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On Bipolar Fuzzy Gradation of Openness

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Abstract: The concept of bipolar fuzziness is of relatively recent origin where in addition to the presence of a property, which is done in fuzzy theory, the presence of its counter-property is also taken into consideration. This seems to be much natural and realistic. In this paper, an attempt has been made to incorporate this bipolar fuzziness in topological perspective. This is done by introducing a notion of bipolar gradation of openness and to redefine the bipolar fuzzy topology. Furthermore, a notion of bipolar gradation preserving map is given. A concept of bipolar fuzzy closure operator is also introduced and its characteristic properties are studied. A decomposition theorem involving our bipolar gradation of openness and Chang type bipolar fuzzy topology is established. Finally, some categorical results of bipolar fuzzy topology (both Chang type and in our sense) are proved.

Keywords: bipolar gradation of openness; bipolar gradation of closedness; bipolar fuzzy topology; bipolar gradation preserving map

1. Introduction

From the very beginning of the invention of fuzzy sets by Zadeh [1], many authors have contributed towards fuzzifying the topological concept. Fuzzy topology was first introduced by Chang [2] in 1968. Since then, fuzzy topology had drawn the attention of many mathematicians and a foundation of systematic research began. Fuzzy topology, L-fuzzy topology, interval-valued fuzzy topology, and intuitionistic fuzzy topology ([3–6]) laid the foundation of new topological structures on some non-crisp sets. The lack of fuzziness in fuzzy topology was still a drawback to some extent. The Chang fuzzy topology is a crisp family of fuzzy subsets satisfying the properties of topology over some domain. However, a crisp collection never looked good for a proper justification for fuzzifying the topological concept. This absence of fuzziness in Chang fuzzy topology was pointed out by Sostak [7], Ying [8], Chattopadhyay et al. [9], Gregoroi [10], and Mondal [11]. Chattopadhyay et al. [12] introduced a notion of gradation, where every fuzzy set was associated with some grade of openness or closedness. With the concept of gradation of openness, they further studied fuzzy closure operator, gradation preserving maps, fuzzy compactness, and fuzzy connectedness ([9,12,13]). This concept of gradation has been used widely instead of direct fuzzification of some mathematical structures mainly in the field of topology by many researchers. Samanta [14] and Ghanim et al. [15] introduced gradation of uniformity and gradation of proximity, Thakur et al. [16] studied gradation of continuity, and Mondal et al. ([11,17–19]) introduced intuitionistic gradation and L-fuzzy gradation.

Bipolar fuzzy set ($\mathcal{BFS}$), a generalized concept of fuzzy set, has already found its way in the field of research as bipolarity in decisions often occurs in many practical problems. Unlike fuzzy set, the range of membership lies in $[-1,1]$, where the range of membership $(0,1]$ for some element is an indication of the satisfaction of the property, whereas the range of membership $[-1,0)$ is an indication of the satisfaction of the counter-property. Some basic operations on bipolar fuzzy sets can be found in ([20,21]). Applications of bipolar fuzzy sets can be found in ([22–24]). Bipolar fuzzy topology ($\mathcal{BFT}$) studied by Azhagappan et al. [25] and Kim et al [26] are of Chang type. For a universal set X, $\mathcal{BF}(X)$ is the collection of all bipolar fuzzy sets of X and a bipolar fuzzy topology τ on X is a collection from $\mathcal{BF}(X)$ containing the null bipolar fuzzy set, absolute bipolar fuzzy set, finite intersection, and arbitrary union. Thus, for a bipolar fuzzy topological space ($\mathcal{BFTS}$) (X,τ), $\tau^+ = \{\mu^+ \in I^X;\ \mu \in \tau\}$ and $\tau^- = \{-\mu^- \in I^X;\ \mu \in \tau\}$ are fuzzy topologies of Chang type. In addition, (X, τ^+, τ^-) is a fuzzy bitopological space deduced from the bipolar fuzzy topology τ. Therefore, the study on bipolar fuzzy topology looks quite logical in the context of fuzzy topology as fuzzy topology can be considered as a special case of $\mathcal{BFT}$ and a $\mathcal{BFT}$ induces a special type of fuzzy bitopology. However the definition of a $\mathcal{BFTS}$ introduced in [25] looks similar to the definition of Chang fuzzy topological space where the bipolar fuzzy open sets are considered as a crisp collection over some universe. This looks to be a drawback in proper bifuzzification of the topological concept. Fuzzy set is a particular case of bipolar fuzzy set where the counter-property is absent i.e., counter-property takes the value 0 only—for example, "sweet and sour", "good and bad", "beauty and ugly", "matter and anti-matter", etc. By incorporating a bipolar gradation in the openness and closedness, we tried to rectify the previous drawbacks in bifuzzification of topological concept and thus introduce a modified definition of bipolar fuzzy topological space.

In this paper, we introduce a definition of bipolar gradation of openness of bipolar fuzzy subsets of X and give a new definition of bipolar fuzzy topological spaces. In our definition of bipolar fuzzy topology, each bipolar fuzzy subset is associated with a definite *bipolar gradation of openness* and non-openness. We have shown that the set of all bipolar fuzzy topologies in our sense form a complete lattice with an order relation defined in Definition 9. We also introduce *bipolar gradation preserving maps* and a decomposition theorem involving bipolar fuzzy topology in our sense and the same in Chang's sense is proved. Bipolar fuzzy closure operator is introduced and some of their characteristic properties are dealt with. Lastly, it is shown that the bipolar fuzzy topologies in our sense and the bipolar gradation preserving mapping is a topological category.

2. Preliminaries

Throughout the paper, the fuzzy topological space ($\mathcal{FTS}$) is considered in Chang's sense. Gradation of openness, gradation of closedness, and gradation preserving map will be called $\mathcal{GO}$, $\mathcal{GC}$, and $\mathcal{GP}$ map, respectively. Some straightforward proofs are omitted and some preliminary results related to this work are not discussed, which can be found in ([2,25–27]).

Definition 1 ([27])**.** *Let X be a non-empty set. Then, a pair $\mu = (\mu^-, \mu^+)$ is called a $\mathcal{BFS}$ in X, where $\mu^- : X \to [-1,0]$ and $\mu^+ : X \to [0,1]$ are two mappings. The positive membership function $\mu^+(x)$ denotes the satisfaction degree of an element x corresponding to the $\mathcal{BFS}$ μ and the negative membership function $\mu^-(x)$ denotes the satisfaction degree of an element x to the counter-property corresponding to the $\mathcal{BFS}$ μ. In particular, a $\mathcal{BFS}$ is said to be a null-$\mathcal{BFS}$ [25], denoted by $\tilde{0}$, where $\tilde{0} = (0^-, 0^+)$ and $0^-(x) = 0$, $0^+(x) = 0$, for all $x \in X$. A $\mathcal{BFS}$ is said to be an absolute $\mathcal{BFS}$ [25], denoted by $\tilde{1}$, where $\tilde{1} = (1^-, 1^+)$ and $1^-(x) = -1$, $1^+(x) = 1$, for all $x \in X$.*

Definition 2 ([27])**.** *Let X be a non-empty set and $\mu, \lambda \in \mathcal{BF}(X)$.*

(1) μ is said to be a subset of λ, denoted by $\mu \subset \lambda$, if, for each $x \in X$, $\mu^+(x) \leq \lambda^+(x)$ and $\mu^-(x) \geq \lambda^-(x)$.

(2) The complement of μ, denoted by $\mu^c = ((\mu^c)^-, (\mu)^c)^+)$, is a bipolar fuzzy set in X, defined as for each $x \in X$, $\mu^c(x) = (-1 - \mu^-(x), 1 - \mu^+(x))$.

