mathematics

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On Coefficient Functionals for Functions with Coefficients Bounded by 1

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Received: 31 January 2020; Accepted: 25 March 2020; Published: 1 April 2020

Abstract: In this paper, we discuss two well-known coefficient functionals $a_2a_4 - {a_3}^2$ and $a_4 - a_2a_3$. The first one is called the Hankel determinant of order 2. The second one is a special case of Zalcman functional. We consider them for functions in the class $\mathcal{Q}_{\mathbb{R}}(\frac{1}{2})$ of analytic functions with real coefficients which satisfy the condition $\mathrm{Re}\,\frac{f(z)}{z} > \frac{1}{2}$ for z in the unit disk Δ. It is known that all coefficients of $f \in \mathcal{Q}_{\mathbb{R}}(\frac{1}{2})$ are bounded by 1. We find the upper bound of $a_2a_4 - {a_3}^2$ and the bound of $|a_4 - a_2a_3|$. We also consider a few subclasses of $\mathcal{Q}_{\mathbb{R}}(\frac{1}{2})$ and we estimate the above mentioned functionals. In our research two different methods are applied. The first method connects the coefficients of a function in a given class with coefficients of a corresponding Schwarz function or a function with positive real part. The second method is based on the theorem of formulated by Szapiel. According to this theorem, we can point out the extremal functions in this problem, that is, functions for which equalities in the estimates hold. The obtained estimates significantly extend the results previously established for the discussed classes. They allow to compare the behavior of the coefficient functionals considered in the case of real coefficients and arbitrary coefficients.

Keywords: coefficient problems; analytic functions; Schwarz functions; starlike functions; functions convex in one direction

MSC: 30C45; 30C50

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1. Introduction

Let Δ be the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{A}$ denote the class of all functions f analytic in Δ with the typical normalization $f(0) = f'(0) - 1 = 0$. This means that the function $f \in \mathcal{A}$ has the following representation

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \,. \tag{1}$$

Additionally, we denote by $\mathcal{A}_{\mathbb{R}}$ the class of those functions $f \in \mathcal{A}$ whose all coefficients are real.

In this paper we discuss two functionals

$$a_2a_4 - {a_3}^2 \tag{2}$$

and

$$a_4 - a_2a_3 \tag{3}$$

considered for functions of the form (1) in a given class $A \subset \mathcal{A}$.

Recently, these functionals have been widely discussed. The research mainly focused on estimating so called Hankel determinants. Pommerenke (see [1,2]) defined the k-th Hankel determinant for a function f of the form (1) and $n, k \in \mathbb{N}$ as

$$H_k(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+k-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+k-1} & a_{n+k} & \cdots & a_{n+2k-2} \end{vmatrix}.$$

In a view of this definition, $a_2a_4 - a_3^2$ is the second Hankel determinant (more precisely, $H_2(2)$).

The sharp bounds of $|a_2a_4 - a_3{}^2|$ for almost all important subclasses of the class $\mathcal{S}$ of analytic univalent functions were found (see, for example, [3–8]). It is worth noting that we still do not know the exact bound of this expression for $\mathcal{S}$, nor for $\mathcal{C}$ consisting of all close-to-convex functions (see [9]). On the other hand, finding the bounds, upper and lower, for classes of analytic functions with real coefficients is a much more complicated task. For this reason, only a few papers were devoted to solving this problem. Such result for univalent starlike functions was obtained by Kwon and Sim ([10]). Furthermore, similar problems for functions which are typically real were discussed in [11].

The functional $a_4 - a_2a_3$ is a special case of the so-called generalized Zalcman functional which was studied, among others, in [12,13]. The generalized version of this functional, that is, $a_4 - \mu a_2 a_3$, was discussed in [14].

We start with considering (2) and (3) in the class $\mathcal{Q}_{\mathbb{R}}(\frac{1}{2})$ of analytic functions given by (1) which satisfy the condition

$$\mathrm{Re}\,\frac{f(z)}{z} > \tfrac{1}{2} \quad , \quad z \in \Delta\,. \tag{4}$$

It is known that all coefficients of $f \in \mathcal{Q}_{\mathbb{R}}(\frac{1}{2})$ are real and bounded by 1.

The class $\mathcal{Q}_{\mathbb{R}}(\frac{1}{2})$ contains three well-known, important subclasses of univalent functions: $\mathcal{K}_{\mathbb{R}}$ of convex functions, $\mathcal{S}^*_{\mathbb{R}}(\frac{1}{2})$ of starlike functions of order $1/2$, $\mathcal{K}_{\mathbb{R}}(i)$ of functions that are convex in the direction of the imaginary axis. Two other classes $\mathcal{T}(\frac{1}{2})$ and $\mathcal{W}$ consisting of functions defined by specific Riemann-Stieltjes integrals are also included in $\mathcal{Q}_{\mathbb{R}}(\frac{1}{2})$. The precise definitions of these classes will be given in Section 3. In this section we show the partial ordering of the mentioned above subclasses of $\mathcal{Q}_{\mathbb{R}}(\frac{1}{2})$ with respect to the relation of inclusion. Clearly, the coefficients of functions in each subset of $\mathcal{Q}_{\mathbb{R}}(\frac{1}{2})$ are bounded by 1. What is interesting, this number cannot be improved. Finding the estimates of $a_2a_4 - a_3{}^2$ and $a_4 - a_2a_3$ gives additional information about the richness of these classes (compare [15]).

All functions in $\mathcal{Q}_{\mathbb{R}}(\frac{1}{2})$ and in other classes discussed in this paper have real coefficients. For this reason, it is interesting to find not only the bounds of moduli of (2) and (3), but also their upper and lower bounds. On the other hand, it is clear that if the following property

$$f \in A \quad \text{if and only if} \quad -f(-z) \in A \tag{5}$$

holds for all functions f in a given class A, then

$$\min\{a_4 - a_2a_3 : f \in A\} = -\max\{a_4 - a_2a_3 : f \in A\}\,. \tag{6}$$

The same property is not true for the functional $a_2a_4 - a_3{}^2$.

