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A Rational Approximation for the Complete Elliptic Integral of the First Kind †

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Abstract: Let $\mathcal{K}(r)$ be the complete elliptic integral of the first kind. We present an accurate rational lower approximation for $\mathcal{K}(r)$. More precisely, we establish the inequality $\frac{2}{\pi}\mathcal{K}(r) > \frac{5(r')^2+126r'+61}{61(r')^2+110r'+21}$ for $r \in (0,1)$, where $r' = \sqrt{1-r^2}$. The lower bound is sharp.

Keywords: complete integrals of the first kind; arithmetic-geometric mean; rational approximation

MSC: Primary 33E05, 41A20; Secondary 26E60, 40A99

1. Introduction

The complete elliptic integral of the first kind $\mathcal{K}(r)$ is defined on $(0,1)$ by

$$\mathcal{K}(r) = \int_0^{\pi/2} \frac{1}{\sqrt{1-r^2\sin^2 t}}dt,$$

which can be also represented by the Gaussian hypergeometric function

$$\frac{2}{\pi}\mathcal{K}(r) = F\left(\frac{1}{2},\frac{1}{2};1;r^2\right) = \sum_{n=0}^{\infty}\frac{(1/2)_n^2}{(n!)^2}r^{2n}, \tag{1}$$

where $(a)_0 = 1$ for $a \neq 0$, $(a)_n = a(a+1)\cdots(a+n-1) = \Gamma(a+n)/\Gamma(a)$ is the shifted factorial function and $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ $(x > 0)$ is the gamma function [1,2]. The famous Landen identities [3], p. 507 show that

$$\mathcal{K}\left(\frac{2\sqrt{r}}{1+r}\right) = (1+r)\mathcal{K}(r) \quad\text{and}\quad \mathcal{K}\left(\frac{1-r}{1+r}\right) = \frac{1+r}{2}\mathcal{K}(r') \tag{2}$$

for all $r \in (0,1)$. Moreover, an asymptotic formula for $\mathcal{K}(r)$ as $r \to 1^-$ is given by

$$\mathcal{K}(r) \sim \ln\frac{4}{r'} \quad \text{as } r \to 1^- \tag{3}$$

(see [3], p. 299), where and in what follows $r' = \sqrt{1-r^2}$.

As we all know, analytic inequality plays a very important role in mathematics and in other parts of science (see for example, [4–6]) since it offers certain computable and accurate bounds for given complicated functions. As far as the complete elliptic integral of the first kind is concerned, there are at least two kinds of bounds for $\mathcal{K}(r)$. The first kind of bounds for $\mathcal{K}(r)$ are in terms of the inverse hyperbolic tangent function, that is,

$$\operatorname{artanh} r = \frac{1}{2}\ln\frac{1+r}{1-r}.$$

In 1992, Anderson, Vamanamurthy and Vuorinen [7] mentioned such bound, and they presented the double inequality

$$\left(\frac{\operatorname{artanh} r}{r}\right)^{1/2} < \frac{2}{\pi}\mathcal{K}(r) < \frac{\operatorname{artanh} r}{r} \tag{4}$$

for $r \in (0,1)$. This was improved by Alzer and Qiu in [8] (Theorem 19) as

$$\left(\frac{\operatorname{artanh} r}{r}\right)^{\alpha_1} < \frac{2}{\pi}\mathcal{K}(r) < \left(\frac{\operatorname{artanh} r}{r}\right)^{\beta_1} \tag{5}$$

for $r \in (0,1)$ with the best weights $\alpha_1 = 3/4$ and $\beta_1 = 1$. In the same paper [8] (Theorem 19), the authors gave another better double inequality for $\mathcal{K}(r)$:

$$1-\alpha_2+\alpha_2\frac{\operatorname{artanh} r}{r} < \frac{2}{\pi}\mathcal{K}(r) < 1-\beta_2+\beta_2\frac{\operatorname{artanh} r}{r} \tag{6}$$

for $r \in (0,1)$ with the best weights $\alpha_2 = 2/\pi$ and $\beta_2 = 3/4$. Recently, Yang, Qian, Chu and Zhang [9] (Theorem 3.2) obtain more accurate bounds:

$$2\frac{(3\pi-7)r'+1}{(5\pi-12)r'+\pi}\frac{\operatorname{artanh} r}{r} < \frac{2}{\pi}\mathcal{K}(r) < 3\frac{17r'+23}{31r'+89}\frac{\operatorname{artanh} r}{r} \tag{7}$$

for $r \in (0,1)$.