(3) *The intersection of μ and λ, denoted by $\mu \cap \lambda$, is a bipolar fuzzy set in X, defined as for each $x \in X$, $(\mu \cap \lambda)(x) = (\mu^-(x) \vee \lambda^-(x), \mu^+(x) \wedge \lambda^+(x))$.*

(4) *The union of μ and λ, denoted by $\mu \cup \lambda$ is a bipolar fuzzy set in X, defined for each $x \in X$, $(\mu \cup \lambda)(x) = (\mu^-(x) \wedge \lambda^-(x), \mu^+(x) \vee \lambda^+(x))$.*

Definition 3 ([25])**.** *Let X be a non-empty set. A collection of bipolar fuzzy subsets τ of $\mathcal{BF}(X)$ is said to be a $\mathcal{BFT}$ on X, if it satisfies the following conditions:*

(1) $\tilde{0}, \tilde{1} \in \tau$,

(2) *if* $\mu, \lambda \in \tau$, *then* $\mu \cap \lambda \in \tau$,

(3) *if* $\mu_i \in \tau$, *for each* $i \in \Delta$, *then* $\underset{i \in \Delta}{\cup} \mu_i \in \tau$.

Definition 4 ([26])**.** *Let (X, τ_1) and (Y, τ_2) be two bipolar fuzzy topological spaces. Then, a mapping $f : (X, \tau_1) \to (Y, \tau_2)$ is said to be continuous, if $f^{-1}(V) \in \tau_1$ for each $V \in \tau_2$.*

Definition 5 ([12])**.** *Let X be a non-empty set and $\tau : I^X \to [0,1]$ be a mapping. Then, τ is said to be a $\mathcal{GO}$ on X, if it satisfies the following conditions:*

(1) $\tau(\tilde{0}) = \tau(\tilde{1}) = 1$,

(2) $\tau(\mu_1 \cap \mu_2) \geq \tau(\mu_1) \wedge \tau(\mu_2)$,

(3) $\tau(\underset{i \in \Delta}{\cup} \mu_i) \geq \underset{i \in \Delta}{\wedge} \tau(\mu_i)$.

Definition 6 ([12])**.** *Let X be a non-empty set and $\mathfrak{F} : I^X \to [0,1]$ be a mapping. Then, $\mathfrak{F}$ is said to be a $\mathcal{GC}$ on X, if it satisfies the following conditions:*

(1) $\mathfrak{F}(\tilde{0}) = \mathfrak{F}(\tilde{1}) = 1$,

(2) $\mathfrak{F}(\mu_1 \cup \mu_2) \geq \mathfrak{F}(\mu_1) \wedge \mathfrak{F}(\mu_2)$,

(3) $\mathfrak{F}(\underset{i \in \Delta}{\cap} \mu_i) \geq \underset{i \in \Delta}{\wedge} \mathfrak{F}(\mu_i)$.

Remark 1 ([12])**.** The set of all $\mathcal{FTS}$ on X along with the order relation "$\leq$" forms a complete lattice.

Definition 7 ([12])**.** *Let (X, τ) and (Y, τ') be two $\mathcal{FTS}$ and $f : X \to Y$ be a mapping. Then, f is said to be a $\mathcal{GP}$ map if for each $\mu \in I^Y$, $\tau'(\mu) \leq \tau(f^{-1}(\mu))$*

Definition 8 ([13])**.** *Let $(X, \mathfrak{F})$ be a $\mathcal{FTS}$ with $\mathfrak{F}$ being a $\mathcal{GC}$ on X. For each $r \in [0,1]$ and for each $\lambda \in I^X$, the fuzzy closure of λ is defined as follows:*

$$cl(\lambda, r) = \cap\{\mu \in I^X \ : \ \mu \supseteq \lambda, \ \mathfrak{F}(\mu) \geq r\}.$$

3. Bipolar Gradation of Openness

In this section, we define bipolar gradation of openness (bipolar $\mathcal{GO}$), bipolar gradation of closedness (bipolar $\mathcal{GC}$) and prove some subsequent results.

Definition 9. *For any $(r_1, s_1), (r_2, s_2) \in [-1,0] \times [0,1]$, and for $\{(r_i, s_i),\ i \in \Delta\}$, define*

(1) $(r_1, s_1) \succeq (r_2, s_2)$ *if* $r_1 \leq r_2$ *and* $s_1 \geq s_2$,

(2) $(r_1, s_1) \succ (r_2, s_2)$ *if* $r_1 < r_2$ *and* $s_1 > s_2$,

(3) $(r_1, s_1) \prec (r_2, s_2)$ *if* $r_1 > r_2$ *and* $s_1 < s_2$,

(4) $\underset{i \in \Delta}{\wedge} (r_i, s_i) = (\underset{i \in \Delta}{\vee} r_i, \underset{i \in \Delta}{\wedge} s_i)$,

(5) $\underset{i \in \Delta}{\vee} (r_i, s_i) = (\underset{i \in \Delta}{\wedge} r_i, \underset{i \in \Delta}{\vee} s_i)$

Definition 10. *Let X be a non-empty set. Then, a mapping $\tau : \mathcal{BF}(X) \to [-1,0] \times [0,1]$ is said to be a bipolar $\mathcal{GO}$ on X, if it satisfies the following properties:*

(1) $\tau(\tilde{0}) = \tau(\tilde{1}) = (-1,1)$,

(2) $\tau(\mu_1 \cap \mu_2) \succeq \tau(\mu_1) \wedge \tau(\mu_2)$,

(3) $\tau(\bigcap_{i\in\Delta} \mu_i) \succeq \bigwedge_{i\in\Delta} \tau(\mu_i)$.

Example 1. *Let $X = \mathbb{R}$ be the set of all real numbers. Let T be the usual topology on $\mathbb{R}$ and T' be the topology generated by $\mathfrak{B} = \{(a,b] : a < b\}$. For $A \subseteq \mathbb{R}$ let χ_A denote the characteristic function of A. Define $\chi_A^* = (-\chi_A, \chi_A)$. Define a mapping $\tau : \mathcal{BF}(X) \to [-1,0] \times [0,1]$ by for each $\chi_A^* \in \mathcal{BF}(X)$,*

$$\tau(\chi_A^*) = \begin{cases} (-1,1) & \text{if } A \in T \\ (-\frac{1}{2}, \frac{1}{2}) & \text{if } A \in T' \backslash T \\ (0,0) & \text{otherwise.} \end{cases}$$

Then, τ is a bipolar $\mathcal{GO}$ on X.

Definition 11. *A mapping $\mathfrak{F} : \mathcal{BF}(X) \to [-1,0] \times [0,1]$ is said to be a bipolar $\mathcal{GC}$, if it satisfies the following properties:*

(1) $\mathfrak{F}(\tilde{0}) = \mathfrak{F}(\tilde{1}) = (-1,1)$,

(2) $\mathfrak{F}(\mu_1 \cup \mu_2) \succeq \mathfrak{F}(\mu_1) \wedge \mathfrak{F}(\mu_2)$,

(3) $\mathfrak{F}(\bigcap_{i\in\Delta} \mu_i) \succeq \bigwedge_{i\in\Delta} \mathfrak{F}(\mu_i)$.

