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Differentiation of the Mittag-Leffler Functions with Respect to Parameters in the Laplace Transform Approach

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Abstract: In this work, properties of one- or two-parameter Mittag-Leffler functions are derived using the Laplace transform approach. It is demonstrated that manipulations with the pair direct–inverse transform makes it far more easy than previous methods to derive known and new properties of the Mittag-Leffler functions. Moreover, it is shown that sums of infinite series of the Mittag-Leffler functions can be expressed as convolution integrals, while the derivatives of the Mittag-Leffler functions with respect to their parameters are expressible as double convolution integrals. The derivatives can also be obtained from integral representations of the Mittag-Leffler functions. On the other hand, direct differentiation of the Mittag-Leffler functions with respect to parameters produces an infinite power series, whose coefficients are quotients of the digamma and gamma functions. Closed forms of these series can be derived when the parameters are set to be integers.

Keywords: derivatives with respect to parameters; Mittag-Leffler functions; Laplace transform approach; infinite power series; integral representations; convolution integrals; quotients of digamma and gamma functions

1. Introduction

At the beginning of the previous century, the exponential function was generalized by the Swedish mathematician G.M. Mittag-Leffler, who introduced a new power series that is named after him today [1]. Quite unexpectedly, enormous interest has developed regarding the Mittag-Leffler functions over the last four decades because of their ability to describe diverse physical phenomena far more easily than other approaches in a host of scientific and engineering disciplines. Consequently, the Mittag-Leffler functions have become one of the most important special functions in mathematics. Examples where they appear include kinetics of chemical reactions, time and space fractional diffusion, nonlinear waves, viscoelastic systems, neural networks, electric field relaxations, and statistical distributions [2–8]. In mathematics, the Mittag-Leffler functions play an important role in fractional calculus, solution of systems with fractional differential, and integral equations [9,10]. As a result of all this activity, there is now extensive literature on their properties and history [11–13]. A number of reviews have been produced [14–16], and of these, the monograph by Gorenflo, Kilbas, Mainardi, and Rogosin [17] occupies a special place.

The one-parameter, classical Mittag-Leffler function $E_\alpha(z)$ is defined in the whole complex plane by the following power series:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad (1)$$

where $\mathrm{Re}\alpha > 0$.

Later, Wiman [18] introduced the two-parameter Mittag-Leffler function $E_{\alpha,\beta}(z)$, which is given by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \tag{2}$$

where $\mathrm{Re}\alpha > 0$ and $\mathrm{Re}\beta > 0$. Only these two functions, not generalizations thereafter, will be studied here.

There are two main aims in this work. The first is to show that many well-known and new functional relations can be easily derived via the Laplace transform theory and the second is to consider differentiation with respect to the parameters α and β. Throughout this paper, all mathematical operations or manipulations with functions, series, integrals, integral representations, and transforms will be formal. There will be no proofs of validity of given expressions, though they are, without doubt, correct. The following sections present many results that have been derived independently by other methods, while the new results are verified by two different numerical procedures. Thus, in the framework of applied operational calculus, the reported results are only valid for real positive values of arguments and parameters.

My previous involvement with the Mittag-Leffler functions has been limited only to establishing their connections to the Volterra functions. In my monograph devoted to the Volterra functions [19], I presented in Appendix A some representations of the Mittag-Leffler functions in terms of other special functions. They can also be derived directly using the Laplace transform technique when applied to $E_\alpha(\pm t^\alpha)$ functions. Evidently, this restricts the transform–inverse pair only to the positive real axis. New results, together with some from [19], are presented below.

According to the definitions of the Mittag-Leffler functions, there is a clear distinction between the argument, z, and the parameters, α and β, as the latter appear in the coefficients. Nevertheless, $E_\alpha(z) = f(\alpha, z)$ and $E_{\alpha,\beta}(z) = f(\alpha, \beta, z)$ can be regarded as the bivariate and trivariate functions, respectively.

As this is the first investigation dealing with mathematical operations with respect to variables α and β, its scope is only limited to derivatives of the Mittag-Leffler functions. The special forms of the Laplace transforms of $E_\alpha(\pm t^\alpha)$ and $E_{\alpha,\beta}(\pm t^\alpha)$ functions will be studied extensively to establish known properties of the Mittag-Leffler functions and to derive new functional relations. As will be demonstrated, the differentiation operations will lead to power series with coefficients being quotients of psi and gamma functions. In some cases, these series can be evaluated in a closed form, i.e., in terms of elementary and special functions. Computation methods used in this investigation to obtain the Mittag-Leffler functions and their derivatives with respect to α differ from those reported in the literature. This results from the fact that the Mittag-Leffler functions are available as the build-in functions in the MATHEMATICA program.

2. Properties of the Mittag-Leffler Functions in the Laplace Transform Approach

The Laplace transform of the Mittag-Leffler function $E_\alpha(t^\rho)$ is given by

$$L\{E_\alpha(t^\rho)\} = \frac{1}{s}\sum_{k=0}^{\infty} \frac{\Gamma(\rho k + 1)}{\Gamma(\alpha k + 1)}\left(\frac{1}{s^\rho}\right)^k, \tag{3}$$

which is not valid to all values of ρ and α as discussed in [17].

For $\rho = \alpha$, (3) becomes

$$L\{E_\alpha(t^\alpha)\} = \frac{s^{\alpha-1}}{s^\alpha - 1}, \tag{4}$$

where $\mathrm{Re}\alpha > 0$ and $\mathrm{Re}s > 1$ and for negative t^α is

$$L\{E_\alpha(-t^\alpha)\} = \frac{s^{\alpha-1}}{s^\alpha + 1}. \tag{5}$$

In a similar manner, the Laplace transforms of two-parameter Mittag-Leffler functions, $t^{\beta-1}E_{\alpha,\beta}(\pm\lambda t^{\alpha})$, in [17] are found to be

$$L\Big\{t^{\beta-1}E_{\alpha,\beta}(\pm\lambda t^{\alpha})\Big\} = \frac{s^{\alpha-\beta}}{s^{\alpha}-\lambda}\,, \tag{6}$$

where Re$\alpha > 0$, Re$\beta > 0$ and Re$s > |\lambda|^{1/\alpha}$.

Not only are the inverse transforms simple to derive from them results, but one is able to identify functions for particular values of α and β. Carrying this out will require algebraic manipulations, the similarity properties of the Laplace transformation, the Heaviside expansion theorem, the convolution (product) theorem, some substitution formulas, and other techniques and rules of the operational calculus.

In the first application of the Laplace transform theory, we consider positive integer values of α from 1 to 4. Then, the Mittag-Leffler functions reduce to elementary or special functions due to the simple inverse transforms.

For $\alpha = 1$, one finds that

$$E_1(t) = L^{-1}L\{E_1(t)\} = L^{-1}\Big\{\frac{1}{s-1}\Big\} = e^{t}\,. \tag{7}$$

For $\alpha = 2$, one obtains

$$E_2(t^2) = L^{-1}L\Big\{E_2(t^2)\Big\} = L^{-1}\Big\{\tfrac{s}{s^2-1}\Big\} = L^{-1}\Big\{\tfrac{s}{(s-1)(s+1)}\Big\} = \cosh t\,, \tag{8}$$

where the dominator has been decomposed into partial fractions. However, the more expedient method is to evaluate the contributions from the residues at $s = \pm 1$.

Carrying out this procedure for $-t^2$ yields

$$E_2(-t^2) = L^{-1}L\Big\{E_2(-t^2)\Big\} = L^{-1}\Big\{\tfrac{s}{s^2+1}\Big\} = L^{-1}\Big\{\tfrac{s}{(s-i)(s+i)}\Big\} = \tfrac{se^{it}}{s+i}\Big|_{s=+i} + \tfrac{se^{-it}}{s-i}\Big|_{s=-i} = \tfrac{e^{it}}{2} + \tfrac{e^{-it}}{2} = \cos t\,. \tag{9}$$

For $\alpha = 3$, one finds that

$$\begin{aligned} E_3(t^3) &= L^{-1}L\Big\{E_3(t^3)\Big\} = L^{-1}\Big\{\tfrac{s^2}{s^3-1}\Big\} = L^{-1}\Big\{\tfrac{s^2}{(s-1)(s^2+s+1)}\Big\} = \\ &L^{-1}\Bigg\{\frac{s^2}{(s-1)(s+\frac{1+i\sqrt{3}}{2})(s+\frac{1-i\sqrt{3}}{2})}\Bigg\} = \\ &\frac{s^2\,e^{t}}{(s+\frac{1+i\sqrt{3}}{2})(s+\frac{1-i\sqrt{3}}{2})}\Bigg|_{s\,=\,1} + \frac{s^2\,e^{-t(1+i\sqrt{3})/2}}{(s-1)(s+\frac{1-i\sqrt{3}}{2})}\Bigg|_{s\,=-\frac{1+i\sqrt{3}}{2}} \\ &+ \frac{s^2\,e^{-t(1-i\sqrt{3})/2}}{(s-1)(s+\frac{1+i\sqrt{3}}{2})}\Bigg|_{s=-\frac{1-i\sqrt{3}}{2}} = \tfrac{1}{3}[e^{t} + 2e^{-t/2}\cos(\tfrac{\sqrt{3}}{2}t)]\,. \end{aligned} \tag{10}$$

Similarly, for negative t^{α}, one arrives at

$$E_3(-t^3) = L^{-1}L\Big\{E_3(-t^3)\Big\} = L^{-1}\Big\{\tfrac{s^2}{s^3+1}\Big\} = \tfrac{1}{3}[e^{-t} + 2e^{t/2}\cos(\tfrac{\sqrt{3}}{2}t)]\,. \tag{11}$$

The calculations become more tedious as α increases. However, for $\alpha = n$, an integer, we obtain in general case

$$E_n(\pm t^{n}) = L^{-1}L\{E_n(\pm t^{n})\} = L^{-1}\Bigg\{\frac{s^{n}\,{}^{-1}}{s^{n}-1}\Bigg\}\,. \tag{12}$$

It is obvious that for integer values of α, the Mittag-Leffler functions can be expressed in terms of elementary functions, such as combination of exponential, hyperbolic, and trigonometric functions.

