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Maximizing the Minimal Satisfaction—Characterizations of Two Proportional Values

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Abstract: A class of solutions are introduced by lexicographically minimizing the complaint of coalitions for cooperative games with transferable utility. Among them, the nucleolus is an important representative. From the perspective of measuring the satisfaction of coalitions with respect to a payoff vector, we define a family of optimal satisfaction values in this paper. The proportional division value and the proportional allocation of non-separable contribution value are then obtained by lexicographically maximizing two types of satisfaction criteria, respectively, which are defined by the lower bound and the upper bound of the core from the viewpoint of optimism and pessimism respectively. Correspondingly, we characterize these two proportional values by introducing the equal minimal satisfaction property and the associated consistency property. Furthermore, we analyze the duality of these axioms and propose more approaches to characterize these two values on basis of the dual axioms.

Keywords: cooperative game; satisfaction criteria; proportional value; axiomatization

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1. Introduction

In the process of economic globalization, multinational corporations usually reach a cooperative agreement and form a cooperative coalition in order to gain more benefits. It is a central problem of how to allocate the overall profits of cooperation among these multinational corporations. Cooperative game theory provides general mathematical methods to solve the allocation problems. The solution concepts, such as the Shapley value [1] and the nucleolus [2], offer concrete schemes of allocating the overall profits among players.

The nucleolus, introduced by Schmeidler [2], is a classical solution concept of cooperative games. The nucleolus is obtained by lexicographically minimizing the maximal excess of coalition over the non-empty imputation set. Here, the excess is an important criterion to describe the dissatisfaction with respect to the payoff vector. Thus, a positive excess of a coalition with respect to a payoff vector represents the loss that the coalition suffers from the payoff vector. Several central solutions of cooperative games are defined according to the idea of excess, for example, the nucleolus [2], the core, the kernel [3], and the τ value [4]. In particular, the core is the set of all payoff vectors with non-positive excesses for all coalitions. Besides the excess criterion, Hou et al. [5] proposed two other criteria to measure the dissatisfaction of coalition with respect to a payoff vector.

On the contrary, the satisfaction is a significant criterion to measure the preference degree of coalitions for a payoff vector. Thus, from the perspective of the satisfaction, we define a family of optimal satisfaction values in this paper. Two special optimal satisfaction values are given in terms of the optimistic satisfaction and the pessimistic satisfaction respectively. For a cooperative game with transferable utility (for short, TU-game), the individual worth vector is the lower bound of the core while the marginal contribution vector is the upper bound of the core. Thus, the individual worth vector and the marginal contribution vector can be viewed as the least potential payoff vector and the ideal payoff vector respectively. There are two representative biases in social comparisons [6], a comparative optimism bias (i.e., a tendency for people to evaluate themselves in a more positive light) and a comparative pessimism bias (i.e., a tendency for people to evaluate themselves in a more negative light). The optimistic satisfaction and the pessimistic satisfaction are defined by the individual worth vector and the marginal contribution vector from the viewpoints of optimism and pessimism respectively. On the optimistic side, players always take the individual worth of themselves into consideration and think of the ratio between the real payoff and the individual worth as the measure of satisfaction. The optimistic satisfaction of a coalition is defined by the ratio between the real payoff of the coalition and the sum of their individual worths with respect to a payoff vector. Conversely, pessimists always take the ideal payoff of themselves into consideration. The pessimistic satisfaction of a coalition is the ratio between the real payoff of the coalition and the sum of the marginal contributions of players in the coalition. Thus, the optimistic optimal satisfaction value and the pessimistic optimal satisfaction value are determined by maximizing the minimal optimistic satisfaction and the minimal pessimistic satisfaction in the lexicographic order over the non-empty pre-imputation set, respectively. Interestingly, the two values are coincident with the proportional division value (PD value) and the proportional allocation of non-separable contribution value (PANSC value), respectively.

The proportional principle is a relatively fair and reasonable allocation criterion in many economic situations. It is a norm of distributed justice rooted in law and custom [7]. Moulin's survey [8] of cost and surplus sharing opens by emphasizing the importance of the proportional principle. The PD value and the PANSC value are defined based on the idea of proportionality. The PD value, introduced by Banker [9], distributes the overall worth of the grand coalition in proportion to player's individual worth among all players. As the dual value of the PD value, the PANSC value distributes the overall worth in proportion to their marginal contributions with respect to the grand coalition. Moreover, some other proportional values have been studied in the literature, such as the proper Shapley value [10,11], the proportional value [12,13], and the proportional Shapley value [14,15]. In this paper, we mainly study the PD value and the PANSC value and propose several new axiomatizations of the PD value and the PANSC value.

Axiomatization is one of the main ways to characterize the reasonability of solutions in cooperative games. For the PD value, Zou et al. [16] proposed several characterizations on the basis of the equal treatment of equals, monotonicity and reduced game consistency. In this paper, we first propose the equal minimal optimistic satisfaction property and equal minimal pessimistic satisfaction property, which are inspired by the kernel concept [17]. The equal minimal optimistic satisfaction property states that for a pair of players $\{i, j\}$ and a payoff vector x, the minimal optimistic satisfaction of coalitions containing i and not j with respect to x should equal that of coalitions containing j and not i under the optimistic satisfaction criterion, while the equal minimal pessimistic satisfaction property describe this situation under the pessimistic satisfaction criterion. Then, the PD value and the PANSC value are characterized by these two properties with efficiency, respectively.

Associated consistency is also an important characteristic of solutions for TU-games. A solution satisfies associated consistency if it allocates the same payoff to players in the associated game as that in the initial game. The concept of associated consistency was firstly introduced by Hamiache [18] to characterize the Shapley value. Driessen [19] characterized the family of efficient, symmetric, and linear values by associated consistency on the basis of Hamiache's axiomatization system. Associated consistency is quite popular in the literature on the axiomatization of solutions for TU-games, for instance, the EANS value and the CIS value [20], linear and symmetric values [21] and the core [22]. We propose two associated consistency properties, optimistic associated consistency and pessimistic associated consistency, to characterize the PD value and the PANSC value in this paper.

Furthermore, we also study the dual axioms of the two associated consistency properties and propose more approaches to characterize these two values.

This paper is organized as follows—in Section 2, some basic definitions and notation are introduced. We determine the PD value and the PANSC value by maximizing the minimal optimistic satisfaction and the minimal pessimistic satisfaction in the lexicographic order in Section 3. In Section 4, we propose two types of axioms, the equal minimal satisfaction property and the associated consistency property, to characterize the PD value and the PANSC value, and analyze the dual axioms of associated consistency. Finally, we give a brief conclusion in Section 5.

2. Preliminaries

Let $\mathcal{U} \subseteq \mathbb{N}$ be the set of potential players, where $\mathbb{N}$ is the set of natural numbers. A cooperative game with transferable utility or simply a TU-game is a pair $\langle N, v\rangle$, where $N \subseteq \mathcal{U}$ is a finite set of n players and $v : 2^N \to \mathbb{R}$ is a characteristic function that assigns to each coalition $S \in 2^N$, the worth $v(S)$ with $v(\emptyset) = 0$. Denote the set of all TU-games on player set N by $\mathcal{G}^N$. Hereafter, a TU-game $\langle N, v\rangle$ is simply denoted by v, the cardinality of a finite set S is denoted by s, and the set of all non-empty coalitions is denoted by Ω.

A TU-game $v \in \mathcal{G}^N$ is individually positive (or negative) if $v(\{i\}) > 0$ (or $v(\{i\}) < 0$) for all $i \in N$. Denote the set of all individually positive (or negative) TU-games on player set N by $\mathcal{G}^N_+$ (or $\mathcal{G}^N_-$). Without ambiguity, let $b^v(\{i\}) \equiv v(N) - v(N\backslash\{i\})$ be the marginal contribution of player i with respect to the grand coalition N. For all $v \in \mathcal{G}^N$ and $S \in \Omega$, let $b^v(S) \equiv \sum_{i\in S} b^v(\{i\})$. A TU-game $v \in \mathcal{G}^N$ is marginally positive (or negative) if $b^v(\{i\}) > 0$ (or $b^v(\{i\}) < 0$) for all $i \in N$. Denote the set of all marginally positive (or negative) TU-games on player set N by $\mathcal{G}^N_\oplus$ (or $\mathcal{G}^N_\ominus$). For convenience, we focus on the family of all individually positive TU-games $\mathcal{G}^N_+$ and the family of all marginally positive TU-games $\mathcal{G}^N_\oplus$ in the rest of this paper.

For any TU-game $v \in \mathcal{G}^N$, its dual game v^d is given as follows, for all $S \subseteq N$,

$$v^d(S) \equiv v(N) - v(N\backslash S), \tag{1}$$

where $v^d(S)$ represents the marginal worth of coalition S with respect to N. Obviously, the dual of a individually positive (or negative) TU-game is marginally positive (or negative). Thus, the duality operator is not closed on the class of individually positive (or negative) TU-games. Given any $\mathcal{A} \subseteq \mathcal{G}^N$, let $\mathcal{A}^d$ be the set of dual of TU-games in $\mathcal{A}$.