Proposition 1. *Let τ be a bipolar $\mathcal{GO}$ on X. Then, a mapping $\mathfrak{F}_\tau : \mathcal{BF}(X) \to [-1,0] \times [0,1]$ defined by $\mathfrak{F}_\tau(\mu) = \tau(\mu^c)$, for all $\mu \in \mathcal{BF}(X)$, is a bipolar $\mathcal{GC}$ on X.*

Proof. We have $\mathfrak{F}_\tau(\tilde{0}) = \tau((\tilde{0})^c) = \tau(\tilde{1}) = (-1,1)$. Similarly, $\mathfrak{F}_\tau(\tilde{1}) = (-1,1)$.

$$\begin{aligned} \mathfrak{F}_\tau(\mu_1 \cup \mu_2) &= \tau((\mu_1 \cup \mu_2)^c) \\ &= \tau(\mu_1^c \cap \mu_2^c) \\ &\succeq \tau(\mu_1^c) \wedge \tau(\mu_2^c) \\ &= \mathfrak{F}_\tau(\mu_1) \wedge \mathfrak{F}_\tau(\mu_2), \end{aligned}$$

$$\begin{aligned} \mathfrak{F}_\tau(\bigcap_{i\in\Delta} \mu_i) &= \tau((\bigcap_{i\in\Delta} \mu_i)^c) \\ &= (\tau(\bigcup_{i\in\Delta} \mu_i^c)) \\ &\succeq \bigwedge_{i\in\Delta} \tau(\mu_i^c) \\ &= \bigwedge_{i\in\Delta} \mathfrak{F}_\tau(\mu_i). \end{aligned}$$

Consequently, the proof completes.

For a mapping $f : \mathcal{BF}(X) \to [-1,0] \times [0,1]$, let $f^- = \pi_1 \circ f$ and $f^+ = \pi_2 \circ f$. Then, f is a *bipolar* $\mathcal{GO}$, ($\mathcal{GC}$) iff $f^+, -f^-$ are $\mathcal{GO}$, ($\mathcal{GC}$) on X. □

Proposition 2. *Let $\mathfrak{F}$ be a bipolar $\mathcal{GC}$ on X. Then, a mapping $\tau_{\mathfrak{F}} : \mathcal{BF}(X) \to [-1,0] \times [0,1]$ defined by $\tau_{\mathfrak{F}}(\mu) = \mathfrak{F}(\mu^c)$, for all $\mu \in \mathcal{BF}(X)$, is a bipolar $\mathcal{GO}$ on X.*

Definition 12. *Let $\{\tau_k : k \in \Delta\}$ be a family of bipolar $\mathcal{GO}$ on X. Then, $\tau = \bigcap_{k\in\Delta} \tau_k$ is defined as, $\tau(\mu) = \bigwedge_{k\in\Delta} \tau_k(\mu)$.*

Proposition 3. *Arbitrary intersection of a family of bipolar $\mathcal{GO}$ is a bipolar $\mathcal{GO}$.*

Proof. Suppose that $\{\tau_k : k \in \Delta\}$ is a family of *bipolar* $\mathcal{GO}$ on X and $\tau = \bigcap_{k\in\Delta} \tau_k$. Clearly, we have $\tau(\tilde{0}) = \tau = (\tilde{1}) = (-1, 1)$:

$$\begin{aligned}
\tau(\mu_1 \cap \mu_2) &= \bigcap_{k\in\Delta} \tau_k(\mu_1 \cap \mu_2) \\
&\succeq \bigcap_{k\in\Delta} \big(\tau_k(\mu_1) \wedge \tau_k(\mu_2)\big) \\
&\succeq \bigcap_{k\in\Delta} \tau_k(\mu_1) \wedge \bigcap_{k\in\Delta} \tau_k(\mu_2) \\
&= \tau(\mu_1) \wedge \tau(\mu_2)
\end{aligned}$$

and

$$\begin{aligned}
\tau(\bigcup_i \mu_i) &= \bigcap_k \tau_k(\bigcup_i \mu_i) \\
&\succeq \bigcap_k \bigwedge_i \tau_k(\mu_i) \\
&= \bigwedge_i \bigcap_k \tau_k(\mu_i) \\
&= \bigwedge_i \tau(\mu_i).
\end{aligned}$$

Hence, τ is a *bipolar* $\mathcal{GO}$ on X. □

Remark 2. *Let X be a non-empty set. Define* $\tau_\circ, \tau_1 : \mathcal{BF}(X) \to [-1, 0] \times [0, 1]$ *by* $\tau_\circ(\tilde{0}) = \tau_\circ(\tilde{1}) = (-1, 1)$, $\tau_\circ(\mu) = (0, 0)$, *for all* $\mu \in \mathcal{BF}(X)\backslash\{\tilde{0}, \tilde{1}\}$ *and* $\tau_1(\mu) = (-1, 1)$, $\forall \mu \in \mathcal{BF}(X)$. *Then,* $\tau_\circ, \tau_1$ *are bipolar* $\mathcal{GO}$ *on X such that, for any bipolar* $\mathcal{GO}$ τ *on X,* $\tau_1 \succeq \tau \succeq \tau_\circ$ *i.e for any* $\mu \in \mathcal{BF}(X)$, $\tau_1(\mu) \succeq \tau(\mu) \succeq \tau_\circ(\mu)$.

Proposition 4. *Let* $\mathcal{M}_{\mathcal{BF}}(X)$ *denote the collection of all bipolar* $\mathcal{GO}$ *on X. Then,* $(\mathcal{M}_{\mathcal{BF}}(X), \succeq)$ *is a complete lattice.*

The proof follows from Proposition 3 and Remark 2.

Proposition 5. *Let* (X, τ) *be a* $\mathcal{BFTS}$, *where* τ *is a bipolar* $\mathcal{GO}$ *on X. Then, for each* $(r, s) \in [-1, 0] \times [0, 1]$, $\tau_{r,s} = \{\mu \in \mathcal{BF}(X) : \tau(\mu) \succeq (r, s)\}$ *is a is a Chang type* $\mathcal{BFT}$ *on X.*

Proof. We have $\tau(\tilde{0}) = \tau(\tilde{1}) = (-1, 1) \succeq (r, s)$, for all $(r, s) \in [-1, 0] \times [0, 1]$. Therefore, we get $\tilde{0}, \tilde{1} \in \tau_{r,s}$. Let $\mu_1, \mu_2 \in \tau_{r,s}$. Then, we have

$$\begin{aligned}
&\tau(\mu_1) \succeq (r, s) \text{ and } \tau(\mu_2) \succeq (r, s) \\
&\tau(\mu_1 \cap \mu_2) \succeq \tau(\mu_1) \wedge \tau(\mu_2) \succeq (r, s) \wedge (r, s) = (r, s).
\end{aligned}$$

Hence, we obtain $\mu_1 \cap \mu_2 \in \tau_{r,s}$. Similarly, it can be shown that $\tau_{r,s}$ is closed under arbitrary union. Therefore, for each $(r, s) \in [-1, 0] \times [0, 1]$, $\tau_{r,s}$ is a Chang type $\mathcal{BFT}$ on X. □

Definition 13. *For each* $(r, s) \in [-1, 0] \times [0, 1]$, $\tau_{r,s}$ *is called the (r-s)-th level* $\mathcal{BFT}$ *on X with respect to the bipolar* $\mathcal{GO}$ τ.

Definition 14. *The family* $\{\tau_{r,s} : (r, s) \in [-1, 0] \times [0, 1]\}$ *is said to be a descending family if any* $(r_1, r_2) \succeq (s_1, s_2)$ *implies* $\tau_{r_1,r_2} \subset \tau_{s_1,s_2}$.