When α is not an integer, special functions are involved. Then, one must use a combination of tables of inverse Laplace transforms, substitution formulas, the convolution theorem, and other rules. For example, from the table of inverse transforms [20], we have

$$\begin{aligned}&L^{-1}\left\{\frac{1}{\sqrt{s}}\right\} = \frac{1}{\sqrt{\pi t}}\,,\\ &L^{-1}\left\{\frac{1}{\sqrt{s}-1}\right\} = \frac{1}{\sqrt{\pi t}} \pm e^{t}erfc(-\sqrt{t})\,,\\ &erfc(-t^{1/2}) = -erfc(t^{1/2}) = erf(t^{1/2}) - 1\end{aligned} \tag{13}$$

Hence, we find that

$$\begin{aligned}&E_{1/2}(\pm\sqrt{t}) = L^{-1}L\left\{E_{1/2}(\pm\sqrt{t})\right\} = L^{-1}\left\{\frac{1}{\sqrt{s}(\sqrt{s}-1)}\right\} =\\ &L^{-1}\left\{-\frac{1}{\sqrt{s}} \pm \frac{1}{(\sqrt{s}-1)}\right\} = e^{t}[1 - erf(\sqrt{t})]\,.\end{aligned} \tag{14}$$

The cases with $\alpha = \pm 1/4$ are more complex. Therefore, only the final result for $\alpha = 1/4$ from [19] is presented here. This is

$$\begin{aligned}&E_{1/4}(\pm t^{1/4}) = L^{-1}L\left\{E_{1/4}(\pm t^{1/4})\right\} = L^{-1}\left\{\frac{1}{s^{3/4}(s^{1/4}-1)}\right\} =\\ &L^{-1}\left\{\frac{1}{\sqrt{s}(\sqrt{s}-1)} \pm \frac{1}{s^{1/4}(s-1)} \pm \frac{1}{s^{3/4}(s-1)}\right\} =\\ &e^{t}\left\{1 + erf(\sqrt{t}) \pm \frac{\gamma(\frac{1}{4},t)}{\Gamma(\frac{1}{4})} \pm \frac{\gamma(\frac{3}{4},t)}{\Gamma(\frac{3}{4})}\right\},\\ &\gamma(a,t) = \Gamma(a) - \Gamma(a,t) = \int_0^t x^{a-1}e^{-x}dx\,,\end{aligned} \tag{15}$$

where the last equation in (15) is the integral representation for the incomplete gamma function.

We can also determine relations between the Mittag-Leffler functions using the Laplace transformation. Putting $\beta = \alpha + 1$ in (6) yields

$$L\left\{t^{\alpha}E_{\alpha,\alpha+1}(t^{\alpha})\right\} = \frac{1}{s(s^{\alpha}-1)}\,. \tag{16}$$

However, noting that

$$L\{E_{\alpha}(t^{\alpha}) - 1\} = \frac{s^{\alpha-1}}{s^{\alpha}-1} - \frac{1}{s} = \frac{1}{s(s^{\alpha}-1)}\,, \tag{17}$$

we can derive the well-known relation for the Mittag-Leffler functions

$$E_{\alpha}(t^{\alpha}) - 1 = t^{\alpha}E_{\alpha,\alpha+1}(t^{\alpha})\,. \tag{18}$$

A similar result for the two-parameter Mittag-Leffler function can be derived from

$$L\left\{t^{\alpha+\beta-1}E_{\alpha,\alpha+\beta}(t^{\alpha})\right\} = \frac{1}{s^{\beta}(s^{\alpha}-1)}\,, \tag{19}$$

and

$$L\left\{t^{\beta-1}E_{\alpha,\beta}(t^{\alpha}) - \frac{t^{\beta-1}}{\Gamma(\beta)}\right\} = \frac{s^{\alpha-\beta}}{(s^{\alpha}-1)} - \frac{1}{s^{\beta}} = \frac{1}{s^{\beta}(s^{\alpha}-1)}\,. \tag{20}$$

Hence, we arrive at

$$E_{\alpha,\beta}(\,t^{\alpha}) = \frac{1}{\Gamma(\beta)} + t^{\alpha}\, E_{\alpha,\alpha+\,\beta}(\,t^{\alpha})\,. \tag{21}$$

For α and β integers, (21) can be written as

$$\begin{aligned} E_{1,\beta}(t) &= \tfrac{1}{\Gamma(\beta)} + t\, E_{1,\alpha+1}(t)\,, \\ E_{n,\beta}(t^n) &= \tfrac{1}{\Gamma(\beta)} + t^n\, E_{n,n\,+\,\beta}(t^n)\,, \\ E_{n,n}(t^n) &= \tfrac{1}{(n-1)!} + t^n\, E_{n,2n}(t^n)\,, \\ E_{n,m}(t^n) &= \tfrac{1}{(m-1)!} + t^n\, E_{n,m\,+\,n}(t^n) \end{aligned} \tag{22}$$

Of the many substitution formulas in the Laplace transform theory, only three will be employed here. From [21] we have

$$\begin{aligned} &L\{f(t)\} = F(s)\,, \\ &L^{-1}\Big\{\tfrac{1}{\sqrt{s}}F(\sqrt{s})\Big\} = \tfrac{1}{\sqrt{\pi t}}\int_0^{\infty} e^{-u^2/4t} f(u)\, du \end{aligned} \tag{23}$$

By wring the Laplace transform of $E_{\alpha}(t^{\alpha})$ as

$$L\{E_{\alpha}(t^{\alpha})\} = \frac{s^{\alpha-1}}{s^{\alpha}-1} = \frac{1}{\sqrt{s}}\,\frac{(\sqrt{s})^{2\alpha-1}}{[(\sqrt{s})^{2\alpha}-1]}\,, \tag{24}$$

we find that the Mittag-Leffler function can be represented by

$$E_{\alpha}(t^{\alpha}) = \frac{1}{\sqrt{\pi t}}\int_0^{\infty} e^{-\,u^2/4t} E_{2\alpha}(u^{2\alpha})\, du\,. \tag{25}$$

The operational rule for the Macdonald function $K_{1/3}(z)$ is

$$L^{-1}\Big\{\frac{1}{s^{2/3}}F(s^{1/3})\Big\} = \frac{1}{\pi}\int_0^{\infty} \sqrt{\frac{u}{t}}\; K_{1/3}\Big(\frac{2u^{3/2}}{\sqrt{27t}}\Big)\, f(u)\, du\,. \tag{26}$$

Writing the Laplace transform of $E_{\alpha}(t^{\alpha})$ as

$$L\{E_{\alpha}(t^{\alpha})\} = \frac{s^{\alpha-1}}{s^{\alpha}-1} = \frac{(s^{1/3})^{3\alpha-1}}{s^{2/3}[(s^{1/3})^{3\alpha}-1]}\,, \tag{27}$$

gives

$$E_{\alpha}(t^{\alpha}) = \frac{1}{\pi}\int_0^{\infty} \sqrt{\frac{u}{t}}\; K_{1/3}\Big(\frac{2u^{3/2}}{\sqrt{27\,t}}\Big)\, E_{3\,\alpha}(u^{3\,\alpha})\, du\,. \tag{28}$$

For specific values of α, the Mittag-Leffler functions in the integrands of (25) and (28) can be expressed as elementary or special functions. Then, the Mittag-Leffler functions on the left-hand side will be represented by definite integrals over infinity.

The third substitution formula is

$$L^{-1}\Big\{\frac{1}{s^2}F(\frac{1}{s})\Big\} = \int_0^{\infty} \sqrt{\frac{t}{u}}\; J_1\big(2\sqrt{tu}\big)\, f(u)\, du\,, \tag{29}$$

where $J_1(z)$ is the Bessel function of the first kind and of the first order

From

$$L\{1-E_\alpha(t^\alpha)\}=\frac{1}{s}-\frac{s^{\alpha-1}}{s^\alpha-1}=\frac{1}{s^2}\frac{\left(\frac{1}{s}\right)^{\alpha-1}}{\left[\left(\frac{1}{s}\right)^{\alpha}-1\right]}\,, \tag{30}$$

it follows that

$$\frac{E_\alpha(t^\alpha)-1}{\sqrt{t}}=\int_0^\infty J_1(2\sqrt{tu})\,E_\alpha(u^\alpha)\,\frac{du}{\sqrt{u}}\,. \tag{31}$$

Many properties and functional relations for the Mittag-Leffler functions can be obtained from the convolution theorem. These are found by expressing the Laplace transforms of $E_\alpha(t^\alpha)$ in various forms and then evaluating the inverses via convolution integrals. For example, using

$$L\{E_\alpha(t^\alpha)\}=\frac{s^{\alpha-1}}{s^\alpha-1}=\frac{s^{2\alpha-1}}{s^{2\alpha}-1}+\frac{s^{2\alpha-1}}{s^{2\alpha}-1}\cdot\frac{1}{s^\alpha}\,, \tag{32}$$

immediately yields

$$\begin{aligned}&E_\alpha(t^\alpha)=E_{2\alpha}(t^{2\alpha})+E_{2\alpha}(t^{2\alpha})*\tfrac{t^{\alpha-1}}{\Gamma(\alpha)}=\\&E_{2\alpha}(t^{2\alpha})+\int_0^t E_{2\alpha}(u^{2\alpha})\tfrac{(t-u)^{\alpha-1}}{\Gamma(\alpha)}\,du\,.\end{aligned} \tag{33}$$

All convolution integrals can be transformed into finite trigonometric integrals by a suitable change of variable. Therefore, putting $u = t[cos\theta]^2$ in (33) yields

$$\begin{aligned}&\tfrac{1}{\Gamma(\alpha)}\int_0^t E_{2\alpha}(u^{2\alpha})(t-u)^{\alpha-1}\,du=\\&\tfrac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^{\pi/2}\sin(2\theta)\,\left[(\sin\theta)^2\right]^{\alpha-1}E_{2\alpha}[t^{2\alpha}(\cos\theta)^{4\alpha}]\,d\theta\end{aligned} \tag{34}$$

Similarly, from

$$L\left\{t^{\beta-1}E_{\alpha,\beta}(t^\alpha)\right\}=\frac{s^{\alpha-\beta}}{s^\alpha-1}=\frac{s^{2\alpha-\beta}}{s^{2\alpha}-1}+\frac{s^{2\alpha-\beta}}{s^{2\alpha}-1}\cdot\frac{1}{s^\alpha}\,, \tag{35}$$

it follows that

$$E_{\alpha,\beta}(t^\alpha)=E_{2\alpha,\beta}(t^{2\alpha})+\int_0^t\left(\frac{u}{t}\right)^{\beta-1}E_{2\alpha,\beta}(u^{2\alpha})\frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)}\,du\,. \tag{36}$$

A different convolution integral can be derived from

$$\frac{1}{s^{\beta+1}}=\frac{s^{\alpha-\beta}}{s^\alpha-1}\cdot\left[\frac{1}{s}-\frac{1}{s^{\alpha+1}}\right], \tag{37}$$

whose inverse Laplace transform is

$$\frac{t^\beta}{\Gamma(\beta+1)}=\int_0^t u^{\beta-1}E_{\alpha,\beta}(u^\alpha)\left[1-\frac{(t-u)^\alpha}{\Gamma(\alpha+1)}\right]du\,. \tag{38}$$

Introducing the Laplace transform of $E_{\alpha,\beta}(\pm t^\alpha)$ in the form

$$L\left\{t^{\beta-1}E_{\alpha,\beta}(\pm t^\alpha)\right\}=\frac{s^{\alpha-\beta}}{s^\alpha-1}=\frac{s^{\alpha-1}}{s^\alpha-1}\cdot\frac{1}{s^{\beta-1}}\,, \tag{39}$$

gives

$$t^{\beta-1}E_{\alpha,\beta}(\pm t^{\alpha}) = E_{\alpha}(\pm t^{\alpha}) * \frac{t^{\beta-2}}{\Gamma(\beta-1)} = \int_0^t E_{\alpha}(\pm u^{\alpha})\frac{(t-u)^{\beta-2}}{\Gamma(\beta-1)}\,du\,. \tag{40}$$

For $\beta = \alpha$, this becomes

$$\begin{aligned} &t^{\alpha-1}\,E_{\alpha,\alpha}(\pm t^{\alpha}) = E_{\alpha}(\pm t^{\alpha}) * \tfrac{t^{\alpha-2}}{\Gamma(\alpha-1)} = \\ &\int_0^t E_{\alpha}(\pm u^{\alpha})\tfrac{(t-u)^{\alpha-2}}{\Gamma(\alpha-1)}\,du\,. \end{aligned} \tag{41}$$

For α and β, positive integers, (40) reduces to

$$t^{m-1}E_{n,m}(\pm t^{n}) = \int_0^t E_{n}(\pm u^{n})\frac{(t-u)^{m-2}}{(m-2)!}\,. \tag{42}$$

where $n = 1, 2, 3, \ldots$ and $m = 2, 3, 4, \ldots$.