A payoff vector for a TU-game $v \in \mathcal{G}^N$ is an n-dimensional vector $x \in \mathbb{R}^n$ assigning a payoff $x_i \in \mathbb{R}$ to every player $i \in N$. Let $x(S) = \sum_{i\in S} x_i$ for all $S \in \Omega$. A payoff vector x satisfies efficiency if $x(N) = v(N)$ for all $v \in \mathcal{G}^N$, satisfies individual rationality if $x_i \geq v(\{i\})$ for all $v \in \mathcal{G}^N$ and $i \in N$, and satisfies group rationality if $x(S) \geq v(S)$ for all $v \in \mathcal{G}^N$ and $S \in \Omega$. According to these properties, the pre-imputation set $I^*(v)$ and the imputation set $I(v)$ are given by $I^*(v) = \{x \in \mathbb{R}^n | x(N) = v(N)\}$ and $I(v) = \{x \in I^*(v) | x_i \geq v(\{i\})$ for all $i \in N\}$.

A value on $\mathcal{G}^N$ is a function φ which assigns to every game $v \in \mathcal{G}^N$ a payoff vector $\varphi(v) \in \mathbb{R}^n$. Given any $\mathcal{A} \subseteq \mathcal{G}^N$ and a value φ on $\mathcal{A}$, its dual value φ^d is defined as, for all $v \in \mathcal{A}^d$, $\varphi^d(v) \equiv \varphi(v^d)$. The PD value, denoted by PD, assigns to every player the payoff in proportion to their singleton worths. For any $v \in \mathcal{G}^N_+$ and $i \in N$,

$$PD_i(v) = \frac{v(\{i\})}{\sum_{k\in N} v(\{k\})} v(N).$$

The PANSC value, denoted by PANSC, assigns to every player the payoff in proportion to their marginal contributions with respect to the grand coalition. For any $v \in \mathcal{G}^N_\oplus$ and $i \in N$,

$$PANSC_i(v) = \frac{b^v(\{i\})}{b^v(N)} v(N).$$

Obviously, the PD value is the dual of the PANSC value.

3. The Optimal Satisfaction Value

The core is one of the most important set solutions for TU-games, which is defined through efficiency and group rationality. The core of a TU-game v is given by

$$C(v) = \{x \in I^*(v) | x(S) \geq v(S) \text{ for all } S \in \Omega\}.$$

Let $e(S, x, v) = v(S) - x(S)$ be the excess of a coalition $S \in \Omega$ with respect to x in a TU-game v. The excess $e(S, x, v)$ is usually used to measure the dissatisfaction degree of a coalition S with respect to x. Obviously, a payoff vector in the core only generates non-positive excesses for all coalitions. The larger the excess $e(S, x, v)$ is, the more unsatisfied the coalition S feel with respect to x. Conversely, the larger the minus excess $-e(S, x, v)$ is, the more satisfied the coalition S feel with respect to x.

From the perspective of measuring the satisfaction of coalitions for a payoff vector, we aim to introduce a family of optimal satisfaction values for TU-games. For a payoff vector $x \in \mathbb{R}^n$ and a TU-game $v \in \mathcal{G}^N$, let $\theta^v(x)$ be the $(2^n - 1)$-tuple vector whose components are the satisfactions of all coalitions $S \in \Omega$ with respect to x in non-decreasing order, that is, $\theta^v_t(x) \leq \theta^v_{t+1}(x)$ for all $t \in \{1, 2, \cdots, 2^n - 2\}$. For any $v \in \mathcal{G}^N$ and $x, y \in \mathbb{R}^n$, we call $\theta^v(x) \geq_L \theta^v(y)$ if and only if $\theta^v(x) = \theta^v(y)$, or there exists an $t \in \{1, 2, \cdots, 2^n - 2\}$ such that $\theta^v_l(x) = \theta^v_l(y)$ for all $l \in \{1, 2, \cdots, t - 1\}$ and $\theta^v_t(x) > \theta^v_t(y)$.

Definition 1. *For any $v \in \mathcal{G}^N$, an optimal satisfaction value φ^{os} is a payoff vector y in the pre-imputation set satisfying $\theta^v(y) \geq_L \theta^v(x)$ for all $x \in I^*(v)$, that is,*

$$\varphi^{os}(v) = \{y \in I^*(v) | \theta^v(y) \geq_L \theta^v(x) \textit{ for all } x \in I^*(v)\}.$$

The optimal satisfaction value can be viewed as a solution for an optimization problem aiming to maximize the minimal satisfaction with respect to the payoff vector over the pre-imputation set in the lexicographic order. It is easy to obtain that the optimal satisfaction value is consistent with the pre-nucleolus of a TU-game v under the satisfaction criterion of the minus excess $-e(S, x, v)$. Hou et al. [5] defined two linear complaint criteria which are given by $e^E(S, x, v) = b^v(S) - x(S)$ and $e^C(S, x, v) = x(N \backslash S) - \sum_{k \in N \backslash S} v(\{k\})$ for any $v \in \mathcal{G}^N$, $S \in \Omega$ and $x \in \mathbb{R}^n$. Conversely, $-e^E(S, x, v)$ and $-e^C(S, x, v)$ can be regarded as two different satisfaction criteria. Thus, two optimal satisfaction values are obtained by Definition 1, which coincide with the ENSC value and the CIS value according to Theorem 3.8 and Theorem 3.14 in Reference [5], respectively.

In this section, we define two special satisfaction criteria, the optimistic satisfaction and the pessimistic satisfaction, from the viewpoint of optimism and pessimism respectively. Given any $v \in \mathcal{G}^N$ and $x \in C(v)$, then it holds that $v(\{i\}) \leq x_i \leq b^v(\{i\})$ for all $i \in N$. Thus, the vector $(v(\{k\}))_{k \in N}$ can be regarded as the least potential payoff vector while $(b^v(\{k\}))_{k \in N}$ can be regarded as the ideal payoff vector of a TU-game v. On the optimistic side, the players prefer taking the least potential payoff of themselves into consideration and think of the ratio between the real payoff of coalition and their least potential payoff as the measure of satisfaction of the coalition. Conversely, pessimists prefer taking the ideal payoff of themselves into consideration. Formally, the optimistic satisfaction and the pessimistic satisfaction are defined as follows.

Definition 2. *For any payoff vector $x \in \mathbb{R}^n$ and $v \in \mathcal{G}^N_+$, $w \in \mathcal{G}^N_\oplus$, the optimistic satisfaction of a coalition $S \in \Omega$ with respect to x is given by*

$$e^o(S, x, v) = \frac{x(S)}{\sum_{k \in S} v(\{k\})}, \tag{2}$$

and the pessimistic satisfaction of a coalition $S \in \Omega$ with respect to x is given by

$$e^p(S, x, w) = \frac{x(S)}{b^w(S)}. \tag{3}$$

With respect to the two satisfaction criteria, we have two corresponding optimal satisfaction values, namely the optimistic optimal satisfaction value and the pessimistic optimal satisfaction value. In the following, we show that they are in coincidence with the PD value and the PANSC value, respectively.

3.1. The Optimistic Optimal Satisfaction Value and the PD Value

Formally, the optimistic optimal satisfaction value is given as follows.

Definition 3. *For any $v \in \mathcal{G}_+^N$, the optimistic optimal satisfaction value φ^o is the unique payoff vector y in the pre-imputation set satisfying $\theta^v(y) \geq_L \theta^v(x)$ for all $x \in I^*(v)$, that is,*

$$\varphi^o(v) = \{y \in I^*(v) | \theta^v(y) \geq_L \theta^v(x) \text{ for all } x \in I^*(v)\},$$

where θ^v is the satisfaction vector with respect to the optimistic satisfaction.

Next we show that the PD value is also obtained by lexicographically maximizing the minimal optimistic satisfaction, and coincides with the optimistic optimal satisfaction value.

Lemma 1. *Given any $v \in \mathcal{G}_\oplus^N$ and a payoff vector $x \in \mathbb{R}^n$, let $l = \arg\min_{l \in N}\{e^o(\{l\}, x, v)\}$. Then, we have $e^o(\{l\}, x, v) = \min_{S \in \Omega}\{e^o(S, x, v)\}$.*

Proof. Let $p = \min_{k \in N}\{e^o(\{k\}, x, v)\}$, then $p = e^o(\{l\}, x, v)$ and $x_k \geq p \cdot v(\{k\})$ for all $k \in N$. Then, we have

$$\begin{aligned} e^o(\{l\}, x, v) &\geq \min_{S \in \Omega}\{e^o(S, x, v)\} = \min_{S \in \Omega}\{\frac{x(S)}{\sum_{k \in S} v(\{k\})}\} \\ &\geq \min_{S \in \Omega}\{\frac{p \cdot \sum_{k \in S} v(\{k\})}{\sum_{k \in S} v(\{k\})}\} = p = e^o(\{l\}, x, v). \end{aligned}$$

Therefore, all inequalities are equalities and then $e^o(\{l\}, x, v) = \min_{S \in \Omega}\{e^o(S, x, v)\}$. □

Lemma 2. *Given any $v \in \mathcal{G}_+^N$ and a payoff vector $x \in \mathbb{R}^n$, let $l = \arg\min_{l \in N}\{e^o(\{l\}, x, v)\}$. If there exists a player $m \in N$ such that $e^o(\{m\}, x, v) > e^o(\{l\}, x, v)$, define a new payoff vector x^* given by*

$$x_k^* = \begin{cases} x_k, & \text{for } k \in N \setminus \{l, m\}; \\ x_l + \triangle, & \text{for } k = l; \\ x_m - \triangle, & \text{for } k = m, \end{cases}$$

where $\triangle = \frac{x_m \cdot v(\{l\}) - x_l \cdot v(\{m\})}{v(\{l\}) + v(\{m\})}$. Then the following five statements hold.

