma_{2,s}}(g) = g + 4e_0x,\quad \theta^{-1}S(g)\theta = (1 - 2e_1x)g(1 + 2e_1x) = g + 4e_1x.$$

It follows that $S^{\sigma_{2,s}}(g) \neq \theta^{-1}S(g)\theta$ and H_z is not a Hopf algebra by Theorem 1.

(c) For the twisted homomorphism $(\sigma_{3,s}, J_3)$, where

$$\sigma_{3,s}(g) = g + 2x,\ \sigma_{3,s}(x) = s\,gx(s \neq 0),$$
$$J_3 = 1 \otimes 1 - 2e_1 \otimes e_1 + 2e_1 \otimes x.$$

Similarly, H_z also is not a Hopf algebra.

The proof is completed. □

Now, we consider Hopf algebra structure on the quotient

$$\mathbb{H}_4[z;\sigma]/\left\langle z^n - t\right\rangle,$$

where $\mathbb{H}_4[z;\sigma]$ is the BOEA for $\mathbb{H}_4$ satisfying

$$\Delta(t) = \prod_{i=0}^{n-1}(\sigma^i \otimes \sigma^i)(J)(t \otimes t)$$

and

$$th = \sigma^n(h)t, \text{ for all } h \in \mathbb{H}_4.$$

The following result is one of the main results.

Theorem 5. *Let $H = \mathbb{H}_4[z;\sigma]$ be the BOEA for $\mathbb{H}_4$ and $\overline{H} := H/\left\langle z^n - t\right\rangle$, where $\sigma \in \mathrm{Aut}(\mathbb{H}_4)$ and $t \in \mathbb{H}_4$ satisfy*

$$\Delta(t) = \prod_{i=0}^{n-1}(\sigma^i \otimes \sigma^i)(J)(t \otimes t)$$

and

$$th = \sigma^n(h)t, \text{ for all } h \in \mathbb{H}_4.$$

If $\overline{H}$ is a Hopf algebra, then it is one of the following lists up to isomorphism.

(a) $H^1_{8m}(m \geq 1)$, $H^2_{4(2m+1)}(m \geq 1)$;

(b) $H^1_{8mr}(2m, r, d)(m \geq 1)$, $H^2_{4(2m+1)r}(2m+1, r, d)(m \geq 0)$, *where ω is $2r$-th root of unity with $r > 1$.*

Proof. By the proof of Theorem 4, we see that only for the twisted homomorphism $(\sigma_{1,s}, J_{1,d})$, θ_l and θ_r enjoy the conditions (1) and (2) in Theorem 1.

Now, we assume that $d = 0$ if $\omega^2 = 1$, and d is arbitrary if $\omega^2 \neq 1$ and $\omega^{2n} = 1$. Note that

$$\prod_{i=0}^{n-1}\left(\sigma^i \otimes \sigma^i\right)(J_{1,d}) = \prod_{i=0}^{n-1}\left(1 \otimes 1 + d\,\omega^{2i} gx \otimes x\right) = 1 \otimes 1 + d\left(\sum_{i=0}^{n-1}\omega^{2i}\right) gx \otimes x.$$

Hence

$$\prod_{i=0}^{n-1}\left(\sigma^i \otimes \sigma^i\right)(J_{1,d}) = 1 \otimes 1.$$

Therefore, the remaining conditions in Theorem 1 also hold. By Theorem 1, we get that $H^1_{8m}(m \geq 1)$, $H^2_{4(2m+1)}(m \geq 1)$; $H^1_{8mr}(2m, r, d)(m \geq 1)$, $H^2_{4(2m+1)r}(2m+1, r, d)(m \geq 0)$, where ω is $2r$-th root of unity with $r > 1$, are all Hopf algebras. The antipodes S can be easily given by Theorem 1.

The remaining two cases are referred to Theorem 3 and the proof of Theorem 4.

This completes the proof. □

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