The second kind of bounds for $\mathcal{K}(r)$ are related to the asymptotic Equation (3). In 1985, Carlson and Guatafson [10] showed that

$$1 < \frac{\mathcal{K}(r)}{\ln(4/r')} < \frac{4}{3+r^2} \tag{8}$$

for $r \in (0,1)$. The first inequality of Equation (8) was improved in [7] as

$$\frac{9}{8.5+r^2} < \frac{\mathcal{K}(r)}{\ln(4/r')} \tag{9}$$

for $r \in (0,1)$. Qiu and Vamanamurthy [11] and Alzer [12] proved the double inequality

$$1+\left(\frac{\pi}{\ln 16}-1\right)(r')^2 < \frac{\mathcal{K}(r)}{\ln(4/r')} < 1+\frac{1}{4}(r')^2 \tag{10}$$

for $r \in (0,1)$, where the constant factors $1/4$ and $\pi/\ln 16 - 1$ in Equation (10) are the best possible. The recent advance on such inequalities can be found in [13–15].

The aim of this paper is to provide the third kind of bounds (rational bounds) for $\mathcal{K}(r)$. More precisely, we will prove the following theorem.

Theorem 1. *Let $q \geq 2$. The inequality*

$$\frac{2}{\pi}\mathcal{K}(r) > \frac{1}{8}\frac{(16-3q)(r')^2+(96+22q)r'+(61q-112)}{10(r')^2+(7q-4)r'+(3q-6)} = L_q(r') \tag{11}$$

holds for all $r \in (0,1)$ if and only if $q \geq q_0 = 192/61$. In particular, we have

$$\frac{2}{\pi}\mathcal{K}(r) > \frac{5(r')^2+126r'+61}{61(r')^2+110r'+21} = L_{q_0}(r')$$

for $r \in (0,1)$.

2. Proof of Theorem 1

To prove Theorem 1, we need a sign rule for a type of special power series and polynomials, which has been proven in [9,15,16] by Yang et al. It should be noted that this sign rule plays an important role in the study for certain special functions, see, for example, [17–21].

Lemma 1. *Let $\{a_k\}_{k=0}^{\infty}$ be a nonnegative real sequence with $a_m > 0$ and $\sum_{k=m+1}^{\infty} a_k > 0$ and let*

$$S(t) = -\sum_{k=0}^{m} a_k t^k + \sum_{k=m+1}^{\infty} a_k t^k$$

be a convergent power series on the interval $(0,r)$ $(r>0)$. (i) If $S(r^-) \leq 0$, then $S(t) < 0$ for all $t \in (0,r)$. (ii) If $S(r^-) > 0$, then there is a unique $t_0 \in (0,r)$ such that $S(t) < 0$ for $t \in (0,t_0)$ and $S(t) > 0$ for $t \in (t_0, r)$.

Remark 1. *If $r = \infty$, then Lemma 1 is reduced to [22] (Lemma 6.3). Furthermore, if $a_k = 0$ for $k \geq k_2 + 1$, then Lemma 1 yields [16] (Lemma 7).*

Remark 2. *It follows from Lemma 1 that if there is a $t_1 \in (0,r)$ such that $S(t_1) < 0$, then we have $S(t) < 0$ for all $t \in (0,t_1)$; if there is a $t_2 \in (0,r)$ such that $S(t_2) > 0$, then we have $S(t) > 0$ for all $t \in (t_2, r)$.*