Proposition 6. *Let* (X, τ) *be a* $\mathcal{BFTS}$, *where* τ *is a bipolar* $\mathcal{GO}$ *on X and* $\{\tau_{r,s} : (r, s) \in [-1, 0] \times [0, 1]\}$ *be the family of all (r-s)-th level* $\mathcal{BFT}$ *on X with respect to the bipolar* $\mathcal{GO}$ τ. *Then, this family is descending family and and for each* $(r_1, r_2) \in [-1, 0] \times [0, 1]$,

$$\tau_{r_1,r_2} = \bigcap_{(r_1,r_2)\succ(s_1,s_2)} \tau_{s_1,s_2}.$$

Proof. Clearly, if $(r_1, r_2) \succeq (s_1, s_2)$, then $\tau_{r_1,r_2} \subset \tau_{s_1,s_2}$. Hence, $\{\tau_{r,s} : (r,s) \in [-1,0] \times [0,1]\}$ is a descending family of $\mathcal{BFT}$s on X.

Obviously, $\tau_{r_1,r_2} \subseteq \bigcap_{(r_1,r_2)\succ(s_1,s_2)} \tau_{s_1,s_2}$.

Next, let $\mu \in \cap\, \tau_{s_1,s_2}$, $\forall(r_1, r_2) \succ (s_1, s_2)$. Then, $\tau(\mu) \succeq (s_1, s_2)$, $\forall(r_1, r_2) \succ (s_1, s_2)$. Then, $\tau(\mu) \succeq \vee\{(s_1, s_2);\ (r_1, r_2) \succ (s_1, s_2)\} \Rightarrow \tau(\mu) \succeq (r_1, r_2) \Rightarrow \mu \in \tau_{r_1,r_2}$. Therefore, $\bigcap_{(r_1,r_2)\succ(s_1,s_2)} \tau_{s_1,s_2} \subseteq \tau_{r_1,r_2}$. Hence, $\tau_{r_1,r_2} = \bigcap_{(r_1,r_2)\succ(s_1,s_2)} \tau_{s_1,s_2}$. □

Proposition 7. *Let* $\{T_{r,s} : (r,s) \in [-1,0] \times [0,1]\backslash\{(0,0)\}$ *be a non-empty descending family of Chang type* $\mathcal{BFT}$*s on X. Let* $\tau : \mathcal{BF}(X) \to [-1,0] \times [0,1]$ *be a mapping defined by* $\tau(\mu) = \vee\{(r,s) \in [-1,0] \times [0,1]\backslash\{(0,0)\};\ \mu \in T_{r,s}\}$. *Then,* τ *is a bipolar* $\mathcal{GO}$ *on X. Furthermore, if, for any* $(r_1, r_2) \in [-1,0] \times [0,1]\backslash\{(0,0)\}$

$$T_{r_1,r_2} = \bigcap_{(r_1,r_2)\succ(s_1,s_2)} T_{s_1,s_2}, \tag{1}$$

then $\tau_{r,s} = T_{r,s}$ *holds for all* $(r,s) \in [-1,0] \times [0,1]\backslash\{(0,0)\}$.

Proof. From the definition of τ, it is clear that $\tau(\tilde{0}) = \tau(\tilde{1}) = (-1,1)$. Let $\mu_1, \mu_2 \in \mathcal{BF}(X)$ and let $\tau(\mu_i) = (l_i, k_i)$, $i = 1,2$. If $(l_i, k_i) = (0,0)$ for some i, then $\tau(\mu_1 \cap \mu_2) \succeq \tau(\mu_1) \wedge \tau(\mu_2)$. Without loss of generality, suppose $l_i < 0$ and $k_i > 0$. Let $l_i \leq s_1$ and $k_i \geq s_2$, $i = 1,2$. Then, for any $\epsilon > 0$ with $l_i + \epsilon > 0$, there exist $r_1, r_2 \in [-1,0)$ and $t_1, t_2 \in (0,1]$ such that $\mu_i \in T_{r_i,t_i}$ and $l_i \leq r_i < l_i + \epsilon$ and $k_i - \epsilon < t_i \leq k_i$ and $k_i - \epsilon > 0$ for $i = 1,2$. Now, let

$$\begin{aligned} r &= \max\{r_1, r_2\}, \quad l = \max\{l_1, l_2\}, \\ t &= \min\{t_1, t_2\}, \quad k = \min\{k_1, k_2\}. \end{aligned}$$

Then, $\mu_1 \cap \mu_2 \in T_{r,t}$ implies that $\tau(\mu_1 \cap \mu_2) \succeq (r,t) \succeq (l+\epsilon,\ k-\epsilon)$. Since $\epsilon > 0$ is arbitrary, it follows that $\tau(\mu_1, \mu_2) \succeq \tau(\mu_1) \wedge \tau(\mu_2)$.

Let $\mu_i \in \mathcal{BF}(X)$, for all $i \in \Delta$. Suppose that $\tau(\mu_i) = (l_i, k_i)$, for all $i \in \Delta$. Let $l = \bigvee_{i\in\Delta} l_i$, $k = \bigwedge_{i\in\Delta} k_i$. W.l.o.g, suppose $l < 0$ and $k > 0$. Let $\epsilon > 0$ be any number such that $k > \epsilon$ and $l + \epsilon < 0$. Then, $0 < k - \epsilon < k_i$ and $l + \epsilon > l_i$ for all $i \in \Delta$. Therefore, we have $\mu_i \in T_{l+\epsilon,\, k-\epsilon}$, for all $i \in \Delta$. Then, $\tau(\bigcup_{i\in\Delta} \mu_i) \succeq (l+\epsilon,\ k-\epsilon)$. Since $\epsilon > 0$ is arbitrary, it follows that $\tau(\bigcup_{i\in\Delta} \mu_i) \succeq (l,\ k)$. This implies that τ is a bipolar $\mathcal{GO}$ on X.

In order to show the next part, assume that $\{T_{r,s} : (r,s) \in [-1,0] \times [0,1]\backslash\{(0,0)\}$ satisfies the condition (1). Let $\mu \in T_{r_1,r_2}$. Then, $\tau(\mu) \succeq (r_1, r_2)$, so $\mu \in \tau_{r_1,r_2}$ and, consequently, $T_{r_1,r_2} \subset \tau_{r_1,r_2}$. Next, suppose that $\mu \in \tau_{r_1,r_2}$. Then, $\tau(\mu) \succeq (r_1, r_2)$. Let $\wedge\{l : \mu \in T_{l,k}\} = s_1 \leq r_1$ and $\vee\{k : \mu \in T_{l,k}\} = s_2 \geq r_2$. If $r_1 = 0$, $r_2 > 0$, then, for $\epsilon > 0$ with $r_2 - \epsilon > 0$, $\mu \in T_{r_1,r_2-\epsilon}$. Since $\epsilon > 0$ is arbitrary, $\mu \in \bigcap_{\epsilon>0} T_{r_1,r_2-\epsilon} = T_{r_1,r_2}$. Similarly, other cases can be dealt with. Thus, $\tau_{r,s} = T_{r,s}$ □

Remark 3. *The family* $\{\tau_{r,s} : (r,s) \in [-1,0] \times [0,1]\backslash\{(0,0)\}\}$ *of Proposition 7 is called the family of* $\mathcal{BFT}$*s associated with the bipolar* $\mathcal{GO}$, τ.

Remark 4. *Two bipolar* $\mathcal{GO}$ τ *and* τ' *on X is equal iff* $\tau_{r,s} = \tau'_{r,s}$, *for all* $(r,s) \in [-1,0] \times [0,1]\backslash\{(0,0)\}$.