1. *$e^o(S, x^*, v) = e^o(S, x, v)$ for any $S \in \Omega$ and $S \not\ni l, m$.*
2. *$e^o(S, x^*, v) = e^o(S, x, v)$ for any $S \in \Omega$ and $S \ni l, m$.*
3. *$e^o(S, x^*, v) > e^o(S, x, v)$ for any $S \in \Omega$, $S \ni l$ and $S \not\ni m$.*
4. *$e^o(S, x^*, v) > e^o(\{l\}, x, v)$ for any $S \in \Omega$, $S \not\ni l$ and $S \ni m$.*
5. *$\theta^v(x^*) >_L \theta^v(x)$, where θ^v is the satisfaction vector with respect to the optimistic satisfaction.*

Proof.

1. It is obvious that $e^o(S, x^*, v) = e^o(S, x, v)$ for any $S \in \Omega$ and $S \not\ni l, m$ because $x^*(S) = x(S)$ for any $S \in \Omega$ and $S \not\ni l, m$.
2. It is trivial that $e^o(S, x^*, v) = e^o(S, x, v)$ for any $S \in \Omega$ and $S \ni l, m$.
3. It is easy to obtain that $\triangle > 0$ since $e^o(\{m\}, x, v) > e^o(\{l\}, x, v)$. Then for any $S \in \Omega$, $S \ni l$ and $S \not\ni m$,
$$e^o(S, x^*, v) = \frac{x^*(S)}{\sum_{k\in S} v(\{k\})} = \frac{x(S) + \triangle}{\sum_{k\in S} v(\{k\})} > \frac{x(S)}{\sum_{k\in S} v(\{k\})} = e^o(S, x, v).$$
4. Since $e^o(\{m\}, x, v) > e^o(\{l\}, x, v)$, we have
$$\begin{aligned} e^o(\{m\}, x^*, v) =& \frac{x^*_m}{v(\{m\})} = \frac{x_m - \triangle}{v(\{m\})} = \frac{x_m + x_l}{v(\{m\}) + v(\{l\})} \\ >& \frac{\frac{x_l}{v(\{l\})} v(\{m\}) + x_l}{v(\{m\}) + v(\{l\})} = \frac{x_l}{v(\{l\}))} = e^o(\{l\}, x, v). \end{aligned}$$
   For any $S \in \Omega$, $S \not\ni l$ and $S \ni m$, we have
$$\begin{aligned} e^o(S, x^*, v) =& \frac{x(S\backslash\{m\}) + x^*_m}{\sum_{k\in S\backslash\{m\}} v(\{k\}) + v(\{m\})} \\ >& \frac{\frac{x_l}{v(\{l\})} \sum_{k\in S\backslash\{m\}} v(\{k\}) + \frac{x_l}{v(\{l\})} v(\{m\})}{\sum_{k\in S\backslash\{m\}} v(\{k\}) + v(\{m\})} \\ =& \frac{x_l}{v(\{l\}))} = e^o(\{l\}, x, v), \end{aligned}$$
   where the second inequality holds because $e^o(\{m\}, x^*, v) > e^o(\{l\}, x, v)$ and $e^o(S\backslash\{m\}, x, v) \geq e^o(\{l\}, x, v)$ by Lemma 1.
5. It holds that $\theta^v(x^*) >_L \theta^v(x)$ by 1–4. $\square$

Theorem 3. *For any $v \in \mathcal{G}^N_+$, the following two statements hold.*

1. $\frac{\varphi^o_i(v)}{v(\{i\})} = \frac{\varphi^o_j(v)}{v(\{j\})}$ *for all $i, j \in N$.*
2. $\varphi^o_i(v) = PD_i(v)$ *for all $i \in N$.*

Proof.

1. We will prove that $\frac{\varphi^o_i(v)}{v(\{i\})} = \frac{\varphi^o_j(v)}{v(\{j\})}$ for all $i, j \in N$ by reduction to absurdity. Given any $v \in \mathcal{G}^N_+$, suppose there exists $m, j \in N$ such that $\frac{\varphi^o_m(v)}{v(\{m\})} \neq \frac{\varphi^o_j(v)}{v(\{j\})}$. Without loss of generality, suppose that $\frac{\varphi^o_m(v)}{v(\{m\})} > \frac{\varphi^o_j(v)}{v(\{j\})}$. Let $y = \varphi^o(v)$ and $l = \arg\min_{l\in N}\{e^o(\{l\}, y, v)\}$, then we have $e^o(\{m\}, y, v) > e^o(\{l\}, y, v)$. By Lemma 2, there exists $x \in I^*(v)$ such that $\theta^v(x) >_L \theta^v(y)$, where θ^v is the satisfaction vector with respect to the optimistic satisfaction, which contradicts with $\theta^v(y) \geq_L \theta^v(x)$ for all $x \in I^*(v)$. Therefore, $\frac{\varphi^o_i(v)}{v(\{i\})} = \frac{\varphi^o_j(v)}{v(\{j\})}$ for all $i, j \in N$.
2. It is immediate to deduce that $\varphi^o_i(v) = PD_i(v) = \frac{v(\{i\})}{\sum_{k\in N} v(\{k\})} v(N)$ by the statement 1 and efficiency. $\square$

Obviously, if $\sum_{k\in N} v(\{k\}) \leq v(N)$, then the optimistic optimal satisfaction value φ^o satisfies individual rationality, that is, $\varphi^o_k(v) \geq v(\{k\})$ for all $k \in N$. The following corollary is immediate for the reason that $C(v) \neq \emptyset$ implies $\sum_{k\in N} v(\{k\}) \leq v(N)$.

Corollary 4. *For any TU-game $v \in \mathcal{G}^N_+$ with $C(v) \neq \emptyset$, the optimistic optimal satisfaction value φ^o satisfies individual rationality, that is, $\varphi^o_k(v) \geq v(\{k\})$ for all $k \in N$.*

3.2. The Pessimistic Optimal Satisfaction Value and the PANSC Value

The pessimistic optimal satisfaction value is defined by lexicographically maximizing the minimal pessimistic satisfaction. We show that the PANSC value is also in coincidence with the pessimistic optimal satisfaction value in this subsection.

Definition 4. *For any $v \in \mathcal{G}_{\oplus}^{N}$, the pessimistic optimal satisfaction value φ^p is the unique payoff vector z in the pre-imputation set satisfying $\theta^v(z) \geq_L \theta^v(x)$ for all $x \in I^*(v)$, that is,*

$$\varphi^p(v) = \{z \in I^*(v) | \theta^v(z) \geq_L \theta^v(x) \text{ for all } x \in I^*(v)\},$$

where θ^v is the satisfaction vector with respect to the pessimistic satisfaction.

Next, we will verify that the PANSC value coincides with the pessimistic optimal satisfaction value. The proofs of Lemma 5, Lemma 6 and Theorem 7 are similar to those of Lemma 1, Lemma 2 and Theorem 3, and are omitted here.

Lemma 5. *Given any $v \in \mathcal{G}_{\oplus}^{N}$ and a payoff vector $x \in \mathbb{R}^n$, let $l = \arg\min_{l \in N}\{e^p(\{l\}, x, v)\}$. Then we have $e^p(\{l\}, x, v) = \min_{S \in \Omega}\{e^p(S, x, v)\}$.*

Lemma 6. *Given any $v \in \mathcal{G}_{\oplus}^{N}$ and a payoff vector $x \in \mathbb{R}^n$, let $l = \arg\min_{l \in N}\{e^p(\{l\}, x, v)\}$. If there is one player $m \in N$ such that $e^p(\{m\}, x, v) > e^p(\{l\}, x, v)$, define a new payoff vector x^* given by*

$$x_k^* = \begin{cases} x_k, & \text{for } k \in N \backslash \{l, m\}; \\ x_l + \triangle, & \text{for } k = l; \\ x_m - \triangle, & \text{for } k = m, \end{cases}$$

where $\triangle = \frac{x_m \cdot b_l^v - x_l \cdot b_m^v}{b_l^v + b_m^v}$. Then the following five statements hold.