These convolution integrals are easily evaluated because the Mittag-Leffler functions reduce to elementary functions. For example, for $n = 1$ and $m = 2$ and 3 and noting that $E_1(t) = \mathrm{e}^t$, it follows that

$$\begin{aligned} &t\,E_{1,2}(t) = \int_0^t e^{u}\,du = e^{t} - 1\,, \\ &t^2\,E_{1,3}(t) = \int_0^t e^{u}\,(t-u)\,du = e^{t} - t - 1 \end{aligned} \tag{43}$$

The Mittag-Leffler functions for $n = 1$ to 4 and $m = 2$ to 4 are presented in [19].

The operational rules of the Laplace transformation enable us to obtain representations for derivatives of the Mittag-Leffler functions $t^{\beta-1}E_{\alpha,\beta}(t^{\alpha})$. It is obvious from (2) that the derivative for any order is zero at the origin. In this case, differentiation of the Mittag-Leffler function is equivalent to multiplying the Laplace transform by powers of s. Because

$$\begin{aligned} &L\left\{f^{(n)}(t)\right\} = s^{n}F(s)\,, \\ &f(0) = f'(0) = f''(0) = \ldots = f^{(n)}(0)\,, \\ &n = 1, 2, 3, \ldots\,, \end{aligned} \tag{44}$$

we find that for $\mathrm{Re}\alpha > 0$, $\mathrm{Re}\beta \geq n+1$ and $\mathrm{Re}s > 1$

$$L\left\{\frac{d^{n}}{dt^{n}}\left[t^{\beta-1}E_{\alpha,\beta}(t^{\alpha})\right]\right\} = s^{n}\left(\frac{s^{\alpha-\beta}}{s^{\alpha}-1}\right) = \frac{s^{\alpha-(\beta-n)}}{s^{\alpha}-1}\,. \tag{45}$$

Hence, the Laplace inverse transform becomes

$$\frac{d^{n}}{dt^{n}}\left[t^{\beta-1}E_{\alpha,\beta}(t^{\alpha})\right] = t^{\beta-n-1}E_{\alpha,\beta-n}(t^{\alpha})\,. \tag{46}$$

In case of $E_{\alpha}(t^{\alpha})$ function, its value is unity at the origin. Only the first derivative has a simple Laplace transform, which is

$$L\left\{\frac{d}{dt}[E_{\alpha}(t^{\alpha})]\right\} = s\left(\frac{s^{\alpha-1}}{s^{\alpha}-1}\right) - 1 = \frac{1}{s^{\alpha}-1} = \left(\frac{s^{\alpha-1}}{s^{\alpha}-1}\right)\cdot\frac{1}{s^{\alpha-1}}\,, \tag{47}$$

the inverse transform of (47) is

$$\frac{d}{dt}[E_\alpha(t^\alpha)] = E_\alpha(t^\alpha) * \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}\,, \tag{48}$$

However, according to (41), this convolution integral is also given by

$$\frac{d}{dt}[E_\alpha(t^\alpha)] = t^{\alpha-1}E_{\alpha,\alpha}(\ t^\alpha)\,. \tag{49}$$

The n-dimensional integrals of the Mittag-Leffler functions are easily evaluated because this is equivalent to dividing the Laplace transform, $F(s)$, by s^n

$$L\left\{\int\limits_0^t\int\limits_0^{u_{n-1}}\cdots\int\limits_0^{u_1} f(u_1)\,du_1\ \ du_2\cdots du_n\right\} = \frac{1}{s^n}F(s)\,, \tag{50}$$

Then, we obtain

$$L\left\{\int\limits_0^t\int\limits_0^{u_{n-1}}\cdots\int\limits_0^{u_1} u_1^{\beta-1}E_{\alpha,\beta}(u_1^\alpha)\,du_1\,du_2\cdots du_n\right\} = \frac{1}{s^n}\left(\frac{s^{\alpha-\beta}}{s^\alpha-1}\right) = \frac{s^{\alpha-(\beta+\ n)}}{s^\alpha-1}\,, \tag{51}$$

The inverse transform of (51) is

$$\int\limits_0^t\int\limits_0^{u_{n-1}}\cdots\int\limits_0^{u_1} u_1^{\beta\,-\,1}E_{\alpha,\beta}(u_1^\alpha)\,du_1\,du_2\cdots du_n = t^{\beta+\ n-1}E_{\alpha,\beta+\ n}(t^\alpha)\,. \tag{52}$$

For $n = 1$ and $\beta = 1$,

$$\begin{aligned} &\int\limits_0^t u^{\beta-1}E_{\alpha,\beta}(u^\alpha)\,du = t^\beta\,E_{\alpha,\beta+1}(t^\alpha)\,,\\ &\int\limits_0^t E_\alpha(u^\alpha)\,du = t\,E_{\alpha,2}(t^\alpha)\,. \end{aligned} \tag{53}$$

Together with the linearity property of the Laplace transformation, operational calculus is able to determine the sums of the Mittag-Leffler functions as power series. Consider the infinite and finite geometrical series, namely,

$$\begin{aligned} &1 + x + x^2 + \ldots\ + x^k + \ldots = \tfrac{1}{1-x}\,,\\ &1 + x + x^2 + \ldots\ x^{n-1} + x^n = \tfrac{x^{n\,+\,1}-1}{x-1} \end{aligned} \tag{54}$$

where $0 < x < 1$.

By taking the Laplace transforms of all the terms in the power series of the corresponding Mittag-Leffler function, one obtains for $s > 1$,

$$F(s) = \frac{s^{\alpha-1}}{s^\alpha-1} + \frac{s^{\alpha-2}}{s^\alpha-1} + \frac{s^{\alpha-3}}{s^\alpha-1} + \ldots + \frac{s^{\alpha-k}}{s^\alpha-1} + \ldots\,, \tag{55}$$

The inverse transform of $F(s)$ is given by the following series of the Mittag-Leffler functions:

$$\begin{aligned} &L^{-1}\{F(s)\} = E_\alpha(t^\alpha) + t\,E_{\alpha,2}(t^\alpha) + t^2\,E_{\alpha,3}(t^\alpha) + \ldots\\ &+t^k\,E_{\alpha,k+\ 1}(t^\alpha) + \ldots = \sum_{k\,=\,1}^{\infty} t^{k-1}E_{\alpha,k}(t^\alpha)\,. \end{aligned} \tag{56}$$

In order to invert *F*(s), one must express (55) as

$$F(s) = \frac{s^{\alpha-1}}{s^{\alpha}-1} + \frac{s^{\alpha-1}}{s^{\alpha}-1}\left\{\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3}\ldots + \frac{1}{s^k} + \ldots\right\}, \tag{57}$$

The series inside the brackets is merely the geometric series. Using (54) one finds that

$$F(s) = \frac{s^{\alpha-1}}{s^{\alpha}-1} + \frac{s^{\alpha-1}}{s^{\alpha}-1}\left[\frac{1}{1-(1/s)} - 1\right] = \frac{s^{\alpha-1}}{s^{\alpha}-1} + \frac{s^{\alpha-1}}{s^{\alpha}-1}\cdot\frac{1}{s-1}, \tag{58}$$

Finally, inverting *F*(*s*) yields

$$\begin{gathered}\sum_{k=1}^{\infty} t^{k-1}E_{\alpha,k}(t^{\alpha}) = E_{\alpha}(t^{\alpha}) + E_{\alpha}(t^{\alpha}) * e^{t} = \\ E_{\alpha}(t^{\alpha}) + \int_{0}^{t} e^{(t-u)}E_{\alpha}(u^{\alpha})\,du\,.\end{gathered} \tag{59}$$

For the case of a finite series of the Mittag-Leffler functions, one requires the second result in (54) to determine the Laplace transform *F*(*s*), which is given by

$$\begin{gathered}F(s) = \frac{s^{\alpha-1}}{s^{\alpha}-1} + \frac{s^{\alpha-1}}{s^{\alpha}-1}\left[\frac{(1/s)^{n}-1}{(1/s)-1} - 1\right] = \\ \frac{s^{\alpha-1}}{s^{\alpha}-1} + \left\{\frac{s^{\alpha-1}}{s^{\alpha}-1} - \frac{s^{\alpha-(n+1)}}{s^{\alpha}-1}\right\}\cdot\frac{1}{s-1}\,,\end{gathered} \tag{60}$$

According to the convolution theorem, the inverse transform of this finite sum is

$$\begin{gathered}\sum_{k=1}^{n} t^{k-1}E_{\alpha,k}(t^{\alpha}) = E_{\alpha}(t^{\alpha}) + e^{t} * \left\{E_{\alpha}(t^{\alpha}) - t^{n}E_{\alpha,n+1}(t^{\alpha})\right\} = \\ E_{\alpha}(t^{\alpha}) + \int_{0}^{t} e^{(t-u)}\left\{E_{\alpha}(u^{\alpha}) - u^{n}E_{\alpha,n+1}(u^{\alpha})\right\}du\,.\end{gathered} \tag{61}$$

Similarly, we can use (54) for negative value of x

$$1 - x + x^2 - \ldots + x^k - \ldots = \frac{1}{1+x}, \tag{62}$$

Then, the corresponding Laplace transform becomes

$$\begin{gathered}F(s) = \frac{s^{\alpha-1}}{s^{\alpha}-1} + \frac{s^{\alpha-1}}{s^{\alpha}-1}\left\{-\frac{1}{s} + \frac{1}{s^2} - \frac{1}{s^3} + \ldots + \frac{1}{s^k} + \ldots\right\} = \\ \frac{s^{\alpha-1}}{s^{\alpha}-1} + \frac{s^{\alpha-1}}{s^{\alpha}-1}\left(\frac{s}{s+1} - 1\right) = \frac{s^{\alpha-1}}{s^{\alpha}-1} - \frac{s^{\alpha-1}}{s^{\alpha}-1}\cdot\frac{1}{s+1}\,,\end{gathered} \tag{63}$$

Inversion of this result yields

$$\begin{gathered}\sum_{k=1}^{\infty} (-1)^{k-1}\, t^{k-1}E_{\alpha,k}(t^{\alpha}) = E_{\alpha}(t^{\alpha}) - E_{\alpha}(t^{\alpha}) * e^{-t} = \\ E_{\alpha}(t^{\alpha}) - \int_{0}^{t} e^{-(t-u)}E_{\alpha}(u^{\alpha})\,du\,.\end{gathered} \tag{64}$$

According to the binomial theorem for $x < 1$, we have

$$P(x) = 1 - 2x + 3\,x^2 - 4x^3 + \ldots = \sum_{k=1}^{\infty}(-1)^{k-1}k\,x^{k-1} = \frac{1}{(1+x)^2}, \tag{65}$$

