

Article

Regularity for Semilinear Neutral Hyperbolic Equations with Cosine Families

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Abstract: The purpose of this paper is to obtain the regularity for solutions of semilinear neutral hyperbolic equations with the nonlinear convolution. The principal operator is the infinitesimal generator of a cosine and sine families. In order to show a variation of constant formula for solutions, we make use of the nature of cosine and sine families.

Keywords: semilinear hyperbolic equations; existence; regularity; cosine family; sine family

1. Introduction

This paper is to establish the regularity of solutions of the following abstract semilinear neutral hyperbolic equation in a Banach space X :

$$\begin{cases} \frac{d}{dt}[w'(t) + g(t, w(t))] = Aw(t) + F(t, w) + f(t), & 0 < t \leq T, \\ w(0) = x_0, \quad w'(0) = y_0. \end{cases} \quad (1)$$

The principal operator A is the infinitesimal generator of a cosine family $C(t)$ ($t \in \mathbb{R}$). The nonlinear part is given by

$$F(t, w) = \int_0^t k(t-s)h(s, w(s))ds.$$

Here, $k \in L^2(0, T)$ and the mapping $h : [0, T] \times D(A) \rightarrow X$ satisfies that $w \mapsto h(t, w)$ satisfies Lipschitz continuous. The nonlinear mapping $g : [0, T] \times X \rightarrow X$ will be explained in detail in Section 3.

Semilinear neutral differential equations have been considered by many authors [1,2] and reference therein. We refer to [3,4] for partial neutral integro-differential equations. The existence of solutions for neutral differential equations with state-dependence delay has been studied in the literature in [5,6]. In [7,8] a hyperbolic equation of convolution type is treated. In [9,10] oscillatory properties of solutions for certain nonlinear impulsive hyperbolic partial differential equation of neutral type are investigated and new sufficient conditions and a necessary and sufficient condition for oscillation of the equations are established. The regularity of solutions of parabolic type equations under some general conditions of the nonlinear terms is considered in [11,12], which is reasonable for application in nonlinear systems. It is worth giving several examples of the use of such and other classes of differential equations in engineering and scientific tasks, for instance, almost automorphic mild solutions of hyperbolic evolution equations with stepanov-like almost automorphic forcing term have studied in [13], and the local well-posedness to the Cauchy problem of the 2D compressible Navier–Stokes–Smoluchowski equations with vacuum was considered in [14]. Recently, regularity problems of second order differential equations have been discussed in [15–17]. The fixed point of a locally asymptotically nonexpansive cosine family is introduced in [17,18].

In this paper, we take an approach different to that of previous works (see [2,19–21]) to discuss some kind of solutions of Cauchy initial problems. By means of L^2 -regularity results, we obtain global

existence of semilinear neutral hyperbolic equations under more general hypotheses of nonlinear terms. Motivated by the L^2 -regularity problems of the linear cases of [19,22], we show to remain valid those under the above the semilinear neutral problem (1).

The summary of the content is as follows. In Section 2, we introduce some notations and preliminaries. In Section 3, we devote to study the regularity and existence of a solution for Equation (1). We prove that the existence for each $T > 0$ of a solution $w \in L^2(0, T; D(A)) \cap W^{1,2}(0, T; E)$ when $f : \mathbb{R} \rightarrow X$ is continuously differentiable, $(x_0, y_0, k) \in D(A) \times E \times W^{1,2}(0, T)$. Here, the space E is an intermediate space between $D(A)$ and X . We are based on some basic ideas of cosine and sine families referred to [23,24] to develop our consequence. We will make general assumptions about the nonlinear terms to obtain the regularity of solutions of Equation (1). An example as application of our results in the last section is given.

2. Preliminaries

We first introduce some notations, definitions and preliminaries. If X is a Banach space with norm denoted by $\|\cdot\|$, and $1 < p < \infty$, $L^p(0, T; X)$ is the collection of all strongly measurable functions from $(0, T)$ into X the p -th powers whose of norms are integrable and $W^{m,p}(0, T; X)$ is the set of all functions f whose derivatives $D^\alpha f$ up to degree m in the distribution sense belong to $L^p(0, T; X)$.

Definition 1 ([23]). A bounded linear family $C(t)(t \in \mathbb{R})$ in X is called a cosine family if

- c(1) $C(0) = I$, $C(s+t) + C(s-t) = 2C(s)C(t)$, for all $s \in \mathbb{R}$,
- c(2) $C(t)x$ is continuous in t on $\mathbb{R}$ for each fixed $x \in X$.

Let $C(t)(t \in \mathbb{R})$ is a cosine family in X . Then the sine family $S(t)(t \in \mathbb{R})$ is defined by

$$S(t)x = \int_0^t C(s)x ds, \quad x \in X. \quad (2)$$

Thus, $S(t)x$ is also strong continuous, that is, $S(t)x$ is continuous in t on $\mathbb{R}$ for each fixed $x \in X$.