1. $e^p(S, x^*, v) = e^p(S, x, v)$ *for any $S \in \Omega$ and $S \not\ni l, m$.*
2. $e^p(S, x^*, v) = e^p(S, x, v)$ *for any $S \in \Omega$ and $S \ni l, m$.*
3. $e^p(S, x^*, v) > e^p(S, x, v)$ *for any $S \in \Omega$, $S \ni l$ and $S \not\ni m$.*
4. $e^p(S, x^*, v) > e^p(\{l\}, x, v)$ *for any $S \in \Omega$, $S \not\ni l$ and $S \ni m$.*
5. $\theta^v(x^*) >_L \theta^v(x)$, *where θ^v is the satisfaction vector with respect to the pessimistic satisfaction.*

Theorem 7. *For any $v \in \mathcal{G}_{\oplus}^{N}$, the following two statements hold.*

1. $\frac{\varphi_i^p(v)}{b^v(\{i\})} = \frac{\varphi_j^p(v)}{b^v(\{j\})}$ *for all $i, j \in N$.*
2. $\varphi_i^p(v) = PANSC_i(v)$ *for all $i \in N$.*

By Theorem 7, it holds that $\varphi_k^p(v) \leq b^v(\{k\})$ for all $k \in N$ if $b^v(N) \geq v(N)$. Then the following corollary is immediate for the reason that $C(v) \neq \emptyset$ implies $b^v(N) \geq v(N)$.

Corollary 8. *For any $v \in \mathcal{G}_{\oplus}^{N}$ with $C(v) \neq \emptyset$, the pessimistic optimal satisfaction value φ^p is bounded by the ideal payoff vector $(b^v(\{k\}))_{k \in N}$, that is, $\varphi_k^p(v) \leq b^v(\{k\})$ for all $k \in N$.*

4. Axiomatizations of the PD Value and the PANSC Value

In this section, we propose two types of axioms, the equal minimal satisfaction property and the associated consistency property, to characterize the PD value and the PANSC value.

4.1. Equal Minimal Satisfaction Property

In this subsection, we introduce the equal minimal optimistic satisfaction property and the equal minimal pessimistic satisfaction property, inspired by the kernel concept [17]. The PD value and the PANSC value are characterized by these two properties with efficiency, respectively. Moreover, we study the dual relationship between the equal minimal optimistic satisfaction property and the equal minimal pessimistic satisfaction property.

Given any pair of axioms of TU-games, if whenever a value satisfies one of these axioms, the dual of the value satisfies the other, then these two axioms are dual to each other. More accurately, for any pair of axioms A and A^d, A and A^d are dual to each other, if for any value that satisfies A, its dual value satisfies A^d, and on the contrary, for any value that satisfies A^d, its dual value satisfies A. An axiom is called self-dual if the dual of the axiom is itself. Obviously, efficiency is a self-dual axiom.

For any payoff vector $x \in \mathbb{R}^n$ and any $v \in \mathcal{G}_+^N$, $w \in \mathcal{G}_\oplus^N$, the optimistic minimal satisfaction $m_{ij}^o(v,x)$ and the pessimistic minimal satisfaction $m_{ij}^p(w,x)$ of player $i \in N$ over player $j \in N\backslash\{i\}$ with respect to x are given as follows

$$m_{ij}^o(v,x) = \min\{e^o(S,x,v)|S \in \Omega, i \in S, j \notin S\},$$
$$m_{ij}^p(w,x) = \min\{e^p(S,x,w)|S \in \Omega, i \in S, j \notin S\}.$$

Definition 5. *Given any $v \in \mathcal{G}_+^N$, $w \in \mathcal{G}_\oplus^N$, a payoff vector x satisfies*

1. *equal minimal optimistic satisfaction property if for every $i, j \in N$, $m_{ij}^o(v,x) = m_{ji}^o(v,x)$.*
2. *equal minimal pessimistic satisfaction property if for every $i, j \in N$, $m_{ij}^p(w,x) = m_{ji}^p(w,x)$.*

The equal minimal optimistic satisfaction property states that for any $i, j \in N$, the minimal optimistic satisfaction of all coalitions containing i and not j should equal that of all coalitions containing j and not i with respect to a payoff vector under the optimistic satisfaction criterion. On the contrary, the equal minimal pessimistic satisfaction property describe this situation under the pessimistic satisfaction criterion.

Proposition 9. *The equal minimal optimistic satisfaction property and the equal minimal pessimistic satisfaction property are dual to each other.*

Proof. Given a value φ on $\mathcal{G}_+^N$, let φ^d be the dual of φ. It is sufficient to prove that φ satisfies the equal minimal optimistic satisfaction property if and only if φ^d satisfies the equal minimal pessimistic satisfaction property.

Suppose that φ satisfies the equal minimal optimistic satisfaction property. Given any $v \in \mathcal{G}_\oplus^N$ and its dual game $v^d \in \mathcal{G}_+^N$, by the equal minimal optimistic satisfaction property, we have $m_{ij}^o(v^d, \varphi(v^d)) = m_{ji}^o(v^d, \varphi(v^d))$ for all $i, j \in N$, and then it holds that

$$\min\{\frac{\sum_{k\in S}\varphi_k(v^d)}{\sum_{k\in S} v^d(\{k\})}|S \in \Omega, i \in S, j \notin S\} = \min\{\frac{\sum_{k\in S}\varphi_k(v^d)}{\sum_{k\in S} v^d(\{k\})}|S \in \Omega, j \in S, i \notin S\}.$$

By the duality theory, it holds that

$$\min\{\frac{\sum_{k\in S}\varphi_k^d(v)}{b^v(S)}|S \in \Omega, i \in S, j \notin S\} = \min\{\frac{\sum_{k\in S}\varphi_k^d(v)}{b^v(S)}|S \in \Omega, j \in S, i \notin S\}.$$

Then, we have $m_{ij}^p(v, \varphi^d(v)) = m_{ji}^p(v, \varphi^d(v))$ for all $i, j \in N$. Therefore, φ^d satisfies the equal minimal pessimistic satisfaction property.

Similarly, we can prove that φ satisfies the equal minimal optimistic satisfaction property if φ^d satisfies the equal minimal pessimistic satisfaction property, which is similar to the above proof.

Therefore, we can conclude that the equal minimal optimistic satisfaction property and the equal minimal pessimistic satisfaction property are dual to each other. □

Theorem 10.

1. *The PD value satisfies the equal minimal optimistic satisfaction property on* $\mathcal{G}_{+}^{N}$.
2. *The PANSC value satisfies the equal minimal pessimistic satisfaction property on* $\mathcal{G}_{\oplus}^{N}$.

Proof.

1. For any $v \in \mathcal{G}_{+}^{N}$, let $x = PD(v)$. Then for any $S \in \Omega$, $e^{o}(S,x,v) = \frac{v(N)}{\sum_{k\in N} v(\{k\})}$. Therefore, for every $i,j \in N$, we have
$$m_{ij}^{o}(v,x) = \frac{v(N)}{\sum_{k\in N} v(\{k\})} = m_{ji}^{o}(v,x).$$
2. For any $v \in \mathcal{G}_{\oplus}^{N}$, let $x = PANSC(v)$. Then for any $S \in \Omega$, $e^{p}(S,x,v) = \frac{v(N)}{b^{v}(N)}$. Therefore, we have
$$m_{ij}^{p}(v,x) = \frac{v(N)}{b^{v}(N)} = m_{ji}^{p}(v,x),$$
for every $i,j \in N$. □

Theorem 11. *The PD value is the unique value on* $\mathcal{G}_{+}^{N}$ *satisfying efficiency and the equal minimal optimistic satisfaction property.*

Proof. Firstly, it is easy to show that the PD value satisfies efficiency. Then the equal minimal optimistic satisfaction property follows from Theorem 10. It is left to show the uniqueness.

Suppose that x is a payoff vector of a TU-game $v \in \mathcal{G}_{+}^{N}$ which satisfies efficiency and the equal minimal optimistic satisfaction property. Now suppose that $x \neq PD(v)$, and then there must exist $i,j \in N$ such that $x_i > PD_i(v)$ and $x_j < PD_j(v)$ by the efficiency. Let $l = \arg\min_{l\in N}\{e^{o}(\{l\},x,v)\}$, it holds that $e^{o}(\{l\},x,v) = \min_{S\in\Omega}\{e^{o}(S,x,v)\}$ by Lemma 1. Then we have

$$e^{o}(\{l\},x,v) \leq e^{o}(\{j\},x,v) < \frac{v(N)}{\sum_{k\in N} v(\{k\})} < e^{o}(\{i\},x,v).$$

Without loss of generality, let $S_0 \subseteq N\backslash\{l\}$ be a coalition containing i such that $m_{il}^{o}(v,x) = e^{o}(S_0,x,v)$. Thus, we have

$$m_{il}^{o}(v,x) = \frac{x(S_0\backslash\{i\}) + x_i}{\sum_{k\in S_0\backslash\{i\}} v(\{k\}) + v(\{i\})} > e^{o}(\{l\},x,v) = m_{li}^{o}(v,x),$$

where the first inequality holds because $e^{o}(S_0\backslash\{i\},x,v) \geq e^{o}(\{l\},x,v)$ and $e^{o}(\{i\},x,v) > e^{o}(\{l\},x,v)$, and the last equality holds because $e^{o}(\{l\},x,v) = \min_{S\in\Omega}\{e^{o}(S,x,v)\}$. But $m_{il}^{o}(v,x) > m_{li}^{o}(v,x)$ contradicts with the equal minimal optimistic satisfaction property. Therefore, the PD value is the unique value on $\mathcal{G}_{+}^{N}$ that satisfies efficiency and the equal minimal optimistic satisfaction property. □

In TU-games, the duality operator is a very useful tool to derive new axiomatizations of solutions. If there is an axiomatization of solution φ, then we can get one of axiomatization of its dual solution φ^{d} by determining the dual axioms of the axioms which are included in the axiomatization of φ. Oishi et al. [23] derived new axiomatizations of several classical solutions for TU-games by the duality theory. Since the equal minimal optimistic satisfaction property and the equal minimal pessimistic satisfaction property are dual to each other and efficiency is self-dual, we can obtain the following theorem.

