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Limiting Values and Functional and Difference Equations †

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Abstract: Boundary behavior of a given important function or its limit values are essential in the whole spectrum of mathematics and science. We consider some tractable cases of limit values in which either a difference of two ingredients or a difference equation is used coupled with the relevant functional equations to give rise to unexpected results. As main results, this involves the expression for the Laurent coefficients including the residue, the Kronecker limit formulas and higher order coefficients as well as the difference formed to cancel the inaccessible part, typically the Clausen functions. We establish these by the relation between bases of the Kubert space of functions. Then these expressions are equated with other expressions in terms of special functions introduced by some difference equations, giving rise to analogues of the Lerch-Chowla-Selberg formula. We also state Abelian results which not only yield asymptotic formulas for weighted summatory function from that for the original summatory function but assures the existence of the limit expression for Laurent coefficients.

Keywords: Lerch-Chowla-Selberg formula; modular relation; Laurent coefficients; Lerch zeta-function; Hurwitz zeta-function

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1. Introduction

In a sense, whole mathematics especially number theory has been centered around the Laurent coefficients of the relevant zeta-and L-functions since Dirichlet, Kronecker, Lerch, Hecke, Siegel, Choela-Selberg. It is no wonder that there have appeared enormous amount of papers on the Laurent coefficients of a large class of zeta-, L- and special functions. They are referred to as generalized Euler constants or Euler-Stieltjes constants [1–9], [10] (pp. 71–81), [11–14] to name a few. Reference [15] is concerned with a series expression for the Euler digamma function

$$\psi(x) = \frac{\Gamma'}{\Gamma}(x) \tag{1}$$

with Hurwitz zeta-function values coefficients, where the Hurwitz zeta-function is defined by (10) below. References [5,6] are concerned with the Gaussian field.

We shall establish the far-reaching theorem, Theorem 4 below, as a consequence of the relation between bases of the Kubert space of functions and the fact that the polylogarithm function is the limit function.

In literature one finds isolated-looking formulas without references to their origin, e.g., ([16], (10), p. 10)

$$\psi(z) - \psi(1-z) = -\pi \cot \pi z \tag{2}$$

or ([17], (2.6))

$$-\pi i \left(1 + 2\sum_{n=1}^{\infty} e^{2\pi i n\tau}\right) = \pi \cot \pi\tau = \lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1}{n+\tau}, \tag{3}$$

the partial fraction expansion for the contangent, which is known to be equivalent to the functional equation for the Riemann zeta-function (see [18], Chapter 5). In this paper we refer to the results in [19,20] freely. Especially [19] (§4.3), [8] (§1.2, Chapter 4), etc. Equation (2) underlies (81) below. We shall elucidate the source of them in Section 8.

Let χ be a Dirichlet character mod M and let

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \quad \sigma = \mathrm{Re}\, s > 1 \tag{4}$$

be the Dirichlet L-function associated to χ. We may include the Riemann zeta-function as a special case with the trivial character mod 1; $\chi_0^*(n) = 1$.

$$\zeta(s) = L(s,\chi_0^*) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \sigma > 1. \tag{5}$$

These are continued meromorphically over the whole plane with a simple pole at $s = 1$ for the principal character. $L(s,\chi)$ with a non-principal character χ can be continued analytically over $\sigma > 0$ and the point $s = 1$ is a regular point with the Taylor coefficients $L^{(k)}(s,\chi)$, $k \geq 0$ as well as special values $K(k,\chi)$ (see Section 4).

Indeed, in ([10], pp. 71–81), almost all (buds of) ingredients are given for the study of generalized Euler constants and is worth being called genesis of the study of generalized Euler constants including the Kronecker limit formula ([10], pp. 161–182). It gives an explicit evaluation of the Laurent constant of the Epstein zeta-function.