The $A : X \rightarrow X$ is defined by

$$Ax = \frac{d^2}{dt^2} C(0)x,$$

which is called the infinitesimal generator of a cosine family $C(t)(t \in \mathbb{R})$. The domain

$$D(A) = \{x \in X : \frac{d}{dt} C(t)x \text{ is continuous}\}$$

with norm

$$\|x\|_{D(A)} = \|x\| + \sup\{\|\frac{d}{dt} C(t)x\| : t \in \mathbb{R}\} + \|Ax\|.$$

We introduce the set

$$E = \{x \in X : C(t)x \text{ is a once continuously differentiable function of } t\}$$

with norm

$$\|x\|_E = \|x\| + \sup\{\|\frac{d}{dt} C(t)x\| : t \in \mathbb{R}\}.$$

We know that $D(A)$ and E with given norms are Banach spaces.

The following Lemma is a summary of Proposition 2.1 and Proposition 2.2 of [23].

Lemma 1. Let $C(t)(t \in \mathbb{R})$ be a cosine family in X . The following terms are satisfied:

c(3) there are constants $\omega \geq 0$ and $K \geq 1$ satisfying

$$\begin{aligned} \|C(t)\| &\leq Ke^{\omega|t|} \text{ for all } t \in \mathbb{R}, \\ \|S(t_1) - S(t_2)\| &\leq K \left| \int_{t_2}^{t_1} e^{\omega|s|} ds \right| \text{ for all } t_1, t_2 \in \mathbb{R}, \end{aligned}$$

c(4) let $x \in E$. Then

$$S(t)x \in D(A), \quad \frac{d}{dt}C(t)x = AS(t)x = S(t)Ax = \frac{d^2}{dt^2}S(t)x,$$

c(5) let $x \in D(A)$. Then

$$C(t)x \in D(A), \quad \frac{d^2}{dt^2}C(t)x = AC(t)x = C(t)Ax,$$

c(6) let $x \in X$ and $r, s \in \mathbb{R}$. Then

$$\int_r^s S(\tau)x d\tau \in D(A) \quad \text{and} \quad A\left(\int_r^s S(\tau)x d\tau\right) = C(s)x - C(r)x,$$

The following Lemma is from Proposition 2.4 of [23].

Lemma 2. Let A is the infinitesimal generator of cosine family $C(t)$ ($t \in \mathbb{R}$). If $f : \mathbb{R} \rightarrow X$ is continuously differentiable, $(x_0, y_0) \in D(A) \times E$ then

$$w(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)f(s)ds, \quad t \in \mathbb{R},$$

belongs to $D(A)$, w is twice continuously differentiable. Moreover, w satisfies

$$\begin{cases} w''(t) = Aw(t) + f(t), & t \in \mathbb{R}, \\ w(0) = x_0, \quad w'(0) = y_0. \end{cases} \quad (3)$$

Conversely, if $f : \mathbb{R} \rightarrow X$ is continuous, w satisfies (3) and $w(t) \in D(A)$ is twice continuously differentiable,

$$w(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)f(s)ds, \quad t \in \mathbb{R}. \quad (4)$$

Proposition 1. Let f be continuously differentiable, $(x_0, y_0) \in D(A) \times E$. Then $w(t)$ defined by (4) is a solution of the linear Equation (3). Moreover, $W(t)$ belongs to $L^2(0, T; D(A)) \cap W^{1,2}(0, T; E)$ and there exists a positive constant C_1 such that for any $T > 0$,

$$\|w\|_{L^2(0,T;D(A))} \leq C_1(1 + \|x_0\|_{D(A)} + \|y_0\|_E + \|f\|_{W^{1,2}(0,T;X)}). \quad (5)$$

Proof. By virtue of Lemma 2, we have that w satisfies Equation (3), $w(t) \in D(A)$ and w is twice continuously differentiable. It is easily seen that there is a constant $C > 0$ such that

$$\|w\|_{L^2(0,T;X)} \leq C(\|x_0\|_{D(A)} + \|y_0\|_E + \|f\|_{L^2(0,T;X)}). \quad (6)$$

Now, we will prove that $w \in L^2(0, T; D(A))$. Using c(3) and c(4), it holds

$$\int_0^T \|AC(t)x_0\|^2 dt \leq K(e^{2\omega T} - 1)\|x_0\|_{D(A)}^2, \quad (7)$$

and if $y_0 \in E$, by c(4), we have

$$\int_0^T \|AS(t)y_0\|^2 dt = \int_0^T \left\| \frac{d}{dt} C(t)y_0 \right\|^2 dt \leq T \|y_0\|_E^2. \quad (8)$$

It is proved in Proposition 2.4 of [23] that

$$A \int_0^t S(t-s)f(s)ds = C(t)f(0) - f(0) + \int_0^t (C(t-s) - I)f'(s)ds.$$

So, since

$$\begin{aligned} \int_0^T \left\| \int_0^t C(t-s)f'(s)ds \right\|^2 dt &\leq K^2(1 - e^{\omega T})^2 \int_0^T \left(\int_0^t \|f'(s)\| ds \right)^2 dt \\ &\leq K^2(1 - e^{\omega T})^2 \int_0^T t \int_0^t \|f'(s)\|^2 ds dt \\ &\leq K^2(1 - e^{\omega T})^2 \frac{T^2}{2} \int_0^T \|f'(s)\|^2 ds, \end{aligned}$$

we have

$$\begin{aligned} \int_0^T \left\| A \int_0^t S(t-s)f(s)ds \right\|^2 dt &\leq \int_0^T \|C(t)f(0)\|^2 dt \\ &\quad + T\|f(0)\|^2 + \int_0^T \left\| \int_0^t C(t-s)f'(s)ds \right\|^2 dt + \int_0^T \left\| \int_0^t f'(s)ds \right\|^2 dt \\ &\leq K^2 e^{2\omega T} T \|f(0)\|^2 + T\|f(0)\|^2 + \{K^2(1 - e^{\omega T})^2 + 1\} \frac{T^2}{2} \int_0^T \|f'(s)\|^2 ds. \end{aligned} \quad (9)$$

