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An Application of the Fixed Point Theory to the Study of Monotonic Solutions for Systems of Differential Equations

Adrian Nicolae Branga *,† and **Ion Marian Olaru** †

Department of Mathematics and Informatics, Lucian Blaga University of Sibiu, Dr. I. Raţiu Street, No. 5–7, 550012 Sibiu, Romania; imolaru@yahoo.com

* Correspondence: adrian.branga@ulbsibiu.ro or adrian.branga@gmail.com

† These authors contributed equally to this work.

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Abstract: In this paper, we establish some conditions for the existence and uniqueness of the monotonic solutions for nonhomogeneous systems of first-order linear differential equations, by using a result of the fixed points theory for sequentially complete gauge spaces.

Keywords: systems of differential equations; monotonic solutions; existence and uniqueness theorems; fixed point theory; sequentially complete gauge spaces

1. Introduction

Demidovich [1] proved an important result regarding the boundedness property of the monotonic solutions for homogeneous systems of linear differential equations of first order. Iseki [2] extended this result for nonhomogeneous systems and showed that under certain conditions any monotonic solution is bounded and its limit exists to $+\infty$. The weak point of the papers of Demidovich and Iseki consists of the fact that they demonstrate the boundedness of monotonous solutions, without specifying whether such solutions exist. Regarding the existence of monotonic solutions of differential equations, important results were obtained by Rovderová [3], Tóthová and Palumbíny [4], Rovder [5], Li and Fan [6], Yin [7], Ertem and Zafer [8], Aslanov [9], Chu [10], and Sanhan et al. [11].

The purpose of this article is to study the existence and uniqueness of the monotonic solutions for nonhomogeneous systems of first-order linear differential equations with variable coefficients. The novelty and originality of our article consists of us proving the existence and uniqueness of the monotonic solution, and finding the conditions under which these properties take place. To prove the theorem of existence and uniqueness of a monotonic solution we rely on the theory of gauge spaces. Dugundji [12] showed that any family of pseudometrics (gauge structure) on a nonempty set induces a uniform structure on that set, and conversely, any uniform structure on a nonempty set is generated by a family of pseudometrics. Moreover, the uniform structure is separating (Hausdorff) if and only if the gauge structure is separating. In this way, the gauge spaces (separating gauge structures) can be identified with Hausdorff uniform spaces. Colojoara [13] and Gheorghiu [14] extended the Banach contraction principle to the gauge spaces. Similar fixed point results were obtained by Knill [15] and Tarafdar [16] in the case of Hausdorff uniform spaces.

2. Preliminaries

Throughout this paper we follow the standard terminology and notation for systems of ordinary differential equations.

Further, we denote by $\mathbb{R}_+$ the real interval $[0, +\infty)$.

Let us consider a nonhomogeneous system of first-order linear differential equations with variable coefficients:

$$x'(t) = A(t)x(t) + b(t),\ t \in \mathbb{R}_+, \tag{1}$$

where $x(t) = (x_i(t))_{i=\overline{1,n}} \in \mathcal{M}_{n\times 1}(\mathbb{R})$, $A(t) = (a_{ij}(t))_{i,j=\overline{1,n}} \in \mathcal{M}_{n\times n}(\mathbb{R})$ and $b(t) = (b_i(t))_{i=\overline{1,n}} \in \mathcal{M}_{n\times 1}(\mathbb{R})$.

Definition 1. *([12]) Let X be a nonempty set. A map $p : X \times X \to \mathbb{R}_+$ is called a pseudometric (gauge) on X if the following conditions are satisfied:*

(1) $p(x, x) = 0$, *for all* $x \in X$*;*
(2) $p(x, y) = p(y, x)$, *for all* $x, y \in X$*;*
(3) $p(x, z) \le p(x, y) + p(y, z)$, *for all* $x, y, z \in X$*.*

Definition 2. *([12]) Let X be a nonempty set. We say that:*

(i) *A family $\mathcal{P} = (p_k)_{k\in I}$ of pseudometrics on X is named a gauge structure on X;*
(ii) *A gauge structure $\mathcal{P} = (p_k)_{k\in I}$ on X is called separating if for each pair of points $x, y \in X$, with $x \neq y$, there is $p_k \in \mathcal{P}$ such that $p_k(x, y) \neq 0$;*
(iii) *A pair $(X, \mathcal{P})$ of a nonempty set X and a separating gauge structure $\mathcal{P}$ on X is named a gauge space.*