The generalized Euler constants $\gamma_k(a,M)$ in (53) for an arithmetic progression is naturally a highlighted subject and after [4,9,11,21], Shirasaka [12] is a culmination providing the genuine generating function for them, based on the theory of Hurwitz zeta-function. In another direction, References [8,22] are another summit of the study on periodic Dirichlet series, appealing to the Deninger-Meyer method, based on the theory of Lerch zeta-function. One of our main results is Theorem 4 which give the evaluation of the generalized Euler constants in terms of Hurwitz and Lerch zeta-functions. This reflects the base change of the Kubert space $\mathcal{K}_s$ as described by Milnor. The other main result of ours is an Abelian theorem in the spirit of [2,10,23] entailing [24]. Under the assumption of the functional equation, this depends on [25] and partly [26] and without it, depends on [23]. Thus these two theorems pave a promenade to the study on Laurent series coefficients of a large class of zeta- and special functions starting from the Abelian theorem.

2. Lerch Zeta-Function

Here we assemble various facts about the Lerch zeta-function, see also Proposition 2 below and the subsequent passage. The Lerch zeta-function is the polylogarithm function of order s with complex exponential argument

$$\ell_s(x) = \sum_{n=1}^{\infty} \frac{e^{2\pi inx}}{n^s}. \tag{6}$$

absolutely convergent for $\sigma = \operatorname{Re} s > 1$.

This is a special case of (9) with $\tau = x \in \mathbb{R}$, $\sigma > 1$ and is a boundary function thereof and we sometimes refer to it as the boundary Lerch zeta-function.

In general the polylogarithm function of order s is defined by

$$\mathrm{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}, \tag{7}$$

for $s \in \mathbb{C}$, $|z| < 1$ or $\operatorname{Re} \sigma > 1$, $|z| \le 1$ cf. [27,28], ([16], pp. 114–127),

The domain of convergence—unit disc $|z| < 1$ is mapped onto the upper half-plane by the familiar transformation

$$q = e^{2\pi i\tau}, \quad \tau \in \mathcal{H} \tag{8}$$

with $\mathcal{H}$ denoting the upper half-plane $\operatorname{Im} \tau > 0$. Under (8),

$$\ell_s(\tau) = \mathrm{Li}_s\left(e^{2\pi i\tau}\right) = \sum_{n=1}^{\infty} \frac{e^{2\pi in\tau}}{n^s}, \tag{9}$$

which is absolutely convergent in $\mathcal{H}$ for all $s \in \mathbb{C}$. (9) is the q-expansion, i.e., the Laurent expansion around ∞. The unit circle corresponds to the boundary of $\mathcal{H}$, the real line $\operatorname{Im} \tau = 0$, i.e., $\tau = x \in \mathbb{R}$.

See [29] for boundary functions in which a looser condition of almost convergence is assumed. In our case it is analyticity and it produces rich results.

In the last a few decades, the most fundamental and influential work related to the Lerch zeta-function are [30–33] which are partly incorporated in [34]. Milnor [33] gives a very clear description of the 2-dimensional vector space of Kubert functions and above all things elucidates the functional Equations (11) and (12) for the Hurwitz and the Lerch zeta-function as the relation between basis elements. Yamamoto [30] established the vector space structure of periodic arithmetic functions $C(M)$ or the corresponding Dirichlet series $D(M)$ and evaluated the special values $L(k, \chi)$ by providing sound basics for the Lerch zeta-function. The main results are the expressions of short character sums in terms of these special values (see [13]). The method of discrete Fourier transform DFT (or finite Fourier series) has been also developed therein. For examples, see [13,35,36].

As a continuation of these, the second author [9] applied the Deninger-Meyer method to evaluate the three types of quantities $L(k, \chi)$, $L^{(k)}(1, \chi)$ and the generalized Euler constants $\gamma_k(a, M)$ (see Section 4).

The boundary Lerch zeta-function (6) has its counterpart, the Hurwitz zeta-function

$$\zeta(s, x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}, \quad \sigma > 1. \tag{10}$$

This is continued meromorphically over the whole plane with a simple pole at $s = 1$.

