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Common Fixed Point Theorems in Intuitionistic Generalized Fuzzy Cone Metric Spaces

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Abstract: In the present work, we study many fixed point results in intuitionistic generalized fuzzy cone metric space. Precisely, we prove new common fixed point theorems for three self mappings that do not require any commutativity or continuity but a generalized contractive condition. Our results are generalizations for many results in the literature. Some examples are given to support these results.

Keywords: fixed point; cone; triangular; fuzzy contractive; symmetric

1. Introduction

In the year 1965, Zadeh [1] introduced the concept of fuzzy sets which permit the gradual assessment of the membership of the elements in a set. In contrast to classical sets, these sets are serving good in describing the vague and imprecise expressions in a formal way. As these sets have no means of incorporating the hesitation, Atanassov [2] brought out a possible solution with intuitionistic fuzzy sets in the year 1983. These sets serve as a powerful tool to deal with vagueness. In addition, in the year 1975, Kramosil and Michalek [3] first introduced a metric on fuzzy sets. Subsequently, kinds of fuzzy metrics [4–6] were introduced over fuzzy sets. In the year 1994, George and Veeramani [7] modified the definition of fuzzy metric space that was given by Kramosil and Michalek [3] and obtained a metrizable Hausdorff topology. As a consequence of these findings, several authors came up with generalized versions of these spaces in various settings. In 2007, Huang and Zhang [8] introduced cone and cone metric space, and, after that, Tarkan Oner et al. [9] defined fuzzy cone metric space as a generalization of fuzzy metric space [7]. Mohamed and Ranjith [10] came up with intuitionistic fuzzy cone metric space in the year 2017.

In 2019, Jeyaraman and Sowndrarajan [11] defined intuitionistic generalized fuzzy cone metric space as a generalization in the sense of Sedghi and Shobe [12] and proved some common fixed point theorems for (ϕ, ψ)-weak contractions in these spaces. The idea of fuzzy contractive mapping was introduced by Gregori and Sapena [13], and they have also extended the Banach fixed point theorem with fuzzy contractive mappings. Many researchers have established many common fixed point theorems in these spaces and in their extended versions, see [13–24].

In the present paper, we study the Banach Contraction theorem in the setting of intuitionistic generalized fuzzy cone metric space [11] and to construct some common fixed point theorems for three self mappings which satisfy generalized contractive conditions in the intuitionistic generalized fuzzy cone metric spaces. The significant advantage of these theorems is that they work well where the Banach contraction theorem fails. Examples are provided to exhibit the novelty of the results given here.

2. Preliminaries

Let us begin the section with triangular norms which are kinds of binary operations introduced by Karl Menger and later revised by Schweizer and Sklar [25] with stronger axioms, as stated here.

Definition 1. *A binary operation* $* : [0,1] \times [0,1] \to [0,1]$ *is a continuous t-norm if it satisfies the following conditions:*

[n1] $*$ *is commutative, associative, and continuous,*
[n2] $a * 1 = a$ *for all* $a \in [0,1]$,
[n3] $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$, *and* $a, b, c, d \in [0,1]$.

Definition 2. *A binary operation* $\diamond : [0,1] \times [0,1] \to [0,1]$ *is a continuous t-conorm if it satisfies the following conditions:*

[cn1] $\diamond$ *is commutative, associative and continuous,*
[cn2] $a \diamond 0 = a$ *for all* $a \in [0,1]$,
[cn3] $a \diamond b \leq c \diamond d$ *whenever* $a \leq c$ *and* $b \leq d$, *and* $a, b, c, d \in [0,1]$.

Definition 3 ([9])**.** *Let* $\mathfrak{B}$ *be a real Banach space and* $\mathfrak{C}$ *be a subset of* $\mathfrak{B}$. $\mathfrak{C}$ *is called a closed cone if and only if:*

[C1] $\mathfrak{C}$ *is nonempty, closed and* $\mathfrak{C} \neq \{0\}$,
[C2] $\rho, \sigma \in \mathbb{R}, \rho, \sigma \geq 0, c_1, c_2 \in \mathfrak{C}$ *imply* $\rho c_1 + \sigma c_2 \in \mathfrak{C}$,
[C3] $c \in \mathfrak{C}$ *and* $-c \in \mathfrak{C}$ *imply* $c = 0$.

The closed cones considered here are subsets of a real Banach space $\mathfrak{B}$ *and are with nonempty interiors.*

Definition 4 ([11])**.** *An Intuitionistic Generalized Fuzzy Cone Metric Space (briefly, IGFCM Space) is a 5-tuple* $(\mathcal{Z}, \mathfrak{M}, \mathfrak{N}, *, \diamond)$ *where* $\mathcal{Z}$ *is an arbitrary set,* $*$ *is a continuous t-norm,* $\diamond$ *is a continuous t-conorm,* $\mathfrak{C}$ *is a closed cone and* $\mathfrak{M}, \mathfrak{N}$ *are fuzzy sets in* $\mathcal{Z}^3 \times int(\mathfrak{C})$ *satisfying the following conditions: For all* $\zeta, \eta, \omega, \mathsf{u} \in \mathcal{Z}$ *and* $c, c' \in int(\mathfrak{C})$,

(1) $\mathfrak{M}(\zeta, \eta, \omega, c) + \mathfrak{N}(\zeta, \eta, \omega, c) \leq 1$,
(2) $\mathfrak{M}(\zeta, \eta, \omega, c) > 0$,
(3) $\mathfrak{M}(\zeta, \eta, \omega, c) = 1$ *if and only if* $\zeta = \eta = \omega$,
(4) $\mathfrak{M}(\zeta, \eta, \omega, c) = \mathfrak{M}(p\{\zeta, \eta, \omega\}, c)$, *where p is a permutation function,*
(5) $\mathfrak{M}(\zeta, \eta, \omega, c + c') \geq \mathfrak{M}(\zeta, \eta, \mathsf{u}, c) * \mathfrak{M}(\mathsf{u}, \omega, \omega, c')$,
(6) $\mathfrak{M}(\zeta, \eta, \omega, \cdot) : int(\mathfrak{C}) \to [0,1]$ *is continuous,*
(7) $\mathfrak{N}(\zeta, \eta, \omega, c) < 1$,
(8) $\mathfrak{N}(\zeta, \eta, \omega, c) = 0$ *if and only if* $\zeta = \eta = \omega$,
(9) $\mathfrak{N}(\zeta, \eta, \omega, c) = \mathfrak{N}(p\{\zeta, \eta, \omega\}, c)$, *where p is a permutation function,*
(10) $\mathfrak{N}(\zeta, \eta, \omega, c + c') \leq \mathfrak{N}(\zeta, \eta, \mathsf{u}, c) \diamond \mathfrak{N}(\mathsf{u}, \omega, \omega, c')$,
(11) $\mathfrak{N}(\zeta, \eta, \omega, \cdot) : int(\mathfrak{C}) \to [0,1]$ *is continuous.*

The pair $(\mathfrak{M}, \mathfrak{N})$ *is called Intuitionistic Generalized Fuzzy Cone Metric on* $\mathcal{Z}$. *The functions* $\mathfrak{M}(\zeta, \eta, \omega, c)$ *and* $\mathfrak{N}(\zeta, \eta, \omega, c)$ *denote, respectively, the degree of nearness and the degree of non nearness between* ζ, η *and* ω *with respect to c.*

Remark 1. *It is to be noted that:*

(i) The intuitionistic fuzzy setting provides both a membership degree and a nonmembership degree for an element, whereas the fuzzy settings provide only the membership degree alone and thus the space considered here will definitely provide a better environment than the latter to work with the applications.
(ii) Reference [2] Every fuzzy setting can be generalized to intuitionistic fuzzy setting but not the converse.