Proposition 8. *Let* (X, T) *be a Chang type* $\mathcal{BFTS}$. *For each* $(r,s) \in [-1,0] \times [0,1]\backslash(0,0)$, *define a mapping* $T^{r,s} : \mathcal{BF}(X) \to [-1,0] \times [0,1]$ *by the rule*

$$T^{r,s}(\mu) = \begin{cases} (-1,1) & if\ \mu = \tilde{0}, \tilde{1} \\ (r,s) & if\ \mu \in T\backslash\{\tilde{0}, \tilde{1}\} \\ (0,0) & otherwise. \end{cases}$$

Then, $T^{r,s}$ is a bipolar $\mathcal{GO}$ on X such that $(T^{r,s})_{r,s} = T$

Definition 15. *Let T be a Chang type $\mathcal{BFT}$ on X; then, $T^{r,s}$ is called an (r-s)-th bipolar $\mathcal{GO}$ on X and $(X, T^{r,s})$ is called the (r-s)-th graded $\mathcal{BFTS}$.*

4. Bipolar Gradation Preserving Mapping

In a bipolar fuzzy setting, the continuity concept of a mapping is formulated in this section by introducing bipolar gradation preserving maps. Some of its properties are also studied.

Definition 16. *Let (X, τ) and (Y, τ') be two $\mathcal{BFTS}$s, where τ and τ' are bipolar $\mathcal{GO}$ on X and Y, respectively, and $f : X \to Y$ be a mapping. Then, f is called a bipolar gradation preserving map (bipolar $\mathcal{GP}$ map) if, for each $\mu \in \mathcal{BF}(Y)$, $\tau\big(f^{-1}(\mu)\big) \succeq \tau'(\mu)$.*

In the following Proposition, a relation between *bipolar gradation preserving* property with the continuity for a mapping over bipolar fuzzy topological spaces is established.

Proposition 9. *Let (X, τ) and (Y, τ') be two $\mathcal{BFTS}$s, where τ and τ' are bipolar $\mathcal{GO}$ on X and Y, respectively. Then, a mapping $f : X \to Y$ is a bipolar $\mathcal{GP}$ map iff $f : (X, \tau_{r,s}) \to (Y, \tau'_{r,s})$ is continuous for all $(r, s) \in [-1, 0] \times [0, 1]\backslash\{(0, 0)\}$.*

Proof. Suppose that f is a bipolar $\mathcal{GP}$ map and $\mu \in \tau'_{r,s}$. Then, $\tau'(\mu) \succeq (r, s)$. Since f is a bipolar $\mathcal{GP}$ map, it follows that $\tau\big(f^{-1}(\mu)\big) \preceq \tau'(\mu) \succeq (r, s)$. Hence, we get $f^{-1}(\mu) \in \tau_{r,s}$. Thus, $f : (X, \tau_{r,s}) \to (Y, \tau'_{r,s})$ is continuous for all $(r, s) \in [-1, 0] \times [0, 1]\backslash\{(0, 0)\}$.

Conversely, suppose that f is continuous for all $(r, s) \in [-1, 0] \times [0, 1]\backslash\{(0, 0)\}$. Let $\mu \in \mathcal{BF}(Y)$. If $\tau'(\mu) = (0, 0)$, then $\tau\big(f^{-1}(\mu)\big) \succeq \tau'(\mu)$. Let $\tau'(\mu) = (r, s)$, where $(r, s) \in [-1, 0] \times [0, 1]\backslash\{(0, 0)\}$. Then, $\mu \in \tau'_{r,s}$. Since f is continuous, it follows that $f^{-1}(\mu) \in \tau_{r,s}$. This implies that $\tau\big(f^{-1}(\mu)\big) \succeq (r, s) = \tau'(\mu)$. Consequently, f is a bipolar $\mathcal{GP}$ map. □

Proposition 10. *Let (X, T) and (Y, T') be two Chang type $\mathcal{BFTS}$s and $f : X \to Y$ be a mapping. Then, f is continuous iff $f : (X, T^{r,s}) \to (Y, (T')^{r,s})$ is a bipolar $\mathcal{GP}$ map for all $(r, s) \in [-1, 0] \times [0, 1]\backslash\{(0, 0)\}$.*

Proof. Suppose that $f : (X, T) \to (Y, T')$ is continuous. Take $\mu \in \mathcal{BF}(Y)$. Then, we have the following possibilities:

Case (1) If $\mu = \tilde{0}$ or $\tilde{1}$, then $f^{-1}(\tilde{0}) = \tilde{0}$ and $f^{-1}(\tilde{1}) = \tilde{1}$ and hence $(T^{r,s})\big(f^{-1}(\mu)\big) \succeq (T')^{r,s}(\mu)$.

Case (2) If $\mu \in T'$, then $(T')^{r,s}(\mu) = (r, s)$. By continuity of $f : (X, T) \to (Y, T')$, $f^{-1}(\mu) \in T$. Therefore, we get $(T^{r,s})\big(f^{-1}(\mu)\big) = (r, s)$. Thus, $(T^{r,s})\big(f^{-1}(\mu)\big) \succeq (T')^{r,s}(\mu)$.

Case (3) If $\mu \notin T'$, then $(T')^{r,s}(\mu) = (0, 0)$ and so $(T^{r,s})\big(f^{-1}(\mu)\big) \succeq (T')^{r,s}(\mu)$. Hence, $f : (X, T^{r,s}) \to (Y, (T')^{r,s})$ is a *bipolar $\mathcal{GP}$ map*.

The converse follows from Propositions 8 and 9. □

Proposition 11. *Let (X, τ), (Y, τ'), (Z, τ'') be three $\mathcal{BFTS}$s, where τ, τ', τ'' are bipolar $\mathcal{GO}$ on X, Y and Z respectively. If $f : (X, \tau) \to (Y, \tau')$ and $g : (Y, \tau') \to (Z, \tau'')$ are bipolar $\mathcal{GP}$ map, then $g \circ f : (X, \tau) \to (Z, \tau'')$ is a bipolar $\mathcal{GP}$ map.*

Proposition 12. *Let (X, τ) be a $\mathcal{BFTS}$ and $f : X \to Y$ be a mapping. Let $\{\tau'_{r,s} : (r, s) \in [-1, 0] \times [0, 1]\backslash\{(0, 0)\}\}$ be a descending family of Chang type $\mathcal{BFTS}$s on Y. Let τ' be the bipolar $\mathcal{GO}$ generated by this family. Suppose that, for each $(r, s) \in [-1, 0] \times [0, 1]\backslash\{(0, 0)\}$, $\mathfrak{B}_{r,s}$ be the base and $\xi_{r,s}$ be the subbase of $\tau'_{r,s}$. Then,*

(1) $f : (X, \tau) \to (Y, \tau')$ *is a bipolar $\mathcal{GP}$ map iff $\tau\big(f^{-1}(\mu)\big) \succeq (r, s)$, for all $\mu \in \tau'_{r,s}$ and $(r, s) \in [-1, 0] \times [0, 1]\backslash\{(0, 0)\}$.*

(2) $f : (X, \tau) \to (Y, \tau')$ *is a bipolar* $\mathcal{GP}$ *map iff* $\tau(f^{-1}(\mu)) \succeq (r,s)$, *for all* $\mu \in \mathfrak{B}_{r,s}$ *and* $(r,s) \in [-1,0] \times [0,1] \backslash \{(0,0)\}$.
(3) $f : (X, \tau) \to (Y, \tau')$ *is a bipolar* $\mathcal{GP}$ *map iff* $\tau(f^{-1}(\mu)) \succeq (r,s)$, *for all* $\mu \in \zeta_{r,s}$ *and* $(r,s) \in [-1,0] \times [0,1] \backslash \{(0,0)\}$.

5. Bipolar Fuzzy Closure Operator

A concept of bipolar fuzzy closure operator is introduced in this section and its characteristic properties are studied. As in the classical case of Kuratowski's closure operator, here it is shown that the bipolar fuzzy topology and the bipolar $\mathcal{GP}$ map are completely characterized by a bipolar fuzzy closure operator.