Noting that from c(4)

$$\begin{aligned} \frac{d}{dt} C(t) \int_0^t S(t-s)f(s)ds \\ = AS(t) \int_0^t S(t-s)f(s)ds = S(t)A \int_0^t S(t-s)f(s)ds, \end{aligned} \quad (10)$$

we can show the relation of (5) from (6)–(10). Combining (2) and c(3), we also obtain that an analogous estimate to (5) holds for $w \in W^{1,2}(0, T; E)$. $\square$

Remark 1. Let $(x_0, y_0) \in D(A) \times E$, and let f be continuously differentiable. Let us remark that if w is a solution of (3) in an interval $[0, t_1 + t_2]$ with $t_1, t_2 > 0$. Then when $t \in [0, t_1 + t_2]$, from c(11)–c(14), we have

$$\begin{aligned} w(t) &= C(t - t_1)w(t_1) + S(t - t_1)w'(t_1) + \int_{t_1}^t S(t-s)f(s)ds \\ &= C(t - t_1)\{C(t_1)x_0 + S(t_1)y_0 + \int_0^{t_1} S(t_1-s)f(s)ds\} \\ &\quad + S(t - t_1)\{AS(t_1)x_0 + C(t_1)y_0 + \int_0^{t_1} C(t_1-s)f(s)ds\} \\ &\quad + \int_{t_1}^t S(t-s)f(s)ds \\ &= C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)f(s)ds, \end{aligned}$$

here, we used the following basic properties of $C(t)$

$$S(t)AS(s) = AS(t)S(s) = \frac{1}{2}C(t+s) - \frac{1}{2}C(t-s) = C(t+s) - C(t)C(s)$$

for all $s, t \in \mathbb{R}$. This mean the mapping $t \mapsto w(t_1 + t)$ is a solution of (3) in $[0, t_1 + t_2]$ with initial data $(w(t_1), w'(t_1)) \in D(A) \times E$.

3. Semilinear Neutral Equations

This section is to deal with the regularity of solutions of a semilinear neutral second order initial value Equation (1) in a Banach space X : The nonlinear part of Equation (1) is given by we set

$$F(t, w) = \int_0^t k(t-s)h(s, w(s))ds \quad (11)$$

for $w \in L^2(0, T; D(A))$ and $k \in L^2(0, T)$.

Assumption (A). Let $h : [0, T] \times D(A) \rightarrow X$ be a nonlinear operator such that

$$(h1) \quad \|h(t, w_1) - h(t, w_2)\|_{D(A)} \leq L\|w_1 - w_2\|,$$

$$(h2) \quad h(t, 0) = 0$$

for a positive constant L .

Assumption (B). Let $g : [0, T] \times X \rightarrow X$ be a nonlinear operator such that there is a constant L_g satisfying the following conditions hold:

- (i) For any $x \in X$, the mapping $g(\cdot, x)$ is strongly measurable function;
- (ii) There exists a positive constant L_g such that

$$\|Ag(t, 0)\| \leq L_g, \quad \|Ag(t, x) - Ag(t, \hat{x})\| \leq L_g\|x - \hat{x}\|,$$

for all $t \in [0, T]$, and $x, \hat{x} \in X$.

We will find a mild solution of Equation (1), which is represented as the integral equation

$$\begin{aligned} w(t) = & C(t)x_0 + S(t)[y_0 + g(0, x_0)] + \int_0^t S(t-s)\{F(s, w) + f(s)\}ds \\ & - \int_0^t C(t-s)g(s, x(s))ds. \end{aligned} \quad (12)$$

Remark 2. In [25], the approximate controllability of Equation (1) has investigated as a general assumption that $h(t, \cdot)$ is continuous mapping of X into itself satisfying

$$\|h(t, w_1) - h(t, w_2)\| \leq L\|w_1 - w_2\|$$

for a positive constant L .

For short, we assume that $0 \in \rho(A)$ and the closed half plane $\{\lambda : \operatorname{Re} \lambda \geq 0\} \subset \rho(A)$. We remake that $A : X \rightarrow X$ is unbounded, but we can assume that $\|A^{-1}\| \leq M$ for a positive constant $M > 0$.

Lemma 3. Let Assumption (A) be satisfied. Then for $w \in L^2(0, T; D(A))$, $T > 0$.

$$\|F(\cdot, w)\|_{L^2(0, T; X)} \leq L\|k\|_{L^2(0, T)}\sqrt{T}\|w\|_{L^2(0, T; D(A))}.$$

Moreover,

$$\|F(\cdot, w_1) - F(\cdot, w_2)\|_{L^2(0, T; X)} \leq L\|k\|_{L^2(0, T)}\sqrt{T}\|w_1 - w_2\|_{L^2(0, T; D(A))}$$

for $w_1, w_2 \in L^2(0, T; D(A))$.