Example 1. *Let* $\mathfrak{B} = \mathbb{R}^2$ *and consider the closed cone* $\mathcal{C} = \{(c_1, c_2) \in \mathbb{R}^2 : c_1 \geq 0, c_2 \geq 0\}$ *in* $\mathfrak{B}$*. Let the t-norm* $*$ *be defined by* $\rho * \sigma = min\{\rho, \sigma\}$ *and the t-conorm* $\diamond$ *be defined by* $\rho \diamond \sigma = max\{\rho, \sigma\}$*. Define the functions* $\mathfrak{M}, \mathfrak{N} : \mathbb{R}^3 \times int(\mathcal{C}) \to [0, 1]$ *by*

$$\mathfrak{M}(\zeta, \eta, \omega, c) = e^{-\frac{|\zeta-\eta|+|\eta-\omega|+|\omega-\zeta|}{\|c\|}},$$
$$\mathfrak{N}(\zeta, \eta, \omega, c) = e^{-\frac{|\zeta-\eta|+|\eta-\omega|+|\omega-\zeta|}{\|c\|}}(e^{\frac{|\zeta-\eta|+|\eta-\omega|+|\omega-\zeta|}{\|c\|}} - 1),$$

for all $\zeta, \eta, \omega \in \mathbb{R}$ *and* $c \in int(\mathcal{C})$*. Then,* $(\mathbb{R}, \mathfrak{M}, \mathfrak{N}, *, \diamond)$ *is an IGFCM Space.*

3. Main Results

Let us begin the section by introducing the following ideas in the IGFCM Space. These special features play vital roles in building the results we intend to present here.

Definition 5. *A symmetric IGFCM Space is an IGFCM Space* $(\mathcal{Z}, \mathfrak{M}, \mathfrak{N}, *, \diamond)$ *satisfying*

$$\mathfrak{M}(\eta, \omega, \omega, c) = \mathfrak{M}(\omega, \eta, \eta, c) \text{ and } \mathfrak{N}(\eta, \omega, \omega, c) = \mathfrak{N}(\omega, \eta, \eta, c)$$

for all $\eta, \omega \in \mathcal{Z}$ *and* $c \in int(\mathcal{C})$.

Remark 2. *An IGFCM Space is symmetric.*

Definition 6. *Let* $(\mathcal{Z}, \mathfrak{M}, \mathfrak{N}, *, \diamond)$ *be an IGFCM Space. A self mapping* $\mathcal{P} : \mathcal{Z} \to \mathcal{Z}$ *is said to be k-Fuzzy Cone Contractive (briefly, k-FCC) if there exists* $k \in (0, 1)$ *such that*

$$\left(\frac{1}{\mathfrak{M}(\mathcal{P}(\zeta), \mathcal{P}(\eta), \mathcal{P}(\omega), c)} - 1\right) \leq k \left(\frac{1}{\mathfrak{M}(\zeta, \eta, \omega, c)} - 1\right),$$
$$\mathfrak{N}(\mathcal{P}(\zeta), \mathcal{P}(\eta), \mathcal{P}(\omega), c) \leq k\mathfrak{N}(\zeta, \eta, \omega, c),$$

for all $\zeta, \eta, \omega \in \mathcal{Z}$ *and* $c \in int(\mathcal{C})$.

Definition 7. *In an IGFCM Space* $(\mathcal{Z}, \mathfrak{M}, \mathfrak{N}, *, \diamond)$*, the pair* $(\mathfrak{M}, \mathfrak{N})$ *is said to be triangular if, for all* $\zeta, \eta, \omega, \mathrm{u} \in \mathcal{Z}$ *and* $c \in int(\mathcal{C})$,

$$\left(\frac{1}{\mathfrak{M}(\zeta, \eta, \omega, c)} - 1\right) \leq \left(\frac{1}{\mathfrak{M}(\zeta, \eta, \mathrm{u}, c)} - 1\right) + \left(\frac{1}{\mathfrak{M}(\mathrm{u}, \omega, \omega, c)} - 1\right),$$
$$\mathfrak{N}(\zeta, \eta, \omega, c) \leq \mathfrak{N}(\zeta, \eta, \mathrm{u}, c) + \mathfrak{N}(\mathrm{u}, \omega, \omega, c).$$

Definition 8. *Let* $(\mathcal{Z}, \mathfrak{M}, \mathfrak{N}, *, \diamond)$ *be an IGFCM Space,* $\zeta' \in \mathcal{Z}$ *and* $\{\zeta_n\}$ *be a sequence in* $\mathcal{Z}$.

(i) $\{\zeta_n\}$ *is said to converge to* ζ' *if, for all* $c \in int(\mathcal{C})$,

$$\lim_{n\to+\infty}\left(\frac{1}{\mathfrak{M}(\zeta_n, \zeta', \zeta', c)} - 1\right) = 0 \quad \text{and } \lim_{n\to+\infty}\mathfrak{N}(\zeta_n, \zeta', \zeta', c) = 0.$$

It is denoted by $\lim_{n\to+\infty}\zeta_n = \zeta'$ *or by* $\zeta_n \to \zeta'$ *as* $n \to +\infty$.

(ii) $\{\zeta_n\}$ *is said to be a Cauchy sequence if, for all* $c \in int(\mathcal{C})$ *and* $m \in \mathbb{N}$,

$$\lim_{n\to+\infty}\left(\frac{1}{\mathfrak{M}(\zeta_{n+m},\zeta_n,\zeta_n,c)}-1\right)=0 \quad \textit{and } \lim_{n\to+\infty}\mathfrak{N}(\zeta_{n+m},\zeta_n,\zeta_n,c)=0.$$

(iii) $(\mathcal{Z},\mathfrak{M},\mathfrak{N},*,\diamond)$ *is called a complete IGFCM space if every Cauchy sequence in* $\mathcal{Z}$ *converges.*

Definition 9. *Let* $(\mathcal{Z},\mathfrak{M},\mathfrak{N},*,\diamond)$ *be an IGFCM Space. A sequence* $\{\zeta_n\}$ *in* $\mathcal{Z}$ *is k-Fuzzy Cone Contractive (briefly, k-FCC) if there exists* $k \in (0,1)$ *such that*

$$\left(\frac{1}{\mathfrak{M}(\zeta_n,\zeta_{n+1},\zeta_{n+1},c)}-1\right)\le k\left(\frac{1}{\mathfrak{M}(\zeta_{n-1},\zeta_n,\zeta_n,c)}-1\right), \textit{and}$$
$$\mathfrak{N}(\zeta_n,\zeta_{n+1},\zeta_{n+1},c)\le k\mathfrak{N}(\zeta_{n-1},\zeta_n,\zeta_n,c),$$

for all $c \in int(\mathcal{C})$.

The following theorem gives the extension of Banach Contraction Principle in the IGFCM Space.

Theorem 1. *Let* $(\mathcal{Z},\mathfrak{M},\mathfrak{N},*,\diamond)$ *be a complete IGFCM Space in which k-FCC sequences are Cauchy. Let* $\mathcal{P}:\mathcal{Z}\to\mathcal{Z}$ *be a k-FCC mapping. Then,* $\mathcal{P}$ *has a unique fixed point.*

Remark 3. *The proof of Theorem 1 follows from Theorem 2 when* $\mathcal{P}=\mathcal{Q}=\mathcal{R}$ *and* $k_2=k_3=k_4=0$.

Next, let us prove some common fixed point theorems for three self mappings satisfying generalized contractive conditions in a complete IGFCM Space.