Let $(X, \mathfrak{F})$ be a $\mathcal{BFTS}$, where $\mathfrak{F}$ is a bipolar $\mathcal{GC}$ on X. For each $(r,s) \in [-1,0] \times [0,1] \backslash \{(0,0)\}$ and for $\lambda \in \mathcal{BF}(X)$, the $(r$-$s)$-th graded bipolar fuzzy closure ($\mathcal{BFC}$) of λ is defined by

$$Cl(\lambda, (r,s)) = \cap \{\mu \in \mathcal{BF}(X) \, : \, \mu \supseteq \lambda, \, \mathfrak{F}(\mu) \succeq (r,s)\}.$$

Proposition 13. *Let* $(X, \mathfrak{F})$ *be a* $\mathcal{BFTS}$, *where* $\mathfrak{F}$ *is a bipolar* $\mathcal{GC}$ *on* X *and let* $Cl : \mathcal{BF}(X) \times [-1,0] \times [0,1] \backslash \{(0,0)\} \to \mathcal{BF}(X)$ *be a* $\mathcal{BFC}$ *operator on* $(X, \mathfrak{F})$. *Then,*

(1) $Cl(\tilde{0}, (r,s)) = \tilde{0}$, $Cl(\tilde{1}, (r,s)) = \tilde{1}$, *for all* $(r,s) \in [-1,0] \times [0,1] \backslash \{(0,0)\}$.
(2) $\lambda \subseteq Cl(\lambda, (r,s))$, *for all* $\lambda \in \mathcal{BF}(X)$.
(3) $Cl(\lambda, (r_1, s_1)) \subseteq Cl(\lambda, (r_2, s_2))$ *if* $(r_2, s_2) \succeq (r_1, s_1)$.
(4) $Cl(\lambda_1 \cup \lambda_2, (r,s)) = Cl(\lambda_1, (r,s)) \cup Cl(\lambda_2, (r,s))$, *for all* $(r,s) \in [-1,0] \times [0,1] \backslash \{(0,0)\}$.
(5) $Cl(Cl(\lambda, (r,s)), (r,s)) = Cl(\lambda, (r,s))$, *for all* $(r,s) \in [-1,0] \times [0,1] \backslash \{(0,0)\}$.
(6) *If* $(r,s) = \bigvee_{i \in \Delta} \{(r_i, s_i); \, Cl(\lambda, (r_i, s_i)) = \lambda\}$, *then* $Cl(\lambda, (r,s)) = \lambda$.

Proposition 14. *Let* $Cl : \mathcal{BF}(X) \times [-1,0] \times [0,1] \backslash \{(0,0)\} \to \mathcal{BF}(X)$ *be a mapping satisfying* $(1) - (4)$ *of Proposition* 13. *Let* $\mathfrak{F} : \mathcal{BF}(X) \to [-1,0] \times [0,1]$ *be a mapping defined by* $\mathfrak{F}(\lambda) = \vee\{(r,s); \, Cl(\lambda, (r,s)) = \lambda\}$ *then* $\mathfrak{F}$ *is a bipolar* $\mathcal{GC}$ *on* X. *Again,* $Cl = Cl_{\mathfrak{F}}$ *iff the conditions* (5) *and* (6) *of Proposition* 13 *are satisfied by* Cl.

Proof. Clearly, $\mathfrak{F}(\tilde{0}) = \mathfrak{F}(\tilde{1}) = (-1,1)$ by (1).

Let $\lambda_1, \lambda_2 \in \mathcal{BF}(X)$ and $\mathfrak{F}(\lambda_1) = (l_1, k_1)$, $\mathfrak{F}(\lambda_2) = (l_2, k_2)$. For $\epsilon > 0$, $\exists \, (r_i, s_i) \in [-1,0] \times [0,1] \backslash \{(0,0)\}$ such that $l_i \leq r_i < l_i + \epsilon$, $k_i - \epsilon < s_i \leq k_i$ and $Cl(\lambda_i, (r_i, s_i)) = \lambda_i$, $i = 1,2$. Let $r = r_1 \vee r_2$, $s = s_1 \wedge s_2$. Then, $(r,s) \preceq (r_i, s_i)$, $i = 1,2$ and hence $Cl(\lambda_1 \cup \lambda_2, (r,s)) = Cl(\lambda_1, (r,s)) \cup Cl(\lambda_2, (r,s)) = \lambda_1 \cup \lambda_2$ (By (iii)). Hence, $Cl(\lambda_1 \cup \lambda_2, (r,s)) = \lambda_1 \cup \lambda_2$. Thus, $\mathfrak{F}(\lambda_1 \cup \lambda_2) \succeq (r,s) \succeq (r_1, s_1) \wedge (r_2, s_2) \succeq (l_1 \vee l_2 + \epsilon, k_1 \wedge k_2 - \epsilon)$. Since $\epsilon > 0$ is arbitrary, $\mathfrak{F}(\lambda_1 \cup \lambda_2) \succeq (l_1 \vee l_2, k_1 \wedge k_2) = (l_1, k_1) \wedge (l_2, k_2) = \mathfrak{F}(\lambda_1) \wedge \mathfrak{F}(\lambda_2)$.

Let $\lambda_i \in \mathcal{BF}(X)$ and $\mathfrak{F}(\lambda_i) = (a_i, b_i)$, $\bigwedge_{i \in \Delta} \mathfrak{F}(\lambda_i) = (l,k)$ for all $i \in \Delta$ for all $i \in \Delta$. Without loss of generality, assume that $(l,k) \neq (0,0)$. For $\epsilon > 0$, $\exists \, (r_i, s_i) \in [-1,0] \times [0,1] \backslash \{(0,0)\}$ with $a_i \leq r_i < a_i + \epsilon$, $b_i - \epsilon < s_i \leq b_i$ such that $Cl(\lambda_i, (r_i, s_i)) = \lambda_i$, $\forall i \in \Delta$ and $(\bigvee_{i \in \Delta} r_i, \bigwedge_{i \in \Delta} s_i) \neq (0,0)$. Let $r = \bigvee_{i \in \Delta} r_i$, $s = \bigwedge_{i \in \Delta} s_i$. Then, $Cl(\lambda_i, (r,s)) = \lambda_i$, $\forall i \in \Delta$ (since $(r_i, s_i) \succeq (r,s)$, $i \in \Delta$). Thus, $Cl(\bigcap_{i \in \Delta} \lambda_i, (r,s)) \subset Cl(\lambda_i, (r,s)) = \lambda_i$, $\forall i \in \Delta$ (by (iv)) and hence $Cl(\bigcap_{i \in \Delta} \lambda_i, (r,s)) = \bigcap_{i \in \Delta} \lambda_i$. Thus, $\mathfrak{F}(\bigcap_{i \in \Delta} \lambda_i) \succeq (r,s) \succeq (l + \epsilon, k - \epsilon)$, since $\epsilon > 0$ is arbitrary $\mathfrak{F}(\bigcap_{i \in \Delta} \lambda_i) \succeq (l,k) \succeq \bigwedge_{i \in \Delta} \mathfrak{F}(\lambda_i)$.