2. Estimates for the class $\mathcal{Q}_{\mathbb{R}}(\frac{1}{2})$

The coefficients of $f \in \mathcal{Q}_{\mathbb{R}}(\frac{1}{2})$ can be expressed in terms of the coefficients of a relative function p in the class $\mathcal{P}_{\mathbb{R}}$ or in terms of the coefficients of a relative function ω in the class $(\mathcal{B}_0)_{\mathbb{R}}$. Recall that

$\mathcal{P}$ and $(\mathcal{B}_0)$ denote the class of functions with positive real part and the class of functions such that $\omega(0) = 0$ and $|\omega(z)| < 1$.

If $f \in \mathcal{Q}_{\mathbb{R}}(\frac{1}{2})$ is of the form (1), $p \in \mathcal{P}_{\mathbb{R}}$ and $\omega \in (\mathcal{B}_0)_{\mathbb{R}}$ are of the form

$$p(z) = 1 + p_1 z + p_2 z^2 + \ldots \tag{7}$$

and

$$\omega(z) = c_1 z + c_2 z^2 + \ldots , \tag{8}$$

then

$$a_n = \tfrac{1}{2} p_{n-1}$$

and

$$a_2 = c_1 \, , \; a_3 = c_2 + {c_1}^2 \, , \; a_4 = c_3 + 2c_1 c_2 + {c_1}^3 \, .$$

For this reason, we have

$$a_2 a_4 - {a_3}^2 = \tfrac{1}{4}(p_1 p_3 - {p_2}^2) = c_1 c_3 - {c_2}^2 \tag{9}$$

and

$$a_4 - a_2 a_3 = \tfrac{1}{2}(p_3 - \tfrac{1}{2} p_1 p_2) = c_3 + c_1 c_2 \, . \tag{10}$$

To obtain our results we need the estimates for initial coefficients of Schwarz functions.

Lemma 1 ([16]). *If $\omega \in \mathcal{B}_0$ is of the form (8), then*

$$|c_1| \le 1 \, , \; |c_2| \le 1 - |c_1|^2 \, , \; |c_3(1 - |c_1|^2) + \overline{c_1} {c_2}^2| \le (1 - |c_1|^2)^2 - |c_2|^2 \, .$$

If, additionally, ω has real coefficients, we have the following fact.

Corollary 1. *If $\omega \in (\mathcal{B}_0)_{\mathbb{R}}$ is of the form (8), then*

$$c_3 \le 1 - {c_1}^2 - \frac{{c_2}^2}{1 - c_1} \, .$$

We also need the generalized Livingstone result obtained by Hayami and Owa.

Lemma 2 ([17]). *If $p \in \mathcal{P}$ is of the form (7) and $\mu \in [0, 1]$, then*

$$|p_3 - \mu p_1 p_2| \le 2 \, .$$

The following sharp result was also proved by Hayami and Owa.

Theorem 1 ([17]). *If $f \in \mathcal{Q}_{\mathbb{R}}(\frac{1}{2})$, then*

$$|a_2 a_4 - {a_3}^2| \le 1 \, . \tag{11}$$

Equality holds for $f(z) = \frac{z}{1-z^2}$.

Observe that for $f(z) = \frac{z}{1-z^2}$ we have $a_2 a_4 - {a_3}^2 = -1$, which means that

$$\min \left\{ a_2 a_4 - {a_3}^2 : f \in \mathcal{Q}_{\mathbb{R}}\left(\tfrac{1}{2}\right) \right\} = -1 \, .$$

Now, we shall derive an upper bound of this functional for $\mathcal{Q}_{\mathbb{R}}(\frac{1}{2})$.

Theorem 2. *If $f \in \mathcal{Q}_{\mathbb{R}}(\frac{1}{2})$, then*

$$a_2a_4 - {a_3}^2 \leq \frac{2\sqrt{3}}{9} = 0.384\ldots\,. \tag{12}$$

Proof. Applying (9) and Corollary 1,

$$a_2a_4 - {a_3}^2 = c_1c_3 - {c_2}^2 \leq c_1(1 - {c_1}^2) - \frac{{c_2}^2}{1 - c_1} =: h_1(c_1, c_2)\,.$$

After simple calculation,

$$h_1(c_1, c_2) \leq h_1(\tfrac{1}{\sqrt{3}}, 0) = \tfrac{2\sqrt{3}}{9}\,.$$

□

The estimate of $|a_4 - a_2a_3|$ for $\mathcal{Q}_{\mathbb{R}}(\frac{1}{2})$ is also easy to obtain. It is enough to apply (10) and Lemma 2.

Theorem 3. *If $f \in \mathcal{Q}_{\mathbb{R}}(\frac{1}{2})$, then*

$$|a_4 - a_2a_3| \leq 1\,. \tag{13}$$

Equality holds for $f(z) = \frac{z}{1-z^3}$.

3. Other Classes with Coefficients Bounded by 1

We know a few other subclasses of $\mathcal{A}_{\mathbb{R}}$ consisting of functions with real coefficients bounded by 1: the class $\mathcal{K}_{\mathbb{R}}$ of convex functions, the class $\mathcal{S}^*_{\mathbb{R}}(\frac{1}{2})$ of starlike functions of order $1/2$, and the class $\mathcal{K}_{\mathbb{R}}(i)$ of functions that are convex in the direction of the imaginary axis. The same property also holds for $\mathcal{T}(\frac{1}{2})$ and $\mathcal{W}$ defined as follows

$$\mathcal{T}(\tfrac{1}{2}) = \left\{ f(z) = \int_{-1}^{1} \frac{z}{\sqrt{1 - 2tz + z^2}} d\mu(t)\,,\ t \in [-1, 1],\ \mu \in P_{[-1,1]} \right\} \tag{14}$$

and

$$\mathcal{W} = \left\{ f(z) = \int_{-1}^{1} \frac{z}{1 - tz} d\mu(t)\,,\ t \in [-1, 1],\ \mu \in P_{[-1,1]} \right\}\,, \tag{15}$$

where $P_{[-1,1]}$ is the set of probability measures on the interval $[-1, 1]$.