Definition 3. *([12]) Let $(X, \mathcal{P})$ be a gauge space, where $\mathcal{P} = (p_k)_{k\in I}$. We say that:*

(i) *A sequence $(x_n)_{n\in\mathbb{N}} \subset X$ is called convergent in X if: there exists a point $x \in X$ with the property that for every $\varepsilon > 0$ and $k \in I$ there is a number $n(\varepsilon, k) \in \mathbb{N}$ such that for all $n \ge n(\varepsilon, k)$ we have $p_k(x_n, x) < \varepsilon$;*
(ii) *A sequence $(x_n)_{n\in\mathbb{N}} \subset X$ is named a Cauchy sequence if: for every $\varepsilon > 0$ and $k \in I$ there is a number $n(\varepsilon, k) \in \mathbb{N}$ such that for all $n \ge n(\varepsilon, k)$ and $p \in \mathbb{N}$ we have $p_k(x_n, x_{n+p}) < \varepsilon$;*
(iii) *The gauge space $(X, \mathcal{P})$ is called sequentially complete if: any Cauchy sequence of points in X is convergent in X.*

Theorem 1. *([13,14]) Let $(X, \mathcal{P})$ be a sequentially complete gauge space, where $\mathcal{P} = (p_k)_{k\in I}$, and $T : X \to X$ is an operator. We suppose that: for every $k \in I$ there exists $\alpha_k \in (0, 1)$ such that*

$$p_k(T(x), T(y)) \le \alpha_k p_k(x, y), \text{ for all } x, y \in X, \text{ for all } k \in I.$$

Then, T has a unique fixed point on X.

Theorem 2. *([17]) Let X be a nonempty set, (Y, ρ) a metric space, and $f_n : X \to Y$, $n \in \mathbb{N}$, a sequence of functions. Then, the following statements are true:*

(i) *If f_n, $n \in \mathbb{N}$ is uniformly convergent on X to a function $f : X \to Y$, then f_n, $n \in \mathbb{N}$, is a uniformly Cauchy sequence;*
(ii) *If f_n, $n \in \mathbb{N}$, is a uniformly Cauchy sequence and (Y, ρ) is complete, then there is a function $f : X \to Y$ such that f_n, $n \in \mathbb{N}$, is uniformly convergent on X to f.*

Theorem 3. *([17]) Let (X, τ) be a topological space, (Y, ρ) a complete metric space, and $f_n : X \to Y$, $n \in \mathbb{N}$, a sequence of functions. If f_n, $n \in \mathbb{N}$, is uniformly convergent on X to a function $f : X \to Y$, and every function f_n, $n \in \mathbb{N}$, is continuous on X, then f is continuous on X.*

Theorem 4. *([17]) Let us consider $\mathbb{R}^n = \{x = (x_1, ..., x_n) \mid x_i \in \mathbb{R},\ i = \overline{1, n}\}$. The following properties are valid:*

(i) $(\mathbb{R}^n, \|\cdot\|_1)$ *is a complete normed linear space, where* $\|\cdot\|_1 : \mathbb{R}^n \to \mathbb{R}_+$, $\|x\|_1 = \sum_{i=1}^{n} |x_i|$*;*
(ii) $(\mathbb{R}^n, \rho_1)$ *is a complete metric space, where* $\rho_1 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$, $\rho_1(x, y) = \|x - y\|_1$*.*

3. The Existence and Uniqueness of the Monotonic Solutions

Theorem 5. *Let us consider* $C(\mathbb{R}_+, \mathbb{R}^n) = \{x : \mathbb{R}_+ \to \mathbb{R}^n \mid x \text{ is continuous on } \mathbb{R}_+\}$. *Then the following statements are valid:*

(i) *The maps* $p_k : C(\mathbb{R}_+, \mathbb{R}^n) \times C(\mathbb{R}_+, \mathbb{R}^n) \to \mathbb{R}_+$, $p_k(x, y) = \sup\limits_{t \in [0,k]} \|x(t) - y(t)\|_1 e^{-\tau t}$, $k \in \mathbb{N}^*$, *are pseudometrics on* $C(\mathbb{R}_+, \mathbb{R}^n)$, *where* $\tau > 0$;

(ii) $(C(\mathbb{R}_+, \mathbb{R}^n), \mathcal{P})$ *is a sequentially complete gauge space, where* $\mathcal{P} = (p_k)_{k \in \mathbb{N}^*}$;