Proof. From Assumption (A) and by using the Hölder inequality, we have

$$\begin{aligned}\|F(\cdot, w)\|_{L^2(0,T;X)}^2 &\leq \int_0^T \left\| \int_0^t k(t-s)h(s, w(s))ds \right\|^2 dt \\ &\leq \|k\|_{L^2(0,T)}^2 \int_0^T \int_0^t L^2 \|w(s)\|^2 ds dt \\ &\leq L^2 \|k\|_{L^2(0,T)}^2 T \|w\|_{L^2(0,T;D(A))}^2.\end{aligned}$$

The proof of the second paragraph is obtained similarly. $\square$

Lemma 4. If k belongs to $W^{1,2}(0, T)$, then

$$\begin{aligned}A \int_0^t S(t-s)F(s, w)ds &= \int_0^t (C(t-s) - I)k(0)h(s, w(s))ds \\ &+ \int_0^t (C(t-s) - I) \int_0^s \frac{d}{ds} k(s-\tau)h(\tau, w(\tau))d\tau ds.\end{aligned}\quad (13)$$

Proof. The proof of (13) is easily obtained from the following formula

$$A \int_0^t S(t-s)F(s, w)ds = \int_0^t (C(t-s) - I) \frac{d}{ds} F(s, w)ds$$

and

$$\frac{d}{ds} F(s, w) = \int_0^s \frac{d}{ds} k(s-\tau)h(\tau, w(\tau))d\tau + k(0)h(s, w(s)).$$

$\square$

First of all, we give the following result on a local solvability of (11).

Theorem 1. Let that Assumptions (A) and (B) be satisfied. If $(x_0, y_0, k) \in D(A) \times E \times W^{1,2}(0, T)$ ($T > 0$), where k is the function defined by (11), and $f: \mathbb{R} \rightarrow X$ is continuously differentiable, then there exists a time $T_0 \leq T$ such that the Equation (1) guarantees a unique solution w in $L^2(0, T_0; D(A)) \cap W^{1,2}(0, T_0; E)$.

Proof. Let us fix $T_0 > 0$ satisfying

$$\begin{aligned}C_2 \equiv &\omega^{-1}KL T_0^{3/2}(e^{\omega T_0} - 1)\|k\|_{L^2(0,T_0)} \\ &+ \{\omega^{-1}K(e^{\omega T_0} - 1) + 1\}T_0^{3/2}/\sqrt{3}L|Ke^{\omega T_0} + 1|\|k\|_{W^{1,2}(0,T_0)} \\ &+ \{\omega^{-1}K(e^{\omega T_0} - 1) + 1\}T_0/\sqrt{2}L(Ke^{\omega T_0} + 1)\|k(0)\| \\ &+ \frac{KMLg}{2\omega}\sqrt{e^{2\omega T_0} - 1} + \{\omega^{-1}K(e^{\omega T_0} - 1) + 1\}\frac{KLg}{2\omega}\sqrt{e^{2\omega T_0} - 1} \\ &< 1,\end{aligned}\quad (14)$$

where K and L are constants in c(4) and (h1), respectively. For any $v \in L^2(0, T_0; D(A))$, let J be the operator on $L^2(0, T_0; D(A))$ defined by

$$\begin{aligned}J(v)(t) = &C(t)x_0 + S(t)[y_0 + g(0, x_0)] + \int_0^t S(t-s)\{F(s, v) + f(s)\}ds \\ &- \int_0^t C(t-s)g(s, v(s))ds.\end{aligned}\quad (15)$$

Then for each $v_1, v_2 \in L^2(0, T_0; D(A))$,

$$\begin{aligned} J(v_1)(t) - J(v_2)(t) &= \int_0^t S(t-s) \{F(s, v_1) - F(s, v_2)\} ds \\ &\quad - \int_0^t C(t-s) [g(s, v_1(s)) - g(s, v_2(s))] ds \\ &= I_1(v_1 - v_2) - I_2(v_1 - v_2), \end{aligned}$$

where

$$\begin{aligned} I_1 &= \int_0^t S(t-s) \{F(s, v_1) - F(s, v_2)\} ds, \\ I_2 &= \int_0^t C(t-s) [g(s, v_1(s)) - g(s, v_2(s))] ds. \end{aligned}$$

Now, we will show that $I_i \in L^2(0, T_0; D(A))$ ($i = 1, 2$). From Lemmas 3 and 4, it follows that for $0 \leq t \leq T_0$,

$$\begin{aligned} \|I_1\| &= \left\| \int_0^t S(t-s) \{F(s, v_1) - F(s, v_2)\} ds \right\| \\ &\leq \omega^{-1} KLT_0(e^{\omega T_0} - 1) \|k\|_{L^2(0, T_0)} \|v_1 - v_2\|_{L^2(0, T_0; D(A))}, \end{aligned} \quad (16)$$

and

$$\begin{aligned} \|AI_1\| &= \left\| A \int_0^t S(t-s) \{F(s, v_1) - F(s, v_2)\} ds \right\| \\ &\leq \left\| \int_0^t (C(t-s) - I) \int_0^s \frac{d}{ds} k(s-\tau) (h(\tau, v_1(\tau)) - h(\tau, v_2(\tau))) d\tau ds \right\| \\ &\quad + \left\| \int_0^t (C(t-s) - I) k(0) (h(s, v_1(s)) - h(s, v_2(s))) ds \right\| \\ &\leq tL(Ke^{\omega t} + 1) \|k\|_{W^{1,2}(0, T_0)} \|v_1 - v_2\|_{L^2(0, T_0; D(A))} \\ &\quad + \sqrt{t}L \|Ke^{\omega t} + 1\| \|k(0)\| \|v_1 - v_2\|_{L^2(0, T_0; D(A))}. \end{aligned} \quad (17)$$