In order to prove the next part, first suppose that Cl satisfies the conditions (1)–(6) of Proposition 13. Then,

$$\begin{aligned}
&Cl_{\mathfrak{F}}(\lambda,(r,s))\\
&=\cap\{\mu\supseteq\lambda:\ \mathfrak{F}(\mu)\succeq(r,s)\}\\
&=\cap\{\mu\supseteq\lambda:\ \bigvee_{i\in\Delta}\{(r_i,s_i);\ Cl(\mu,(r_i,s_i))=\mu\}\succeq(r,s)\}\\
&=\cap\{\mu\supseteq\lambda:\ \forall\epsilon>0,\ Cl(\mu,(r+\epsilon,s-\epsilon))=\mu\}\\
&\subseteq Cl(\lambda,(r,s)).
\end{aligned}$$

Again, by (2) $\lambda\subseteq Cl(\lambda,(r,s))$ and $Cl(\lambda,(r,s))=Cl\Big(Cl(\lambda,(r,s)),(r+\epsilon,s-\epsilon)\Big)$ (by (2), (3), and (5)). Again, $Cl(\mu,(r+\epsilon,s-\epsilon))=\mu\supseteq\lambda$, for all $\epsilon>0$, implies, by (6), $\mu=Cl(\mu,(r,s))\supseteq Cl(\lambda,(r,s))$. Thus,

$$Cl_{\mathfrak{F}}(\lambda,(r,s))=\cap\{\mu\supseteq\lambda\ :\ \forall\epsilon>0,Cl(\mu,(r+\epsilon,s-\epsilon))=\mu\}\supseteq Cl(\lambda,(r,s)).$$

Therefore, we conclude that $Cl_{\mathfrak{F}}(\lambda,(r,s))=Cl(\lambda,(r,s))$.

Next, suppose that $Cl_{\mathfrak{F}}(\lambda,(r,s))=Cl(\lambda,(r,s))$ holds $\forall\lambda\in\mathcal{BF}(X)$. Since $Cl_{\mathfrak{F}}$ is the $\mathcal{BFC}$ operator generated by the bipolar $\mathcal{GC}$ $\mathfrak{F}$, it follows that $Cl_{\mathfrak{F}}$ satisfies conditions (1)–(6) of Proposition 13. Thus, by assumption, Cl also satisfies conditions (1)–(6) of Proposition 13. This completes the proof. □

Remark 5. *It can be easily verified that, if* $Cl:\mathcal{BF}(X)\times[-1,0]\times[0,1]\backslash\{(0,0)\to\mathcal{BF}(X)$ *is a* $\mathcal{BFC}$ *operator on* X*, then, for each* $(r,s)\in[-1,0]\times[0,1]\backslash\{(0,0)\}$*,* $Cl_{r,s}:\mathcal{BF}(X)\to\mathcal{BF}(X)$ *defined by* $Cl_{r,s}(\lambda)=Cl(\lambda,(r,s))$ *is a* $\mathcal{BFC}$ *operator of Chang type.*

Proposition 15. *Let* (X,τ) *be a Chang type* $\mathcal{BFTS}$*. Then,* $Cl:\mathcal{BF}(X)\times[-1,0]\times[0,1]\backslash\{(0,0)\}\to\mathcal{BF}(X)$ *is a* $\mathcal{BFC}$ *operator iff* $Cl_{r,s}:\mathcal{BF}(X)\to\mathcal{BF}(X)$ *is a Chang type* $\mathcal{BFC}$ *operator for the Chang type* $\mathcal{BFTS}$ $(X,\tau_{r,s})$ *for all* $(r,s)\in[-1,0]\times[0,1]\backslash\{(0,0)\}$*.*

Proof. Clearly, if Cl is a $\mathcal{BFC}$ operator for the $\mathcal{BFTS}$ (X,τ), then $Cl_{r,s}$ is a Chang type $\mathcal{BFC}$ operator for all $(r,s)\in[-1,0]\times[0,1]\backslash\{(0,0)\}$.

Conversely, suppose that $Cl_{r,s}$ is a Chang type $\mathcal{BFC}$ operator for the Chang type $\mathcal{BFTS}$ $(X,\tau_{r,s})$ for all $(r,s)\in[-1,0]\times[0,1]\backslash\{(0,0)\}$. Thus, the conditions (1), (2), (4), and (5) of Proposition 13 are satisfied. If $(r_1,s_1)\succeq(r_2,s_2)$, then, $\tau_{r_1,s_1}\subseteq\tau_{r_2,s_2}$. Therefore, condition (3) of Proposition 13 is satisfied. In order to prove condition (6), suppose that

$$(r,s)=\vee\{(u,v);\ Cl(\lambda,(u,v))=\lambda.$$

Then, $\lambda^c\in\tau_{r+\epsilon,s-\epsilon}$ for all $\epsilon>0$. Thus, we have $\lambda^c\in\bigcap_{\epsilon>0}\tau_{r+\epsilon,s-\epsilon}$, i.e., $\lambda^c\in\tau_{r,s}$. Therefore, we have $\lambda\in\mathfrak{F}_{r,s}$ and hence we conclude that $Cl(\lambda,(r,s))=\lambda$. This completes the proof. □

Proposition 16. *Let* $f:(X,\tau)\to(Y,\tau')$ *be a mapping between two* $\mathcal{BFTS}$*s. Then,* f *is a bipolar* $\mathcal{GP}$ *map iff* $f\big(Cl(\lambda,(r,s))\big)\subseteq Cl(f(\lambda),(r,s))$*.*

Proof. By Proposition 9, f is a *bipolar* $\mathcal{GP}$ *map* iff $f:(X,\tau_{r,s})\to(Y,\tau'_{r,s})$ is continuous for all $(r,s)\in[-1,0]\times[0,1]\backslash\{(0,0)\}$ iff $f\big(Cl(\lambda,(r,s))\big)\subseteq Cl(f(\lambda),(r,s))$. □

6. Category of Bipolar Fuzzy Topology

In this section, categorical behavior of bipolar fuzzy topological spaces is studied.

Let $\mathcal{C}_{\mathcal{BFT}}$ denote the category of all Chang type $\mathcal{BFTS}$s and continuous functions; $\mathcal{F}_{Top}$ denotes the category of all $\mathcal{BFTS}$s and bipolar $\mathcal{GP}$ maps in our sense; for each $(r,s)\in[-1,0]\times[0,1]\backslash\{(0,0)\}$, $\mathcal{F}^{r,s}_{Top}$ denotes the category of $(r$-$s)$-th graded $\mathcal{BFTS}$s and *bipolar* $\mathcal{GP}$ *maps*.

Proposition 17.

(1) $\mathcal{F}^{r,s}_{Top}$ *is a full subcategory of* $\mathcal{F}_{Top}$.
(2) *For each* $(r,s) \in [-1,0] \times [0,1] \backslash \{(0,0)\}$, $\mathcal{C}_{\mathcal{BFT}}$ *and* $\mathcal{F}^{r,s}_{Top}$ *are isometric.*
(3) $\mathcal{F}^{r,s}_{Top}$ *is a bireflective full subcategory of* $\mathcal{F}_{Top}$.