The Laplace transform corresponding to this series is

$$F(s) = \frac{s^{\alpha-1}}{s^{\alpha}-1} + \frac{s^{\alpha-1}}{s^{\alpha}-1}\left\{-\frac{2}{s} + \frac{3}{s^2} - \frac{4}{s^3}\ldots\right\} = \frac{s^{\alpha-1}}{s^{\alpha}-1} + \\ \frac{s^{\alpha-1}}{s^{\alpha}-1}\left[\left(\frac{s}{s+1}\right)^2 - 1\right] = \frac{s^{\alpha-1}}{s^{\alpha}-1} - \frac{s^{\alpha-1}}{s^{\alpha}-1}\left[\frac{1}{(s+1)^2} + \frac{2s}{(s+1)^2}\right], \tag{66}$$

The inverse transform of the second term in (66) is

$$L^{-1}\left\{-\frac{s^{\alpha-1}}{s^{\alpha}-1}\left[\frac{1}{(s+1)^2} + \frac{2s}{(s+1)^2}\right]\right\} = \\ -E_{\alpha}(t^{\alpha}) * \left[t\, e^{-t} + 2e^{-t}(1-t)\right] = E_{\alpha}(t^{\alpha}) * \left[(t-2)\, e^{-t}\right], \tag{67}$$

Thus, the infinite series of the Mittag-Leffler functions in (65) and (67) is

$$E_{\alpha,1}(t^{\alpha}) - 2t\, E_{\alpha,2}(t^{\alpha}) + 3t^2\, E_{\alpha,3}(t^{\alpha}) - 4t^3\, E_{\alpha,4}(t^{\alpha})\ + \ldots = \\ \sum_{k\,=\,1}^{\infty} (-1)^{k-1} k t^{k-1} E_{\alpha,k}(t^{\alpha}) = \\ E_{\alpha}(t^{\alpha}) + \int_0^t \left[(t-u-2)\, e^{-(t-u)}\right] E_{\alpha}(u^{\alpha})\, du\ . \tag{68}$$

From the preceding examples, it is obvious that if the function $f(t)$ is expanded into the Taylor series,

$$f(x) = \sum_{k\,=\,0}^{\infty} \frac{f^{(k)}(0)}{k!}\, x^k \tag{69}$$

Then, the sum of the corresponding series of the Mittag-Leffler functions can be expressed in terms of convolution integrals. This is only possible if the inverse Laplace transforms, $L^{-1}[f(1/s) - 1]$, are known.

Now, consider the binomial series with the power of 1/2. Then, we have some derivatives of the function $f(t)$, which are equal to zero at the origin

$$f(x) = \sqrt{1+x^2} = 1 + \frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{16} - \frac{5x^8}{128} + \ldots\ , \tag{70}$$

The corresponding series of the Mittag-Leffler functions is

$$S(t^{\alpha}) = \\ E_{\alpha,1}(t^{\alpha}) + \frac{t^2}{2}E_{\alpha,3}(t^{\alpha}) - \frac{t^4}{8}E_{\alpha,5}(t^{\alpha}) + \frac{t^6}{16}E_{\alpha,7}(t^{\alpha}) - \frac{5t^8}{128}E_{\alpha,9}(t^{\alpha}) - \ldots\ , \tag{71}$$

while the Laplace transform of $S(t^{\alpha})$ after few manipulations is given by

$$F(s) = \frac{s^{\alpha-1}}{s^{\alpha}-1} + \frac{s^{\alpha-1}}{s^{\alpha}-1}\left\{\frac{1}{2s^2} - \frac{1}{8s^4} + \frac{1}{16s^6} - \frac{5}{128s^8} + \ldots\right\} = \\ \frac{s^{\alpha-1}}{s^{\alpha}-1} + \frac{s^{\alpha-1}}{s^{\alpha}-1}\left[\sqrt{1+\left(\frac{1}{s}\right)^2} - 1\right] = \frac{s^{\alpha-1}}{s^{\alpha}-1} + \frac{s^{\alpha-1}}{s^{\alpha}-1}\left[\frac{s}{\sqrt{s^2+1}} - 1\right] \\ = \frac{s^{\alpha-1}}{s^{\alpha}-1} - \frac{s^{\alpha-1}}{s^{\alpha}-1} \cdot \frac{1}{\sqrt{s^2+1}\ [s+\sqrt{s^2+1}]}\ , \tag{72}$$

Noting that the inverse Laplace transform of the Bessel function of the first kind and of the first order is

$$L^{-1}\left\{\frac{1}{\sqrt{s^2+1}\ [s+\ \sqrt{s^2+1}]}\right\} = J_1(t)\ , \tag{73}$$

one finds that the series of the Mittag-Leffler functions in (74) can be expressed as

$$\begin{aligned}
&E_{\alpha,1}(t^{\alpha}) + \tfrac{t^2}{2} E_{\alpha,3}(t^{\alpha}) - \tfrac{t^4}{8} E_{\alpha,5}(t^{\alpha}) + \tfrac{t^6}{16} E_{\alpha,7}(t^{\alpha}) - \tfrac{5\,t^8}{128} E_{\alpha,9}(t^{\alpha}) - \ldots \\
&= E_{\alpha,1}(t^{\alpha}) - E_{\alpha,1}(t^{\alpha}) * J_1(t) = E_{\alpha,1}(t^{\alpha}) - \int_0^t E_{\alpha,1}(u^{\alpha})\, J_1(t-u)\, du \\
&= E_{\alpha}(t^{\alpha}) - \int_0^{\pi/2} t \sin(2\theta)\, E_{\alpha}[t^{\alpha} (\cos\theta)^{2\alpha}]\ J_1[t(\sin\theta)^2]\, d\theta\,.
\end{aligned} \tag{74}$$

3. Differentiation and Integration of the Mittag-Leffler Functions with Respect to Parameters in the Laplace Transform Approach

The operational rules of the Laplace transformation are also appropriate in the evaluation of derivatives of the Mittag-Leffler functions with respect to parameters. Differentiation under the integral transform sign is permissible if the function $f(t,\alpha)$ is continuous with respect to the variable t and the parameter α. Then, we have

$$\begin{aligned}
&L\{f(t,\alpha)\} = F(s,\alpha)\,, \\
&L\left\{\tfrac{\partial f(t,\alpha)}{\partial \alpha}\right\} = \tfrac{\partial F(s,\alpha)}{\partial \alpha} = G(s,\alpha)\,, \\
&L^{-1}\left\{\tfrac{\partial F(s,\alpha)}{\partial \alpha}\right\} = L^{-1}\{G(s,\alpha)\} = \tfrac{\partial f(t,\alpha)}{\partial \alpha}
\end{aligned} \tag{75}$$

The Laplace transform $G(s,\alpha)$ of the derivative of the Mittag-Leffler function $E_{\alpha}(t^{\alpha})$ is

$$\begin{aligned}
&G(s,\alpha) = L\left\{\tfrac{\partial E_{\alpha}(t^{\alpha})}{\partial \alpha}\right\} = \tfrac{\partial}{\partial \alpha}\left(\tfrac{s^{\alpha-1}}{s^{\alpha}-1}\right) = \left[\tfrac{s^{\alpha-1}\ln s}{s^{\alpha}-1} - \tfrac{s^{2\alpha-1}\ln s}{(s^{\alpha}-1)^2}\right] \\
&= -\tfrac{s^{\alpha-1}}{s^{\alpha}-1} \cdot \tfrac{\ln s}{s^{\alpha}-1} = -\tfrac{s^{\alpha-1}}{s^{\alpha}-1} \cdot \tfrac{s^{\alpha-1}}{s^{\alpha}-1} \cdot \tfrac{\ln s}{s^{\alpha-1}}\,.
\end{aligned} \tag{76}$$

In order to avoid evaluating a complex integral in the inversion process, $G(s,\alpha)$ is expressed as the product of three Laplace transforms. The convolution theorem can be applied for $G(s,\alpha)$ because inverse of the third term in (76) is given for $\mathrm{Re}\lambda > 0$ in [20]

$$L^{-1}\left\{\frac{\ln s}{s^{\lambda}}\right\} = \frac{t^{\lambda-1}}{\Gamma(\lambda)}[\psi(\lambda) - \ln t]\,, \tag{77}$$

From (76) and (77) it follows that

$$\frac{\partial\, E_{\alpha}(t^{\alpha})}{\partial\, \alpha} = E_{\alpha}(t^{\alpha}) * E_{\alpha}(t^{\alpha}) * \left\{\frac{t^{\alpha-2}}{\Gamma(\alpha-1)}[\ln t - \psi(\alpha-1)]\right\} \tag{78}$$

where $\alpha > 1$.

Thus, due to two convolutions, the derivative with respect to α is expressed by a double convolution integral. If the Laplace transform in (76) is written as

$$G(s,\alpha) = -\frac{s^{\alpha-1}}{s^{\alpha}-1} \cdot \frac{s^{\alpha-\lambda}}{s^{\alpha}-1} \cdot \frac{\ln s}{s^{\alpha-\lambda}}\ , \tag{79}$$

the inverse transform of (79) becomes

$$\begin{aligned}
&\tfrac{\partial E_{\alpha}(t^{\alpha})}{\partial \alpha} = \\
&E_{\alpha}(t^{\alpha}) * \left[t^{\lambda\ -1} E_{\alpha,\lambda}(t^{\alpha})\right] * \left\{\tfrac{t^{\alpha\ -\lambda\ \ -1}}{\Gamma(\alpha-\lambda)}[\ln t - \psi(\alpha-\lambda)]\right\}
\end{aligned} \tag{80}$$

where $0 < \lambda < \alpha < 1$.

The case $\alpha = 1$ will be considered in the next section.

In a similar manner, the Laplace transform of derivative of the Mittag-Leffler function $t^{\beta-1}E_{\alpha,\beta}(t^{\alpha})$ with respect to α is

$$\begin{aligned} G(s,\alpha,\beta) &= L\left\{\frac{\partial[t^{\beta\ -1}E_{\alpha,\beta}(t^{\alpha})]}{\partial\alpha}\right\} = \frac{\partial}{\partial\alpha}\left(\frac{s^{\alpha-\beta}}{s^{\alpha}-1}\right) = \\ &= -\frac{s^{\alpha-\beta}}{s^{\alpha}-1}\cdot\frac{\ln s}{s^{\alpha}-1} = -\frac{s^{\alpha-1}}{s^{\alpha}-1}\cdot\frac{s^{\alpha-\beta}}{s^{\alpha}-1}\cdot\frac{\ln s}{s^{\alpha-1}}\ , \end{aligned} \tag{81}$$

This gives

$$\begin{aligned} &\frac{\partial[t^{\beta-1}E_{\alpha,\beta}(t^{\alpha})]}{\partial\alpha} = \\ &E_{\alpha}(t^{\alpha}) * \left\{t^{\beta-1}E_{\alpha,\beta}(t^{\alpha})\right\} * \left\{\frac{t^{\alpha-2}}{\Gamma(\alpha-1)}[\ln t - \psi(\alpha-1)]\right\} \end{aligned} \tag{82}$$

where $\alpha > 1$.

As expected, for $\beta = 1$, (82) reduces to (78).