For convenience of the reader, let us recall that an analytic and normalized function belongs to $\mathcal{K}_{\mathbb{R}}$ and $\mathcal{S}^*_{\mathbb{R}}(\frac{1}{2})$ when the following conditions are satisfied, respectively

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \quad , \quad z \in \Delta \tag{16}$$

and

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > \tfrac{1}{2} \quad , \quad z \in \Delta\,. \tag{17}$$

The class $\mathcal{K}_{\mathbb{R}}(i)$ is related to the class $\mathcal{T}$ of typically real functions. Namely, for all $z \in \Delta$ there is

$$f \in \mathcal{K}_{\mathbb{R}}(i) \iff zf'(z) \in \mathcal{T}\,. \tag{18}$$

The classes defined above can be ordered in the following chains of inclusions:

$$\mathcal{K}_{\mathbb{R}} \subset \mathcal{S}^*_{\mathbb{R}}(1/2) \subset \mathcal{T}(1/2) \subset \mathcal{Q}(1/2)\,,$$

$$\mathcal{K}_{\mathbb{R}} \subset \mathcal{K}_{\mathbb{R}}(i) \subset \mathcal{T}(1/2) \subset \mathcal{Q}(1/2)\,,$$

$$\mathcal{W} \subset \mathcal{K}_{\mathbb{R}}(i) \subset \mathcal{T}(1/2) \subset \mathcal{Q}(1/2)\,.$$

The first inclusion in the first chain is the famous theorem of Marx-Strohhäcker ([18]). In [19] Hallenbeck proved that $\mathcal{S}^*_{\mathbb{R}}(1/2) \subset \mathcal{T}(1/2)$. In fact, he proved that $\mathcal{T}(1/2)$ is a closed convex hull fo $\mathcal{S}^*_{\mathbb{R}}(1/2)$. Robertson proved in [20] that if $f \in \mathcal{K}_{\mathbb{R}}(i)$ then $\mathrm{Re}(f(z)/z) > 1/2$, or, in other words, $f \in \mathcal{Q}(1/2)$.

The inclusion $\mathcal{K}_{\mathbb{R}} \subset \mathcal{K}_{\mathbb{R}}(i)$ is obvious. The proof of the relation $\mathcal{K}_{\mathbb{R}}(i) \subset \mathcal{T}(1/2)$ can be found in [21].

To prove the third chain of inclusions, observe that $\mathcal{W} \subset \mathcal{K}_{\mathbb{R}}(i)$. Indeed, $\{z/(1-tz) : t \in [-1,1]\} \subset \mathcal{K}_{\mathbb{R}}$, so the closed convex hull of the set $\{z/(1-tz) : t \in [-1,1]\}$ is included in the closed convex hull of $\mathcal{K}_{\mathbb{R}}$ equal to $\mathcal{K}_{\mathbb{R}}(i)$. The successive inclusions have already been shown.

Moreover,

$$\mathcal{W} \not\subset \mathcal{K}_{\mathbb{R}} \quad \text{and} \quad \mathcal{K}_{\mathbb{R}} \not\subset \mathcal{W}\,.$$

The first statement follows from the fact that $f_1(z) = \frac{z}{1-z^2}$, as a convex combination of the functions $\frac{z}{1-z}$ and $\frac{z}{1-z}$, belongs to $\mathcal{W}$, but it does not belong to $\mathcal{K}_{\mathbb{R}}$. To show the second statement, it is enough to consider $f_2(z) = z - \frac{1}{9}z^3$. Since

$$\mathrm{Re}\left(1 + \frac{zf_2''(z)}{f_2'(z)}\right) = \mathrm{Re}\left(\frac{1-z^2}{1-\frac{1}{3}z^2}\right) > 0\,,$$

this function is in $\mathcal{K}_{\mathbb{R}}$. From the formula for the coefficients of functions in $\mathcal{W}$ it follows that $a_3 \geq 0$ for each $f \in \mathcal{W}$, but $a_3 = -1/9$ for f_2. Consequently, f_2 does not belong to $\mathcal{W}$.

It is easy to check that

$$\mathcal{K}_{\mathbb{R}}(i) \not\subset \mathcal{S}^*_{\mathbb{R}}(\tfrac{1}{2}) \quad \text{and} \quad \mathcal{S}^*_{\mathbb{R}}(\tfrac{1}{2}) \not\subset \mathcal{K}_{\mathbb{R}}(i)\,.$$

It is clear that for $f_1(z) = \frac{z}{1-z^2} \in \mathcal{K}_{\mathbb{R}}(i)$ there is $\frac{zf_1'(z)}{f_1(z)} = \frac{1+z^2}{1-z^2}$. Consequently, $f_1 \notin \mathcal{S}^*_{\mathbb{R}}(\frac{1}{2})$. On the other hand, for the function $f_3(z) = \frac{z}{\sqrt{1+z^2}}$ we have $\frac{zf_3'(z)}{f_3(z)} = \frac{1}{1+z^2}$. Hence, $\mathrm{Re}\,\frac{zf_3'(z)}{f_3(z)} > 1/2$ and $f_3 \in \mathcal{S}^*_{\mathbb{R}}(\frac{1}{2})$. The image set $f_3(\Delta)$ coincides with the domain lying between two branches of the hyperbola $(\mathrm{Re}\,w)^2 - (\mathrm{Im}\,w)^2 < 1/2$. This means that f_3 is not in $\mathcal{K}_{\mathbb{R}}(i)$.

From the argument given above and Theorem 1 we obtain

Corollary 2. *Let A denote one of the classes: $\mathcal{T}(\frac{1}{2})$, $\mathcal{K}_{\mathbb{R}}(i)$ and $\mathcal{W}$ and let $f \in A$. Then*

$$|a_2a_4 - a_3{}^2| \leq 1\,.$$

Equality holds for $f(z) = \frac{z}{1-z^2}$.

In our research two different methods are applied. The first method connects the coefficients of a function in a given class with coefficients with a corresponding Schwarz function or a function with positive real part. The second method is based on the Szapiel theorem. According to this theorem, we can point out the extremal functions in this problem, that is, functions for which equalities in the estimates hold.