For convenience, we introduce a notation:

$$W_n = \frac{(1/2)_n}{n!} = \frac{\Gamma(n+1/2)}{\Gamma(1/2)\Gamma(n+1)}. \tag{12}$$

Then

$$\begin{aligned}\frac{(-1/2)_n}{n!} &= \frac{(-1/2)}{n-1/2}\frac{\Gamma(n+1/2)}{n!\Gamma(1/2)} = -\frac{1}{2n-1}W_n,\\ \frac{(-3/2)_n}{n!} &= \frac{(-3/2)(-1/2)}{(n-3/2)(n-1/2)}\frac{\Gamma(n+1/2)}{n!\Gamma(1/2)} = \frac{3W_n}{(2n-1)(2n-3)}.\end{aligned}$$

Clearly, W_n satisfies the recurrence relation:

$$W_{n-1} = \frac{2n}{2n-1}W_n. \tag{13}$$

Using this notation, the Gaussian hypergeometric Equation (1) can be written as

$$\frac{2}{\pi}\mathcal{K}(r) = F\left(\frac{1}{2},\frac{1}{2};1;r^2\right) = \sum_{n=0}^{\infty} W_n^2 r^{2n}. \tag{14}$$

In addition, the functions $r' = \sqrt{1-r^2}$ and $(r')^3 = (1-r^2)^{3/2}$ can be expanded in power sereis

$$r' = \sqrt{1-r^2} = \sum_{n=0}^{\infty} \frac{(-1/2)_n}{n!} r^{2n} = -\sum_{n=0}^{\infty} \frac{W_n}{2n-1} r^{2n}, \tag{15}$$

$$(r')^3 = \left(1-r^2\right)^{3/2} = \sum_{n=0}^{\infty} \frac{(-3/2)_n}{n!} r^{2n} = \sum_{n=0}^{\infty} \frac{3W_n}{(2n-1)(2n-3)} r^{2n}. \tag{16}$$

We are now in a position to prove our main result.

Proof. (i) Necessity. Expanding in power series yields

$$\begin{aligned} \frac{2}{\pi}\mathcal{K}(r) &= F\left(\frac{1}{2},\frac{1}{2};1;r^2\right) = 1 + \frac{1}{4}r^2 + \frac{9}{64}r^4 + \frac{25}{256}r^6 + \frac{1225}{16384}r^8 + O\left(r^{10}\right) \\ L_q(r') &= 1 + \frac{1}{4}r^2 + \frac{9}{64}r^4 + \frac{25}{256}r^6 + \frac{379q+12}{5120q}r^8 + O\left(r^{10}\right), \end{aligned}$$

which gives

$$\frac{2}{\pi}\mathcal{K}(r) - L_q(r') = \frac{61q-192}{81920q} r^8 + O\left(r^{10}\right). \tag{17}$$

If the inequality seen in Equation (11) holds for all $r \in (0,1)$, then

$$\lim_{r\to 0^+} \frac{(2/\pi)\mathcal{K}(r) - L_q(r')}{r^8} = \frac{61q-192}{81920q} \geq 0,$$

which implies $q \geq 192/61 = q_0$. This proves the necessity.

(ii) Sufficiency. Differentiation yields

$$\frac{\partial}{\partial q} L_q(r') = -\frac{15}{4} \frac{(1-r')^4}{\left[10(r')^2 + (7q-4)r' + (3q-6)\right]^2} < 0$$

for $r \in (0,1)$. That is, the lower bound $L_q(r')$ is decreasing with respect to q on $[2,\infty)$. Thus, to prove the sufficiency, it is enough to prove the inequality in Equation (11) holds for $r \in (0,1)$ and $q = q_0$.