Proof. The first two results follow from the facts: $(\tau_{r,s})^{r,s} = \tau$ if τ is a $(r$-$s)$-th *bipolar* $\mathcal{GO}$; $(T^{r,s})_{r,s} = T$ if T is a Chang type $\mathcal{BFT}$ and $f : (X,T) \to (Y,T')$ is continuous w.r.t the Chang type $\mathcal{BFT}$ iff $f : (X, T^{r,s}) \to (Y, (T')^{r,s})$ is a *bipolar* $\mathcal{GP}$ *map*, for all $(r,s) \in [-1,0] \times [0,1] \backslash \{(0,0)\}$. To prove (3), let us take a member (X,τ) of $\mathcal{F}_{Top}$. Then, for each $(r,s) \in [-1,0] \times [0,1] \backslash \{(0,0)\}$, $(X, (\tau_{r,s})^{r,s})$ is a $\mathcal{F}^{r,s}_{Top}$ member and also $I_X : (X,\tau) \to (X, (\tau_{r,s})^{r,s})$ is a *bipolar* $\mathcal{GP}$ *map*. Let (Y,τ') be a member of $\mathcal{F}^{r,s}_{Top}$ and $f : (X,\tau) \to (Y,\tau')$ be a *bipolar* $\mathcal{GP}$ *map*. Now, we only need to check whether $f : (X, (\tau_{r,s})^{r,s}) \to (Y,\tau')$ is a *bipolar* $\mathcal{GP}$ *map*. If $\mu = \tilde{0}$, then $\tau\big(f^{-1}(\tilde{0})\big) = \tau'(\tilde{0})$. Then, $(\tau_{r,s})^{r,s}\big(f^{-1}(\tilde{0})\big) = (\tau_{r,s})^{r,s}(\tilde{0}) \succeq \tau'(\tilde{0})$. Similarly, $(\tau_{r,s})^{r,s}\big(f^{-1}(\tilde{1})\big) \succeq \tau'(\tilde{1})$. If $\tau'(\mu) = (0,0)$, then, obviously $(\tau_{r,s})^{r,s}\big(f^{-1}(\mu)\big) \succeq \tau'(\mu)$. Let $\tau'(\mu) = (r,s)$. Then, $\tau\big(f^{-1}(\mu)\big) \succeq \tau'(\mu) \Rightarrow f^{-1}(\mu) \in \tau_{r,s}$. Then, $(\tau_{r,s})^{r,s}\big(f^{-1}(\mu)\big) \succeq (r,s) = \tau'(\mu)$. Thus, $f : (X, (\tau_{r,s})^{r,s}) \to (Y,\tau')$ is a *bipolar* $\mathcal{GP}$ *map*. □

$$\begin{tikzcd}
 & (X,(\tau_{r,s})^{r,s}) \arrow[d, dash] \\
(X,\tau)\ \mathcal{F}_{Top}\text{ object} \arrow[ur, dash] \arrow[r, dash] & (Y,\tau')\ \mathcal{F}^{r,s}_{Top}\text{ object}
\end{tikzcd}$$

Remark 6. *From (2), (3) in Proposition 17* $\mathcal{C}_{\mathcal{BFT}}$ *may be called a bireflective full subcategory of* $\mathcal{F}_{Top}$.

Proposition 18. *Let* $\{(X_i, \tau'_i) : i \in \Delta\}$ *be a family of* $\mathcal{BFTS}$*s and* X *be a set such that* $f : X \to X_i$ *is a map for each* $i \in \Delta$. *Then, there exists a bipolar* $\mathcal{GO}$ τ *on* X *such that the following condition holds:*

(1) *for each* $i \in \Delta$, $f_i : (X,\tau) \to (X_i, \tau'_i)$ *is a bipolar* $\mathcal{GP}$ *map.*
(2) *If* (Z,τ'') *is a* $\mathcal{BFTS}$, *then* $g : (Z,\tau'') \to (X,\tau)$ *is a bipolar* $\mathcal{GP}$ *map iff* $f_i \circ g$ *is a bipolar* $\mathcal{GP}$ *map for each* $i \in \Delta$.

Proof. (1) For each $(r,s) \in [-1,0] \times [0,1] \backslash \{(0,0)\}$ and for each $i \in \Delta$, we define

$$T^{r,s}_i = \{f_i^{-1}(\mu) \ : \ \mu \in (\tau'_i)_{r,s}\},$$

where $(\tau'_i)_{r,s} = \{\mu \in \mathcal{BF}(X_i) \ : \ \tau'_i(\mu) \succeq (r,s)\}$ is the $(r$-$s)$-th level $\mathcal{BFT}$ on X_i w.r.t τ'_i. It can be shown that $T^{r,s}_i$ is a $\mathcal{BFT}$ on X. Clearly, $\{T^{r,s}_i \ : \ (r,s) \in [-1,0] \times [0,1] \backslash \{(0,0)\}\}$ is a descending family. For each $(r,s) \in [-1,0] \times [0,1] \backslash \{(0,0)\}$, we define

$$\prod_{r,s} = \bigcup_{j \in \Delta} T^{r,s}_i.$$

Let $T_{r,s}$ be the $\mathcal{BFT}$ on X generated by $\prod\limits_{r,s}$ as a subbase. Then, $\{T_{r,s} \ : \ (r,s) \in [-1,0] \times [0,1] \backslash \{(0,0)\}\}$ is a descending family. Then, there exists a *bipolar* $\mathcal{GO}$ τ on X associated with the family $\{T_{r,s} \ : \ (r,s) \in [-1,0] \times [0,1] \backslash \{(0,0)\}\}$, where $\tau(\mu) = \vee\{(r,s) \in [-1,0] \times [0,1] \backslash \{(0,0)\};\ \mu \in T_{r,s}\}$.

First, we show that for each $i \in \Delta$, $f_i : (X, \tau) \to (X_i, \tau_i')$ is a *bipolar* $\mathcal{GP}$ *map*. Let $\mu \in \mathcal{BF}(X_i)$ and $\tau_i'(\mu) = (r, s)$, where $(r, s) \succ (0, 0)$. Then, $f_i^{-1}(\mu) \in T_i^{r,s} \subset \prod_{r,s} \subset T_{r,s}$. Thus, $\tau(f_i^{-1}(\mu)) \succeq (r, s) = \tau_i'(\mu)$. Consequently, $f_i : (X, \tau) \to (X_i, \tau_i')$ is a *bipolar* $\mathcal{GP}$ *map*.

(2) If $g : (Z, \tau'')$ is a *bipolar* $\mathcal{GP}$ *map* and since, for each $i \in \Delta$, $f_i : (X, \tau) \to (X_i, \tau_i')$ is a *bipolar* $\mathcal{GP}$ *map*, by Proposition 11, the composition of two *bipolar* $\mathcal{GP}$ *map* $f_i \circ g$ is a *bipolar* $\mathcal{GP}$ *map* for each $i \in \Delta$.

Conversely, we have to show that $g : (Z, \tau'') \to (X, \tau)$ is a *bipolar* $\mathcal{GP}$ *map*. Let $(r, s) \in [-1, 0] \times [0, 1] \backslash \{(0, 0)\}$ and $\mu \in \xi_{r,s}$. Then, $\mu \in T_i^{r,s}$ for some $i \in \Delta$. Then, there exists $\lambda \in (\tau_i')_{r,s}$ such that $f_i^{-1}(\lambda) = \mu$. Since $f_i \circ g$ is a *bipolar* $\mathcal{GP}$ *map* for each $i \in \Delta$, it follows that

$$\tau''((f_i \circ g)^{-1}(\lambda)) \succeq (r, s) \Rightarrow \tau''(g^{-1}(f_i^{-1}(\lambda))) \succeq (r, s) \Rightarrow \tau''(g^{-1}(\mu)) \succeq (r, s).$$

Hence, the result follows from Proposition 12. □

7. Conclusions

The notion of a bipolar fuzzy set is a generalization of a fuzzy set in the sense that a fuzzy set describes some property in a graded manner from its existence to its non existence by assigning values from 1 to 0, whereas a bipolar fuzzy set describes the same from the existence to the reverse existence through non-existence by taking values from 1 to –1 through 0. In this article, this idea of bipolarity is formalized in the topological sense by introducing a concept of *bipolar gradation of openness* to redefine bipolar fuzzy topology. Consequently, we introduce bipolar $\mathcal{GO}$ and bipolar $\mathcal{GC}$ and studied their properties. The relation between Chang type $\mathcal{BFT}$ and $\mathcal{BFT}$ in our sense is established successfully. The bipolar $\mathcal{GP}$ map and bipolar $\mathcal{FC}$ operator are studied. In addition, we have shown that the Chang type $\mathcal{BFT}$ and continuous function is a bireflective full subcategory of the topological category of $\mathcal{BFT}$ and bipolar $\mathcal{GP}$ maps in our sense. In the upcoming papers, we will study various topological properties including the compactness and connectedness in this setting.

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