4. Estimates for $\mathcal{K}_{\mathbb{R}}$ and $\mathcal{S}^*_{\mathbb{R}}(\frac{1}{2})$

We know that the estimate given in Corollary 2 is true but not sharp for $\mathcal{K}_{\mathbb{R}}$ and $\mathcal{S}^*_{\mathbb{R}}(\frac{1}{2})$, because the function $f(z) = \frac{z}{1-z^2}$ does not belong to either of them. In [5] it was shown that the sharp bound of $|a_2a_4 - a_3{}^2|$ in $\mathcal{K}_{\mathbb{R}}$ is $1/8$ and the extremal function is

$$f(z) = \frac{2}{\sqrt{3}}\arctan\left(\frac{\sqrt{3}z}{2-z}\right) = z + \frac{1}{2}z^2 - \frac{1}{4}z^4 + \dots\,.$$

On the other hand, $|a_2a_4 - a_3{}^2| \leq 1/4$ in $\mathcal{S}^*_{\mathbb{R}}(\frac{1}{2})$ (see [3]) and the extremal function is

$$f(z) = \frac{z}{\sqrt{1-z^2}} = z + \tfrac{1}{2}z^3 + \tfrac{3}{8}z^5 + \dots .$$

Let $f \in \mathcal{K}_{\mathbb{R}}$ and $\omega \in (\mathcal{B}_0)_{\mathbb{R}}$ be given by (1) and (8), respectively. From the correspondence between $\mathcal{K}_{\mathbb{R}}$ and $(\mathcal{B}_0)_{\mathbb{R}}$, that is,

$$1 + \frac{zf''(z)}{f'(z)} = \frac{1+\omega(z)}{1-\omega(z)} \tag{19}$$

we obtain

$$\Big(1 + 4a_2z + 9a_3z^2 + 16a_4z^3 \dots\Big) \cdot \Big(1 - c_1z - c_2z^2 - \dots\Big) = \\ (1 + 2a_2z + 3a_3z^2 + 4a_4z^3 \dots) \cdot \Big(1 + c_1z + c_2z^2 + \dots\Big) \;. \tag{20}$$

Consequently,

$$a_2 = c_1 \;,\; a_3 = \tfrac{1}{3}c_2 + c_1{}^2 \;,\; a_4 = \tfrac{1}{6}c_3 + \tfrac{5}{6}c_1c_2 + c_1{}^3 \;. \tag{21}$$

Putting these formulae into (9) and (10) we have

$$a_2a_4 - a_3{}^2 = \tfrac{1}{6}c_1c_3 + \tfrac{1}{6}c_1{}^2c_2 - \tfrac{1}{9}c_2{}^2 \tag{22}$$

and

$$a_4 - a_2a_3 = \tfrac{1}{6}(c_3 + 3c_1c_2) \;. \tag{23}$$

Theorem 4. *If $f \in \mathcal{K}_{\mathbb{R}}$, then*

1. $a_2a_4 - a_3{}^2 \leq 0.066\dots$,
2. $|a_4 - a_2a_3| \leq \frac{4\sqrt{3}}{27} = 0.256\dots$.

Proof. Let $f \in \mathcal{K}_{\mathbb{R}}$.
I. By Lemma 1,

$$a_2a_4 - a_3{}^2 \leq \tfrac{1}{6}\left(c_1(1-c_1{}^2) + c_1{}^2c_2 - \tfrac{1}{3}\frac{2+c_1}{1-c_1}c_2{}^2\right) =: h_1(c_1, c_2) \;.$$

The critical points of h_1 are the solutions of the system

$$\begin{cases} c_2 = \frac{3(1-c_1)}{2(2+c_1}c_2{}^2 \\ 1 - 3c_1{}^2 + 2c_1c_2 - \left(\frac{c_2}{1-c_1}\right)^2 = 0 \,. \end{cases}$$

From this system we have

$$12c_1{}^5 + 33c_1{}^4 + 24c_1{}^3 + 44c_1{}^2 - 16c_1 - 16 = 0 \,.$$

Hence, the critical points, satisfying the first two inequalities of Lemma 1, are as follows $(-0.468\dots, 0.315\dots)$, $(0.593\dots, 0.082\dots)$ and

$$h_1(-0.468\dots, 0.315\dots) = 0.055\dots \quad , \quad h_1(0.593\dots, 0.082\dots) = 0.066\dots .$$

Since $c_1 \in [-1, 1]$,

$$h_1(c_1, 1 - c_1{}^2) = -\tfrac{1}{9}(1 - c_1{}^2)^2 \leq 0$$

and

$$h_1(c_1, -1 + c_1{}^2) = -\tfrac{1}{9}(1 - c_1{}^2)(1 + 2c_1{}^2) \leq 0 \,.$$

This means that $a_2a_4 - a_3{}^2 \le 0.066\ldots$.

II. From Lemma 1,

$$a_4 - a_2a_3 \le \tfrac{1}{6}\left(1 - c_1{}^2 - \frac{c_2{}^2}{1-c_1} + 3c_1c_2\right) =: h_2(c_1, c_2)\,.$$

The points $(0,0)$ and $(10/27, 85/243)$ are the only critical points of h_2. For these points $h_2(0,0) = 1/6$ and $h_2(10/27, 85/243) = 1156/6561 = 0.176\ldots$. Moreover,

$$h_2(c_1, 1 - c_1{}^2) = \tfrac{1}{3}(c_1 - c_1{}^3) \le \tfrac{2}{9\sqrt{3}}$$

and

$$h_2(c_1, -1 + c_1{}^2) = -\tfrac{2}{3}(c_1 - c_1{}^3) \le \tfrac{4}{9\sqrt{3}}\,.$$

The relation in (6) and the above results give the declared bound. □

From the definition of the class $\mathcal{S}^*_{\mathbb{R}}(\frac{1}{2})$ we can represent a function f of this class as follows

$$\frac{zf'(z)}{f(z)} = \frac{1}{1-\omega(z)}\,,\ \omega \in (\mathcal{B}_0)_{\mathbb{R}}\,. \tag{24}$$