Let

$$f_0(r) = \left(61(r')^2 - 110r' + 21\right)\left(61(r')^2 + 110r' + 21\right)\left(\frac{2}{\pi}\mathcal{K}(r) - L_{q_0}(r')\right).$$

Obviously, $f_0(r)$ has at least one zero on $(0,1)$ which satisfies

$$61(r')^2 - 110r' + 21 = 0.$$

Solving the equation yields

$$r_0 = \sqrt{\frac{440\sqrt{109} - 1048}{3721}} = 0.97617\ldots. \tag{18}$$

On the other hand, using the power series representations in Equations (14)–(16), $f_0(r)$ can be expressed as

$$\begin{aligned} f_0(r) &= \left[3721(r')^4 - 9538(r')^2 + 441\right]\frac{2}{\pi}\mathcal{K}(r) \\ &\quad - \left[305(r')^4 + 7136(r')^3 - 10034(r')^2 - 4064r' + 1281\right] \end{aligned}$$

$$= \left(3721r^4 + 2096r^2 - 5376\right) \sum_{n=0}^{\infty} W_n^2 r^{2n} - \left[305r^4 + 9424r^2 - 8448 \right.$$
$$\left. +7136 \sum_{n=0}^{\infty} \frac{3W_n}{(2n-1)(2n-3)} r^{2n} + 4064 \sum_{n=0}^{\infty} \frac{W_n}{2n-1} r^{2n} \right],$$

which, by the recurrence relations in Equation (13), is simplified to

$$= \sum_{n=3}^{\infty} \frac{16\left(441n^4 + 7774n^3 - 21131n^2 + 16128n - 3024\right)}{(2n-1)^2 (2n-3)^2} W_n^2 r^{2n}$$
$$- \sum_{n=3}^{\infty} \frac{64(127n+144)}{(2n-1)(2n-3)} W_n r^{2n} = \sum_{n=3}^{\infty} a_n r^{2n},$$

where

$$a_n = \frac{64(127n+144) W_n}{(2n-1)(2n-3)} a_n^*, \tag{19}$$

$$a_n^* = \frac{441n^4 + 7774n^3 - 21131n^2 + 16128n - 3024}{4(127n+144)(2n-1)(2n-3)} W_n - 1. \tag{20}$$

We now check that there is an integer $n_0 > 3$ such that $a_n^* \leq 0$ for $3 \leq n \leq n_0$ and $a_n^* > 0$ for $n > n_0$. It is easy to verify that $a_3^* = a_4^* = 0$ and

$$\frac{a_{n+1}^* - a_n^*}{W_n} = \frac{1}{8} \frac{(n-3)\, p_4(n)}{(n+1)(2n-1)(2n-3)(127n+144)(127n+271)},$$

where

$$p_4(n) = 56007n^4 - 740779n^3 + 1451470n^2 + 1793252n - 519264. \tag{21}$$

Since

$$p_4(m+4) = 56007m^4 + 155333m^3 - 2061206m^2 - 7814588m - 3194800$$

and $p_4(6+4) = -18148744 < 0$, $p_4(7+4) = 28856016 > 0$, by Lemma 1 we see that $p_4(n) < 0$ for $5 \leq n \leq 10$ and $p_4(n) > 0$ for $n \geq 11$. This shows that the sequence $\{a_n^*\}_{n \geq 5}$ is decreasing for $5 \leq n \leq 10$ and increasing for $n \geq 11$. Due to $a_5^* = -245/199424 < 0$ and $a_\infty^* = \infty$, there is an $n_1 > 5$ such that $a_n^* < 0$ for $5 \leq n \leq n_1$ and $a_n^* > 0$ for $n \geq n_1 + 1$.

Thus, as a special power series, those coefficients of $f_0(r)$ satisfy the conditions of Lemma 1, and clearly, $f_0(1^-) = \infty$. From Lemma 1, it follows that there is a unique $r_0^* \in (0,1)$ such that $f_0(r) < 0$ for $r \in (0, r_0^*)$ and $f_0(r) > 0$ for $r \in (r_0^*, 1)$, that is to say, $f_0(r)$ has a unique zero r_0^*, and therefore, $r_0^* = r_0$, where r_0 is given by Equation (18).