Theorem 2. *Let* $(\mathcal{Z},\mathfrak{M},\mathfrak{N},*,\diamond)$ *be a complete IGFCM Space where* $(\mathfrak{M},\mathfrak{N})$ *is triangular. If* $\mathcal{P},\mathcal{Q},\mathcal{R}:\mathcal{Z}\to\mathcal{Z}$ *are such that for all* $\zeta,\eta,\omega\in\mathcal{Z}$ *and* $c\in int(\mathcal{C})$,

$$\left(\frac{1}{\mathfrak{M}(\mathcal{P}\zeta,\mathcal{Q}\eta,\mathcal{R}\omega,c)}-1\right)\le\left\{\begin{array}{c} k_1\left(\frac{1}{\mathfrak{M}(\zeta,\eta,\omega,c)}-1\right)+k_2\left(\frac{1}{\mathfrak{M}(\zeta,\eta,\mathcal{R}\omega,c)}-1\right)\\ +k_3\left(\frac{1}{\mathfrak{M}(\zeta,\mathcal{Q}\eta,\omega,c)}-1\right)+k_4\left(\frac{1}{\mathfrak{M}(\mathcal{P}\zeta,\eta,\omega,c)}-1\right)\end{array}\right\}, \tag{1}$$

$$\mathfrak{N}(\mathcal{P}\zeta,\mathcal{Q}\eta,\mathcal{R}\omega,c)\le\left\{\begin{array}{c} k_1\mathfrak{N}(\zeta,\eta,\omega,c)+k_2\mathfrak{N}(\zeta,\eta,\mathcal{R}\omega,c)\\ +k_3\mathfrak{N}(\zeta,\mathcal{Q}\eta,\omega,c)+k_4\mathfrak{N}(\mathcal{P}\zeta,\eta,\omega,c)\end{array}\right\}, \tag{2}$$

where $k_i\in[0,+\infty]$, $i=1,\ldots,4$ *and* $k_1+2(k_2+k_3)+k_4<1$. *Then,* $\mathcal{P},\mathcal{Q}$ *and* $\mathcal{R}$ *have a unique common fixed point.*

Proof. Let $\zeta_0\in\mathcal{Z}$ be arbitrary. Let the sequence $\{\zeta_n\}$ be defined by

$$\begin{aligned}\zeta_{3n+1}&=\mathcal{P}\zeta_{3n},\\ \zeta_{3n+2}&=\mathcal{Q}\zeta_{3n+1}\text{ and}\\ \zeta_{3n+3}&=\mathcal{R}\zeta_{3n+2},\quad\text{for } n\ge 0.\end{aligned}$$

From (1),

$$\begin{aligned}&\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1},\zeta_{3n+2},\zeta_{3n+2},c)}-1\right)\le\left(\frac{1}{\mathfrak{M}(\mathcal{P}\zeta_{3n},\mathcal{Q}\zeta_{3n+1},\mathcal{Q}\zeta_{3n+1},c)}-1\right)\\ &\le\left\{\begin{array}{c} k_1\left(\frac{1}{\mathfrak{M}(\zeta_{3n},\zeta_{3n+1},\zeta_{3n+1},c)}-1\right)+k_2\left(\frac{1}{\mathfrak{M}(\zeta_{3n},\zeta_{3n+1},\mathcal{Q}\zeta_{3n+1},c)}-1\right)\\ +k_3\left(\frac{1}{\mathfrak{M}(\zeta_{3n},\mathcal{Q}\zeta_{3n+1},\zeta_{3n+1},c)}-1\right)+k_4\left(\frac{1}{\mathfrak{M}(\mathcal{P}\zeta_{3n},\zeta_{3n+1},\zeta_{3n+1},c)}-1\right)\end{array}\right\}\end{aligned}$$

$$= \left\{ \begin{array}{c} k_1\left(\frac{1}{\mathfrak{M}(\zeta_{3n},\zeta_{3n+1},\zeta_{3n+1},c)}-1\right)+k_2\left(\frac{1}{\mathfrak{M}(\zeta_{3n},\zeta_{3n+1},\zeta_{3n+2},c)}-1\right) \\ +k_3\left(\frac{1}{\mathfrak{M}(\zeta_{3n},\zeta_{3n+2},\zeta_{3n+1},c)}-1\right)+k_4\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1},\zeta_{3n+1},\zeta_{3n+1},c)}-1\right) \end{array} \right\}$$
$$= \left\{ \begin{array}{c} k_1\left(\frac{1}{\mathfrak{M}(\zeta_{3n},\zeta_{3n+1},\zeta_{3n+1},c)}-1\right) \\ +k_2\left(\frac{1}{\mathfrak{M}(\zeta_{3n},\zeta_{3n+1},\zeta_{3n+2},c)}-1\right)+k_3\left(\frac{1}{\mathfrak{M}(\zeta_{3n},\zeta_{3n+2},\zeta_{3n+1},c)}-1\right) \end{array} \right\}$$
$$\leq \left\{ \begin{array}{c} k_1\left(\frac{1}{\mathfrak{M}(\zeta_{3n},\zeta_{3n+1},\zeta_{3n+1},c)}-1\right) \\ +k_2\left[\left(\frac{1}{\mathfrak{M}(\zeta_{3n},\zeta_{3n+1},\zeta_{3n+1},c)}-1\right)+\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1},\zeta_{3n+2},\zeta_{3n+2},c)}-1\right)\right] \\ +k_3\left[\left(\frac{1}{\mathfrak{M}(\zeta_{3n},\zeta_{3n+1},\zeta_{3n+1},c)}-1\right)+\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1},\zeta_{3n+2},\zeta_{3n+2},c)}-1\right)\right] \end{array} \right\}$$
$$= \left\{ \begin{array}{c} (k_1+k_2+k_3)\left(\frac{1}{\mathfrak{M}(\zeta_{3n},\zeta_{3n+1},\zeta_{3n+1},c)}-1\right) \\ +(k_2+k_3)\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1},\zeta_{3n+2},\zeta_{3n+2},c)}-1\right) \end{array} \right\}.$$

Therefore,

$$\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1},\zeta_{3n+2},\zeta_{3n+2},c)}-1\right) \leq \frac{k_1+k_2+k_3}{1-(k_2+k_3)}\left(\frac{1}{\mathfrak{M}(\zeta_{3n},\zeta_{3n+1},\zeta_{3n+1},c)}-1\right).$$

Similarly,

$$\left(\frac{1}{\mathfrak{M}(\zeta_{3n+2},\zeta_{3n+3},\zeta_{3n+3},c)}-1\right) \leq \frac{k_1+k_2+k_3}{1-(k_2+k_3)}\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1},\zeta_{3n+2},\zeta_{3n+2},c)}-1\right),$$
$$\left(\frac{1}{\mathfrak{M}(\zeta_{3n+3},\zeta_{3n+4},\zeta_{3n+4},c)}-1\right) \leq \frac{k_1+k_2+k_3}{1-(k_2+k_3)}\left(\frac{1}{\mathfrak{M}(\zeta_{3n+2},\zeta_{3n+3},\zeta_{3n+3},c)}-1\right).$$

Putting $\mathfrak{M}_n = \left(\frac{1}{\mathfrak{M}(\zeta_n,\zeta_{n+1},\zeta_{n+1},c)}-1\right)$ and $k = \frac{k_1+k_2+k_3}{1-(k_2+k_3)}$, we obtain the inequalities:
For $n = 0, 1, 2, \ldots,$

$$\mathfrak{M}_{3n+1} \leq k\mathfrak{M}_{3n},$$
$$\mathfrak{M}_{3n+2} \leq k\mathfrak{M}_{3n+1} \text{ and}$$
$$\mathfrak{M}_{3n+3} \leq k\mathfrak{M}_{3n+2}.$$

These inequalities together result in

$$\mathfrak{M}_{n+1} \leq k\mathfrak{M}_n, \quad \text{for } n = 0, 1, 2, \ldots, \tag{3}$$