Theorem 12. *The PANSC value is the unique value on $\mathcal{G}_{\oplus}^N$ satisfying efficiency and the equal minimal pessimistic satisfaction property.*

4.2. Associated Consistency Property

In the framework of the axiomatic system for TU-games, associated consistency is a significant characteristic of feasible and stable solutions. Associated consistency states that the solution should be invariant when the game changes into its associated game.

Throughout this subsection we deal with two types of associated games, the optimistic associated game and the pessimistic associated game. In these two associated games, every coalition reevaluates its own worth. Every coalition S just considers the players in $N\backslash S$ as individual elements and ignores the connection among players in $N\backslash S$. On the optimistic side, every coalition S always thinks that players in $N\backslash S$ should just receive their least potential payoff $(v\{k\})_{k\in N\backslash S}$. The amount $v(N)-v(S)-\sum_{k\in N\backslash S} v(\{k\})$ can be regarded as the optimistic surplus arising from mutual cooperation between S itself and all $j \in N\backslash S$. On the pessimistic side, every coalition S takes into consideration the ideal payoff vector and thinks that players in $N\backslash S$ can obtain their ideal payoff $(b^v(\{k\}))_{k\in N\backslash S}$. The amount $v(N)-v(S)-b^v(N\backslash S)$ is considered as the pessimistic surplus. Every coalition S believes that the appropriation of at least a part of the surpluses is within reach. Thus, every coalition S reevaluates its own worth $v_{\lambda,O}(S)$ in the optimistic associated game as the sum of its initial worth $v(S)$ and a percentage $\lambda \in (0,1)$ of a part $\frac{\sum_{k\in S} v(\{k\})}{\sum_{k\in N} v(\{k\})}$ of the optimistic surplus $v(N)-v(S)-\sum_{k\in N\backslash S} v(\{k\})$. Similarly, the pessimistic surplus is taken into account in the pessimistic associated game.

Definition 6. *Given any $v \in \mathcal{G}_{+}^N$ with $v(N) > 0$, and a real number $\lambda, 0 < \lambda < 1$, the optimistic associated game, denoted by $\langle N, v_{\lambda,O}\rangle$, is given by $v_{\lambda,O}(\emptyset) = 0$ and*

$$v_{\lambda,O}(S) = v(S) + \lambda \frac{\sum_{k\in S} v(\{k\})}{\sum_{k\in N} v(\{k\})}[v(N) - v(S) - \sum_{k\in N\backslash S} v(\{k\})], \text{ for all } S \in \Omega. \quad (4)$$

The purpose of making $v(N) > 0$ is in order to ensure $v_{\lambda,O} \in \mathcal{G}_{+}^N$. For convenience, let $\mathcal{G}_{++}^N = \{v \in \mathcal{G}_{+}^N | v(N) > 0\}$. It is easy to obtain that $v_{\lambda,O} \in \mathcal{G}_{++}^N$ if $v \in \mathcal{G}_{++}^N$. Moveover, let $\mathcal{G}_{\oplus\oplus}^N = \{v \in \mathcal{G}_{\oplus}^N | v(N) > 0\}$. Obviously, $\mathcal{G}_{++}^N$ and $\mathcal{G}_{\oplus\oplus}^N$ are dual to each other.

Definition 7. *Given any $v \in \mathcal{G}_{\oplus}^N$ and a real number $\lambda, 0 < \lambda < 1$, the pessimistic associated game, denoted by $\langle N, v_{\lambda,P}\rangle$, is given by $v_{\lambda,P}(\emptyset) = 0$ and*

$$v_{\lambda,P}(S) = v(S) + \lambda \frac{b^v(S)}{b^v(N)}[v(N) - v(S) - b^v(N\backslash S)], \text{ for all } S \in \Omega. \quad (5)$$

Obviously, $v_{\lambda,P} \in \mathcal{G}_{\oplus}^N$ if $v \in \mathcal{G}_{\oplus}^N$.

Definition 8.

1. *A value φ on $\mathcal{G}_{++}^N$ satisfies optimistic associated consistency if $\varphi(v) = \varphi(v_{\lambda,O})$ for any $v \in \mathcal{G}_{++}^N$.*
2. *A value φ on $\mathcal{G}_{\oplus}^N$ satisfies pessimistic associated consistency if $\varphi(v) = \varphi(v_{\lambda,P})$ for any $v \in \mathcal{G}_{\oplus}^N$.*

Next, let us consider the dual relation between these two associated consistency. Given any $v \in \mathcal{G}_{++}^N$ and its dual game $v^d \in \mathcal{G}_{\oplus\oplus}^N$, we only need to verify whether $(v^d)_{\lambda,O}$ is equal to $(v_{\lambda,P})^d$, to determine the dual relation between optimistic associated consistency and pessimistic associated consistency.

Remark 1. *Optimistic associated consistency and pessimistic associated consistency are not dual to each other.*

Theorem 13.

1. *The PD value satisfies optimistic associated consistency on* $\mathcal{G}^N_{++}$.
2. *The PANSC value satisfies pessimistic associated consistency on* $\mathcal{G}^N_{\oplus}$.

Proof.

1. By Definition 6, $v_{\lambda,O}(N) = v(N)$ and for all $i \in N$,

$$\begin{aligned} v_{\lambda,O}(\{i\}) =& v(\{i\}) + \frac{\lambda v(\{i\})}{\sum_{k\in N} v(\{k\})}[v(N) - \sum_{k\in N} v(\{k\})] \\ =&(1-\lambda)v(\{i\}) + \frac{\lambda v(\{i\})}{\sum_{k\in N} v(\{k\})} v(N) > 0. \end{aligned}$$

Then we have, for all $i \in N$

$$PD_i(v_{\lambda,O}) = \frac{v_{\lambda,O}(\{i\})}{\sum_{k\in N} v_{\lambda,O}(\{k\})} v_{\lambda,O}(N) = \frac{v(\{i\})}{\sum_{k\in N} v(\{k\})} v(N) = PD_i(v).$$

Therefore, the PD value satisfies optimistic associated consistency.

2. By Definition 7, $v_{\lambda,P}(N) = v(N)$ and $v_{\lambda,P}(N\backslash\{i\}) = v(N\backslash\{i\})$ for all $i \in N$. Then, for all $i \in N$, we have

$$PANSC_i(v_{\lambda,P}) = \frac{b^{v_{\lambda,P}}(\{i\})}{b^{v_{\lambda,P}}(N)} v_{\lambda,P}(N) = \frac{b^v(\{i\})}{b^v(N)} v(N) = PANSC_i(v).$$

Therefore, the PANSC value satisfies pessimistic associated consistency. □

Next we recall some classical properties of solutions for TU-games. A value φ satisfies

1. continuity, if for any convergent sequence of games $\{\langle N, v^k\rangle\}_{k=1}^{\infty}$ and its limit game $\langle N, \tilde{v}\rangle$ (i.e., for all $S \in \Omega$, $\lim_{k\to\infty} v^k(S) = \tilde{v}(S)$), the corresponding sequence of the values $\{\varphi(v^k)\}_{k=1}^{\infty}$ converges to the payoff vector $\varphi(\tilde{v})$.
2. inessential game property, if $\varphi_i(v) = v(\{i\})$ for any inessential game $v \in \mathcal{G}^N$ and $i \in N$. A game v is inessential if $v(S) = \sum_{k\in S} v(\{k\})$ for all $S \in \Omega$.
3. proportional constant additivity, if $\varphi_i(v + w) = \varphi(v) + \frac{b^v(\{i\})}{b^v(N)} w(N)$ for any $v \in \mathcal{G}^N_{\oplus}$, any constant game $w \in \mathcal{G}^N$ and $i \in N$. A game w is a constant game if $w(S) = \alpha$ for all $S \in \Omega$ and some $\alpha \in \mathbb{R}$.

The following lemma states the convergence of the sequence of repeated optimistic associated games, and its detailed proof is in Appendix A.

Lemma 14. *For any* $v \in \mathcal{G}^N_{++}$, *the sequence of repeated optimistic associated games* $\{\langle N, v^t_{\lambda,O}\rangle\}_{t=1}^{\infty}$ *converges, and its limit game* $\langle N, \hat{v}\rangle$ *is inessential, where* $v^1_{\lambda,O} = v_{\lambda,O}$ *and* $v^{t+1}_{\lambda,O} = (v^t_{\lambda,O})_{\lambda,O}$, $t = 1, 2, \cdots$.