It is readily checked that $61(r')^2 - 110r' + 21 < 0$ for $r \in (0, r_0)$ and $61(r')^2 - 110r' + 21 > 0$ for $r \in (r_0, 1)$, which yields

$$\frac{2}{\pi}\mathcal{K}(r) - L_{q_0}(r') = \frac{f_0(r)}{\left(61(r')^2 - 110r' + 21\right)\left(61(r')^2 + 110r' + 21\right)} > 0$$

for $r \in (0,1)$ with $r \neq r_0$. The continuity at $r = r_0$ of the functions $\mathcal{K}(r)$ and $L_{q_0}(r)$ shows that the inequality in Equation (11) also holds for $r = r_0$.

This completes the proof. $\square$

3. Remarks

In this section, we give some remarks on our result.

Remark 3. *Using the monotonicity of the bound $L_q(r')$, we can obtain a chain of inequalities. For example, taking $q = q_0 = 192/61$, $16/3$ and $q \to \infty$, we obtain*

$$\begin{aligned} \frac{2}{\pi}\mathcal{K}(r) &> \frac{5(r')^2 + 126r' + 61}{61(r')^2 + 110r' + 21} \\ &> 8\frac{r'+1}{(r'+3)(3r'+1)} > \frac{1}{8}\frac{61 + 22r' - 3(r')^2}{7r'+3} \end{aligned}$$

for $r \in (0,1)$.

Remark 4. *Let $E_k(r)$ ($k = 1,2,3$) denote the lower bound in Equation (5), upper bound in Equation (6) and upper bound in Equation (7), respectively. Power series expansions give*

$$\begin{aligned} \frac{2}{\pi}\mathcal{K}(r) - E_1(r) &= \frac{1}{960}r^4 + O\left(r^6\right) \\ \frac{2}{\pi}\mathcal{K}(r) - E_2(r) &= -\frac{3}{320}r^4 + O\left(r^6\right), \\ \frac{2}{\pi}\mathcal{K}(r) - E_3(r) &= -\frac{437}{537600}r^6 + O\left(r^8\right), \\ \frac{2}{\pi}\mathcal{K}(r) - L_{q_0}(r') &= \frac{35}{786432}r^{10} + O\left(r^{12}\right). \end{aligned}$$

Since

$$\lim_{r\to 0}\frac{\left|(2/\pi)\mathcal{K}(r) - L_{q_0}(r')\right|}{\left|(2/\pi)\mathcal{K}(r) - E_k(r)\right|} = 0 < 1, \tag{22}$$

$$\lim_{r\to 1}\frac{\left|(2/\pi)\mathcal{K}(r) - L_{q_0}(r')\right|}{\left|(2/\pi)\mathcal{K}(r) - E_k(r)\right|} = \infty > 1, \tag{23}$$

for $k = 1,2,3$, the absolute errors approximating for $\mathcal{K}(r)$ by $E_k(r)$ ($k = 1,2,3$) are greater (less) than $L_{q_0}(r')$ near $r = 0$ ($r = 1$). These show that the bound $L_{q_0}(r')$ for $\mathcal{K}(r)$ is much more accurate near $r = 0$ than those given in Equations (5)–(7). It is weaker than the latter three bounds near $r = 1$.

Remark 5. *Let $E_k(r)$ ($k = 4,5,6$) denote the upper bound in (8), lower bound in (9) and lower bound in (10), respectively. Similarly, the relations in Equations (22) and (23) also hold for $k = 4,5,6$. We conclude that the bound $L_{q_0}(r')$ for $\mathcal{K}(r)$ is much more accurate near $r = 0$ than $E_k(r)$ ($k = 4,5,6$), but weaker than $E_k(r)$ ($k = 4,5,6$) near $r = 1$.*

Let $a, b > 0$ with $a \neq b$. The Gaussian arithmetic-geometric mean (AGM) is defined by

$$AGM(a,b) = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n,$$

where $a_0 = a, b_0 = b$, and for $n \in \mathbb{N}$,

$$a_{n+1} = A(a_n, b_n) = \frac{a_n + b_n}{2}, b_{n+1} = G(a_n, b_n) = \sqrt{a_n b_n}. \tag{24}$$

An amazing connection between $AGM(a,b)$ and complete elliptic integral of the first kind $\mathcal{K}(r)$ is given by Gauss' formula

$$AGM(1, r') = \frac{\pi}{2\mathcal{K}(r)} \tag{25}$$

(see [23]).