From (2),

$$\mathfrak{N}(\zeta_{3n+1},\zeta_{3n+2},\zeta_{3n+2},c) \leq \mathfrak{N}(\mathcal{P}\zeta_{3n},\mathcal{Q}\zeta_{3n+1},\mathcal{Q}\zeta_{3n+1},c)$$
$$\leq \left\{ \begin{array}{c} k_1\mathfrak{N}(\zeta_{3n},\zeta_{3n+1},\zeta_{3n+1},c)+k_2\mathfrak{N}(\zeta_{3n},\zeta_{3n+1},\mathcal{Q}\zeta_{3n+1},c) \\ +k_3\mathfrak{N}(\zeta_{3n},\mathcal{Q}\zeta_{3n+1},\zeta_{3n+1},c)+k_4\mathfrak{N}(\mathcal{P}\zeta_{3n},\zeta_{3n+1},\zeta_{3n+1},c) \end{array} \right\}$$
$$= \left\{ \begin{array}{c} k_1\mathfrak{N}(\zeta_{3n},\zeta_{3n+1},\zeta_{3n+1},c)+k_2\mathfrak{N}(\zeta_{3n},\zeta_{3n+1},\zeta_{3n+2},c) \\ +k_3\mathfrak{N}(\zeta_{3n},\zeta_{3n+2},\zeta_{3n+1},c)+k_4\mathfrak{N}(\zeta_{3n+1},\zeta_{3n+1},\zeta_{3n+1},c) \end{array} \right\}$$
$$= \left\{ \begin{array}{c} k_1\mathfrak{N}(\zeta_{3n},\zeta_{3n+1},\zeta_{3n+1},c) \\ +k_2\mathfrak{N}(\zeta_{3n},\zeta_{3n+1},\zeta_{3n+2},c))+k_3\mathfrak{N}(\zeta_{3n},\zeta_{3n+2},\zeta_{3n+1},c) \end{array} \right\}$$
$$\leq \left\{ \begin{array}{c} k_1\mathfrak{N}(\zeta_{3n},\zeta_{3n+1},\zeta_{3n+1},c) \\ +k_2\left[\mathfrak{N}(\zeta_{3n},\zeta_{3n+1},\zeta_{3n+1},c)+\mathfrak{N}(\zeta_{3n+1},\zeta_{3n+2},\zeta_{3n+2},c)\right] \\ +k_3\left[\mathfrak{N}(\zeta_{3n},\zeta_{3n+1},\zeta_{3n+1},c)+\mathfrak{N}(\zeta_{3n+1},\zeta_{3n+2},\zeta_{3n+2},c)\right] \end{array} \right\}$$

$$= \left\{ \begin{array}{c} (k_1 + k_2 + k_3)\mathfrak{N}(\zeta_{3n}, \zeta_{3n+1}, \zeta_{3n+1}, c) \\ +(k_2 + k_3)\mathfrak{N}(\zeta_{3n+1}, \zeta_{3n+2}, \zeta_{3n+2}, c) \end{array} \right\}.$$

Therefore,

$$\mathfrak{N}(\zeta_{3n+1}, \zeta_{3n+2}, \zeta_{3n+2}, c) \leq \frac{k_1 + k_2 + k_3}{1 - (k_2 + k_3)} \mathfrak{N}(\zeta_{3n}, \zeta_{3n+1}, \zeta_{3n+1}, c).$$

Similarly,

$$\mathfrak{N}(\zeta_{3n+2}, \zeta_{3n+3}, \zeta_{3n+3}, c) \leq \frac{k_1 + k_2 + k_3}{1 - (k_2 + k_3)} \mathfrak{N}(\zeta_{3n+1}, \zeta_{3n+2}, \zeta_{3n+2}, c),$$
$$\mathfrak{N}(\zeta_{3n+3}, \zeta_{3n+4}, \zeta_{3n+4}, c) \leq \frac{k_1 + k_2 + k_3}{1 - (k_2 + k_3)} \mathfrak{N}(\zeta_{3n+2}, \zeta_{3n+3}, \zeta_{3n+3}, c).$$

Putting $\mathfrak{N}_n = \mathfrak{N}(\zeta_n, \zeta_{n+1}, \zeta_{n+1}, c)$, we have the inequalities:

For $n = 0, 1, 2, \ldots,$

$$\begin{aligned} &\mathfrak{N}_{3n+1} \leq k\mathfrak{N}_{3n}, \\ &\mathfrak{N}_{3n+2} \leq k\mathfrak{N}_{3n+1} \text{ and} \\ &\mathfrak{N}_{3n+3} \leq k\mathfrak{N}_{3n+2}. \end{aligned}$$

These inequalities together result in

$$\mathfrak{N}_{n+1} \leq k\mathfrak{N}_n \quad \text{for } n = 0, 1, 2, \ldots, \tag{4}$$

(3) and (4) make $\{\zeta_n\}$ a k-FCC sequence.

Now, $(\mathfrak{M}, \mathfrak{N})$ is triangular and the space $(\mathcal{Z}, \mathfrak{M}, \mathfrak{N}, *, \diamond)$ is symmetric. Therefore, we have

$$\begin{aligned} \left(\frac{1}{\mathfrak{M}(\zeta_n, \zeta_n, \zeta_m, c)} - 1\right) &\leq \left\{ \begin{array}{c} \left(\frac{1}{\mathfrak{M}(\zeta_n, \zeta_{n+1}, \zeta_{n+1}, c)} - 1\right) \\ + \left(\frac{1}{\mathfrak{M}(\zeta_{n+1}, \zeta_{n+2}, \zeta_{n+2}, c)} - 1\right) + \cdots + \left(\frac{1}{\mathfrak{M}(\zeta_{m-1}, \zeta_m, \zeta_m, c)} - 1\right) \end{array} \right\} \\ &= \mathfrak{M}_n + \mathfrak{M}_{n+1} + \cdots + \mathfrak{M}_{m-1} \\ &\leq k^n \mathfrak{M}_0 + k^{n+1}\mathfrak{M}_0 + \cdots + k^{m-1}\mathfrak{M}_0 \\ &\leq \frac{k^n}{1-k}\mathfrak{M}_0 \to 0 \text{ as } n \to +\infty. \end{aligned}$$

$$\begin{aligned} \mathfrak{N}(\zeta_n, \zeta_n, \zeta_m, c) &\leq \left\{ \begin{array}{c} \mathfrak{N}(\zeta_n, \zeta_{n+1}, \zeta_{n+1}, c) \\ +\mathfrak{N}(\zeta_{n+1}, \zeta_{n+2}, \zeta_{n+2}, c) + \cdots + \mathfrak{N}(\zeta_{m-1}, \zeta_m, \zeta_m, c) \end{array} \right\} \\ &= \mathfrak{N}_n + \mathfrak{N}_{n+1} + \cdots + \mathfrak{N}_{m-1} \\ &\leq k^n \mathfrak{N}_0 + k^{n+1}\mathfrak{N}_0 + \cdots + k^{m-1}\mathfrak{N}_0 \\ &\leq \frac{k^n}{1-k}\mathfrak{N}_0 \to 0 \text{ as } n \to +\infty. \end{aligned}$$

Thus, $\{\zeta_n\}$ is Cauchy. As $\mathcal{Z}$ is complete, there exists $\dot{\zeta} \in \mathcal{Z}$ such that

$$\lim_{n \to +\infty} \left(\frac{1}{\mathfrak{M}(\zeta_n, \dot{\zeta}, \dot{\zeta}, c)} - 1\right) = 0, \ \lim_{n \to +\infty} \mathfrak{N}(\zeta_n, \dot{\zeta}, \dot{\zeta}, c) = 0. \tag{5}$$

From (4) to (5), we obtain that

$$\mathfrak{M}_{n+1} \leq k^n \mathfrak{M}_0, \mathfrak{N}_{n+1} \leq k^n \mathfrak{N}_0 \text{ for } n = 0, 1, 2, \ldots,$$

$$\lim_{n \to +\infty} \mathfrak{M}_n = 0, \lim_{n \to +\infty} \mathfrak{N}_n = 0. \tag{6}$$