Let f and ω be given by (1) and (8), respectively. Comparing the coefficients of both sides in

$$\left(z + 2a_2z^2 + 3a_3z^3 + \ldots\right) \cdot \left(1 - c_1z - c_2z^2 - \ldots\right) = z + a_2z^2 + a_3z^3 + \ldots\,,$$

we obtain

$$a_2 = c_1\,,\ a_3 = \tfrac{1}{2}c_2 + c_1{}^2\,,\ a_4 = \tfrac{1}{3}c_3 + \tfrac{7}{6}c_1c_2 + c_1{}^3\,. \tag{25}$$

Putting these formulae into (9) and (10) we have

$$a_2a_4 - a_3{}^2 = \tfrac{1}{3}c_1c_3 + \tfrac{1}{6}c_1{}^2c_2 - \tfrac{1}{4}c_2{}^2 \tag{26}$$

and

$$a_4 - a_2a_3 = \tfrac{1}{3}(c_3 + 2c_1c_2)\,. \tag{27}$$

Theorem 5. *If $f \in \mathcal{S}^*_{\mathbb{R}}(\frac{1}{2})$, then*

1. $a_2a_4 - a_3{}^2 \le 0.129\ldots$,
2. $|a_4 - a_2a_3| \le \frac{2\sqrt{3}}{9} = 0.384\ldots$.

Proof. Let $f \in \mathcal{S}^*_{\mathbb{R}}(\frac{1}{2})$.

I. By Lemma 1,

$$a_2a_4 - a_3{}^2 \le \tfrac{1}{3}\left(c_1(1 - c_1{}^2) + \tfrac{1}{2}c_1{}^2c_2 - \tfrac{1}{4}\frac{3+c_1}{1-c_1}c_2{}^2\right) =: h_3(c_1, c_2)\,.$$

The critical points of h_3 are the solutions of the system

$$\begin{cases} c_2 = \frac{3+c_1}{1-c_1}c_1{}^2 \\ 1 - 3c_1{}^2 + c_1c_2 - \left(\frac{c_2}{1-c_1}\right)^2 = 0\,. \end{cases}$$

We consider only the points lying in the set

$$\left\{(c_1, c_2) : -1 \le c_1 \le 1, -1 + c_1{}^2 \le c_2 \le 1 - c_1{}^2\right\}$$

(compare, Lemma 1). Hence, we obtain the equation

$$c_1{}^5 + 6c_1{}^4 + 15c_1{}^3 + 26c_1{}^2 - 6c_1 - 9 = 0\,.$$

The critical points are: $(-0.543\ldots, 0.185\ldots)$ and $(0.581\ldots, 0.039\ldots)$. For these points,

$$h_3(-0.543\ldots, 0.185\ldots) = -0.123\ldots \quad \text{and} \quad h_3(0.581\ldots, 0.039\ldots) = 0.129\ldots\,.$$

Moreover,

$$h_3(c_1, 1-c_1{}^2) = -\tfrac{1}{12}(1-c_1{}^2)(3-c_1{}^2) \le 0$$

and

$$h_3(c_1, -1+c_1{}^2) = -\tfrac{1}{4}(1-c_1{}^2)(1+c_1{}^2) \le 0\,.$$

This means that $a_2a_4 - a_3{}^2 \le 0.129\ldots$.

II. From Lemma 1,

$$a_4 - a_2a_3 \le \tfrac{1}{3}\left(1 - c_1{}^2 - \frac{c_2{}^2}{1-c_1} + 2c_1c_2\right) =: h_4(c_1, c_2)\,.$$

The point $(0,0)$ is the only critical point of h_4 and $h_4(0,0) = 1/3$. Moreover,

$$h_4(c_1, 1-c_1{}^2) = \tfrac{1}{3}(c_1 - c_1{}^3) \le \tfrac{2}{9\sqrt{3}}$$

and

$$h_4(c_1, -1+c_1{}^2) = -c_1 + c_1{}^3 \le \tfrac{2\sqrt{3}}{9}\,.$$

The relation in (6) and the above results give the declared bound. □

5. Preliminary Results for $\mathcal{T}(\frac{1}{2})$, $\mathcal{K}_{\mathbb{R}}(i)$ and $\mathcal{W}$

Let X be a compact Hausdorff space and $J_\mu = \int_X J(t)d\mu(t)$. Szapiel in [22] proved the following theorem.

Theorem 6 ([22], Thm.1.40)**.** *Let $J : [\alpha, \beta] \to \mathbb{R}^n$ be continuous. Suppose that there exists a positive integer k, such that for each non-zero p in $\mathbb{R}^n$ the number of solutions of any equation $\langle \vec{J(t)}, \vec{p}\,\rangle = const$, $\alpha \le t \le \beta$ is not greater than k. Then, for every $\mu \in P_{[\alpha,\beta]}$ such that J_μ belongs to the boundary of the convex hull of $J([\alpha, \beta])$, the following statements are true:*

1. if $k = 2m$, then

(a) $|\operatorname{supp}(\mu)| \le m$

or

(b) $|\operatorname{supp}(\mu)| = m+1$ and $\{\alpha, \beta\} \subset \operatorname{supp}(\mu)$,

2. if $k = 2m+1$, then

(a) $|\operatorname{supp}(\mu)| \le m$

or

(b) $|\operatorname{supp}(\mu)| = m+1$ and one of the points α, β belongs to $\operatorname{supp}(\mu)$.

In the above the symbol $\langle \vec{u}, \vec{v}\,\rangle$ denotes the scalar product of vectors $\vec{u}$ and $\vec{v}$, whereas the symbols P_X and $|\operatorname{supp}(\mu)|$ denote the set of probability measures on X and the cardinality of the support of μ, respectively.

Observe that the coefficients a_n of a function f belonging to the classes $\mathcal{T}(\frac{1}{2})$, $\mathcal{K}_{\mathbb{R}}(i)$ and $\mathcal{W}$ can be expressed by

$$a_n = \int_{-1}^{1} A_{n-1}(t)d\mu(t)\,, \tag{28}$$

where $A_n(t)$ is a polynomial of degree n.