We also obtain that

$$\begin{aligned} \left\| \frac{d}{dt} C(t) I_1 \right\| &= \left\| \frac{d}{dt} C(t) \int_0^t S(t-s) \{F(s, v_1) - F(s, v_2)\} ds \right\| \\ &= \left\| AS(t) \int_0^t S(t-s) \{F(s, v_1) - F(s, v_2)\} ds \right\| \\ &= \left\| S(t) A \int_0^t S(t-s) \{F(s, v_1) - F(s, v_2)\} ds \right\|. \end{aligned} \quad (18)$$

Therefore we have

$$\begin{aligned} &\|I_1\|_{L^2(0, T_0; D(A))} \\ &\leq \omega^{-1} KLT_0^{3/2} (e^{\omega T_0} - 1) \|k\|_{L^2(0, T_0)} \|v_1 - v_2\|_{L^2(0, T_0; D(A))} \\ &\quad + \{\omega^{-1} K(e^{\omega T_0} - 1) + 1\} T_0^{3/2} / \sqrt{3} L (Ke^{\omega T_0} + 1) \|k\|_{W^{1,2}(0, T_0)} \|v_1 - v_2\|_{L^2(0, T_0; D(A))} \\ &\quad + \{\omega^{-1} K(e^{\omega T_0} - 1) + 1\} T_0 / \sqrt{2} L (Ke^{\omega T_0} + 1) \|k(0)\| \|v_1 - v_2\|_{L^2(0, T_0; D(A))} \end{aligned}$$

From now on, we show that $I_2 \in L^2(0, T_0; D(A))$, from Assumption(B), (c3), it follows that

$$\begin{aligned}
 \|I_2\| &= \left\| \int_0^t C(t-s) \{g(s, v_1) - g(s, v_2)\} ds \right\| \\
 &\leq \int_0^t \|C(t-s) \{g(s, v_1) - g(s, v_2)\}\| ds \\
 &\leq \frac{K}{\sqrt{2\omega}} \sqrt{e^{2\omega t} - 1} \left[\int_0^t \|A^{-1} A \{g(s, v_1) - g(s, v_2)\}\|^2 ds \right]^{\frac{1}{2}} \\
 &\leq \frac{KMLg}{\sqrt{2\omega}} \sqrt{e^{2\omega t} - 1} \left[\int_0^t \|v_1 - v_2\|^2 ds \right]^{\frac{1}{2}} \\
 &\leq \frac{KMLg}{\sqrt{2\omega}} \sqrt{e^{2\omega t} - 1} \|v_1 - v_2\|_{L^2(0, T_0; D(A))},
 \end{aligned} \tag{19}$$

and

$$\begin{aligned}
 \|AI_2\| &= \left\| A \int_0^t C(t-s) \{g(s, v_1) - g(s, v_2)\} ds \right\| \\
 &\leq \int_0^t \|C(t-s) A \{g(s, v_1) - g(s, v_2)\}\| ds \\
 &\leq \left[\int_0^t \|C(t-s)\|^2 ds \right]^{\frac{1}{2}} \left[\int_0^t \|A \{g(s, v_1) - g(s, v_2)\}\|^2 ds \right]^{\frac{1}{2}} \\
 &\leq \frac{KLg}{\sqrt{2\omega}} \sqrt{e^{2\omega t} - 1} \|v_1 - v_2\|_{L^2(0, T_0; D(A))},
 \end{aligned} \tag{20}$$

We also obtain that

$$\begin{aligned}
 &\frac{d}{dt} C(t) \int_0^t C(t-s) g(s, w(s)) ds \\
 &= S(t) A \int_0^t C(t-s) g(s, w(s)) ds,
 \end{aligned} \tag{21}$$

Hence by (19)–(21), we conclude that

$$\begin{aligned}
 &\|I_2\|_{L^2(0, T_0; D(A))} \\
 &\leq \frac{KMLg}{2\omega} \sqrt{e^{2\omega T_0} - 1} \|v_1 - v_2\|_{L^2(0, T_0; D(A))} \\
 &+ \{\omega^{-1} K(e^{\omega T_0} - 1) + 1\} \frac{KLg}{2\omega} \sqrt{e^{2\omega T_0} - 1} \|v_1 - v_2\|_{L^2(0, T_0; D(A))}
 \end{aligned}$$