Since $(\mathfrak{M}, \mathfrak{N})$ is triangular,

$$\left(\frac{1}{\mathfrak{M}(\dot{\zeta}, \dot{\zeta}, \mathcal{P}\dot{\zeta}, c)} - 1\right) \leq \left(\frac{1}{\mathfrak{M}(\dot{\zeta}, \dot{\zeta}, \zeta_{3n+2}, c)} - 1\right) + \left(\frac{1}{\mathfrak{M}(\zeta_{3n+2}, \mathcal{P}\dot{\zeta}, \mathcal{P}\dot{\zeta}, c)} - 1\right), \tag{7}$$

$$\mathfrak{N}(\dot{\zeta}, \dot{\zeta}, \mathcal{P}\dot{\zeta}, c) \leq \mathfrak{N}(\dot{\zeta}, \dot{\zeta}, \zeta_{3n+2}, c) + \mathfrak{N}(\zeta_{3n+2}, \mathcal{P}\dot{\zeta}, \mathcal{P}\dot{\zeta}, c). \tag{8}$$

From (1),

$$\begin{aligned}
\left(\frac{1}{\mathfrak{M}(\zeta_{3n+2}, \mathcal{P}\dot{\zeta}, \mathcal{P}\dot{\zeta}, c)} - 1\right) &\leq \left(\frac{1}{\mathfrak{M}(\mathcal{Q}\zeta_{3n+1}, \mathcal{P}\dot{\zeta}, \mathcal{P}\dot{\zeta}, c)} - 1\right) \\
&\leq \left\{ \begin{array}{c} k_1\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1}, \dot{\zeta}, \dot{\zeta}, c)} - 1\right) + k_2\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1}, \dot{\zeta}, \mathcal{P}\dot{\zeta}, c)} - 1\right) \\ +k_3\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1}, \mathcal{P}\dot{\zeta}, \dot{\zeta}, c)} - 1\right) + k_4\left(\frac{1}{\mathfrak{M}(\mathcal{Q}\zeta_{3n+1}, \dot{\zeta}, \dot{\zeta}, c)} - 1\right) \end{array} \right\} \\
&= \left\{ \begin{array}{c} k_1\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1}, \dot{\zeta}, \dot{\zeta}, c)} - 1\right) + k_2\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1}, \dot{\zeta}, \mathcal{P}\dot{\zeta}, c)} - 1\right) \\ +k_3\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1}, \mathcal{P}\dot{\zeta}, \dot{\zeta}, c)} - 1\right) + k_4\left(\frac{1}{\mathfrak{M}(\zeta_{3n+2}, \dot{\zeta}, \dot{\zeta}, c)} - 1\right) \end{array} \right\} \\
&\to (k_2 + k_3)\left(\frac{1}{\mathfrak{M}(\dot{\zeta}, \dot{\zeta}, \mathcal{P}\dot{\zeta}, c)} - 1\right) \text{ as } n \to +\infty.
\end{aligned}$$

Therefore,

$$\lim_{n \to +\infty} \sup \left(\frac{1}{\mathfrak{M}(\zeta_{3n+2}, \mathcal{P}\dot{\zeta}, \mathcal{P}\dot{\zeta}, c)} - 1\right) \leq (k_2 + k_3)\left(\frac{1}{\mathfrak{M}(\dot{\zeta}, \dot{\zeta}, \mathcal{P}\dot{\zeta}, c)} - 1\right). \tag{9}$$

From (2),

$$\begin{aligned}
\mathfrak{N}(\zeta_{3n+2}, \mathcal{P}\dot{\zeta}, \mathcal{P}\dot{\zeta}, c) &\leq \mathfrak{N}(\mathcal{Q}\zeta_{3n+1}, \mathcal{P}\dot{\zeta}, \mathcal{P}\dot{\zeta}, c) \\
&\leq \left\{ \begin{array}{c} k_1\mathfrak{N}(\zeta_{3n+1}, \dot{\zeta}, \dot{\zeta}, c) + k_2\mathfrak{N}(\zeta_{3n+1}, \dot{\zeta}, \mathcal{P}\dot{\zeta}, c) \\ +k_3\mathfrak{N}(\zeta_{3n+1}, \mathcal{P}\dot{\zeta}, \dot{\zeta}, c) + k_4\mathfrak{N}(\mathcal{Q}\zeta_{3n+1}, \dot{\zeta}, \dot{\zeta}, c) \end{array} \right\} \\
&= \left\{ \begin{array}{c} k_1\mathfrak{N}(\zeta_{3n+1}, \dot{\zeta}, \dot{\zeta}, c) + k_2\mathfrak{N}(\zeta_{3n+1}, \dot{\zeta}, \mathcal{P}\dot{\zeta}, c) \\ +k_3\mathfrak{N}(\zeta_{3n+1}, \mathcal{P}\dot{\zeta}, \dot{\zeta}, c) + k_4\mathfrak{N}(\zeta_{3n+2}, \dot{\zeta}, \dot{\zeta}, c) \end{array} \right\} \\
&\to (k_2 + k_3)\mathfrak{N}(\dot{\zeta}, \dot{\zeta}, \mathcal{P}\dot{\zeta}, c) \text{ as } n \to +\infty.
\end{aligned}$$

Therefore,

$$\lim_{n \to +\infty} \sup \mathfrak{N}(\zeta_{3n+2}, \mathcal{P}\dot{\zeta}, \mathcal{P}\dot{\zeta}, c) \leq (k_2 + k_3)\mathfrak{N}(\dot{\zeta}, \dot{\zeta}, \mathcal{P}\dot{\zeta}, c). \tag{10}$$

From (7) to (10), we obtain that

$$\left(\frac{1}{\mathfrak{M}(\dot{\zeta}, \dot{\zeta}, \mathcal{P}\dot{\zeta}, c)} - 1\right) \leq (k_2 + k_3)\left(\frac{1}{\mathfrak{M}(\dot{\zeta}, \dot{\zeta}, \mathcal{P}\dot{\zeta}, c)} - 1\right),$$
$$\mathfrak{N}(\dot{\zeta}, \dot{\zeta}, \mathcal{P}\dot{\zeta}, c) \leq (k_2 + k_3)\mathfrak{N}(\dot{\zeta}, \dot{\zeta}, \mathcal{P}\dot{\zeta}, c).$$

Since $k_2 + k_3 < 1$, we have

$$\left(\frac{1}{\mathfrak{M}(\dot{\zeta}, \dot{\zeta}, \mathcal{P}\dot{\zeta}, c)} - 1\right) = 0 \text{ and } \mathfrak{N}(\dot{\zeta}, \dot{\zeta}, \mathcal{P}\dot{\zeta}, c) = 0.$$

Therefore, $\mathcal{P}\dot{\zeta} = \dot{\zeta}$.

In a similar way, we can show that $\mathcal{Q}\dot{\zeta} = \dot{\zeta}$ and $\mathcal{R}\dot{\zeta} = \dot{\zeta}$. Then, $\mathcal{P}\dot{\zeta} = \mathcal{Q}\dot{\zeta} = \mathcal{R}\dot{\zeta} = \dot{\zeta}$.

Suppose $\mathcal{P}\ddot{\zeta} = \mathcal{Q}\ddot{\zeta} = \mathcal{R}\ddot{\zeta} = \ddot{\zeta}$.