Theorem 15. *The PD value is the unique value on* $\mathcal{G}^N_{++}$ *satisfying optimistic associated consistency, continuity and the inessential game property.*

Proof. It is easy to verify that the PD value satisfies continuity and the inessential game property. Optimistic associated consistency follows from Theorem 13. It is left to show the uniqueness.

Now suppose that a value φ on $\mathcal{G}^N_{++}$ satisfies these three axioms. For any $v \in \mathcal{G}^N_{++}$, by Lemma 14, the sequence of repeated optimistic associated games $\{\langle N, v^t_{\lambda,O}\rangle\}_{t=1}^{\infty}$ converges to an inessential game $\langle N, \hat{v}\rangle$. Then by optimistic associated consistency and continuity, it holds that

$$\varphi(v) = \varphi(v^1_{\lambda,O}) = \varphi(v^2_{\lambda,O}) = \cdots = \varphi(\hat{v}).$$

By the inessential game property, we have $\varphi_i(\hat{v}) = v(\{i\}) = \frac{v(\{i\})}{\sum_{k\in N} v(\{k\})} v(N)$ for all $i \in N$. Thus, $\varphi(v) = PD(v)$. $\square$

Next we prove the convergence of the sequence of repeated pessimistic associated games. The detailed proof of the lemma is in Appendix A.

Lemma 16. *For any $v \in \mathcal{G}^N_{\oplus}$, the sequence of repeated pessimistic associated games $\{\langle N, v^t_{\lambda,P}\rangle\}_{t=1}^{\infty}$ converges and its limit game $\langle N, \check{v}\rangle$ is the sum of an inessential game $\langle N, u\rangle$ and a constant game $\langle N, w\rangle$. where $v^1_{\lambda,P} = v_{\lambda,P}$ and $v^{t+1}_{\lambda,P} = (v^t_{\lambda,P})_{\lambda,P}, t = 1, 2, \cdots$.*

Theorem 17. *The PANSC value is the unique value on $\mathcal{G}^N_{\oplus}$ satisfying pessimistic associated consistency, continuity, the inessential game property and proportional constant additivity.*

Proof. It is easy to verify that the PANSC value satisfies continuity, the inessential game property and proportional constant additivity. Pessimistic associated consistency follows from Theorem 13. It is left to show the uniqueness.

Suppose that a value φ on $\mathcal{G}^N_{\oplus}$ satisfies pessimistic associated consistency, continuity, the inessential game property and proportional constant additivity. For any $v \in \mathcal{G}^N_{\oplus}$, by Lemma 16, the sequence of repeated pessimistic associated games $\{\langle N, v^t_{\lambda,P}\rangle\}_{t=1}^{\infty}$ converges to a game $\langle N, \check{v}\rangle$ which is expressed as the sum of a constant game $\langle N, w\rangle$ and an inessential game $\langle N, u\rangle$, where $w(S) = v(N) - b^v(N)$ and $u(S) = b^v(S)$ for all $S \in \Omega$. By continuity and pessimistic associated consistency, we have

$$\varphi(v) = \varphi(v^1_{\lambda,P}) = \varphi(v^2_{\lambda,P}) = \cdots = \varphi(\check{v}).$$

Let $\alpha = v(N) - b^v(N)$. By the inessential game property and proportional constant additivity, for any $i \in N$

$$\begin{aligned} \varphi_i(\check{v}) =& \varphi_i(u + w) = \varphi(u) + \frac{b^u(\{i\})}{b^u(N)} w(N) \\ =& u(\{i\}) + \frac{b^v(\{i\})}{b^v(N)}[v(N) - b^v(N)] = \frac{b^v(\{i\})}{b^v(N)} v(N). \end{aligned}$$

Therefore, $\varphi(v) = \frac{b^v(\{i\})}{b^v(N)} v(N) = PANSC(v)$. $\square$

4.3. Dual Axioms of Associated Consistency

In Remark 1, we mentioned that optimistic associated consistency and pessimistic associated consistency are not dual to each other. Next let us consider the dual axioms of optimistic associated consistency and pessimistic associated consistency.

Definition 9. *Given any $v \in \mathcal{G}^N_{\oplus\oplus}$, and a real number $\lambda, 0 < \lambda < 1$, the dual optimistic associated game $\langle N, v^*_{\lambda,O}\rangle$ is given by*

$$v^*_{\lambda,O}(S) = \begin{cases} v(S) + \lambda \frac{b^v(N\setminus S)}{b^v(N)}[b^v(S) - v(S)], & \text{if } S \subset N, \\ v(N), & \text{if } S = N. \end{cases} \tag{6}$$

Definition 10. *Given any $v \in \mathcal{G}^N_{+}$, and a real number $\lambda, 0 < \lambda < 1$, the dual pessimistic associated game $\langle N, v^*_{\lambda,P}\rangle$ is given by*

$$v^*_{\lambda,P}(S) = \begin{cases} v(S) + \lambda \frac{\sum_{k\in N\setminus S} v(\{k\})}{\sum_{k\in N} v(\{k\})}[\sum_{k\in S} v(\{k\}) - v(S)], & \text{if } S \subset N, \\ v(N), & \text{if } S = N. \end{cases} \tag{7}$$

Obviously, $v^*_{\lambda,O} \in \mathcal{G}^N_{\oplus\oplus}$ if $v \in \mathcal{G}^N_{\oplus\oplus}$, and $v^*_{\lambda,P} \in \mathcal{G}^N_{+}$ if $v \in \mathcal{G}^N_{+}$.

Definition 11.

1. *A value φ on $\mathcal{G}^N_{\oplus\oplus}$ satisfies dual optimistic associated consistency if $\varphi(v) = \varphi(v^*_{\lambda,O})$ for any $v \in \mathcal{G}^N_{\oplus\oplus}$.*
2. *A value φ on $\mathcal{G}^N_{+}$ satisfies dual pessimistic associated consistency if $\varphi(v) = \varphi(v^*_{\lambda,P})$ for any $v \in \mathcal{G}^N_{+}$.*

Lemma 18. *For any $v \in \mathcal{G}^N_{++}$ and $w \in \mathcal{G}^N_{\oplus}$, the following two statements hold.*

1. $(v_{\lambda,O})^d = (v^d)^*_{\lambda,O}$.
2. $(w_{\lambda,P})^d = (w^d)^*_{\lambda,P}$.

Proof.

1. By Equations (4) and (6), for any $v \in \mathcal{G}^N_{++}$ and $S \subset N$, we have

$$\begin{aligned}(v_{\lambda,O})^d(S) =& v_{\lambda,O}(N) - v_{\lambda,O}(N\backslash S)\\ =& v(N) - v(N\backslash S) - \lambda\frac{\sum_{k\in N\backslash S} v(\{k\})}{\sum_{k\in N} v(\{k\})}[v(N) - v(N\backslash S) - \sum_{k\in S} v(\{k\})]\\ =& v^d(S) + \lambda\frac{b^{v^d}(N\backslash S)}{b^{v^d}(N)}[b^{v^d}(S) - v^d(S)]\\ =& (v^d)^*_{\lambda,O}(S).\end{aligned}$$

   For $S = N$, we have $(v_{\lambda,O})^d(N) = v(N) = (v^d)^*_{\lambda,O}(N)$. Thus, $(v_{\lambda,O})^d = (v^d)^*_{\lambda,O}$.
2. By Equations (5) and (7), for any $w \in \mathcal{G}^N_{\oplus}$ and $S \subset N$, we have

$$\begin{aligned}(w_{\lambda,P})^d(S) =& w_{\lambda,P}(N) - w_{\lambda,P}(N\backslash S)\\ =& w(N) - w(N\backslash S) - \lambda\frac{b^w(N\backslash S)}{b^w(N)}[w(N) - w(N\backslash S) - b^w(S)]\\ =& w^d(S) + \lambda\frac{\sum_{k\in N\backslash S} w^d(\{k\})}{\sum_{k\in N} w^d(\{k\})}[\sum_{k\in S} w^d(\{k\}) - w^d(S)]\\ =& (w^d)^*_{\lambda,P}(S).\end{aligned}$$

   For $S = N$, we have $(w_{\lambda,P})^d(N) = w(N) = (w^d)^*_{\lambda,P}(N)$. Thus, $(w_{\lambda,P})^d = (w^d)^*_{\lambda,P}$. □

Proposition 19. *Optimistic associated consistency and dual optimistic associated consistency are dual to each other.*

Proof. Given a value φ on $\mathcal{G}^N_{++}$, let φ^d be the dual of φ. We just prove that φ satisfies optimistic associated consistency if and only if φ^d satisfies dual optimistic associated consistency.