Thus, from I_1 and I_2 , we conclude that

$$\begin{aligned}
 &\|J(v_1) - J(v_2)\|_{L^2(0, T_0; D(A))} \\
 &\leq \|I_1\|_{L^2(0, T_0; D(A))} + \|I_2\|_{L^2(0, T_0; D(A))} \\
 &\leq \omega^{-1} KLT_0^{3/2} (e^{\omega T_0} - 1) \|k\|_{L^2(0, T_0)} \|v_1 - v_2\|_{L^2(0, T_0; D(A))} \\
 &+ \{\omega^{-1} K(e^{\omega T_0} - 1) + 1\} T_0^{3/2} / \sqrt{3} L (Ke^{\omega T_0} + 1) \|k\|_{W^{1,2}(0, T_0)} \|v_1 - v_2\|_{L^2(0, T_0; D(A))} \\
 &+ \{\omega^{-1} K(e^{\omega T_0} - 1) + 1\} T_0 / \sqrt{2} L (Ke^{\omega T_0} + 1) \|k(0)\| \|v_1 - v_2\|_{L^2(0, T_0; D(A))} \\
 &+ \frac{KMLg}{2\omega} \sqrt{e^{2\omega T_0} - 1} \|v_1 - v_2\|_{L^2(0, T_0; D(A))} \\
 &+ \{\omega^{-1} K(e^{\omega T_0} - 1) + 1\} \frac{KLg}{2\omega} \sqrt{e^{2\omega T_0} - 1} \|v_1 - v_2\|_{L^2(0, T_0; D(A))}.
 \end{aligned} \tag{22}$$

So in terms of the condition (14), the contraction mapping principle on J defined by (15) the contraction mapping guarantees that the solution of Equation (1) exists uniquely in $[0, T_0]$. $\square$

Now, we establish the global existence of solution of Equation (1) based on a variation of constant formula for solutions

Theorem 2. Let Assumptions (A) and (B) be satisfied. If $f : \mathbb{R} \rightarrow X$ is continuously differentiable, $(x_0, y_0, k) \in D(A) \times E \times W^{1,2}(0, T)$, then the solution w of Equation (1) exists and is unique in $L^2(0, T; D(A)) \cap W^{1,2}(0, T; E)$ for each $T > 0$, and there is a constant C_3 depending on T such that

$$\|w\|_{L^2(0,T;D(A))} \leq C_3(1 + \|x_0\|_{D(A)} + \|y_0\|_E + \|f\|_{W^{1,2}(0,T;X)}). \quad (23)$$

Proof. Let $w(\cdot)$ be the solution of Equation (1) in $[0, T_0]$ where T_0 is a constant in (14) and let $v(\cdot)$ be a solution of the following linear equation

$$\begin{cases} v''(t) &= Av(t) + f(t), \quad 0 < t, \\ v(0) &= x_0, \quad v'(0) = y_0. \end{cases}$$

Then

$$(w - v)(t) = S(t)g(0, x_0) + \int_0^t S(t-s)F(s, v)ds - \int_0^t C(t-s)g(s, v)ds,$$

and by (22)

$$\|v - w\|_{L^2(0,T_0;D(A))} \leq C_2\|v\|_{L^2(0,T_0;D(A))} + \omega^{-1}KM(e^{\omega T_0} - 1)\|x_0\|_{L^2(0,T_0;D(A))}, \quad (24)$$

where C_2 is the constant defined by (14). Combining (24) with Proposition 1, it holds

$$\begin{aligned} \|v\|_{L^2(0,T_0;D(A))} &\leq \frac{1}{1-C_2}\|v\|_{L^2(0,T_0;D(A))} + \frac{1}{1-C_2}\omega^{-1}KM(e^{\omega T_0} - 1)\|x_0\|_{L^2(0,T_0;D(A))} \\ &\leq \frac{C_1}{1-C_2}(1 + \|x_0\|_{D(A)} + \|y_0\|_E + \|f\|_{W^{1,2}(0,T_0;X)}) \\ &\quad + \frac{1}{1-C_2}\omega^{-1}KM(e^{\omega T_0} - 1)\|x_0\|_{L^2(0,T_0;D(A))}. \end{aligned} \quad (25)$$

In order to obtain the solution of Equation (1) on $[T_0, 2T_0]$, we will show that $w(T_0) \in D(A)$, $w'(T_0) \in E$. Since

$$\begin{aligned} AI_1(w) &= A \int_0^{T_0} S(T_0 - s)\{F(s, w) + f(s)\}ds \\ &= C(T_0)f(0) - f(0) + \int_0^{T_0} (C(T_0 - s) - I)f'(s)ds \\ &\quad + \int_0^{T_0} (C(T_0 - s) - I) \int_0^s \frac{d}{ds}k(s - \tau)h(\tau, w(\tau))d\tau ds \\ &\quad + \int_0^{T_0} (C(T_0 - s) - I)k(0)h(s, w(s))ds \end{aligned}$$

and

$$\begin{aligned} &\frac{d}{dt}C(t) \int_0^t S(t-s)\{F(s, v) + f(s)\}ds \\ &= S(t)A \int_0^t S(t-s)\{F(s, v) + f(s)\}ds, \end{aligned}$$

it holds

$$\begin{aligned}\|I_1(w)\|_{D(A)} &\leq L\|k\|_{L^2(0,T_0)}\|w\|_{L^2(0,T_0;D(A))} \\ &\quad + (\omega^{-1}K(e^{\omega T_0} - 1) + 1)T_0L(Ke^{\omega T_0} + 1)\|k\|_{W^{1,2}(0,T_0)}\|w\|_{L^2(0,T_0;D(A))} \\ &\quad + (\omega^{-1}K(e^{\omega T_0} - 1) + 1)\sqrt{T_0}L(Ke^{\omega T_0} + 1)\|k(0)\|\|w\|_{L^2(0,T_0;D(A))},\end{aligned}$$