From (1),

$$\begin{aligned}
\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\ddot{\zeta},\ddot{\zeta},c)}-1\right) &= \left(\frac{1}{\mathfrak{M}(\mathcal{P}\dot{\zeta},\mathcal{Q}\ddot{\zeta},\mathcal{R}\ddot{\zeta},c)}-1\right) \\
&\leq \left\{\begin{array}{c} k_1\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\ddot{\zeta},\ddot{\zeta},c)}-1\right)+k_2\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\ddot{\zeta},\mathcal{R}\ddot{\zeta},c)}-1\right) \\ +k_3\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\mathcal{Q}\ddot{\zeta},\ddot{\zeta},c)}-1\right)+k_4\left(\frac{1}{\mathfrak{M}(\mathcal{P}\dot{\zeta},\ddot{\zeta},\ddot{\zeta},c)}-1\right) \end{array}\right\} \\
&= \left\{\begin{array}{c} k_1\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\ddot{\zeta},\ddot{\zeta},c)}-1\right)+k_2\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\ddot{\zeta},\ddot{\zeta},c)}-1\right) \\ +k_3\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\ddot{\zeta},\ddot{\zeta},c)}-1\right)+k_4\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\ddot{\zeta},\ddot{\zeta},c)}-1\right) \end{array}\right\} \\
&= (k_1+k_2+k_3+k_4)\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\ddot{\zeta},\ddot{\zeta},c)}-1\right)
\end{aligned}$$

$$\text{That is,}\quad \left(\frac{1}{\mathfrak{M}(\dot{\zeta},\ddot{\zeta},\ddot{\zeta},c)}-1\right) \leq (k_1+k_2+k_3+k_4)\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\ddot{\zeta},\ddot{\zeta},c)}-1\right).$$

$$\text{Therefore,}\quad \left(\frac{1}{\mathfrak{M}(\dot{\zeta},\ddot{\zeta},\ddot{\zeta},c)}-1\right) = 0, \quad \text{since } k_1+k_2+k_3+k_4 < 1.$$

Hence, we can conclude that $\dot{\zeta}$ is the unique common fixed point of $\mathcal{P}, \mathcal{Q}$ and $\mathcal{R}$. □

Corollary 1. *Let $(\mathcal{Z}, \mathfrak{M}, \mathfrak{N}, *, \diamond)$ be a complete IGFCM Space where $(\mathfrak{M}, \mathfrak{N})$ is triangular. If $\mathcal{P} : \mathcal{Z} \to \mathcal{Z}$ is such that, for all $\zeta, \eta, \omega \in \mathcal{Z}$ and $c \in int(\mathcal{C})$,*

$$\left(\frac{1}{\mathfrak{M}(\mathcal{P}\zeta,\mathcal{P}\eta,\mathcal{P}\omega,c)}-1\right) \leq \left\{\begin{array}{c} k_1\left(\frac{1}{\mathfrak{M}(\zeta,\eta,\omega,c)}-1\right)+k_2\left(\frac{1}{\mathfrak{M}(\zeta,\eta,\mathcal{P}\omega,c)}-1\right) \\ +k_3\left(\frac{1}{\mathfrak{M}(\zeta,\mathcal{P}\eta,\omega,c)}-1\right)+k_4\left(\frac{1}{\mathfrak{M}(\mathcal{P}\zeta,\eta,\omega,c)}-1\right) \end{array}\right\},$$

$$\mathfrak{N}(\mathcal{P}\zeta,\mathcal{P}\eta,\mathcal{P}\omega,c) \leq \left\{\begin{array}{c} k_1\mathfrak{N}(\zeta,\eta,\omega,c)+k_2\mathfrak{N}(\zeta,\eta,\mathcal{P}\omega,c) \\ +k_3\mathfrak{N}(\zeta,\mathcal{P}\eta,\omega,c)+k_4\mathfrak{N}(\mathcal{P}\zeta,\eta,\omega,c) \end{array}\right\},$$

where $k_i \in [0, +\infty], i = 1, \ldots, 4$ and $k_1 + 2(k_2 + k_3) + k_4 < 1$. Then, $\mathcal{P}$ has a unique fixed point.

Example 2. *Consider the metric space $\mathcal{Z} = [0, +\infty)$ with metric d given by $d(\zeta, \eta) = |\zeta - \eta|$ for all $\zeta, \eta \in \mathcal{Z}$. Let $\mathcal{C} = \mathbb{R}^+$, $*$ be a continuous t-norm, and $\diamond$ be a continuous t-conorm. Define the $\mathfrak{M}, \mathfrak{N} : \mathcal{Z}^3 \times (0, +\infty) \to [0, 1]$ by*

$$\mathfrak{M}(\zeta,\eta,\omega,c) = \frac{c}{c+(|\zeta-\eta|+|\eta-\omega|+|\omega-\zeta|)}, \ \mathfrak{N}(\zeta,\eta,\omega,c) = \frac{|\zeta-\eta|+|\eta-\omega|+|\omega-\zeta|}{c+(|\zeta-\eta|+|\eta-\omega|+|\omega-\zeta|)},$$

for all $\zeta, \eta, \omega \in \mathcal{Z}$ and $c \in int(\mathcal{C})$. Then, it is clear that $(\mathcal{Z}, \mathfrak{M}, \mathfrak{N}, *, \diamond)$ is a complete IGFCM Space and that $(\mathfrak{M}, \mathfrak{N})$ is triangular. Consider the self mappings $\mathcal{P}, \mathcal{Q}$ and $\mathcal{R}$ from $\mathcal{Z}$ to $\mathcal{Z}$, given by

$$\mathcal{P}\zeta = \begin{cases} \frac{6}{5}\zeta + 3, & \zeta \in [0,1), \\ \frac{3}{4}\zeta + \frac{7}{2}, & \zeta \in [1,+\infty), \end{cases}$$

$$\mathcal{Q}\zeta = \begin{cases} \frac{6}{5}\zeta + 3, & \zeta \in [0,1), \\ \frac{2}{3}\zeta + \frac{14}{3}, & \zeta \in [1,+\infty), \end{cases}$$

$$\mathcal{R}\zeta = \begin{cases} \frac{6}{5}\zeta + 3, & \zeta \in [0,1), \\ \frac{1}{2}\zeta + 7, & \zeta \in [1,+\infty). \end{cases}$$

Here, $\mathcal{P}, \mathcal{Q}$ *and* $\mathcal{R}$ *together satisfy the condition (1) with*

$$k_1 = \frac{3}{4}, k_2 = \frac{1}{21}, k_3 = \frac{1}{21}, k_4 = \frac{1}{21}.$$

Therefore, $\mathcal{P}, \mathcal{Q}$ *and* $\mathcal{R}$ *have a unique common fixed point and it is* $\zeta = 14$.

Remark 4. *In the above example,* $\mathcal{P}, \mathcal{Q}$ *and* $\mathcal{R}$ *are not k-FCC and hence Theorem 1 cannot assure the existence of fixed points of any of* $\mathcal{P}, \mathcal{Q}$ *and* $\mathcal{R}$.

Theorem 3. *Let* $(\mathcal{Z}, \mathfrak{M}, \mathfrak{N}, *, \diamond)$ *be a complete IGFCM Space where* $(\mathfrak{M}, \mathfrak{N})$ *is triangular. If* $\mathcal{P}, \mathcal{Q}, \mathcal{R} : \mathcal{Z} \to \mathcal{Z}$ *is such that for all* $\zeta, \eta, \omega \in \mathcal{Z}$ *and* $c \in int(\mathcal{C})$,

$$\left(\frac{1}{\mathfrak{M}(\mathcal{P}\zeta, \mathcal{Q}\eta, \mathcal{R}\omega, c)} - 1\right) \leq k \left(\frac{1}{\Psi_1(\zeta, \eta, \omega)} - 1\right), \tag{11}$$

$$\mathfrak{N}(\mathcal{P}\zeta, \mathcal{Q}\eta, \mathcal{R}\omega, c) \leq k\Psi_2(\zeta, \eta, \omega), \tag{12}$$

where $k \in (0,1)$ *and*

$$\Psi_1(\zeta, \eta, \omega) = \min\{\mathfrak{M}(\zeta, \mathcal{Q}\eta, \mathcal{R}\omega, c), \mathfrak{M}(\mathcal{P}\zeta, \eta, \mathcal{R}\omega, c), \mathfrak{M}(\mathcal{P}\zeta, \mathcal{Q}\eta, \omega, c)\}, \text{ and}$$

$$\Psi_2(\zeta, \eta, \omega) = \min\{\mathfrak{N}(\zeta, \mathcal{Q}\eta, \mathcal{R}\omega, c), \mathfrak{N}(\mathcal{P}\zeta, \eta, \mathcal{R}\omega, c), \mathfrak{N}(\mathcal{P}\zeta, \mathcal{Q}\eta, \omega, c)\}.$$

Then, $\mathcal{P}, \mathcal{Q}$ *and* $\mathcal{R}$ *have a unique common fixed point.*

Proof. Let $\zeta_0 \in \mathcal{Z}$ be arbitrary. Define the sequence $\{\zeta_n\}$ as in Theorem 2.