(iii) *If the functions* $A : \mathbb{R}_+ \to \mathcal{M}_{n \times n}(\mathbb{R})$, $b : \mathbb{R}_+ \to \mathcal{M}_{n \times 1}(\mathbb{R})$ *are continuous on* $\mathbb{R}_+$ *and there is a number* $L > 0$ *such that* $|a_{i,j}(t)| \leq L$, $i, j = \overline{1, n}$, $t \in \mathbb{R}_+$ *(A is bounded on* $\mathbb{R}_+$*), then the system of first-order linear differential equations ((Equation 1)), with initial condition* $x(0) = x^0 \in \mathbb{R}^n$, *has a unique solution for* $C(\mathbb{R}_+, \mathbb{R}^n)$.

Proof. (i) Let $k \in \mathbb{N}^*$ be an arbitrary number.

We choose $x, y \in C(\mathbb{R}_+, \mathbb{R}^n)$ arbitrary elements. We deduce that $x - y : \mathbb{R}_+ \to \mathbb{R}^n$ is a continuous function on $\mathbb{R}_+$. On the other hand, the norm $\| \cdot \|_1 : \mathbb{R}^n \to \mathbb{R}_+$ is a continuous map on $\mathbb{R}^n$. Consequently, the function $\varphi_1 : \mathbb{R}_+ \to \mathbb{R}_+$, $\varphi_1(t) = \|x(t) - y(t)\|_1$ is continuous on $\mathbb{R}_+$. Additionally, the function $\varphi_2 : \mathbb{R}_+ \to \mathbb{R}_+$, $\varphi_2(t) = e^{-\tau t}$ is continuous on $\mathbb{R}_+$. Therefore, the function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, $\varphi(t) = \varphi_1(t)\varphi_2(t) = \|x(t) - y(t)\|_1 e^{-\tau t}$ is continuous on $\mathbb{R}_+$. It follows that φ is a continuous function on $[0, k]$. Applying the Weierstrass extreme value theorem we find that φ is bounded on $[0, k]$ and there exists $\bar{t} \in [0, k]$ such that $\varphi(\bar{t}) = \sup\limits_{t \in [0,k]} \varphi(t) = \sup\limits_{t \in [0,k]} \|x(t) - y(t)\|_1 e^{-\tau t} \in \mathbb{R}_+$ (φ attains its supremum in $[0, k]$). Therefore, the map p_k is well-defined.

We now prove that the function p_k verifies the properties of a pseudometric. Let $x, y, z \in C(\mathbb{R}_+, \mathbb{R}^n)$ be arbitrary functions. By using the properties of the norm $\| \cdot \|_1$ we get

$$p_k(x, x) = \sup_{t \in [0,k]} \|x(t) - x(t)\|_1 e^{-\tau t} = \sup_{t \in [0,k]} \|0\|_1 e^{-\tau t} = 0,$$

$$p_k(x, y) = \sup_{t \in [0,k]} \|x(t) - y(t)\|_1 e^{-\tau t} = \sup_{t \in [0,k]} \|y(t) - x(t)\|_1 e^{-\tau t} = p_k(y, x),$$

$$\|x(t) - z(t)\|_1 e^{-\tau t} = \|x(t) - y(t) + y(t) - z(t)\|_1 e^{-\tau t} \leq \|x(t) - y(t)\|_1 e^{-\tau t} + \|y(t) - z(t)\|_1 e^{-\tau t}$$
$$\leq \sup_{t \in [0,k]} \|x(t) - y(t)\|_1 e^{-\tau t} + \sup_{t \in [0,k]} \|y(t) - z(t)\|_1 e^{-\tau t} = p_k(x, y) + p_k(y, z), \text{ for all } t \in [0, k],$$

hence

$$p_k(x, z) = \sup_{t \in [0,k]} \|x(t) - z(t)\|_1 e^{-\tau t} \leq p_k(x, y) + p_k(y, z).$$

(ii) The family $\mathcal{P} = (p_k)_{k \in \mathbb{N}^*}$ of pseudometrics defines on $C(\mathbb{R}_+, \mathbb{R}^n)$ a gauge structure. We remark that this gauge structure is separating because for each pair of elements $x, y \in C(\mathbb{R}_+, \mathbb{R}^n)$, with $x \neq y$, and every $k \in \mathbb{N}^*$, we have $p_k(x, y) = \sup\limits_{t \in [0,k]} \|x(t) - y(t)\|_1 e^{-\tau t} \neq 0$. Consequently, $(C(\mathbb{R}_+, \mathbb{R}^n), \mathcal{P})$ is a gauge space.