These are connected by the Hurwitz formula (i.e., the functional equation for the Hurwitz zeta-function), with $x = 1$ being the limiting case (14): for $\sigma > 1, 0 < x \le 1$,

$$\zeta(1-s, x) = \frac{\Gamma(s)}{(2\pi)^s} \left(e^{-\frac{\pi is}{2}}\, \ell_s(x) + e^{\frac{\pi is}{2}}\, \ell_s(1-x)\right), \tag{11}$$

while its reciprocal is

$$\ell_{1-s}(x) = \frac{\Gamma(s)}{(2\pi)^s}\left(e^{\frac{\pi is}{2}}\zeta(s,x) + e^{-\frac{\pi is}{2}}\zeta(s,1-x)\right), \quad 0 < x < 1. \tag{12}$$

By Equation (12), the boundary Lerch zeta-function $\ell_{1-s}(x)$ is continued meromorphically over the whole plane with $s = 0$ a plausible singular point. However, it is also a regular posit since the pole of $\Gamma(s)$ is cancelled by the factor $\zeta(s,x) + \zeta(s,1-x) = 0$ by (75). See ([18], pp. 145–147), ([34], pp. 65–84), [32], etc.

Both of these reduce to the Riemann zeta-function for $x \in \mathbb{Z}$ for the former and $x = 1$ for the latter:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{13}$$

valid for $\sigma > 1$ in the first instance. This is continued meromorphically over the whole place with a simple pole at $s = 1$ by way of the functional equation

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s), \tag{14}$$

which is a direct special case of (19) and indeed is equivalent to (12). Most of known zeta-functions satisfy the functional equation of this kind and the relations equivalent to it, the *modular relation*s, have been developed in [20].

We remark that both the Lerch and Hurwitz zeta-functions are special cases of the Lipschitz-Lerch transcendent $L(\xi,s,x)$ ([34], Chapter 3) which in turn is the boundary function of the Hurwitz-Lerch zeta-function $\Phi(z,s,x)$

$$\Phi(z,s,x) = \sum_{n=0}^{\infty} \frac{z^n}{(n+x)^s}, \quad L(\xi,s,x) = \sum_{n=0}^{\infty} \frac{e^{2\pi inx}}{(n+x)^s}, \quad \xi \in \mathbb{R}. \tag{15}$$

In the paper [34] ubiquity of the Lerch zeta-function, especially the monologarithm $\ell_1(x)$ (21) of the complex exponential argument, has been pursued.

In [13] the ubiquity and omnipotence is shown of the (boundary) Lerch zeta-function by proving above all things that even Yamamoto's decisive results on short character sums [30], in the long run, are consequences of the modular relation, i.e., the functional equation for the Lerch zeta-function. Also mentioned is the closed expressions for the Laurent coefficients, called generalized Euler constants (see Section 4).

Below we assemble various known facts about Lerch and Hurwitz zeta-function that lie scattered around in literature.

For $s = 1$, the series on the right of (6) is uniformly convergent in an interval not containing an integer and defines the polylogarithm function of order 1, which is indeed, the boundary function given as the genuine Fourier series—the monologarithm function (21) below. See [37]. We refer to (9) as the Lerch zeta-function with (6) as its boundary function.

Thus the Lerch zeta-function incorporates three aspects—modular function (9), the boundary function as the zeta-function (6) and the monologarithm function (6) with $s = 1$.

For fixed $s \in \mathbb{C}$, the Lerch zeta-function (9) is a one-valued analytic function on the τ-plane with slit along negative imaginary axis and its translations by integers, i.e.,

$$\mathbb{C}_1 = \mathbb{C} - \{n + iy | n \in \mathbb{Z}, y \leq 0\}. \tag{16}$$

Let $B_\varkappa(x) = \sum_{a=0}^{\varkappa} \binom{\varkappa}{a} B_a x^{\varkappa-a}$ be the $\varkappa$-th Bernoulli polynomial of degree $\varkappa$ with $B_0(x) = 1$ and let $[x]$ be the integer part of x .