For $0 < \alpha < 1$, from (79), it follows that

$$G(s,\alpha,\beta) = -\frac{s^{\alpha-\lambda}}{s^{\alpha}-1}\cdot\frac{s^{\alpha-\beta}}{s^{\alpha}-1}\cdot\frac{\ln s}{s^{\alpha-\lambda}}\ , \tag{83}$$

and

$$\begin{aligned} &\frac{\partial[t^{\beta-1}E_{\alpha,\beta}(t^{\alpha})]}{\partial\alpha} = \\ &\left\{t^{\lambda\ -1}E_{\alpha,\lambda}(t^{\alpha})\right\} * \left\{t^{\beta-1}E_{\alpha,\beta}(t^{\alpha})\right\} * \left\{\frac{t^{\alpha\ -\lambda\ \ -1}}{\Gamma(\alpha-\lambda)}[\ln t - \psi(\alpha-\lambda)]\right\} \end{aligned} \tag{84}$$

$0 < \lambda < \alpha < 1$.

For β, a variable, the Laplace transform of $t^{\beta-1}E_{\alpha,\beta}(t^{\alpha})$ derivative is

$$\begin{aligned} H(s,\alpha,\beta) &= L\left\{\frac{\partial t^{\beta\ -1}E_{\alpha,\beta}(t^{\alpha})}{\partial\beta}\right\} = \frac{\partial}{\partial\beta}\left(\frac{s^{\alpha\ -\beta}}{s^{\alpha}-1}\right) = \\ &= -\frac{s^{\alpha-\beta}\ln s}{s^{\alpha}-1} = -\frac{s^{\alpha-(\beta-\lambda)}}{s^{\alpha}-1}\cdot\frac{\ln s}{s^{\lambda}}\ , \end{aligned} \tag{85}$$

and the inverse transform is

$$\begin{aligned} &\frac{\partial t^{\beta-1}E_{\alpha,\beta}(t^{\alpha})}{\partial\beta} = t^{\beta\ -1}\ln t\ E_{\alpha,\beta}(t^{\alpha}) + t^{\beta-1}\frac{\partial E_{\alpha,\beta}(t^{\alpha})}{\partial\beta} = \\ &\left\{t^{\beta-\lambda-1}E_{\alpha,\beta-\lambda}(t^{\alpha})\right\} * \left\{\frac{t^{\lambda-1}}{\Gamma(\lambda)}[\ln t - \psi(\lambda)\right\}. \end{aligned} \tag{86}$$

where $\beta > \lambda > 0$.

As in the case with differential operations, there are rules in the Laplace transformation for evaluation of integrals. The Laplace transform of the Mittag-Leffler function $t^{\beta-1}E_{\alpha,\beta}(t^{\alpha})$ enables one to derive the following integral

$$I(t,\lambda) = \int_0^{\lambda} t^{\beta-1}E_{\alpha,\beta}(t^{\alpha})\,d\beta\ , \tag{87}$$

The Laplace transform of (87) can be determined by changing the order of integration as follows:

$$\begin{aligned} &\int_0^{\infty} e^{-st}\left\{\int_0^{\lambda} t^{\beta\ -1}E_{\alpha,\beta}(t^{\alpha})\,d\beta\right\}dt = \int_0^{\lambda}\left\{\int_0^{\infty} e^{-\ st}t^{\beta\ -1}E_{\alpha,\beta}(t^{\alpha})\,dt\right\}d\beta = \\ &\int_0^{\lambda}\frac{s^{\alpha-\beta}}{s^{\alpha}-1}\,d\beta = \frac{s^{\alpha}}{s^{\alpha}-1}\cdot\frac{1}{\ln s} - \frac{s^{\alpha-\lambda}}{s^{\alpha}-1}\cdot\frac{1}{\ln s}\ . \end{aligned} \tag{88}$$

The inverse of $(\ln s)^{-1}$ is closely related to a Volterra function [19] as

$$L^{-1}\left\{\frac{1}{\ln s}\right\} = \int_0^{\infty}\frac{t^{u-1}}{\Gamma(u)}\,du\ , \tag{89}$$

It follows from (47) that

$$L^{-1}\left\{\frac{s^{\alpha}}{s^{\alpha}-1}\right\}=\delta(t)+\frac{d}{dt}[E_{\alpha}(t^{\alpha})]\ , \tag{90}$$

whereas (49) gives

$$\frac{d}{dt}[E_{\alpha}(t^{\alpha})]=t^{\alpha-1}E_{\alpha,\alpha}(\ t^{\alpha})\ , \tag{91}$$

The final result in terms of convolution integrals is

$$I(t,\lambda)=\left[\delta(t)+t^{\alpha-1}E_{\alpha,\alpha}(t^{\alpha})-t^{\lambda-1}E_{\alpha,\lambda}(t^{\alpha})\right]*\int_{0}^{\infty}\frac{t^{u-1}}{\Gamma(u)}\,du\ . \tag{92}$$

Two limits of integration in (87) can be altered to

$$\begin{gathered}\int_{0}^{\infty}t^{\beta-1}E_{\alpha,\beta}(t^{\alpha})\,d\beta=\left[\delta(t)+t^{\alpha-1}E_{\alpha,\alpha}(\ t^{\alpha})\right]*\int_{0}^{\infty}\frac{t^{u\ -1}}{\Gamma(u)}\,du\ ,\\ \int_{\lambda}^{\infty}t^{\beta-1}E_{\alpha,\beta}(t^{\alpha})\,d\beta=\left[t^{\lambda\ -1}E_{\alpha,\lambda}(\ t^{\alpha})\right]*\int_{0}^{\infty}\frac{t^{u\ -1}}{\Gamma(u)}\,du\ .\end{gathered} \tag{93}$$

The second term on the right-hand side of (88), written in a different form as inversion of the Volterra function, is as follows

$$\begin{gathered}L^{-1}\left\{\frac{s^{\alpha-\lambda}}{s^{\alpha}-1}\cdot\frac{1}{\ln s}\right\}=L^{-1}\left\{\frac{s^{\alpha-(\lambda\ -1)}}{s^{\alpha}-1}\cdot\frac{1}{s\ln s}\right\}=t^{\lambda-2}E_{\alpha,\lambda\ -1}(t^{\alpha})*\nu(t)\ ,\\ \nu(t)=\int_{0}^{\infty}\frac{t^{u}}{\Gamma(u+1)}\,du\ .\end{gathered} \tag{94}$$

The connection between the Mittag-Leffler functions and the Volterra functions in the Laplace transformation is discussed in detail in [19].

4. Derivatives of the Mittag-Leffler Functions with Respect to Parameters α and β Expressed as Power Series

As it has been shown in the previous section, the differentiation with respect to parameters of the Mittag-Leffler functions can be represented formally, in closed form, in terms of double convolution integrals. Unfortunately, these convolution integrals are not amenable to numerical computations. Hence, an alternative approach is required. Differentiating (1) and (2) with respect to α and β yields

$$\begin{gathered}\frac{\partial E_{\alpha}(t)}{\partial\alpha}=G(\alpha,t)=-\sum_{k\,=\,1}^{\infty}\left(\frac{\psi(\alpha k+1)}{\Gamma(\alpha k+1)}\right)kt^{k}\ ,\\ \frac{\partial E_{\alpha,\beta}(t)}{\partial\alpha}=-\sum_{k\,=\,1}^{\infty}\left(\frac{\psi(\alpha k+\beta)}{\Gamma(\alpha k+\beta)}\right)kt^{k}\ .\end{gathered} \tag{95}$$

and

$$\frac{\partial\ E_{\alpha,\beta}(t)}{\partial\beta}=-\sum_{k\,=\,0}^{\infty}\left(\frac{\psi(\alpha k+\beta)}{\Gamma(\alpha k+\beta)}\right)t^{k}\ . \tag{96}$$

The second derivatives are

$$\frac{\partial^{2}E_{\alpha}(t)}{\partial\alpha^{2}}=G'(\alpha,t)=\sum_{k\,=\,1}^{\infty}\left\{\frac{[\psi(\alpha k+1)]^{2}-\psi^{(1)}\ (\alpha k+1)}{\Gamma(\alpha k+1)}\right\}k^{\,2}t^{k}\ , \tag{97}$$

and

$$\begin{aligned} \frac{\partial^2 E_{\alpha,\beta}(t)}{\partial \beta^2} &= \sum_{k=0}^{\infty} \left\{ \frac{[\psi(\alpha\, k+\beta)]^2 - \psi^{(1)}\,(\alpha k+\beta)}{\Gamma(\alpha k+\beta)} \right\} t^k\,, \\ \frac{\partial^2 E_{\alpha,\beta}(t)}{\partial \alpha \partial \beta} &= \sum_{k=1}^{\infty} \left\{ \frac{[\,\psi(\alpha\, k+\beta)]^2 - \psi^{(1)}(\alpha k+\beta)}{\Gamma(\alpha k+\beta)} \right\} k t^k \end{aligned} \tag{98}$$

Higher derivatives with respect to α and β yield similar summands, only differing in powers of k.

Infinite series with the digamma functions in their summands do not appear often in mathematical investigations [22,23]. This changed in 2008 with the huge collection of results in the book by Brychkov [24]. Nevertheless, in their general form, infinite series with quotients of the digamma and gamma functions in their summands are still unsolved. However, for specific values of α and β, MATHEMATICA is able to determine closed forms for them, although they are rather cumbersome with mixture of elementary and special functions. Their validity was checked by carrying out numerical calculations with (95) and (96). Only a limited number of results will appear in this section, with the remainder appearing in Tables 1 and 2.

Table 1. First derivatives of the Mittag-Leffler functions with respect to the parameter α.