If φ satisfies optimistic associated consistency, for any $v \in \mathcal{G}^N_{\oplus\oplus}$ and its dual game $v^d \in \mathcal{G}^N_{++}$, we have

$$\varphi^d(v) = \varphi(v^d) = \varphi((v^d)_{\lambda,O}) = \varphi((v^*_{\lambda,O})^d) = \varphi^d(v^*_{\lambda,O}),$$

where the third equation holds by Lemma 18. Thus, φ^d satisfies dual optimistic associated consistency.

If φ^d satisfies dual optimistic associated consistency, for any $v \in \mathcal{G}^N_{++}$ and its dual game $v^d \in \mathcal{G}^N_{\oplus\oplus}$, we have

$$\varphi(v) = \varphi^d(v^d) = \varphi^d((v^d)^*_{\lambda,O}) = \varphi^d((v_{\lambda,O})^d) = \varphi(v_{\lambda,O}).$$

Then, φ satisfies optimistic associated consistency. □

The proof of Proposition 20 is similar to that of Proposition 19 and is left to readers.

Proposition 20. *Pessimistic associated consistency and dual pessimistic associated consistency are dual to each other.*

Next, let us identify the dual axioms of other axioms which are included in the axiomatizations of the PD value and the PANSC value appearing in Theorems 15 and 17. It is easy to verify that continuity and the inessential game property are self-dual. A value φ on $\mathcal{G}_+^N$ satisfies dual proportional constant additivity, if $\varphi_i(v+w) = \varphi(v) + \frac{v(\{i\})}{\sum_{k\in N} v(\{k\})} w(N)$ for any $v \in \mathcal{G}_+^N$, any constant game $w \in \mathcal{G}^N$ and $i \in N$. Obviously, proportional constant additivity and dual proportional constant additivity are dual to each other. Thus, it is straightforward to obtain the following two theorems by the duality theory.

Theorem 21. *The PANSC value is the unique value on $\mathcal{G}_{\oplus\oplus}^N$ satisfying dual optimistic associated consistency, continuity and the inessential game property.*

Theorem 22. *The PD value is the unique value on $\mathcal{G}_+^N$ satisfying dual pessimistic associated consistency, continuity, the inessential game property and dual proportional constant additivity.*

5. Conclusions

In this paper, we introduce the family of optimal satisfaction values from the perspective of the satisfaction criteria. According to the optimistic satisfaction criterion and the pessimistic satisfaction criterion, the PD value and the PANSC value are determined by lexicographically maximizing the corresponding minimal satisfaction. Then, we characterize these two proportional values by introducing the equal minimal satisfaction property, associated consistency and the dual axioms of associated consistency. As two representative values of the proportional principle, the PD value and the PANSC value are relatively fair and reasonable allocations applied in many economic situations. For instance, in China's bankruptcy law, the bankruptcy property shall be distributed on a proportional principle when it is insufficient to repay all the repayment needs within a single order of priority. The proportional principle is deeply rooted in law and custom as a norm of distributed justice.

In the future, we will study other characterizations of the PD value and the PANSC value relying on some existing characterizations of classical solutions for TU-games. Coordinating the optimistic satisfaction and the pessimistic satisfaction, we may elicit the combination of the PD value and the PANSC value by an underlying neutral satisfaction criterion, and apply to some real situations.

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Appendix A

Proof of Lemma 14. For any $v \in \mathcal{G}_{++}^N$, we have $v_{\lambda,O}^t(N) = v(N)$, $t = 1, 2, \cdots$. Next, we show the convergence of the sequence of repeated optimistic associated games in two cases.

Case 1 $|S| = 1$. We first show that $v_{\lambda,O}^t(\{i\}) = (1-\lambda)^t v(\{i\}) + [1-(1-\lambda)^t] \frac{v(\{i\})}{\sum_{k\in N} v(\{k\})} v(N)$ for all $i \in N$ and $t \in \{1, 2, \cdots\}$ by induction on t. When $t = 1$, by Definition 6, we have

$v^1_{\lambda,O}(\{i\}) = (1-\lambda)v(\{i\}) + \frac{\lambda v(\{i\})}{\sum_{k\in N} v(\{k\})} v(N)$ for any $i \in N$. Suppose that $v^{t-1}_{\lambda,O}(\{i\}) = (1-\lambda)^{t-1}v(\{i\}) + [1-(1-\lambda)^{t-1}]\frac{v(\{i\})}{\sum_{k\in N} v(\{k\})} v(N)$. Then we have

$$\begin{aligned}
v^t_{\lambda,O}(\{i\}) =&(1-\lambda)v^{t-1}_{\lambda,O}(\{i\}) + \frac{\lambda v^{t-1}_{\lambda,O}(\{i\})}{\sum_{k\in N} v^{t-1}_{\lambda,O}(\{k\})} v(N) \\
=&(1-\lambda)\left\{(1-\lambda)^{t-1}v(\{i\}) + [1-(1-\lambda)^{t-1}]\frac{v(\{i\})}{\sum_{k\in N} v(\{k\})} v(N)\right\} \\
&+ \frac{\lambda\left\{(1-\lambda)^{t-1}v(\{i\}) + [1-(1-\lambda)^{t-1}]\frac{v(\{i\})}{\sum_{k\in N} v(\{k\})} v(N)\right\}}{\sum_{j\in N}\left\{(1-\lambda)^{t-1}v(\{j\}) + [1-(1-\lambda)^{t-1}]\frac{v(\{j\})}{\sum_{k\in N} v(\{k\})} v(N)\right\}} v(N) \\
=&(1-\lambda)^t v(\{i\}) + (1-\lambda)[1-(1-\lambda)^{t-1}]\frac{v(\{i\})}{\sum_{k\in N} v(\{k\})} v(N) \\
&+ \frac{\lambda v(\{i\})}{\sum_{k\in N} v(\{k\})} v(N) \\
=&(1-\lambda)^t v(\{i\}) + [1-(1-\lambda)^t]\frac{v(\{i\})}{\sum_{k\in N} v(\{k\})} v(N).
\end{aligned}$$

Thus, it holds that

$$v^t_{\lambda,O}(\{i\}) = (1-\lambda)^t v(\{i\}) + [1-(1-\lambda)^t]\frac{v(\{i\})}{\sum_{k\in N} v(\{k\})} v(N), \quad t = 1, 2, \cdots \tag{A1}$$

Therefore, for any $0 < \lambda < 1$, we have

$$\hat{v}(\{i\}) = \lim_{t\to\infty} v^t_{\lambda,O}(\{i\}) = \frac{v(\{i\})}{\sum_{k\in N} v(\{k\})} v(N).$$

Case 2 $|S| \geq 2$. For convenience, let $\rho_S = \frac{\sum_{k\in S} v(\{k\})}{\sum_{k\in N} v(\{k\})}$ and $\sigma = v(N) - \sum_{k\in N} v(\{k\})$. Next we will show that

$$\begin{aligned}
v^t_{\lambda,O}(S) =&(1-\lambda\rho_S)^t v(S) + [1-(1-\lambda\rho_S)^t]\rho_S v(N) \\
&+ \lambda\rho_S\sigma(1-\rho_S)[\sum_{m=1}^{t}(1-\lambda)^{m-1}(1-\lambda\rho_S)^{t-m}],
\end{aligned} \tag{A2}$$

for all $S \in \Omega$ and $t \in \{1, 2, \cdots\}$ by induction on t. When $t = 1$, by Definition 6, it holds that $v^1_{\lambda,O}(S) = (1-\lambda\rho_S)v(S) + \lambda\rho_S\rho_S v(N) + \lambda\rho_S\sigma(1-\rho_S)$, and Equation (A2) holds. Without loss of generality, suppose that Equation (A2) holds at $t-1$. Then, by Definition 6 and Equation (A1), we have

$$\begin{aligned}
v^t_{\lambda,O}(S) =& v^{t-1}_{\lambda,O}(S) + \lambda \frac{\sum_{k\in S} v^{t-1}_{\lambda,O}(\{k\})}{\sum_{k\in N} v^{t-1}_{\lambda,O}(\{k\})}[v(N) - v^{t-1}_{\lambda,O}(S) - \sum_{k\in N\setminus S} v^{t-1}_{\lambda,O}(\{k\})] \\
=& (1-\lambda\rho_S)v^{t-1}_{\lambda,O}(S) + \lambda\rho_S[v(N) - (1-\rho_S)\sum_{k\in N} v^{t-1}_{\lambda,O}(\{k\})] \\
=& (1-\lambda\rho_S)v^{t-1}_{\lambda,O}(S) + \lambda\rho_S\rho_S v(N) + \lambda\rho_S\sigma(1-\rho_S)(1-\lambda)^{t-1} \\
=& (1-\lambda\rho_S)^t v(S) + (1-\lambda\rho_S)[1-(1-\lambda\rho_S)^{t-1}]\rho_S v(N) \\
& + \lambda\rho_S\sigma(1-\rho_S)(1-\lambda\rho_S)[\sum_{m=1}^{t-1}(1-\lambda)^{m-1}(1-\lambda\rho_S)^{t-1-m}] \\
& + \lambda\rho_S\rho_S v(N) + \lambda\rho_S\sigma(1-\rho_S)(1-\lambda)^{t-1} \\
=& (1-\lambda\rho_S)^t v(S) + [1-(1-\lambda\rho_S)^t]\rho_S v(N) \\
& + \lambda\rho_S\sigma(1-\rho_S)[\sum_{m=1}^{t}(1-\lambda)^{m-1}(1-\lambda\rho_S)^{t-m}].
\end{aligned}$$

Thus, Equation (A2) holds for all $S \in \Omega$ and $t \in \{1, 2, \cdots\}$.