For $\varkappa \in \mathbb{N}$, the $\varkappa$-th periodic Bernoulli polynomial $\bar{B}_\varkappa(x)$ is defined by

$$\bar{B}_\varkappa(x) = B_\varkappa(\{x\}) = \sum_{a=0}^{\varkappa} \binom{\varkappa}{a} B_a \{x\}^{\varkappa-a}, \tag{17}$$

where $\{x\} = x - [x]$ indicates the fractional part.

The following is well-known and taken as a heaven-sent fact that it has the Fourier expansion, cf. [30]

$$\bar{B}_\varkappa(x) = -\frac{\varkappa!}{(2\pi i)^\varkappa} \sum_{n=-\infty}^{\infty}{}' \frac{e^{2\pi i n x}}{n^\varkappa}, \tag{18}$$

where the prime on the summation sign means that $n = 0$ is excluded and the summation is taken in symmetric sense.

In ([34], (3.7), pp. 48–50) there is a list of equivalent expressions for the Hurwitz and reverse Hurwitz formulas. One of them reads

$$\zeta(s, x) = \frac{\Gamma(1-s)}{(2\pi)^{1-s}} \sum_{n=1}^{\infty} \frac{\sin(2\pi n x + (\pi s)/2)}{n^{1-s}}, \tag{19}$$

which in particular gives the Fourier series for $\zeta(0, x) = -\bar{B}_1(x)$. More generally, see ([34], (1.1)). Hence (18) may be viewed as a consequence of the modular relation (19).

The novel view-point stated for the first time in [13] is that it is the very Definition (6), as the boundary function of the "q-expansion," of the Lerch zeta-function that gives the Fourier expansion (18) and (70) when substituted in (71) and (72) and that for finding radial limits, use of the Lerch zeta-function is in the very nature of things since it does is the limit function.

For the latter, as has been noticed [34], there are many instances of radial limits in which the odd part, the first periodic Bernoulli polynomial, of the polylogarithm function of order 1 appears as a result of eliminating the real part, the log sin function. For examples, see [38,39]. The polylogarithm function of order 1 is indeed the monologarithm function, the ordinary logarithm function extended to the circle of convergence ($|z| = 1, z \neq 1$)

$$-\log(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n} = L_1(z). \tag{20}$$

The series is absolutely convergent for $|z| < 1$ and uniformly convergent for $|z| = 1, z \neq 1$.

We assemble the identities for $\ell_1(s)$ of which use has been made.

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{\cos(2\pi n x)}{n} + i \sum_{n=1}^{\infty} \frac{\sin(2\pi n x)}{n} = \ell_1(x) \\ &= -\log\left(1 - e^{2\pi i x}\right) = \sum_{n=1}^{\infty} \frac{e^{2\pi i n x}}{n} = A_1(x) - \pi i \bar{B}_1(x), \end{aligned} \tag{21}$$

$0 < x < 1$, where

$$A_1(x) = -\log 2|\sin \pi x| = \sum_{n=1}^{\infty} \frac{\cos(2\pi n x)}{n}, \tag{22}$$

is its real part, the first Clausen function (or the loggamma function) and the imaginary part is (71) with $\varkappa = 1$, which reads

$$x - [x] - \frac{1}{2} = \bar{B}_1(x) = -\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2\pi n x)}{n} \tag{23}$$

for $x \notin \mathbb{Z}$, where $[x]$ indicates the greatest integer function.

3. Limit Values in Riemann's Fragment II

We elucidate the main ingredients from the first author's paper [38] and its sequel [39]. Reference [38] condenses the 67 pages long paper [40] into 17 pages.