α	β	$\partial E_{\alpha,\beta}(t)/\partial\alpha$
1	3	$-\sum_{k=1}^{\infty} \frac{k\, t^k\, \psi(k+3)}{\Gamma(k+3)} = \frac{1-t+\gamma(t+2)-e^t+e^t(t-2)[Chi(t)-Shi(t)-\ln t]}{t^2}$
1	4	$-\sum_{k=1}^{\infty} \frac{k\, t^k\, \psi(k+4)}{\Gamma(k+4)} = \frac{4-8t-3t^2+2\gamma\,(t^2+4z=t+6)-4e^t}{4t^3} + \frac{e^t(t-3)[Chi(t)-Shi(t)-\ln t]}{4\,t^3}$
1	5	$-\sum_{k=1}^{\infty} \frac{k\, t^k \psi(k+5)}{\Gamma(k+5)} = \frac{36e^t(t-4)[Chi(t)-Shi(t)-\ln t]}{36t^4} + \frac{6\,\gamma[t^3+6t^2+18t+24]-[11\,t^3+54t^2+108t-36]-36e^t}{36t^4}$
1	6	$-\sum_{k=1}^{\infty} \frac{k\, t^k\, \psi(k+6)}{\Gamma(k+6)} = \frac{e^t(t-5)[Chi(t)-Shi(t)-\ln t]-e^t}{t^5} - \frac{25\,t^4+176t^3+648\,t^2+1158\,t}{288\,t^5} + \frac{12\,\gamma\,[\,t^4+8t^3+36\,t^3+96\,t^2+120\,t+24]}{288\,t^5}$
2	3	$-\sum_{k=1}^{\infty} \frac{k\, t^k\, \psi(2k+3)}{\Gamma(2\,k+3)} = \frac{2+4\,\gamma+\sinh(\sqrt{t})[(2Chi(\sqrt{t})-\ln t)\sqrt{t}+4Shi(\sqrt{t})]}{4t} - \frac{2\cosh(\sqrt{t})[2Chi(\sqrt{t})+\sqrt{t}Shi(\sqrt{t})-\ln t+1]}{4t}$
2	4	$-\sum_{k=1}^{\infty} \frac{k\, t^k\, \psi(2k+4)}{\Gamma(2\,k+4)} = \frac{-\sinh(\sqrt{t})[6Chi(\sqrt{t})+2\sqrt{t}Shi(\sqrt{t})-3\ln t+2]}{4t^{3/2}} + \frac{\cosh(\sqrt{t})[2\sqrt{t}Chi(\sqrt{t})-\sqrt{t}\ln t+6Shi(\sqrt{t})]+4(\gamma-1)\sqrt{t}}{4t^{3/2}}$
2	5	$-\sum_{k=1}^{\infty} \frac{k\, t^k\, \psi(2k+5)}{\Gamma(2\,k+5)} = \frac{\sinh(\sqrt{t})[\,2\sqrt{t}\; Chi(\sqrt{t})-\sqrt{t}\ln t+8Shi(\sqrt{t})]+(2\gamma-3)\,t}{4\,t^2} + \frac{-2\cosh(\sqrt{t})\;[4\; Chi(\sqrt{t})+\sqrt{t}\; Shi(\sqrt{t})-2\ln t+1]+8\gamma+2}{4t^2}$
2	6	$-\sum_{k=1}^{\infty} \frac{kt^k\psi(2k+6)}{\Gamma(2k+6)} = \frac{\sqrt{t}[-11t+6\gamma(t+12)-72]}{36t^{5/2}} + \frac{-9\sinh(\sqrt{t})(10Chi(\sqrt{t})+2\sqrt{t}Shi(\sqrt{t})-5\ln t+2]}{36t^{5/2}} + \frac{9\cosh(\sqrt{t})[\sqrt{t}(2Chi(\sqrt{t})-\ln t)+10Shi(\sqrt{t})]}{36t^{5/2}}$

Table 1. *Cont.*

α	β	$\partial E_{\alpha,\beta}(t)/\partial\alpha$
4	0	$-\sum_{k=1}^{\infty}\frac{k\,t^k\psi(4\,k)}{\Gamma(4\,k)}=\frac{t^{1/4}[\sin(t^{1/4})-\sinh(t^{1/4})]\ln t+4\,t^{1/4}Chi(t^{1/4})\sinh(t^{1/4})}{8}+\frac{t^{1/4}[-Shi(t^{1/4})\;\cosh(t^{1/4})+Si(t^{1/4})\;\cos(t^{1/4})-4\;Ci(t^{1/4})\;\sin(t^{1/4})]}{2}$
4	2	$-\sum_{k=1}^{\infty}\frac{k\,t^k\;\psi(4\,k+2)}{\Gamma(4\,k+2)}=\frac{[\sinh(t^{1/4})-t^{1/4}\cosh(t^{1/4})]\ln t-4\sinh(t^{1/4})}{32\;t^{1/4}}+\frac{4Chi(t^{1/4})[t^{1/4}\cosh(t^{1/4})-\sinh(t^{1/4})]+4\cosh(t^{1/4})Shi(t^{1/4})}{32\;t^{1/4}}+\frac{t^{1/4}\cos(z^{1/4})[4Ci(z^{1/4})-\ln t]-4t^{1/4}\sinh(t^{1/4})Shi(t^{1/4})}{32z^{1/4}}+\frac{\sin(t^{1/4})[-4-4Ci(t^{1/4})+\ln t+4t^{1/4}Si(t^{1/4})]+4Si(t^{1/4})}{32\;t^{1/4}}$

Table 2. First derivatives of the Mittag-Leffler functions with respect to the parameter β.

α	β	$\partial E_{\alpha,\beta}(t)/\partial\beta$
1	3	$-\sum_{k=0}^{\infty}\frac{t^k\psi(k+3)}{\Gamma(k+3)}=\frac{t-\gamma\;(t+1)+e^t[Chi(t)-Shi(t)-\ln t]}{t^2}$ $-\sum_{k=1}^{\infty}\frac{(-1)^k z^k\psi(4\,k)}{\Gamma(4\,k)}=\frac{z^{1/4}[\sin(z^{1/4})-\sinh(z^{1/4})]\ln z}{8}+\frac{4\;z^{1/4}[Chi(z^{1/4})\;\sinh(z^{1/4})-Shi(z^{1/4})\;\cosh(z^{1/4})+}{8}$
1	4	$-\sum_{k=0}^{\infty}\frac{t^k\psi(k+4)}{\Gamma(k+4)}=\frac{e^t[Chi(t)-Shi(t)-\ln t]}{t^3}+\frac{3t^2+4\;t-2\gamma(t^2+2t+2)}{4\;t^3}$
1	5	$-\sum_{k=0}^{\infty}\frac{t^k\psi(k+5)}{\Gamma(k+5)}=\frac{e^t[Chi(t)-Shi(t)-\ln t]}{t^4}+\frac{11t^3+27t^2+36t-6\gamma\;(t^3+3t^2+6t+6)}{36t^4}$
1	6	$-\sum_{k=0}^{\infty}\frac{t^k\psi(k+6)}{\Gamma(k+6)}=\frac{e^t[Chi(t)-Shi(t)-\ln t]}{t^5}+\frac{25t^4+88t^3+216t^2+288t-12\;\gamma\;(t^4+4t^3+12t^2+24t+24)}{288t^5}$
4	0	$-\sum_{k=1}^{\infty}\frac{t^k\psi(4\,k)}{\Gamma(4\,k)}=\frac{t^{1/4}[4Ci(t^{1/4})\;\sin(t^{1/4})+\ln t][\sin(t^{1/4})-\sinh(t^{1/4})]}{8}+\frac{t^{1/4}[Shi(t^{1/4})\;\cos(t^{1/4})-4Chi(t^{1/4})\;\sinh(t^{1/4})-Shi(t^{1/4})\;\cosh(t^{1/4})]}{8}-1$
4	1	$-\sum_{k=1}^{\infty}\frac{t^k\psi(4\,k+1)}{\Gamma(4\,k+1)}=\frac{-Chi(t^{1/4})\;\cosh(t^{1/4})-Ci(t^{1/4})\cos(t^{1/4})}{2}+\frac{Shi(t^{1/4})\;\sinh(t^{1/4})-Si(t^{1/4})\;\sin(t^{1/4})}{2}+\frac{\ln t[\cos(t^{1/4})+\cosh(t^{1/4})]}{8}$
4	2	$-\sum_{k=1}^{\infty}\frac{t^k\psi(4\,k+2)}{\Gamma(4\,k+2)}=\frac{-Ci(t^{1/4})\;\sin(t^{1/4})-Chi(t^{1/4})\sinh(t^{1/4})}{2t^{1/4}}+\frac{Shi(t^{1/4})\;\cosh(t^{1/4})+Si(t^{1/4})\;\cos(t^{1/4})}{2t^{1/4}}+\frac{\ln t[\sin(t^{1/4})+\sinh(t^{1/4})]}{8t^{1/4}}$
4	3	$-\sum_{k=1}^{\infty}\frac{t^k\psi(4\,k+3)}{\Gamma(4\,k+3)}=\frac{Ci(t^{1/4})\cos(t^{1/4})-Chi(t^{1/4})\;\cosh(t^{1/4})}{2\sqrt{t}}+\frac{Shi(t^{1/4})\sinh(t^{1/4})+Si(t^{1/4})\sin(t^{1/4})}{2\sqrt{t}}+\frac{\ln t[\cos(t^{1/4})-\cosh(t^{1/4})]}{8\sqrt{t}}$
4	4	$-\sum_{k=1}^{\infty}\frac{t^k\psi(4\,k+4)}{\Gamma(4\,k+4)}=\frac{Ci(t^{1/4})\sin(t^{1/4})-Chi(t^{1/4})\;\sinh(t^{1/4})}{2t^{1/4}}+\frac{Si(t^{1/4})\cos(t^{1/4})-Shi(t^{1/4})\cosh(t^{1/4})}{2t^{1/4}}+\frac{\ln t[\sin(t^{1/4})-\sinh(t^{1/4})]}{8t^{1/4}}$

Convergence conditions for the power series reported in this section were not established, and therefore t values are in some cases restricted (e.g., in (99) and (100) for $|t|<1$). These summands were obtained from MATHEMATICA, but the validity was numerically checked for only some of them.

The simplest cases occur when α and β equal zero or unity. Then, we find that

$$\frac{\partial\, E_{\alpha,\beta}(t)}{\partial\alpha}|_{\ \alpha=\,0,\beta=\,1} = -\frac{\psi(1)}{\Gamma(1)}\sum_{k\,=\,1}^{\infty} kt^k = \frac{\gamma\; t}{(t-1)^2}\ , \tag{99}$$

$$\frac{\partial\, E_{\alpha,\beta}(t)}{\partial\alpha}|_{\ \alpha=\,0,\beta} = -\frac{\psi(\beta)}{\Gamma(\beta)}\sum_{k\,=\,1}^{\infty} kt^k = -\frac{\psi(\beta)\; t}{\Gamma(\beta)\;(t-1)^2}\ , \tag{100}$$

$$\begin{gathered}\frac{\partial E_{\alpha,\beta}(t)}{\partial\alpha}|_{\ \alpha=\,1,\beta\,=\,0} = -\sum_{k\,=\,1}^{\infty}\left(\frac{\psi(k)}{\Gamma(k)}\right)kt^k \ = \\ t\left\{e^t(1+t)[Chi(t)-Shi(t)-\ln t]+1-e^t\right\}\end{gathered} \tag{101}$$

$$\begin{gathered}\frac{\partial E_{\alpha,\beta}(t)}{\partial\alpha}|_{\ \alpha=\,1,\beta\,=\,1} = -\sum_{k\,=\,1}^{\infty}\left(\frac{\psi(k+1)}{\Gamma(k+1)}\right)kt^k = \\ 1-e^t\{t[\ln t+\Gamma(0,t)]+1\}\ ;\ \Gamma(0,t)] = -Ei(-t)\ ,\end{gathered} \tag{102}$$

and

$$\frac{\partial\, E_{\alpha,\beta}(t)}{\partial\beta}|_{\ \alpha=\,1,\beta=\,1} = -\sum_{k\,=\,0}^{\infty}\left(\frac{\psi(k+1)}{\Gamma(k+1)}\right)t^k \ = -\,e^t[\ln t+\Gamma(0,t)]\ , \tag{103}$$

where $\Gamma(0,t) = -Ei(-t)$, and the hyperbolic sine and cosine integrals and the exponential integral are defined by

$$\begin{aligned} Shi(t) &= \int_0^t \frac{\sinh u}{u}\,du\ , \\ Chi(t) &= -\int_0^t \frac{1-\cosh u}{u}\,du+\gamma+\ln t\ , \\ -Ei(-t) &= \int_t^{\infty}\frac{e^{-u}}{u}\,du\ . \end{aligned} \tag{104}$$

γ represents Euler's constant.