Taking into account the fact that we estimate the functionals $a_2a_4 - a_3{}^2$ and $|a_4 - a_2a_3|$, depending only on 3 coefficients of f, it is enough to consider the vectors $J(t) = [A_1(t), A_2(t), A_3(t)]$, $t \in [-1,1]$ and $\vec{p} = [p_1, p_2, p_3]$. We can observe that

$$p_1A_1(t) + p_2A_2(t) + p_3A_3(t) = const, \quad t \in [-1,1] \tag{29}$$

is a polynomial equation of degree 3. Therefore, the Equation (29) has at most 3 solutions.

In particular, for the classes $\mathcal{T}(\frac{1}{2})$, $\mathcal{K}_{\mathbb{R}}(i)$ and $\mathcal{W}$, it is known that $A_n(t)$ are the Legendre polynomials $P_n(t)$, the Chebyshev polynomials $U_n(t)$ and the monomial t^n, respectively.

For a given class $A \subset \mathcal{A}_{\mathbb{R}}$, we denote by $\Omega(A)$ the region of variability of three succeeding coefficients of functions in A, that is, the set $\{(a_2(f), a_3(f), a_4(f)) : f \in A)\}$. Therefore, $\Omega(\mathcal{T}(\frac{1}{2}))$ is the closed convex hull of the curve

$$\gamma_1 : [-1,1] \ni t \mapsto (P_1(t), P_2(t), P_3(t)) \ ,$$

$\Omega(\mathcal{K}_{\mathbb{R}}(i))$ is the closed convex hull of the curve

$$\gamma_2 : [-1,1] \ni t \mapsto (U_1(t), U_2(t), U_3(t))$$

and $\Omega(\mathcal{W})$ is the closed convex hull of the curve

$$\gamma_3 : [-1,1] \ni t \mapsto \left(t, t^2, t^3\right) \ .$$

According to Theorem 6, the boundary of the convex hulls of $\gamma_k([-1,1])$, $k = 1,2,3$ are determined by atomic measures μ for which the support consists of 2 points at most with one of them being equal to -1 or 1. In this way, we have proved the following lemmas.

Lemma 3. *The boundary of $\Omega(\mathcal{T}(\frac{1}{2}))$ consists of points (a_2, a_3, a_4) that correspond to the following functions*

$$f(z) = \alpha\frac{z}{\sqrt{1-2tz+z^2}} + (1-\alpha)\frac{z}{1-z}\,, \quad \alpha \in [0,1]\,, \quad t \in [-1,1] \tag{30}$$

or

$$f(z) = \alpha\frac{z}{\sqrt{1-2tz+z^2}} + (1-\alpha)\frac{z}{1+z}\,, \quad \alpha \in [0,1]\,, \quad t \in [-1,1]\,. \tag{31}$$

Lemma 4. *The boundary of $\Omega(\mathcal{K}_{\mathbb{R}}(i))$ consists of points (a_2, a_3, a_4) that correspond to the following functions*

$$zf'(z) = \alpha\frac{z}{1-2tz+z^2} + (1-\alpha)\frac{z}{(1-z)^2}\,, \quad \alpha \in [0,1]\,, \quad t \in [-1,1] \tag{32}$$

or

$$zf'(z) = \alpha\frac{z}{1-2tz+z^2} + (1-\alpha)\frac{z}{(1+z)^2}\,, \quad \alpha \in [0,1]\,, \quad t \in [-1,1]\,. \tag{33}$$

Lemma 5. *The boundary of $\Omega(\mathcal{W})$ consists of points (a_2, a_3, a_4) that correspond to the following functions*

$$f(z) = \alpha\frac{z}{1-tz} + (1-\alpha)\frac{z}{1-z}\,, \quad \alpha \in [0,1]\,, \quad t \in [-1,1] \tag{34}$$

or

$$f(z) = \alpha\frac{z}{1-tz} + (1-\alpha)\frac{z}{1+z}\,, \quad \alpha \in [0,1]\,, \quad t \in [-1,1]\,. \tag{35}$$

6. Estimates for the Class $\mathcal{T}(\frac{1}{2})$, $\mathcal{K}_{\mathbb{R}}(i)$ and $\mathcal{W}$

Now, we are ready to derive the sharp bound of (9) and (10) in $\mathcal{T}(\frac{1}{2})$, $\mathcal{K}_{\mathbb{R}}(i)$ and $\mathcal{W}$.

Theorem 7. *If $f \in \mathcal{T}(\frac{1}{2})$, then*

1. $|a_4 - a_2a_3| \leq \frac{256}{405}$,
2. $a_2a_4 - a_3{}^2 \leq 0.204\ldots$.

The results are sharp.

Proof. Function (30) has the Taylor series expansion

$$f(z) = z + \sum_{k=2}^{\infty} \left[\alpha P_{k-1}(t) + (1-\alpha)P_{k-1}(1)\right] z^k \,. \tag{36}$$

Let $a_4 - a_2a_3 = g_1(\alpha, t)$ and $a_2a_4 - a_3{}^2 = g_2(\alpha, t)$. Using (36), we can write

$$\begin{aligned} g_1(\alpha,t) = {} & [\alpha P_3(t) + (1-\alpha)P_3(1)] \\ & - [\alpha P_1(t + (1-\alpha)P_1(1)] \cdot [\alpha P_2(t) + (1-\alpha)P_2(1)] \,, \end{aligned} \tag{37}$$

where

$$P_1(t) = t, \quad P_2(t) = \tfrac{1}{2}(3t^2 - 1) \quad \text{and} \quad P_3(t) = \tfrac{1}{2}(5t^3 - 3t) \,.$$

Therefore,

$$g_1(\alpha,t) = \tfrac{3}{2}\alpha(1-\alpha)(1-t)(1-t^2) - (1-t^3)\alpha + (1-t)\alpha \,. \tag{38}$$

Similarly,

$$\begin{aligned} g_2(\alpha,t) = {} & [\alpha P_1(t) + (1-\alpha)P_1(1)] \cdot [\alpha P_3(t) + (1-\alpha)P_3(1)] \\ & - [\alpha P_2(t) + (1-\alpha)P_2(1)]^2 \,. \end{aligned} \tag{39}$$