Remark 6. *Using the second Landen identity in Equation (2) and the Gauss identity in Equation (25), some inequalities for the complete elliptic integral of the first kind $\mathcal{K}(r)$ can be transformed into corresponding inequalities for $AGM(1,r)$. For example, let $r = (1-x)/(1+x)$. Then*

$$r' = \frac{2\sqrt{x}}{1+x}, \quad \frac{\operatorname{artanh} r}{r} = \frac{1}{2}\frac{1+x}{L(1,x)}, \quad \mathcal{K}(r) = \frac{1+x}{2}\mathcal{K}(x'),$$

where $x' = \sqrt{1-x^2}$ and $L(1,x)$ is the logarithmic mean of 1 and x. Thus, the double inequality for $\mathcal{K}(r)$ in Equation (6) is changed into

$$1 - \frac{2}{\pi} + \frac{2}{\pi}\frac{1}{2}\frac{1+x}{L(1,x)} < \frac{2}{\pi}\frac{1+x}{2}\mathcal{K}(x') < \frac{1}{4} + \frac{3}{4}\frac{1}{2}\frac{1+x}{L(1,x)},$$

which, due to Equation (25), gives

$$\left[\frac{1/4}{A(1,x)} + \frac{3/4}{L(1,x)}\right]^{-1} < AGM(1,x) < \left[\frac{1-2/\pi}{A(1,x)} + \frac{2/\pi}{L(1,x)}\right]^{-1}. \tag{26}$$

Remark 7. *Likewise, the double inequality in Equation (7) can be equivalently changed into*

$$\frac{1}{3}\frac{89A+31G}{23A+17G}L < AGM < \frac{1}{2}\frac{\pi A + (5\pi-12)G}{A+(3\pi-7)G}L \tag{27}$$

for $x \in (0,1)$, where $AGM \equiv AGM(1,x)$, $A \equiv A(1,x)$, $G \equiv G(1,x)$, $L \equiv L(1,x)$.

More inequalities for AGM can be seen in [8,24–27].

In the same way, our Theorem 1 also implies an sharp upper bound for AGM.

Theorem 2. *Let $q \geq 2$. The inequality*

$$AGM < 8A\frac{10G^2 + (7q-4)AG + (3q-6)A^2}{(16-3q)G^2 + (96+22q)AG + (61q-112)A^2}$$

holds for all $x \in (0,1)$ if and only if $q \geq q_0 = 192/61$, where $AGM \equiv AGM(1,x)$, $A \equiv A(1,x)$, $G \equiv G(1,x)$. In particular, we have

$$AGM < A\frac{21A^2 + 110AG + 61G^2}{61A^2 + 126AG + 5G^2} \tag{28}$$

Remark 8. *Correspondingly, the upper bound for AGM given in Equation (28) has the same accuracy near $x = 1$. Numeric computations show that this new upper bound is not comparable with ones given in Equations (26) and (27), but is better than them near $x = 1$.*

Remark 9. *It was proven in [26] (Theorem 1) that*

$$AGM(1,r) < S(1,r) = \frac{1}{5}\frac{1-r^{5/4}}{1-r^{1/4}}$$

for $r \in (0,1)$. Since

$$A\frac{21A^2 + 110AG + 61G^2}{61A^2 + 126AG + 5G^2} - S(1,r) = -\frac{1}{40}\frac{\left(\sqrt[4]{r}-1\right)^4 D\left(\sqrt[4]{r}\right)}{61A^2 + 126AG + 5G^2} < 0,$$

where

$$D(x) = 17x^8 + 190x^7 + 184x^6 + 290x^5 + 174x^4 + 290x^3 + 184x^2 + 190x + 17,$$

The upper bound for AGM given in Equation (28) is superior to $S(1, r)$.

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