Furthermore, we obtain

$$\begin{aligned}\|I_2(w)\| &= \left\| \int_0^{T_0} C(T_0 - s)g(s, w(s))ds \right\| \leq \int_0^{T_0} \|C(T_0 - s)g(s, w(s))\|ds \\ &\leq \frac{K}{\sqrt{2\omega}} \sqrt{e^{2\omega T_0} - 1} \left[\int_0^{T_0} \|A^{-1}Ag(s, w(s))\|^2 ds \right]^{\frac{1}{2}} \\ &\leq \frac{KML_g}{\sqrt{2\omega}} \sqrt{e^{2\omega T_0} - 1} \left[\int_0^{T_0} \|w\|^2 ds \right]^{\frac{1}{2}} \\ &\leq \frac{KML_g}{\sqrt{2\omega}} \sqrt{e^{2\omega T_0} - 1} \|w\|_{L^2(0,T_0;D(A))},\end{aligned}$$

and

$$\begin{aligned}\|AI_2(w)\| &= \left\| A \int_0^{T_0} C(T_0 - s)g(s, w(s))ds \right\| \leq \int_0^{T_0} \|C(T_0 - s)Ag(s, w(s))\|ds \\ &\leq \frac{KL_g}{\sqrt{2\omega}} \sqrt{e^{2\omega T_0} - 1} \|w\|_{L^2(0,T_0;D(A))},\end{aligned}$$

Since

$$\frac{d}{dt}C(t) \int_0^{T_0} C(T_0 - s)g(s, w(s))ds = S(t)A \int_0^{T_0} C(T_0 - s)g(s, w(s))ds,$$

We obtain that

$$\begin{aligned}\|I_2(w)\|_{D(A)} &\leq \frac{KML_g}{\sqrt{2\omega}} \sqrt{e^{2\omega T_0} - 1} \|w\|_{L^2(0,T_0;D(A))} \\ &\quad + (\omega^{-1}K(e^{\omega T_0} - 1) + 1) \frac{KL_g}{\sqrt{2\omega}} \sqrt{e^{2\omega T_0} - 1} \|w\|_{L^2(0,T_0;D(A))},\end{aligned}$$

Therefore, we conclude that

$$\begin{aligned}&\|w(T_0)\|_{D(A)} \\ &= \|C(T_0)x_0 + S(T_0)(y_0 + g(0, x_0)) + I_1(w) - I_2(w) - \int_0^{T_0} C(T_0 - s)g(s, w(s))ds\|_{D(A)} \\ &\leq (\omega^{-1}K(e^{\omega T_0} - 1) + 1)[Ke^{\omega T_0} + L_g\{\omega^{-1}Ke^{\omega T_0} - 1\} + M]\|x_0\|_{D(A)} \\ &\quad + \omega^{-1}K(e^{\omega T_0} - 1)\|y_0\|_E \\ &\quad + \|Ke^{\omega T_0}f(0)\| + \|f(0)\| + K(e^{\omega T_0} + 1)\sqrt{T_0}\|f\|_{W^{1,2}(0,T;X)} \\ &\quad + L\|k\|_{L^2(0,T_0)}\|w\|_{L^2(0,T_0;D(A))} \\ &\quad + T_0L(Ke^{\omega T_0} + 1)\|k\|_{W^{1,2}(0,T_0)}\|w\|_{L^2(0,T_0;D(A))} \\ &\quad + \sqrt{T_0}L(Ke^{\omega T_0} + 1)\|k(0)\|\|w\|_{L^2(0,T_0;D(A))} \\ &\quad + \frac{KML_g}{\sqrt{2\omega}} \sqrt{e^{2\omega T_0} - 1} \|w\|_{L^2(0,T_0;D(A))} \\ &\quad + \frac{KL_g}{\sqrt{2\omega}} \sqrt{e^{2\omega T_0} - 1} \|w\|_{L^2(0,T_0;D(A))}].\end{aligned}$$

Thus, by (25), there is a positive constant $C > 0$ such that

$$\|w(T_0)\|_{D(A)} \leq C(1 + \|x_0\|_{D(A)} + \|y_0\|_E + \|f\|_{W^{1,2}(0,T_0;X)}).$$

From which it is immediately obtain that $w(T_0) \in E$. Hence, we can show that the solution can be extended to the interval $[T_0, 2T_0]$ with the initial $(w(nT_0), w'(nT_0)) \in D(A) \times E$ and that an estimate similar to (25) holds. Since the condition (14) are independent of initial values, the solution can be extended to the interval $[0, nT_0]$ for every natural number n . So the proof is complete. $\square$

Example. We consider the following semilinear neutral partial differential equation in $X = L^2([0, \pi]; \mathbb{R})$:

$$\begin{cases} \frac{d}{dt}[w'(t, x) + g(t, w(t, x))] = Aw(t, x) + F(t, w) + f(t), & 0 < t, \quad 0 < x < \pi, \\ w(t, 0) = w(t, \pi) = 0, & t \in \mathbb{R} \\ w(0, x) = x_0(x), \quad w'(0, x) = y_0(x), & 0 < x < \pi \end{cases} \quad (26)$$