Let $a_t = \sum_{m=1}^{t}(1-\lambda)^{m-1}(1-\lambda\rho_S)^{t-m}$. Since $0 < \lambda < 1$ and $0 < \rho_S < 1$, we have $t(1-\lambda)^{t-1} \le a_t \le t(1-\lambda\rho_S)^{t-1}$. Since $\lim_{t\to\infty} t(1-\lambda)^{t-1} = 0$ and $\lim_{t\to\infty} t(1-\lambda\rho_S)^{t-1} = 0$, then $\lim_{t\to\infty} a_t = 0$. Thus, we have

$$\hat{v}(S) = \lim_{t\to\infty} v^t_{\lambda,O}(S) = \rho_S v(N) = \frac{\sum_{k\in S} v(\{k\})}{\sum_{k\in N} v(\{k\})} v(N).$$

Therefore, the sequence of repeated optimistic associated games $\{\langle N, v^t_{\lambda,O}\rangle\}_{t=1}^{\infty}$ converges and its limit game $\langle N, \hat{v}\rangle$ is given by $\hat{v}(S) = \frac{\sum_{k\in S} v(\{k\})}{\sum_{k\in N} v(\{k\})} v(N)$ for all $S \in \Omega$. □

Proof of Lemma 16. For any $v \in \mathcal{G}^N_{\oplus}$, it holds that $v^t_{\lambda,P}(N) = v(N)$ and $v^t_{\lambda,P}(N\setminus\{i\}) = v(N\setminus\{i\})$ for all $i \in N$ and $t = 1, 2, \cdots$. Then we can obtain that $b^{v^t_{\lambda,P}}(\{i\}) = b^v(\{i\})$ for all $i \in N$ and $t = 1, 2, \cdots$. For convenience, let $\tau_S = \frac{b^v(S)}{b^v(N)}$. Next we will prove that

$$v^t_{\lambda,P}(S) = (1-\lambda\tau_S)^t v(S) + [1-(1-\lambda\tau_S)^t][v(N) - b^v(N\setminus S)] \tag{A3}$$

for all $S \in \Omega$ and $t = 1, 2, \cdots$, by induction on t. When $t = 1$, by Definition 7, we have $v^1_{\lambda,P}(S) = (1-\lambda\tau_S)v(S) + \lambda\tau_S[v(N) - b^v(N\setminus S)]$, and Equation (A3) holds. Suppose that Equation (A3) holds at $t-1$. Then, by Definition 7, we have

$$\begin{aligned}
v^t_{\lambda,P}(S) =& v^{t-1}_{\lambda,P}(S) + \lambda \frac{b^{v^{t-1}_{\lambda,P}}(S)}{b^{v^{t-1}_{\lambda,P}}(N)}[v(N) - v^{t-1}_{\lambda,P}(S) - b^{v^{t-1}_{\lambda,P}}(N\setminus S)] \\
=& (1-\lambda\tau_S)v^{t-1}_{\lambda,P}(S) + \lambda\tau_S[v(N) - b^v(N\setminus S)] \\
=& (1-\lambda\tau_S)\left\{(1-\lambda\tau_S)^{t-1}v(S) + [1-(1-\lambda\tau_S)^{t-1}][v(N) - b^v(N\setminus S)]\right\} \\
& + \lambda\tau_S[v(N) - b^v(N\setminus S)] \\
=& (1-\lambda\tau_S)^t v(S) + [1-(1-\lambda\tau_S)^t][v(N) - b^v(N\setminus S)]
\end{aligned}$$

Thus, Equation (A3) holds for all $S \in \Omega$ and $t \in \{1, 2, \cdots\}$.

Due to $0 < \lambda < 1$ and $0 < \tau_S < 1$, for any $S \in \Omega$, we have

$$\breve{v}(S) = \lim_{t\to\infty} v^t_{\lambda,P}(S) = v(N) - b^v(N \backslash S).$$

Let $u(S) = b^v(S)$ and $w(S) = v(N) - b^v(N)$ for all $S \in \Omega$. Obviously, $\langle N, u\rangle$ is an inessential game and $\langle N, w\rangle$ is a constant game. The limit game $\langle N, \breve{v}\rangle$ is given by $\breve{v}(S) = u(S) + w(S)$. $\square$

References

1. Shapley, L.S. A value for n-person games. In *Contributions to the Theory of Games II*; Kuhn, H.W., Tucker, A.W., Eds.; Princeton University Press: Princeton, NJ, USA, 1953; pp. 307–317.
2. Schmeidler, D. The nucleolus of a characteristic function game. *SIAM J. Appl. Math.* **1969**, *17*, 1163–1170. [CrossRef]
3. Davis, M.; Maschler, M. The kernel of a cooperative game. *Nav. Res. Logist. Q.* **1965**, *12*, 223–259. [CrossRef]
4. Tijs, S.H. An axiomatization of the τ-value. *Math. Soc. Sci.* **1987**, *13*, 177–181. [CrossRef]
5. Hou, D.; Sun, P.; Xu, G.; Driessen, T. Compromise for the complaint: an optimization approach to the ENSC value and the CIS value. *J. Oper. Res. Soc.* **2018**, *69*, 571–579. [CrossRef]
6. Menon, G.; Kyung, E.J.; Agrawal, N. Biases in social comparisons: Optimism or pessimism? *Organ. Behav. Hum. Decis. Process.* **2009**, *108*, 39–52. [CrossRef]
7. Young, H.P. *Equity: In Theory and Practice*; Princeton University Press: Princeton, NJ, USA, 1994.
8. Moulin, H. Chapter 6 Axiomatic cost and surplus sharing. *Handb. Soc. Choice Welf.* **2002**, *1*, 289–357.
9. Banker, R. Equity considerations in traditional full cost allocation practices: An axiomatic perspective. In *Moriarty*; Joint Cost Allocations; Editor, S., Ed.; University of Oklahoma: Norman, OK, USA, 1981; pp. 110–130.
10. Van den Brink, R.; Levínskỳ, R.; Zelenỳ, R. On proper Shapley values for monotone TU-games. *Int. J. Game Theory* **2015**, *44*, 449–471. [CrossRef]
11. Van den Brink, R.; Levínskỳ, R.; Zelenỳ, R. The Shapley value, proper Shapley value, and sharing rules for cooperative ventures. *Oper. Res. Lett.* **2020**, *48*, 55–60. [CrossRef]
12. Ortmann, K.H. The proportional value for positive cooperative games. *Math. Methods Oper. Res.* **2000**, *51*, 235–248. [CrossRef]
13. Kamijo, Y.; Kongo, T. Properties based on relative contributions for cooperative games with transferable utilities. *Theory Decis.* **2015**, *57*, 77–87. [CrossRef]
14. Béal, S.; Ferrières, S.; Rémila, E.; Solal, P. The proportional Shapley value and applications. *Games Econ. Behav.* **2018**, *108*, 93–112. [CrossRef]
15. Besner, M. Axiomatizations of the proportional Shapley value. *Theory Decis.* **2019**, *86*, 161–183. [CrossRef]
16. Zou, Z.; Van den Brink, R.; Chun, Y.; Funaki, Y. Axiomatizations of the Proportional Division Value. Tinbergen Institute Discussion Paper 2019–072/II. Available online: http://dx.doi.org/10.2139/ssrn.3479365 (accessed on 31 October 2019).
17. Maschler, M.; Peleg, B.; Shapley, L.S. The kernel and bargaining set for convex games. *Int. J. Game Theory* **1971**, *1*, 73–93. [CrossRef]
18. Hamiache, G. Associated consistency and Shapley value. *Int. J. Game Theory* **2001**, *30*, 279–289. [CrossRef]
19. Driessen, T.S.H. Associated consistency and values for TU games. *Int. J. Game Theory* **2010**, *39*, 467–482. [CrossRef]
20. Xu, G.; Van den Brink, R.; Van der Laan, G.; Sun, H. Associated consistency characterization of two linear values for TU games by matrix approach. *Linear Algebra Its Appl.* **2015**, *471*, 224–240. [CrossRef]
21. Kleinberg, N.L. A note on associated consistency and linear, symmetric values. *Int. J. Game Theory* **2018**, *47*, 913–925. [CrossRef]
22. Kong, Q.; Sun, H.; Xu, G.; Hou, D. Associated games to optimize the core of a transferable utility game. *J. Optim. Theory Appl.* **2019**, *182*, 816–836. [CrossRef]
23. Oishi, T.; Nakayama, M.; Hokari, T.; Funaki, Y. Duality and anti-duality in TU games applied to solutions, axioms, and axiomatizations. *J. Math. Econ.* **2016**, *63*, 44–53. [CrossRef]