We choose $\varepsilon > 0$ and $k \in \mathbb{N}^*$ arbitrary elements.

We now show that the gauge space $C(\mathbb{R}_+, \mathbb{R}^n)$ is sequentially complete. Let us consider $(x_m)_{m \in \mathbb{N}} \subset C(\mathbb{R}_+, \mathbb{R}^n)$ an arbitrary Cauchy sequence. It follows that for $\varepsilon e^{-\tau k} > 0$ and $k \in \mathbb{N}^*$ there is a number $m(\varepsilon, k) \in \mathbb{N}$ such that for all $m \geq m(\varepsilon, k)$ and $p \in \mathbb{N}$ we have $p_k(x_m, x_{m+p}) < \varepsilon e^{-\tau k}$, i.e., $\sup\limits_{t \in [0,k]} \|x_m(t) - x_{m+p}(t)\|_1 e^{-\tau t} < \varepsilon e^{-\tau k}$. As $e^{-\tau k} \leq e^{-\tau t}$ for all $t \in [0, k]$, we get $\|x_m(t) - x_{m+p}(t)\|_1 e^{-\tau k} \leq \|x_m(t) - x_{m+p}(t)\|_1 e^{-\tau t}$ for all $t \in [0, k]$; hence, $\sup\limits_{t \in [0,k]} \|x_m(t) - x_{m+p}(t)\|_1 e^{-\tau k} \leq$

$\sup\limits_{t\in[0,k]} \|x_m(t) - x_{m+p}(t)\|_1 e^{-\tau t}$. Consequently, for all $m \geq m(\varepsilon, k)$ and $p \in \mathbb{N}$ we have $\sup\limits_{t\in[0,k]} \|x_m(t) - x_{m+p}(t)\|_1 e^{-\tau k} < \varepsilon e^{-\tau k}$, i.e., $\sup\limits_{t\in[0,k]} \|x_m(t) - x_{m+p}(t)\|_1 < \varepsilon$, which implies that $\|x_m(t) - x_{m+p}(t)\|_1 < \varepsilon$ for all $t \in [0,k]$. As the number $\varepsilon > 0$ was chosen arbitrarily, it follows that for every $\varepsilon > 0$ there is a number $m(\varepsilon, k) \in \mathbb{N}$ such that for all $m \geq m(\varepsilon, k)$ and $p \in \mathbb{N}$ we have $\rho_1(x_m(t), x_{m+p}(t)) < \varepsilon$ for all $t \in [0,k]$. Therefore, $x_m : [0,k] \to \mathbb{R}^n$, $m \in \mathbb{N}$, is a uniformly Cauchy sequence. According to Theorem 4 (ii), $(\mathbb{R}^n, \rho_1)$ is a complete metric space and using Theorem 2 (ii) we find that there is a function $x : [0,k] \to \mathbb{R}^n$ such that x_m, $m \in \mathbb{N}$, is uniformly convergent on $[0,k]$ to x. Since every function x_m, $m \in \mathbb{N}$, is continuous on $[0,k]$ and $(\mathbb{R}^n, \rho_1)$ is a complete metric space (Theorem 4 (ii)), by applying Theorem 3 we deduce that x is continuous on $[0,k]$. Consequently, we proved that the sequence x_m, $m \in \mathbb{N}$ is uniformly convergent on $[0,k]$ to a continuous function $x : [0,k] \to \mathbb{R}^n$. As the number $k \in \mathbb{N}^*$ was chosen arbitrarily, it follows that the sequence of continuous functions $x_m : \mathbb{R}_+ \to \mathbb{R}^n$, $m \in \mathbb{N}$, is uniformly convergent on $\mathbb{R}_+$ to a continuous function $x : \mathbb{R}_+ \to \mathbb{R}^n$.