Hence,

$$g_2(\alpha,t) = \tfrac{1}{4}\alpha^2(t^4 - 1) + \alpha(1-\alpha)\left(\tfrac{5}{2}t^3 - 3t^2 - \tfrac{1}{2}t + 1\right) \,. \tag{40}$$

It is easy to conclude that for function (31) we have $a_4 - a_2a_3 = -g_1(\alpha, -t)$ and $a_2a_4 - a_3{}^2 = g_2(\alpha, -t)$. For this reason, it is enough to discuss only the function given by (30).
I. The critical points of g_1 inside the set $[0,1] \times [-1,1]$ coincide with the solutions of the system

$$\begin{cases} \frac{1}{2}(1-t^2)\left[(6\alpha-5)t - 6\alpha + 3\right] = 0 \\ \frac{1}{2}\alpha\left[t^2(15-9\alpha) + 6t(\alpha-1) + 3\alpha - 5\right] = 0 \,. \end{cases}$$

From the first equation we have $t = \frac{6\alpha-3}{6\alpha-5}$. Putting it into the second one, we obtain only one critical point $\alpha = \frac{20}{33}$, $t = -\frac{7}{15}$ and $g_1(\frac{20}{33}, -\frac{7}{15}) = \frac{256}{405}$.

Now we need to verify the behavior of the function $g_1(\alpha, t)$ on the boundary of the set $[0,1] \times [-1,1]$. We have

$$g_1(0,t) = g_1(\alpha,-1) = g_1(\alpha,1) = 0$$

and

$$g_1(1,t) = t^3 - t \,.$$

For $t \in [-1,1]$ there is $\frac{-2\sqrt{3}}{9} \leq g_1(1,t) \leq \frac{2\sqrt{3}}{9}$.

Therefore, we conclude that $g_1(\alpha,t) \leq \frac{256}{405}$ for all $(\alpha,t) \in [0,1] \times [-1,1]$. Taking into account (6), we obtain the same estimates of $|g_1(\alpha,t)| \leq \frac{256}{405}$. The equality holds for the function (30) with $\alpha = \frac{20}{33}$ and $t = -\frac{7}{15}$, so the extremal function has the form

$$f(z) = z + \tfrac{1}{9}z^2 + \tfrac{13}{45}z^3 + \tfrac{269}{405}z^4 + \ldots \,.$$

Thus, the estimate is sharp.

II. Now, we shall derive the greatest value of $g_2(\alpha, t)$ for $\alpha \in [0,1]$ and $t \in [-1,1]$. The critical points of g_2 inside the set $[0,1] \times [-1,1]$ coincide with the solutions of the system of equations

$$\begin{cases} \frac{1}{2}(t-1)\left[\alpha t^3 + t^2(5-9\alpha) + t(3\alpha-1) + 5\alpha - 2\right] = 0 \\ \alpha\left[\alpha t^3 + \frac{1}{2}(1-\alpha)(15t^2 - 12t - 1)\right] = 0\,. \end{cases}$$

From the second equation we have $\alpha = \frac{-15t^2+12t+1}{2t^3-15t^2+12t+1}$. Combining it with the first equation we obtain $\alpha = 0.403\ldots$, $t = -0.076\ldots$ and $g_2(0.403\ldots, -0.076\ldots) = 0.204\ldots$.

If we examine the behavior of $g_2(\alpha, t)$ on the boundary of the set $[0,1] \times [-1,1]$, we have $g_2(0,t) = g_2(\alpha, 1) = 0$,

$$g_2(\alpha, -1) = -4\alpha(1-\alpha) \le 0 \quad \text{and} \quad g_2(1,t) = \tfrac{1}{4}(t^4 - 1) \le 0\,.$$

Therefore, we conclude that for all $(\alpha, t) \in [0,1] \times [-1,1]$ we have $g_2(\alpha, t) \le 0.204\ldots$. The equality holds for function (30) with $\alpha = 0.403\ldots$ and $t = -0.076\ldots$. Thus, the estimate is sharp. □

Theorem 8. *If $f \in \mathcal{K}_{\mathbb{R}}(i)$,*

1. $|a_4 - a_2 a_3| \le \frac{125}{243}$,
2. $a_2 a_4 - {a_3}^2 \le \frac{1}{7}$.

The results are sharp.

Proof. Observe that the function given by (32) has the following Taylor series expansion

$$zf'(z) = z + \frac{1}{k}\sum_{k=2}^{\infty}\left[\alpha U_{k-1}(t) + (1-\alpha)U_{k-1}(1)\right] z^k. \tag{41}$$

Let $a_4 - a_2 a_3 = g_3(\alpha, t)$ and $a_2 a_4 - {a_3}^2 = g_4(\alpha, t)$. Using the same reasoning as in the proof of Theorem 7, we consider the function given by (32).

Notice that

$$\begin{aligned} g_3(\alpha, t) = {} & \tfrac{1}{4}\left[\alpha U_3(t) + (1-\alpha)U_3(1)\right] \\ & - \tfrac{1}{6}\left[\alpha U_1(t) + (1-\alpha)U_1(1)\right] \cdot \left[\alpha U_2(t) + (1-\alpha)U_2(1)\right]\,, \end{aligned} \tag{42}$$

where

$$U_1(t) = 2t, \quad U_2(t) = 4t^2 - 1, \quad U_3(t) = 8t^3 - 4t\,.$$

Hence,

$$g_3(\alpha, t) = \alpha\left(2t^3 - t\right) + (1-\alpha) - \tfrac{1}{3}(\alpha t + 1 - \alpha)\left[\alpha(4t^2 - 1) + 3(1-\alpha)\right]\,. \tag{43}$$

Similarly,

$$\begin{aligned} g_4(\alpha, t) = {} & \tfrac{1}{8}\left[\alpha U_1(t) + (1-\alpha)U_1(1)\right] \cdot \left[\alpha U_3(t) + (1-\alpha)U_3(1)\right] \\ & - \tfrac{1}{9}\left[\alpha U_2(t) + (1-\alpha)U_2(1)\right]^2\,. \end{aligned} \tag{44}$$

Hence,

$$g_4(\alpha, t) = (\alpha t + 1 - \alpha)\left[\alpha(2t^3 - t) + (1-\alpha)\right] - \tfrac{1}{9}\left[\alpha(4t^2 - 1) + 3(1-\alpha)\right]^2\,. \tag{45}$$

I. Similarly, as in the proof of Theorem 7, we find that $(\frac{15}{26}, -\frac{4}{9})$ is the only critical point of g_3 inside the set $[0,1] \times [-1,1]$. Therefore, $g_3(\frac{15}{26}, -\frac{4}{9}) = \frac{125}{243}$.