Then, from (11) and (12), for $n = 0, 1, 2, \ldots$,

$$\begin{aligned}
\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1}, \zeta_{3n+2}, \zeta_{3n+2}, c)} - 1\right) &\leq \frac{k}{1-k}\left(\frac{1}{\mathfrak{M}(\zeta_{3n}, \zeta_{3n+1}, \zeta_{3n+1}, c)} - 1\right), \\
\left(\frac{1}{\mathfrak{M}(\zeta_{3n+2}, \zeta_{3n+3}, \zeta_{3n+3}, c)} - 1\right) &\leq \frac{k}{1-k}\left(\frac{1}{\mathfrak{M}(\zeta_{3n+1}, \zeta_{3n+2}, \zeta_{3n+2}, c)} - 1\right), \\
\left(\frac{1}{\mathfrak{M}(\zeta_{3n+3}, \zeta_{3n+4}, \zeta_{3n+4}, c)} - 1\right) &\leq \frac{k}{1-k}\left(\frac{1}{\mathfrak{M}(\zeta_{3n+2}, \zeta_{3n+3}, \zeta_{3n+3}, c)} - 1\right), \\
\mathfrak{N}(\zeta_{3n+1}, \zeta_{3n+2}, \zeta_{3n+2}, c) &\leq \frac{k}{1-k}\mathfrak{N}(\zeta_{3n}, \zeta_{3n+1}, \zeta_{3n+1}, c), \\
\mathfrak{N}(\zeta_{3n+2}, \zeta_{3n+3}, \zeta_{3n+3}, c) &\leq \frac{k}{1-k}\mathfrak{N}(\zeta_{3n+1}, \zeta_{3n+2}, \zeta_{3n+2}, c), \\
\mathfrak{N}(\zeta_{3n+3}, \zeta_{3n+4}, \zeta_{3n+4}, c) &\leq \frac{k}{1-k}\mathfrak{N}(\zeta_{3n+2}, \zeta_{3n+3}, \zeta_{3n+2}, c).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\left(\frac{1}{\mathfrak{M}(\zeta_{n+1}, \zeta_{n+2}, \zeta_{n+2}, c)} - 1\right) &\leq \frac{k}{1-k}\left(\frac{1}{\mathfrak{M}(\zeta_{n}, \zeta_{n+1}, \zeta_{n+1}, c)} - 1\right), \\
\mathfrak{N}(\zeta_{n+1}, \zeta_{n+2}, \zeta_{n+2}, c) &\leq \frac{k}{1-k}\mathfrak{N}(\zeta_{n}, \zeta_{n+1}, \zeta_{n+1}, c).
\end{aligned}$$

Using these inequalities repeatedly, we obtain that

$$\left(\frac{1}{\mathfrak{M}(\zeta_{n+1},\zeta_{n+2},\zeta_{n+2},c)}-1\right)\leq\frac{k^n}{1-k}\left(\frac{1}{\mathfrak{M}(\zeta_0,\zeta_1,\zeta_1,c)}-1\right),$$
$$\mathfrak{N}(\zeta_{n+1},\zeta_{n+2},\zeta_{n+2},c)\leq\frac{k^n}{1-k}\mathfrak{N}(\zeta_0,\zeta_1,\zeta_1,c).$$

Therefore, $\{\zeta_n\}$ is k-FCC and Cauchy and hence we can find an element $\dot{\zeta}\in\mathcal{Z}$ such that

$$\lim_{n\to+\infty}\left(\frac{1}{\mathfrak{M}(\zeta_n,\dot{\zeta},\dot{\zeta},c)}-1\right)=0,\ \lim_{n\to+\infty}\mathfrak{N}(\zeta_n,\dot{\zeta},\dot{\zeta},c)=0, \tag{13}$$

Since $(\mathfrak{M},\mathfrak{N})$ is triangular,

$$\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\dot{\zeta},\mathcal{P}\dot{\zeta},c)}-1\right)\leq\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\dot{\zeta},\zeta_{3n+2},c)}-1\right)+\left(\frac{1}{\mathfrak{M}(\zeta_{3n+2},\mathcal{P}\dot{\zeta},\mathcal{P}\dot{\zeta},c)}-1\right), \tag{14}$$

$$\mathfrak{N}(\dot{\zeta},\dot{\zeta},\mathcal{P}\dot{\zeta},c)\leq\mathfrak{N}(\dot{\zeta},\dot{\zeta},\zeta_{3n+2},c)+\mathfrak{N}(\zeta_{3n+2},\mathcal{P}\dot{\zeta},\mathcal{P}\dot{\zeta},c). \tag{15}$$

From (11) and (12), we can bring that

$$\lim_{n\to+\infty}\sup\left(\frac{1}{\mathfrak{M}(\zeta_{3n+2},\mathcal{P}\dot{\zeta},\mathcal{P}\dot{\zeta},c)}-1\right)\leq k\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\dot{\zeta},\mathcal{P}\dot{\zeta},c)}-1\right)\text{ and} \tag{16}$$

$$\lim_{n\to+\infty}\sup\mathfrak{N}(\zeta_{3n+2},\mathcal{P}\dot{\zeta},\mathcal{P}\dot{\zeta},c)\leq k\mathfrak{N}(\dot{\zeta},\dot{\zeta},\mathcal{P}\dot{\zeta},c). \tag{17}$$

From (14) to (17), we obtain that

$$\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\dot{\zeta},\mathcal{P}\dot{\zeta},c)}-1\right)\leq k\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\dot{\zeta},\mathcal{P}\dot{\zeta},c)}-1\right),$$
$$\mathfrak{N}(\dot{\zeta},\dot{\zeta},\mathcal{P}\dot{\zeta},c)\leq k\mathfrak{N}(\dot{\zeta},\dot{\zeta},\mathcal{P}\dot{\zeta},c).$$

As $k<1$, we get that

$$\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\dot{\zeta},\mathcal{P}\dot{\zeta},c)}-1\right)=0,\ \mathfrak{N}(\dot{\zeta},\dot{\zeta},\mathcal{P}\dot{\zeta},c)=0.$$

Therefore, $\mathcal{P}\dot{\zeta}=\dot{\zeta}$. In a similar way, we can bring that $\mathcal{Q}\dot{\zeta}=\dot{\zeta}$ and $\mathcal{R}\dot{\zeta}=\dot{\zeta}$.

Suppose $\mathcal{P}\ddot{\zeta}=\mathcal{Q}\ddot{\zeta}=\mathcal{R}\ddot{\zeta}=\ddot{\zeta}$.