Since the sequence $x_m : [0,k] \to \mathbb{R}^n$, $m \in \mathbb{N}$, is uniformly convergent on $[0,k]$ to a function $x : [0,k] \to \mathbb{R}^n$, it follows that for $\frac{\varepsilon}{2} > 0$ there is a number $m'(\varepsilon, k) \in \mathbb{N}$ such that for all $m \geq m'(\varepsilon, k)$ we have $\rho_1(x_m(t), x(t)) < \frac{\varepsilon}{2}$ for all $t \in [0,k]$, i.e., $\|x_m(t) - x(t)\|_1 < \frac{\varepsilon}{2}$ for all $t \in [0,k]$. Therefore, for all $m \geq m'(\varepsilon, k)$ we get $\|x_m(t) - x(t)\|_1 e^{-\tau t} < \frac{\varepsilon}{2} e^{-\tau t} \leq \frac{\varepsilon}{2}$ for all $t \in [0,k]$, which implies that $\sup\limits_{t\in[0,k]} \|x_m(t) - x(t)\|_1 e^{-\tau t} \leq \frac{\varepsilon}{2} < \varepsilon$; i.e., $p_k(x_m, x) < \varepsilon$. As the elements $\varepsilon > 0$ and $k \in \mathbb{N}^*$ were arbitrarily selected, we deduce that for every $\varepsilon > 0$ and $k \in \mathbb{N}^*$ there is a number $m'(\varepsilon, k) \in \mathbb{N}$ such that for all $m \geq m'(\varepsilon, k)$ we have $p_k(x_m, x) < \varepsilon$. Consequently, we proved that there exists a function $x \in C(\mathbb{R}_+, \mathbb{R}^n)$ with the property that for every $\varepsilon > 0$ and $k \in \mathbb{N}^*$ there is a number $m'(\varepsilon, k) \in \mathbb{N}$ such that for all $m \geq m'(\varepsilon, k)$ we have $p_k(x_m, x) < \varepsilon$. Therefore, the sequence of functions $(x_m)_{m\in\mathbb{N}} \subset C(\mathbb{R}_+, \mathbb{R}^n)$ is convergent in $C(\mathbb{R}_+, \mathbb{R}^n)$ to a function $x \in C(\mathbb{R}_+, \mathbb{R}^n)$. Since the Cauchy sequence of functions $(x_m)_{m\in\mathbb{N}} \subset C(\mathbb{R}_+, \mathbb{R}^n)$ was chosen arbitrarily, we find that any Cauchy sequence of functions in $C(\mathbb{R}_+, \mathbb{R}^n)$ is convergent in $C(\mathbb{R}_+, \mathbb{R}^n)$. Consequently, the gauge space $(C(\mathbb{R}_+, \mathbb{R}^n), \mathcal{P})$ is sequentially complete.

(iii) As the functions $x : \mathbb{R}_+ \to \mathcal{M}_{n\times 1}(\mathbb{R})$, $A : \mathbb{R}_+ \to \mathcal{M}_{n\times n}(\mathbb{R})$, $b : \mathbb{R}_+ \to \mathcal{M}_{n\times 1}(\mathbb{R})$ are continuous for $\mathbb{R}_+$, we deduce that the system of first-order linear differential Equations (1), with initial condition $x(0) = x^0 \in \mathbb{R}^n$, is equivalent to the system of integral equations

$$x(t) = x^0 + \int_0^t A(s)x(s)ds + \int_0^t b(s)ds,\ t \in \mathbb{R}_+. \tag{2}$$

Using relation (2), we can define an operator $T : C(\mathbb{R}_+, \mathbb{R}^n) \to C(\mathbb{R}_+, \mathbb{R}^n)$,

$$T(x)(t) = x^0 + \int_0^t A(s)x(s)ds + \int_0^t b(s)ds,\ t \in \mathbb{R}_+. \tag{3}$$

For every $k \in \mathbb{N}^*$, $x, y \in C(\mathbb{R}_+, \mathbb{R}^n)$, $t \in [0,k]$, $i = \overline{1,n}$, we have, successively:

$$|pr_i(T(x)(t)) - pr_i(T(y)(t))|$$

$$= \left| x_i^0 + \int_0^t \sum_{j=1}^n a_{i,j}(s)x_j(s)ds + \int_0^t b_i(s)ds - x_i^0 - \int_0^t \sum_{j=1}^n a_{i,j}(s)y_j(s)ds - \int_0^t b_i(s)ds \right|$$

$$= \left| \int_0^t \sum_{j=1}^n a_{i,j}(s)x_j(s)ds - \int_0^t \sum_{j=1}^n a_{i,j}(s)y_j(s)ds \right| = \left| \int_0^t \sum_{j=1}^n a_{i,j}(s)(x_j(s) - y_j(s))ds \right|$$