Now, we need to study the behavior of $g_3(\alpha, t)$ on the boundary of the set $[0,1] \times [-1,1]$. We have

$$g_3(0,t) = g_3(\alpha,-1) = g_3(\alpha,1) = 0$$

and

$$g_3(1,t) = \tfrac{2}{3}(t^3 - t)\,.$$

For $t \in [-1,1]$, there is $-\frac{4}{9\sqrt{3}} \le g_3(1,t) \le \frac{4}{9\sqrt{3}}$.

Taking into account the previous reasoning, we obtain the desired estimation $|g_3(\alpha,t)| \le \frac{125}{243}$. The equality holds for function (32) with $t = -\frac{4}{9}$ and $\alpha = \frac{15}{26}$, so we obtain the extremal function

$$f(z) = \tfrac{1}{26}\left(\tfrac{135}{\sqrt{65}}\arctan\left(\tfrac{\sqrt{65}z}{4z+9}\right) + \tfrac{11z}{1-z}\right) = z + \tfrac{1}{6}z^2 + \tfrac{31}{81}z^3 + \tfrac{281}{486}z^4 + \ldots\,.$$

Thus, the estimate is sharp.

II. It is easy to verify that $(\frac{3}{7},0)$ is the only critical point of g_4 inside the set $[0,1] \times [-1,1]$. Therefore, $g_4(\frac{3}{7},0) = \frac{1}{7}$.

Analyzing the behavior of the function $g_4(\alpha,t)$ on the boundary of the set $[0,1] \times [-1,1]$, we have

$$g_4(\alpha,1) = g_4(0,t) = 0, \quad g_4(1,t) = \tfrac{1}{9}(2t^4 - t^2 - 1) \le 0$$

and

$$g_4(\alpha,-1) = 4\alpha^2 - 4\alpha \le 0\,.$$

Hence, we conclude that $g_4(\alpha,t) \le \frac{1}{7}$ for all $(\alpha,t) \in [0,1] \times [-1,1]$. The equality holds for function (32) with $\alpha = \frac{3}{7}$ and $t = 0$, so we obtain the extremal function

$$f(z) = \tfrac{1}{7}\left(3\arctan z + \tfrac{4z}{1-z}\right) = z + \tfrac{4}{7}z^2 + \tfrac{3}{7}z^3 + \tfrac{4}{7}z^4 + \ldots$$

Thus, the estimate is sharp. □

For the function given by (34) we have

$$a_2 = \alpha t + 1 - \alpha\,,\ a_3 = \alpha t^2 + 1 - \alpha\,,\ a_4 = \alpha t^3 + 1 - \alpha\,.$$

Therefore,

$$a_4 - a_2 a_3 = \alpha(1-\alpha)(1+t)(1-t)^2$$

and

$$a_2 a_4 - {a_3}^2 = \alpha(1-\alpha)t(1-t)^2\,.$$

For $\alpha \in [0,1]$ there is $\alpha(1-\alpha) \le \frac{1}{4}$, so

$$|a_4 - a_2 a_3| = a_4 - a_2 a_3 \le \tfrac{1}{4}(1-t)(1-t)^2\,.$$

The latter attains the greatest value for $t = -\frac{1}{3}$. Hence, $|a_4 - a_2 a_3| \le \frac{8}{27}$.
The equality holds for (34) with $\alpha = \frac{1}{2}$ and $t = -\frac{1}{3}$, so for

$$f(z) = \tfrac{3z - z^2}{(3+z)(1-z)} = z + \tfrac{1}{3}z^2 + \tfrac{5}{9}z^3 + \tfrac{13}{27}z^4 + \ldots$$

Furthermore, observe that

$$\max\{a_2 a_4 - {a_3}^2 : \alpha \in [0,1], t \in [-1,1]\} = \max\{a_2 a_4 - {a_3}^2 : \alpha \in [0,1], t \in [0,1]\}\,.$$

Hence,

$$a_2a_4 - {a_3}^2 \le \tfrac{1}{4}t(1-t)^2 \le \tfrac{1}{27}\,.$$

In this way we have proved the following theorem.

Theorem 9. *If $f \in \mathcal{W}$, then*

1. $a_2a_4 - {a_3}^2 \le \frac{1}{27}$,
2. $|a_4 - a_2a_3| \le \frac{8}{27}$.

The results are sharp.

7. Concluding Remarks

In this paper we derived the upper estimates of the functionals $a_2a_4 - {a_3}^2$ and $|a_4 - a_2a_3|$ for the functions in the subclasses of $\mathcal{Q}_{\mathbb{R}}(\frac{1}{2})$. In the paper two different methods were applied. In the first method we expressed the coefficients of a function in a given class by coefficients with a corresponding Schwarz function or a function with positive real part. The second method was based on the Szapiel theorem. This theorem allowed us to obtain the sharp bounds of the functionals (9) and (10) and to point out the extremal functions.

It is clear that if $A_1 \subset A_2$, then

$$\max\{a_2a_4 - {a_3}^2 : f \in A_1\} \le \max\{a_2a_4 - {a_3}^2 : f \in A_2\}$$

and

$$\max\{|a_4 - a_2a_3| : f \in A_1\} \le \max\{|a_4 - a_2a_3| : f \in A_2\}\,.$$

The obtained results satisfy the above inequalities and coincide with the inclusions presented in Section 3.

Author Contributions: Conceptualization, P.Z.; Formal analysis, P.Z., A.F. and M.J.; Supervision, P.Z.; Writing–original draft, A.F. and M.J. All authors contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

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