For α, β = 0, 1, and 2, the following sums of infinite series are known:

$$\begin{gathered}\frac{\partial E_{\alpha,\beta}(t)}{\partial\alpha}|_{\ \alpha=\,1,\beta\,=\,2} = -\sum_{k\,=\,1}^{\infty}\left(\frac{\psi(\ k+2)}{\Gamma(\ k+2)}\right)kt^k \ = \\ \frac{1+\gamma+e^t[(t-1)\ Chi(t)+Shi(t)-t\ (Shi(t)+\ln t)+\ln t-1]}{t}\end{gathered} \tag{105}$$

$$\begin{gathered}\frac{\partial E_{\alpha,\beta}(t)}{\partial\alpha}|_{\ \alpha=\,2,\beta\,=\,0} = -\sum_{k\,=\,1}^{\infty}\left(\frac{\psi(2k)}{\Gamma(2k)}\right)kt^k \ = \\ \frac{\sqrt{t}\ [2Chi(\sqrt{t})-\ln t][\sinh(\sqrt{t})+\sqrt{t}\cosh(\sqrt{t})]}{4} - \\ \frac{2\sqrt{t}Shi(\sqrt{t})[\sqrt{t}\ \sinh(\sqrt{t})+\cosh(\sqrt{t})]-2\sqrt{t}\sinh(\sqrt{t})}{4}\ ,\end{gathered} \tag{106}$$

$$\begin{gathered}\frac{\partial E_{\alpha,\beta}(t)}{\partial\alpha}|_{\ \alpha=\,2,\beta\,=\,1} = -\sum_{k\,=\,1}^{\infty}\left(\frac{\psi(2k+1)}{\Gamma(2k+1)}\right)kt^k \ = \\ \frac{\sqrt{t}\sinh(\sqrt{t})[2Chi(\sqrt{t})-\ln t]-2\cosh(\sqrt{t})\ [\sqrt{t}Shi(\sqrt{t})+1]+2}{4}\ ,\end{gathered} \tag{107}$$

$$\begin{gathered}\frac{\partial E_{\alpha,\beta}(t)}{\partial\alpha}|_{\ \alpha=\,2,\beta\,=\,2} = -\sum_{k\,=\,1}^{\infty}\left(\frac{\psi(2k+2)}{\Gamma(2k+2)}\right)kt^k \ = \\ \frac{[2Chi(\sqrt{t})-\ln t][\sqrt{t}\ \cosh(\sqrt{t})-\sinh(\sqrt{t})]-2\sinh(\sqrt{t})}{4\sqrt{t}} + \\ \frac{[2Shi(\sqrt{t})[\cosh(\sqrt{t})-\sqrt{t}\ \sinh(\sqrt{t})]}{4\sqrt{t}}\ ,\end{gathered} \tag{108}$$

and

$$\begin{gathered}\frac{\partial E_{\alpha,\beta}(t)}{\partial\beta}|_{\ \alpha=\,1,\beta\,=\,2} = -\sum_{k\,=\,0}^{\infty}\left(\frac{\psi(k+2)}{\Gamma(k+2)}\right)t^k = \\ -\frac{\gamma+e^t[Shi(t)-Chi(t)+\ln t]}{t}\ ,\end{gathered} \tag{109}$$

$$\frac{\partial E_{\alpha,\beta}(t)}{\partial\beta}|_{\alpha=2,\beta=0} = -\sum_{k=0}^{\infty}\left(\frac{\psi(2k)}{\Gamma(2k)}\right)t^k = 1+ \frac{\sqrt{t}\{\sinh(\sqrt{t})[2Chi(\sqrt{t})-\ln t]-2\cosh(\sqrt{t})Shi(t)\}}{2}, \tag{110}$$

$$\frac{\partial E_{\alpha,\beta}(-t)}{\partial\beta}|_{\alpha=2,\beta=0} = -\sum_{k=0}^{\infty}\left(\frac{\psi(2k)}{\Gamma(2k)}\right)(-t)^k = \frac{\sqrt{t}\{\sin(\sqrt{t})[Ci(\sqrt{t})-\ln t]-2\cos(\sqrt{t})Si(\sqrt{t})\}}{2}, \tag{111}$$

$$\frac{\partial E_{\alpha,\beta}(t)}{\partial\beta}|_{\alpha=2,\beta=1} = -\sum_{k=0}^{\infty}\left(\frac{\psi(2k+1)}{\Gamma(2k+1)}\right)t^k = -\sinh(\sqrt{t})Shi(\sqrt{t}) + \frac{\cosh(\sqrt{t})[2Chi(\sqrt{t})-\ln t]}{2}, \tag{112}$$

$$\frac{dE_{\alpha,\beta}(t)}{d\beta}|_{\alpha=2,\beta=2} = -\sum_{k=0}^{\infty}\left(\frac{\psi(2k+2)}{\Gamma(2k+2)}\right)t^k = \frac{-2\cosh(\sqrt{t})Shi(\sqrt{t})+\sinh(\sqrt{t})[2Chi(\sqrt{t})-\ln t]}{2\sqrt{t}}, \tag{113}$$

and

$$\frac{dE_{\alpha,\beta}(-t)}{d\beta}|_{\alpha=2,\beta=2} = -\sum_{k=0}^{\infty}\left(\frac{\psi(2k+2)}{\Gamma(2k+2)}\right)(-t)^k = \frac{[\sqrt{t}\,\cosh(\sqrt{t})-\sinh(\sqrt{t})][2Chi(\sqrt{t})-\ln t]}{4\sqrt{t}} + \frac{Shi(\sqrt{t})[\cosh(\sqrt{t})-\sqrt{t}\,\sinh(\sqrt{t})]-\sinh(\sqrt{t})}{2\sqrt{t}}, \tag{114}$$

where the sine and cosine integrals are defined by

$$\begin{aligned} Si(t) &= \int_0^t \frac{\sin u}{u}\,du\,, \\ Ci(t) &= -\int_t^{\infty} \frac{\cos u}{u}\,du \end{aligned} \tag{115}$$

A number of numerical methods for evaluating the Mittag-Leffler functions and their derivatives with respect to the argument z are given in the literature [25–27]. Fortunately, the Mittag-Leffler functions are available in MATHEMATICA, which means that the first and the second derivatives with respect to α can also be evaluated. The results for $0.05 < \alpha < 5.0$ and $0 < t < 2.25$ can be obtained from the author on request. Two numerical methods were used to verify the results. In the first method, direct summation of infinite series (95) and (96) was performed in MATHEMATICA module, while in the second method, the calculations were carried out by applying the central differences to $O(h^4)$ with $h = 0.001$.

$$\frac{\partial\, E_\alpha(t)}{\partial\,\alpha} = \frac{-E_{\alpha+2h}(t) + 8E_{\alpha+h}(t) - 8E_{\alpha-h}(t) + E_{\alpha-2h}(t)}{12\,h} \tag{116}$$

and

$$\frac{\partial^2 E_\alpha(t)}{\partial\alpha^2} = \frac{-E_{\alpha+2h}(t)+16\,E_{\alpha+h}(t)-30\,E_\alpha(t)+16\,E_{\alpha-h}(t)-E_{\alpha-2h}(t)}{12\,h^2} \tag{117}$$

The above results of the Mittag-Leffler functions were evaluated in MATHEMATICA.

The Mittag-Leffler functions, $f(\alpha,t) = E_\alpha(t)$, as a function of α for constant t are plotted in Figure 1. The rapid exponential behavior of these functions means that only narrow intervals of the functions can be plotted. As can be seen, they are always positive and become more divergent as t increases. For $0 < \alpha < 1$, they possess a maximum, which moves as t is increased. For large values α and t, they tend to zero.

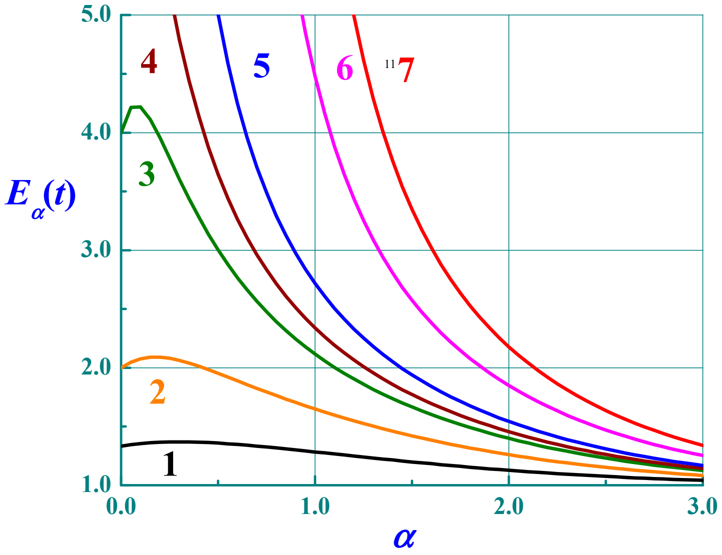

Figure 1. The Mittag-Leffler functions $E_\alpha(t)$ as a function of α at constant values of argument t. 1—0.25; 2—0.50; 3—0.75; 4—0.85; 5—1.0; 6—1.5; 7—2.0.

The first derivatives of the Mittag-Leffler with respect to α or $G(\alpha,t) = \partial E_\alpha(t)/\partial\alpha$ are plotted in Figure 2 Their behavior mirrors $E_\alpha(t)$, except that they are inverted as they are always negative.

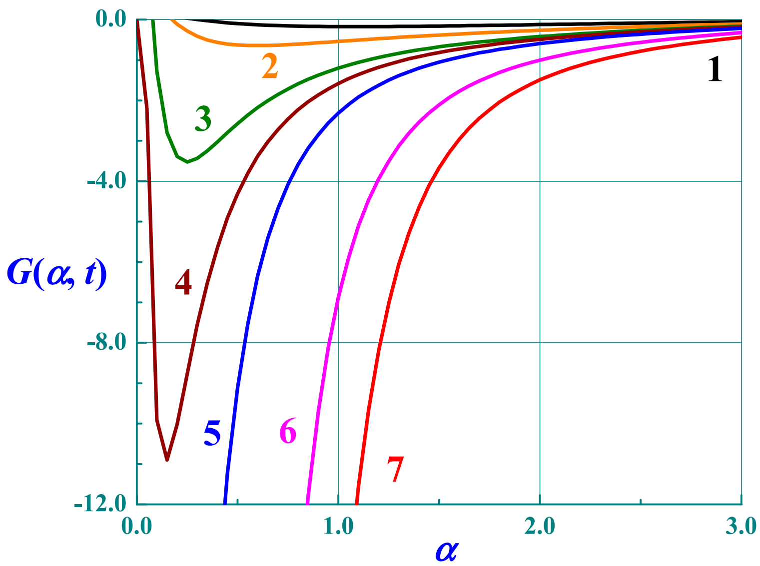

Figure 2. $G(\alpha,t)$—First derivatives of the Mittag-Leffler functions with respect to α plotted at constant values of t. 1—0.25; 2—0.50; 3—0.75; 4—0.85; 5—1.0; 6—1.5; 7—2.0.

The second derivatives with respect to α, $G'(\alpha,t) = \partial^2 E_\alpha(t)/\partial\alpha^2$ are presented in Figure 3. Their behavior resembles that of the Mittag-Leffler functions (Figure 1). However, for small values of t, they move from negative to positive values. The divergent behavior of $G'(\alpha,t)$ also applies for large values of t, but for increasing values of α and t, they tend to zero.