$$\leq \int_0^t \left| \sum_{j=1}^n a_{i,j}(s)(x_j(s) - y_j(s)) \right| ds \leq \int_0^t \sum_{j=1}^n |a_{i,j}(s)(x_j(s) - y_j(s))| ds$$

$$= \int_0^t \sum_{j=1}^n |a_{i,j}(s)| \cdot |x_j(s) - y_j(s)| ds \leq \int_0^t \sum_{j=1}^n L |x_j(s) - y_j(s)| ds$$

$$= L \int_0^t \sum_{j=1}^n |x_j(s) - y_j(s)| ds = L \int_0^t \|x(s) - y(s)\|_1 ds$$

$$= L \int_0^t \|x(s) - y(s)\|_1 e^{-\tau s} e^{\tau s} ds = L \int_0^t (\|x(s) - y(s)\|_1 e^{-\tau s}) e^{\tau s} ds$$

$$\leq L \int_0^t \sup_{s \in [0,k]} (\|x(s) - y(s)\|_1 e^{-\tau s}) e^{\tau s} ds = L \int_0^t p_k(x,y) e^{\tau s} ds$$

$$= L p_k(x,y) \int_0^t e^{\tau s} ds = L p_k(x,y) \frac{e^{\tau s}}{\tau} \Big|_0^t$$

$$= L p_k(x,y) \left(\frac{e^{\tau t}}{\tau} - \frac{1}{\tau} \right) = \frac{L}{\tau} p_k(x,y)(e^{\tau t} - 1) \leq \frac{L}{\tau} p_k(x,y) e^{\tau t}.$$

Hence, for every $k \in \mathbb{N}^*$, $x, y \in C(\mathbb{R}_+, \mathbb{R}^n)$, $t \in [0,k]$, we get

$$\sum_{i=1}^n |pr_i(T(x)(t)) - pr_i(T(y)(t))| \leq \sum_{i=1}^n \frac{L}{\tau} p_k(x,y) e^{\tau t} = \frac{nL}{\tau} p_k(x,y) e^{\tau t},$$

i.e.,

$$\|T(x)(t) - T(y)(t)\|_1 \leq \frac{nL}{\tau} p_k(x,y) e^{\tau t},$$

which is equivalent to

$$\|T(x)(t) - T(y)(t)\|_1 e^{-\tau t} \leq \frac{nL}{\tau} p_k(x,y).$$

Therefore, for every $k \in \mathbb{N}^*$, $x, y \in C(\mathbb{R}_+, \mathbb{R}^n)$, we obtain

$$\sup_{t \in [0,k]} \|T(x)(t) - T(y)(t)\|_1 e^{-\tau t} \leq \frac{nL}{\tau} p_k(x,y),$$

i.e.,

$$p_k(T(x), T(y)) \leq \frac{nL}{\tau} p_k(x,y).$$

Consequently, for $\tau > nL$ and denoting $\alpha_k := \dfrac{nL}{\tau} \in (0,1)$, we have

$$p_k(T(x), T(y)) \leq \alpha_k p_k(x,y), \text{ for all } x, y \in C(\mathbb{R}_+, \mathbb{R}^n), \text{ for all } k \in \mathbb{N}^*.$$

Thus, $(C(\mathbb{R}_+, \mathbb{R}^n), \mathcal{P})$ is a sequentially complete gauge space, where $\mathcal{P} = (p_k)_{k \in \mathbb{N}^*}$, and $T : C(\mathbb{R}_+, \mathbb{R}^n) \to C(\mathbb{R}_+, \mathbb{R}^n)$ an operator with the property that: for every $k \in \mathbb{N}^*$ there exists $\alpha_k \in (0,1)$ such that

$$p_k(T(x), T(y)) \leq \alpha_k p_k(x,y), \text{ for all } x, y \in C(\mathbb{R}_+, \mathbb{R}^n), \text{ for all } k \in \mathbb{N}^*.$$

Applying Theorem 1 it follows that T has a unique fixed point in $C(\mathbb{R}_+, \mathbb{R}^n)$; i.e., there exists an unique element $x^* \in C(\mathbb{R}_+, \mathbb{R}^n)$ such that $T(x^*) = x^*$. Therefore, the system of integral Equation (2) has a unique solution for $C(\mathbb{R}_+, \mathbb{R}^n)$. Consequently, the system of first-order linear differential Equations (1), with initial condition $x(0) = x^0 \in \mathbb{R}^n$, has a unique solution for $C(\mathbb{R}_+, \mathbb{R}^n)$. □