From (11), we have that

$$\left(\frac{1}{\mathfrak{M}(\dot{\zeta},\ddot{\zeta},\ddot{\zeta},c)}-1\right)=\left(\frac{1}{\mathfrak{M}(\mathcal{P}\dot{\zeta},\mathcal{Q}\ddot{\zeta},\mathcal{R}\ddot{\zeta},c)}-1\right)\leq k\left(\frac{1}{\Psi_1(\zeta,\eta,\omega)}-1\right),$$

$$\begin{aligned}\text{where, }\Psi_1(\zeta,\eta,\omega)&=\min\left\{\ \mathfrak{M}(\dot{\zeta},\mathcal{Q}\ddot{\zeta},\mathcal{R}\ddot{\zeta},c),\mathfrak{M}(\mathcal{P}\dot{\zeta},\ddot{\zeta},\mathcal{R}\ddot{\zeta},c),\mathfrak{M}(\mathcal{P}\dot{\zeta},\mathcal{Q}\ddot{\zeta},\ddot{\zeta},c)\ \right\}\\&=\min\left\{\ \mathfrak{M}(\dot{\zeta},\ddot{\zeta},\ddot{\zeta},c),\mathfrak{M}(\dot{\zeta},\ddot{\zeta},\ddot{\zeta},c),\mathfrak{M}(\dot{\zeta},\ddot{\zeta},\ddot{\zeta},c)\ \right\}\\&=\mathfrak{M}(\dot{\zeta},\ddot{\zeta},\ddot{\zeta},c).\end{aligned}$$

From (12),

$$\mathfrak{N}(\dot{\zeta},\ddot{\zeta},\ddot{\zeta},c)=\mathfrak{N}(\mathcal{P}\dot{\zeta},\mathcal{Q}\ddot{\zeta},\mathcal{R}\ddot{\zeta},c)\leq k\Psi_2(\zeta,\eta,\omega),$$
$$\begin{aligned}\text{where, }\Psi_2(\zeta,\eta,\omega)&=\min\left\{\ \mathfrak{N}(\dot{\zeta},\mathcal{Q}\ddot{\zeta},\mathcal{R}\ddot{\zeta},c),\mathfrak{N}(\mathcal{P}\dot{\zeta},\ddot{\zeta},\mathcal{R}\ddot{\zeta},c),\mathfrak{N}(\mathcal{P}\dot{\zeta},\mathcal{Q}\ddot{\zeta},\ddot{\zeta},c)\ \right\}\\&=\min\left\{\ \mathfrak{N}(\dot{\zeta},\ddot{\zeta},\ddot{\zeta},c),\mathfrak{N}(\dot{\zeta},\ddot{\zeta},\ddot{\zeta},c),\mathfrak{N}(\dot{\zeta},\ddot{\zeta},\ddot{\zeta},c)\ \right\}\end{aligned}$$

$$= \mathfrak{N}(\dot{\zeta}, \ddot{\zeta}, \ddot{\zeta}, c).$$

Therefore,

$$\left(\frac{1}{\mathfrak{M}(\dot{\zeta}, \ddot{\zeta}, \ddot{\zeta}, c)} - 1\right) \leq k\left(\frac{1}{\mathfrak{M}(\dot{\zeta}, \ddot{\zeta}, \ddot{\zeta}, c)} - 1\right), \mathfrak{N}(\dot{\zeta}, \ddot{\zeta}, \ddot{\zeta}, c) \leq k\mathfrak{N}(\dot{\zeta}, \ddot{\zeta}, \ddot{\zeta}, c).$$

Hence, $\left(\frac{1}{\mathfrak{M}(\dot{\zeta}, \ddot{\zeta}, \ddot{\zeta}, c)} - 1\right) = 0, \mathfrak{N}(\dot{\zeta}, \ddot{\zeta}, \ddot{\zeta}, c) = 0.$

Therefore, $\dot{\zeta} = \ddot{\zeta}$ and we can conclude that $\mathcal{P}, \mathcal{Q}$ and $\mathcal{R}$ have a unique common fixed point. □

Example 3. *Consider the IGFCM Space given in Examsple 2 and the self mappings $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ from $\mathcal{Z}$ to $\mathcal{Z}$, given by*

$$\mathcal{P}\zeta = \begin{cases} \frac{1}{2}\zeta - \frac{1}{4}, & \zeta \in [0,1), \\ \frac{1}{2}\zeta + \frac{3}{2}, & \zeta \in [1,+\infty), \end{cases}$$

$$\mathcal{Q}\zeta = \begin{cases} \frac{1}{2}\zeta - \frac{1}{4}, & \zeta \in [0,1), \\ \frac{2}{3}\zeta + 1, & \zeta \in [1,+\infty), \end{cases}$$

$$\mathcal{R}\zeta = \begin{cases} \frac{1}{2}\zeta - \frac{1}{4}, & \zeta \in [0,1), \\ \frac{1}{3}\zeta + 2, & \zeta \in [1,+\infty). \end{cases}$$

Then,

$$\left(\frac{1}{\mathfrak{M}(\mathcal{P}\zeta, \mathcal{P}\eta, \mathcal{P}\omega, c)} - 1\right) = \frac{1}{2}\left(\frac{1}{\mathfrak{M}(\zeta, \eta, \omega, c)} - 1\right) \leq \frac{1}{4}\left(\frac{1}{\Psi_1(\zeta, \eta, \omega)} - 1\right) \text{ and}$$
$$\mathfrak{N}(\mathcal{P}\zeta, \mathcal{P}\eta, \mathcal{P}\omega, c) = \frac{1}{2}\mathfrak{N}(\zeta, \eta, \omega, c) \leq \frac{1}{4}\Psi_2(\zeta, \eta, \omega)$$

for all $\zeta, \eta, \omega \in \mathcal{Z}$. Thus, $\mathcal{P}, \mathcal{Q}$ and $\mathcal{R}$ together satisfy the conditions (11) and (12) with $k = \frac{1}{4}$. Therefore, $\mathcal{P}, \mathcal{Q}$ and $\mathcal{R}$ have a unique common fixed point, and it is $\zeta = 3$.

Corollary 2. *Let $(\mathcal{Z}, \mathfrak{M}, \mathfrak{N}, *, \diamond)$ be a complete IGFCM Space where $(\mathfrak{M}, \mathfrak{N})$ is triangular. If $\mathcal{P} : \mathcal{Z} \to \mathcal{Z}$ is such that for all $\zeta, \eta, \omega \in \mathcal{Z}$ and $c \in int(\mathcal{C})$,*

$$\left(\frac{1}{\mathfrak{M}(\mathcal{P}\zeta, \mathcal{P}\eta, \mathcal{P}\omega, c)} - 1\right) \leq k\left(\frac{1}{\Psi_1(\zeta, \eta, \omega)} - 1\right), \mathfrak{N}(\mathcal{P}\zeta, \mathcal{P}\eta, \mathcal{P}\omega, c) \leq k\Psi_2(\zeta, \eta, \omega)$$

where $k \in (0,1)$ and

$$\Psi_1(\zeta, \eta, \omega) = \min\{\mathfrak{M}(\zeta, \mathcal{P}\eta, \mathcal{P}\omega, c), \mathfrak{M}(\mathcal{P}\zeta, \eta, \mathcal{P}\omega, c), \mathfrak{M}(\mathcal{P}\zeta, \mathcal{P}\eta, z, c)\}, \text{ and}$$
$$\Psi_2(\zeta, \eta, \omega) = \min\{\mathfrak{N}(\zeta, \mathcal{P}\eta, \mathcal{P}\omega, c), \mathfrak{N}(\mathcal{P}\zeta, \eta, \mathcal{P}\omega, c), \mathfrak{N}(\mathcal{P}\zeta, \mathcal{P}\eta, z, c)\}.$$

Then, $\mathcal{P}$ has a unique fixed point.

4. Conclusions

This work extended the Banach contraction theorem to intuitionistic generalized fuzzy cone metric spaces. This work also constructed and proved some common fixed point theorems for three self mappings under generalized fuzzy contractive conditions. It is clear from the examples that the common fixed point theorems given here assure the existence of fixed points of the mappings, but the Banach contraction theorem fails to prove the existence of the common fixed point of such examples. The results proved here can be further extended to various spaces in different settings by increasing the number of self mappings, by imposing conditions on/between them and by analyzing the generalized contractive conditions.

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