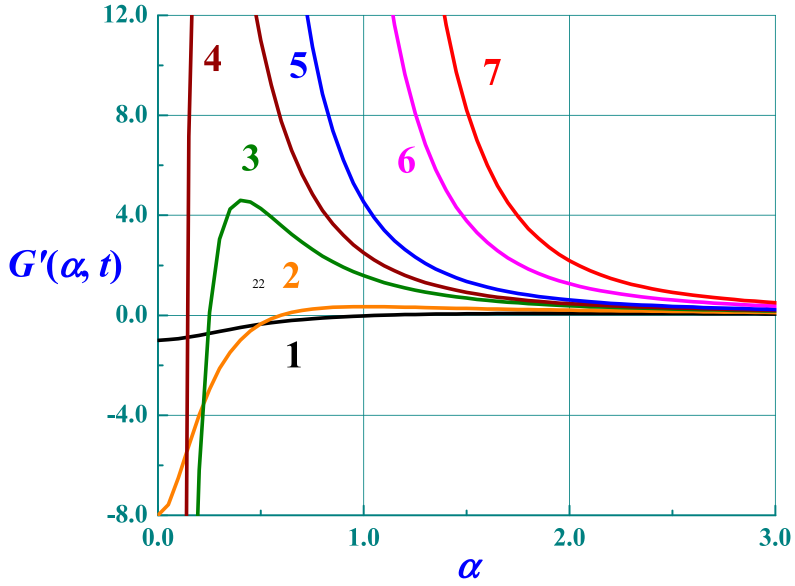

Figure 3. $G'(\alpha,t)$—Second derivatives of the Mittag-Leffler functions with respect to α plotted at constant values of t. 1—0.25; 2—0.50; 3—0.75; 4—0.85; 5—1.0; 6—1.5; 7—2.0.

5. Derivatives of the Mittag-Leffler Functions with Respect to Parameters α and β from Integral Representations

Derivatives with respect to α and β can be determined by direct differentiation of the integrands in integral representations of the Mittag-Leffler functions. Because no general expression exists for integral representations [25,27–33], it is possible to use only those that are valid for real positive and negative values of t and for restricted values of α and β.

For $0 < \alpha < 1$ and $t > 0$, these are

$$E_\alpha(t^\alpha) = \frac{e^t}{\alpha} - \frac{\sin(\pi\alpha)}{\pi} \int_o^\infty \frac{e^{-tu}\ u^{\alpha-1}}{u^{2\alpha} - 2u^\alpha \cos(\pi\alpha) + 1}\, du\,, \tag{118}$$

$$E_\alpha(-t^\alpha) = \frac{\sin(\pi\alpha)}{\pi} \int_o^\infty \frac{e^{-tu}\ u^{\alpha-1}}{u^{2\alpha} + 2u^\alpha \cos(\pi\alpha) + 1}\, du\,. \tag{119}$$

and

$$E_{\alpha,\beta}(t^\alpha) = \frac{e^t}{\alpha} - \frac{1}{\pi} \int_o^\infty \frac{e^{-tu}\ u^{\alpha-\beta}\{u^\alpha \sin(\pi\beta) + \sin[\pi(\alpha-\beta)]\}}{u^{2\alpha} - 2u^\alpha \cos(\pi\alpha) + 1}\, du\,, \tag{120}$$

$$E_{\alpha,\beta}(-\,t^\alpha) = \frac{1}{\pi} \int_o^\infty \frac{e^{-tu}\ u^{\alpha-\beta}\{u^\alpha \sin(\pi\beta) + \sin[\pi(\alpha-\beta)]\}}{u^{2\alpha} + 2u^\alpha \cos(\pi\alpha) + 1}\, du\,. \tag{121}$$

In (120) and (121), $0 < \beta < \alpha + 1$.

Direct differentiation of (118) and (119) with respect to α gives

$$\begin{aligned}\frac{\partial E_\alpha(t^\alpha)}{\partial\alpha} &= -\frac{e^t}{\alpha^2} - \cos(\pi\alpha) \int_o^\infty \frac{e^{-u\ t}\ u^{\alpha-1}}{u^{2\alpha} - 2u^\alpha \cos(\pi\alpha) + 1}\, du - \\ &\frac{\sin(\pi\alpha)}{\pi} \int_o^\infty \frac{e^{-u\ t}\ u^{\alpha-1}\ [(1-u^{2\alpha})\ln u - 2\pi u^\alpha \sin(\pi\alpha)]}{\left[u^{2\alpha} - 2u^\alpha \cos(\pi\alpha) + 1\right]^2}\, du\,,\end{aligned} \tag{122}$$

and

$$\begin{aligned}\frac{\partial E_\alpha(-t^\alpha)}{\partial \alpha} &= \cos(\pi\alpha) \int\limits_0^\infty \frac{e^{-u\ t}\ u^{\alpha-1}}{u^{2\alpha}+2u^\alpha \cos(\pi\alpha)+1}\, du+ \\ &\frac{\sin(\pi\alpha)}{\pi} \int\limits_0^\infty \frac{e^{-u\ t}\ u^{\alpha-1}[(1-u^{2\alpha})\ln u+2\pi u^\alpha \sin(\pi\alpha)]}{[u^{2\alpha}+2u^\alpha \cos(\pi\alpha)+1]^2}\, du\end{aligned} \tag{123}$$

where the first integrals in (122) and (123) can be written in terms of the Mittag-Leffler functions using (118) and (119).

In the same manner, one can obtain derivatives of the Mittag-Leffler functions $E_{\alpha,\beta}(\pm t^\alpha)$ with respect to α and β. Thus, we find that

$$\begin{aligned}\frac{\partial E_{\alpha,\beta}(t^\alpha)}{\partial \alpha} &= -\frac{e^t}{\alpha^2} - \int\limits_0^\infty \frac{e^{-tu}\ u^{\alpha-\beta}\cos[\pi(\alpha-\beta)]}{u^{2\alpha}-2u^\alpha \cos(\pi\alpha)+1}\, du- \\ &\frac{1}{\pi}\int\limits_0^\infty \frac{e^{-tu}\ u^{\alpha-\beta}\ln u\{2u^\alpha \sin(\pi\beta)+\sin[\pi(\alpha-\beta)]\}}{u^{2\alpha}-2u^\alpha \cos(\pi\alpha)+1}\, du+ \\ &\frac{2}{\pi}\int\limits_0^\infty \frac{e^{-tu}\ u^{2\alpha-\beta}\ln u\{u^\alpha \sin(\pi\beta)+\sin[\pi(\alpha-\beta)]\}[u^\alpha-\cos(\pi\alpha)]}{[u^{2\alpha}-2u^\alpha \cos(\pi\alpha)+1]^2}\, du \\ &+2\int\limits_0^\infty \frac{e^{-tu}\ u^{2\alpha-\beta}\sin(\pi\alpha)\{u^\alpha \sin(\pi\beta)+\sin[\pi(\alpha-\beta)]\}}{[u^{2\alpha}-2u^\alpha \cos(\pi\alpha)+1]^2}\, du\ ,\end{aligned} \tag{124}$$

and

$$\begin{aligned}\frac{\partial E_{\alpha,\beta}(t^\alpha)}{\partial \beta} &= \frac{1}{\pi}\int\limits_0^\infty \frac{e^{-tu}\ u^{\alpha-\beta}\ln u\{u^\alpha \sin(\pi\beta)+\sin[\pi(\alpha-\beta)]\}}{u^{2\alpha}-2u^\alpha \cos(\pi\alpha)+1}\, du- \\ &\int\limits_0^\infty \frac{e^{-tu}\ u^{\alpha-\beta}\{u^\alpha \cos(\pi\beta)-\cos[\pi(\alpha-\beta)]\}}{u^{2\alpha}-2u^\alpha \cos(\pi\alpha)+1}\, du\ .\end{aligned} \tag{125}$$

For the negative real axis, one obtains

$$\begin{aligned}t^{\beta-1}\frac{\partial E_{\alpha,\beta}(-t^\alpha)}{\partial \alpha} &= \int\limits_0^\infty \frac{e^{-tu}\ u^{\alpha-\beta}\cos[\pi(\alpha-\beta)]}{u^{2\alpha}+2u^\alpha \cos(\pi\alpha)+1}\, du- \\ &\frac{1}{\pi}\int\limits_0^\infty \frac{e^{-tu}\ u^{\alpha-\beta}\ln u\{2u^\alpha \sin(\pi\beta)+\sin[\pi(\alpha-\beta)]\}}{u^{2\alpha}+2u^\alpha \cos(\pi\alpha)+1}\, du- \\ &\frac{2}{\pi}\int\limits_0^\infty \frac{e^{-tu}\ u^{2\alpha-\beta}\ln u\{u^\alpha \sin(\pi\beta)+\sin[\pi(\alpha-\beta)]\}[u^\alpha+\cos(\pi\alpha)]}{[u^{2\alpha}+2u^\alpha \cos(\pi\alpha)+1]^2}\, du \\ &+2\int\limits_0^\infty \frac{e^{-tu}\ u^{2\alpha-\beta}\sin(\pi\alpha)\{u^\alpha \sin(\pi\beta)+\sin[\pi(\alpha-\beta)]\}}{[u^{2\alpha}+2u^\alpha \cos(\pi\alpha)+1]^2}\, du\ ,\end{aligned} \tag{126}$$

and

$$\begin{aligned}&t^{\beta-1}\ln t\ E_{\alpha,\beta}(-t^\alpha) + t^{\beta-1}\frac{\partial E_{\alpha,\beta}(-t^\alpha)}{\partial \beta} = \\ &-\frac{1}{\pi}\int\limits_0^\infty \frac{e^{-tu}\ u^{\alpha-\beta}\ln u\{u^\alpha \sin(\pi\beta)+\sin[\pi(\alpha-\beta)]\}}{u^{2\alpha}+2u^\alpha \cos(\pi\alpha)+1}\, du+ \\ &\int\limits_0^\infty \frac{e^{-tu}\ u^{\alpha-\beta}\{u^\alpha \cos(\pi\beta)-\cos[\pi(\alpha-\beta)]\}}{u^{2\alpha}+2u^\alpha \cos(\pi\alpha)+1}\, du\ .\end{aligned} \tag{127}$$

The infinite integrals in (122) to (127) are valid for restricted values of α and β. As can be expected, they represent the Laplace transforms and are similar to convolution integrals in Section 3.

6. Conclusions

For the first time, the parameters of the Mittag-Leffler functions in (1) and (2) have been treated as variables, and derivatives with respect to them have consequently been determined and discussed. Thus, it has been shown that operational calculus is a powerful tool for determining the properties of the Mittag-Leffler functions. Using the Laplace transform theory, new functional relations, together with infinite and finite series of the Mittag-Leffler functions, have also been calculated. Moreover, derivatives with respect to α and β have been found to be expressible in terms of convolution integrals. Direct differentiation of (1) and (2) yields infinite power series with quotients of digamma and gamma functions in their coefficients. For small integer values of α and β, closed forms are derived in terms of elementary and special functions. The Mittag-Leffler functions, together with their first and second derivatives, are graphed as functions of α and t. On a final note, it should be mentioned that Biyajima et al. [30,31] have used (102) in their new blackbody radiation law, but not the closed form given here.

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