It is well known that $\{e_n = \sqrt{\frac{2}{\pi}} \sin nx : n = 1, \dots\}$ is an orthonormal base for X . Let $A : X \rightarrow X$ be defined by

$$Aw(t, x) = \frac{\partial^2}{\partial x^2} w(t, x).$$

For short, $w(t, x) \equiv w(x)$, we know $D(A) = \{w \in W^{2,2}(0, \pi) : (w, 0) = w(\pi) = 0\}$, and

$$Aw = \sum_{n=1}^{\infty} -n^2(w, e_n)e_n, \quad w \in D(A),$$

and A is the infinitesimal generator of a cosine family $C(t)(t \in \mathbb{R})$ in X represented as

$$C(t)w = \sum_{n=1}^{\infty} \cos nt(w, e_n)e_n, \quad w \in X.$$

The associated sine family is given by

$$S(t)w = \sum_{n=1}^{\infty} \frac{\sin nt}{n}(w, e_n)e_n, \quad w \in X.$$

Since $\{e_n : n \in \mathbb{N}\}$ is an orthogonal basis of X and

$$e^{At} = \sum_{n=1}^{\infty} e^{-n^2 t}(w, e_n)e_n, \quad \forall w \in H, \quad t > 0.$$

Moreover, there exists a constant M_0 such that $\|e^{At}\| \leq M_0$.

Define $g : [0, T] \times X \rightarrow X$ as

$$g(t, w) = \sum_{n=1}^{\infty} \int_0^t e^{-n^2 t} \left(\int_0^s a_2(t-s)w(s)ds, w_n \right) w_n,$$

where there exists a constant M_1 such that

$$|a_2(s)| \leq M_1, \quad |a_2(s) - a_2(\tau)| \leq M_1(s - \tau), \quad s, \tau \in \mathbb{R}^+.$$

Then it can be checked that Assumption (B) in Section 3 is satisfied. Indeed, for $w \in X$, we know

$$Ag(t, w) = (e^{At} - I) \int_0^t a_2(t-s)w(s)ds,$$

where I is the identity operator from X to itself. Hence, we have

$$\begin{aligned} \|Ag(t, w)\|^2 &\leq (M_0 + 1)^2 \left\{ \left| \int_0^t (a_2(t+s) - a_2(t))w(s)ds \right|^2 + \left| \int_0^t a_2(t)w(s)ds \right|^2 \right\} \\ &\leq (M_0 + 1)^2 M_1^2 \{t^3/3 + t\} \|w\|. \end{aligned}$$

It is immediately seen that Assumption (B) has been satisfied. Let

$$h(t, w)x = h_1(t, w, Dw, D^2w).$$

We consider Equation (26) under the following assumptions:

Assumption (C). There is a continuous $\gamma(t, r) : \mathbb{R}^2 \rightarrow \mathbb{R}^+$ such that

- (h1) $h_1(t, x, 0, 0) = 0$,
- (h2) $|h_1(t, x, w, p) - h_1(t, x, w, q)| \leq \gamma(t, |w|)|p - q|$,
- (h3) $|h_1(t, x, w_1, p) - h_1(t, x, w_2, p)| \leq \gamma(t, |w_1| + |w_2|)|w_1 - w_2|$.

Then since

$$\begin{aligned} \|h(t, w_1) - h(t, w_2)\|_{0,2}^2 &\leq 2 \int_{\Omega} |h_1(t, x, w_1, Dw_1, D^2w_1) - h_1(t, w_2, Dw_2, D^2w_2)|^2 du \\ &\quad + 2 \int_{\Omega} |h_1(t, x, w_1, Dw_1, D^2w_1) - h_1(t, x, w_2, Dw_2, D^2w_2)|^2 du, \end{aligned}$$

it follows from Assumption (C) that

$$\|h(t, w_1) - h(t, w_2)\|_{0,2}^2 \leq L(\|w_1\|_{D(A)}, \|w_2\|_{D(A)}) \|w_1 - w_2\|_{D(A)},$$

where $L(\|w_1\|_{D(A)}, \|w_2\|_{D(A)})$ is a constant depending on $\|w_1\|_{D(A)}$ and $\|w_2\|_{D(A)}$. We set

$$F(t, w) = \int_0^t k(t-s)h(s, w(s))ds$$

where k belongs to $L^2(0, T)$.

Theorem 3. Let Assumption (C) be satisfied for the Equation (26). If $f : \mathbb{R} \rightarrow X$ is continuously differentiable, $(x_0, y_0, k) \in D(A) \times E \times W^{1,2}(0, T)$, then the solution w of Equation (26) exists and is unique in $L^2(0, T; D(A)) \cap W^{1,2}(0, T; E)$ for each $T > 0$, and there is a constant C_3 depending on T such that

$$\|w\|_{L^2(0,T;D(A))} \leq C_3(1 + \|x_0\|_{D(A)} + \|y_0\|_E + \|f\|_{W^{1,2}(0,T;X)}).$$

4. Conclusions

This paper investigates the regularity for solutions of semilinear neutral hyperbolic equations with the nonlinear convolution. The principal operator is the infinitesimal generator of a cosine and sine families. We can obtain a variation of constant formula for solutions under more general hypotheses of nonlinear terms by using the nature of cosine and sine families. To show the regularity results of semilinear neutral hyperbolic equations is to remain valid those under the above the semilinear neutral problem based by the L^2 -regularity problems of the linear cases, which is also applicable to the functional analysis concerning control problems and optimal control theory.

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