Theorem 6. *If the functions $A : \mathbb{R}_+ \to \mathcal{M}_{n\times n}(\mathbb{R})$, $b : \mathbb{R}_+ \to \mathcal{M}_{n\times 1}(\mathbb{R})$ are continuous on $\mathbb{R}_+$ and there is a number $L > 0$ such that $0 \le a_{i,j}(t) \le L$, $i, j = \overline{1, n}$, $t \in \mathbb{R}_+$, $b_i(t) \ge 0$, $i = \overline{1, n}$, $t \in \mathbb{R}_+$, then the system of first-order linear differential Equations (1), with initial condition $x(0) = x^0 \in \mathbb{R}^n_+$, has a unique solution for $C(\mathbb{R}_+, \mathbb{R}^n_+)$ and this solution is monotonic for $t \to +\infty$.*

Proof. Similarly to the proof of Theorem 5, the system of first-order linear differential Equations (1), with initial condition $x(0) = x^0 \in \mathbb{R}^n_+$, has a unique solution for $C(\mathbb{R}_+, \mathbb{R}^n_+)$. Let us denote by $x^* \in C(\mathbb{R}_+, \mathbb{R}^n_+)$ this solution. Therefore,

$$x^{*\prime}(t) = A(t)x^*(t) + b(t) = \begin{pmatrix} \sum\limits_{j=1}^{n} a_{1,j}(t)x_j^*(t) \\ \vdots \\ \sum\limits_{j=1}^{n} a_{n,j}(t)x_j^*(t) \end{pmatrix} + \begin{pmatrix} b_1(t) \\ \vdots \\ b_n(t) \end{pmatrix}$$

$$= \begin{pmatrix} \sum\limits_{j=1}^{n} a_{1,j}(t)x_j^*(t) + b_1(t) \\ \vdots \\ \sum\limits_{j=1}^{n} a_{n,j}(t)x_j^*(t) + b_n(t) \end{pmatrix} \ge \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \text{ for all } t \in \mathbb{R}_+.$$

It follows that x^* is a monotonically increasing function on $\mathbb{R}_+$. Consequently, each function $x_i^*(t)$, $i = \overline{1, n}$, is monotonic on $[0, +\infty)$; i.e., x^* is monotonic for $t \to +\infty$. □

Example 1. *Let us consider the matrix function $A : \mathbb{R}_+ \to \mathcal{M}_{n\times n}(\mathbb{R})$,*

$$A(t) = \begin{pmatrix} \frac{t}{2t+2} & \frac{1}{2}e^{-sint} \\ \frac{1}{2}e^{-cost} & \frac{1}{2} \end{pmatrix},$$

and the vector function $b : \mathbb{R}_+ \to \mathcal{M}_{n\times 1}(\mathbb{R})$,

$$b(t) = \begin{pmatrix} ln(1+t) \\ e^t \end{pmatrix}.$$

We remark that the functions A, b are continuous on $\mathbb{R}_+$ and there is a number $L := \frac{1}{2}e > 0$ such that $0 \le a_{i,j}(t) \le L$, $i, j = \overline{1, 2}$, $t \in \mathbb{R}_+$, $b_i(t) \ge 0$, $i = \overline{1, 2}$, $t \in \mathbb{R}_+$; therefore, the conditions of Theorem 6 are fulfilled. Considering the vector $x^0 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \in \mathbb{R}^2_+$ and applying Theorem 6 it follows that the system of first-order linear differential equations

$$x'(t) = A(t)x(t) + b(t),\ t \in \mathbb{R}_+,$$

with initial condition $x(0) = x^0 \in \mathbb{R}^2_+$, has a unique solution on $C(\mathbb{R}_+, \mathbb{R}^2_+)$ and this solution is monotonic for $t \to +\infty$.

4. Conclusions

In this article we studied the existence and uniqueness of the monotonic solutions for nonhomogeneous systems of first-order linear differential equations with variable coefficients. The novelty and originality of our article consists of us proving the existence and uniqueness of the monotonic solution, and finding the conditions under which these properties take place. An example was presented at the end of the paper which reinforces that our theory is correct. Additionally, the paper established conditions for the existence and uniqueness of the solution of the systems of first-order linear differential equations, with initial condition, defined over an unbounded interval (the positive real axis).

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