In order to remove singularities, it is rather common to use a well-known device of taking the odd part or an alternate sum described by

$$\sum_{2\nmid n} a_n = \sum_n a_n - \sum_{2|n} a_n \tag{24}$$

resp.

$$\sum_n (-1)^n a_n = \sum_{2|n} a_n - \sum_{2\nmid n} a_n = 2\sum_{2|n} a_n - \sum_n a_n, \tag{25}$$

by (24), where n runs over a finite range or the series are absolutely convergent. All sums in Definition 1 are odd parts. Thus the following result follows if only we may take the limit $z \to \xi \in \mathbb{R}$, which is assured by Theorem 2 below.

All the theorems that Riemann considers in the second fragment are rephrases of the results of §40 of Jacobi and we state them as the following

Definition 1. *Let $\tau \in \mathcal{H}$. The elliptic modular functions $k = k(\tau)$, $K = K(\tau)$, $k' = k'(\tau)$ are defined respectively by*

$$\log k - \log 4\sqrt{\tau} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \frac{4\tau^n}{1+\tau^n}, \tag{26}$$

$$\log \frac{2K}{\pi} = \sum_{p=1}^{\infty} \frac{4\tau^p}{p(1+\tau^p)}, \tag{27}$$

and

$$-\log k' = \sum_{p=1}^{\infty} \frac{8z\tau^p}{p(1-\tau^{2p})}, \tag{28}$$

where in the last two sums, following Riemann, p runs through odd integers.

Theorem 1 ([38], Theorem 3)**.** *Let $\xi = \frac{M}{Q}$ be a rational number with M even and $Q > 1$ and let $z = ye^{\pi i \xi}$, $y \in [0, 1)$. Then we have*

$$\log k = \frac{1}{2}\log y + \frac{M\pi}{2Q} i + \omega(y) - 2\sum_{r=1}^{Q-1} (-1)^r \left(l_1\left(\frac{Mr}{Q}\right) - l_1\left(\frac{Mr}{2Q}\right)\right), \tag{29}$$

and

$$\log \frac{2K}{\pi} = -\log(1-y) + \omega(y) + \log\frac{\pi}{Q} + \sum_{r=1}^{Q-1} (-1)^r \left(l_1\left(\frac{Mr}{Q}\right) - 2l_1\left(\frac{Mr}{2Q}\right)\right), \tag{30}$$

and

$$\begin{aligned} -\log k' = {} & \frac{\pi^2}{2Q^2(1-y)} - \frac{\pi^2}{4Q^2} - \log 4 + \omega(y) + 2\sum_{r=0}^{\frac{Q-1}{2}-1} \frac{2r+1}{Q} \times \\ & \left(2l_1\left(\frac{M(2r+1)}{2Q}\right) - l_1\left(\frac{M(2r+1)}{Q}\right) - 2l_1\left(\frac{-M(2r+1)}{2Q}\right) + l_1\left(\frac{-M(2r+1)}{Q}\right)\right), \end{aligned} \tag{31}$$

for Q odd. Similar but more involved results hold for Q even.

To state the Dirichlet-Abel theorem ([38], Theorem 1) we need rudiments of the Discrete Fourier Transform (DFT). Its theory is stated in many literature, [35,36], ([41], pp. 89–109) etc. The theory of

DFT for arithmetic functions has been developed in [42] in the case of periodic functions. See also ([18], §8.1), ([34], §§4.1, 4.3). Let

$$\varepsilon_j(a) = e^{2\pi i j a/M}, \quad 1 \le j \le M, \tag{32}$$

where a is an integer variable. Then the set $\{\varepsilon_j(a) | 1 \le j \le M\}$ forms a basis of the vector space $C(M)$ of all periodic functions f with period M. We define the DFT $\hat{f}$ (or the bth Fourier coefficient) of $f \in C(M)$ by

$$\hat{f}(b) = \frac{1}{\sqrt{M}} \sum_{a=1}^{M} \varepsilon_b(-a) f(a). \tag{33}$$

Then the Fourier inversion or Fourier expansion formula holds true:

$$f(a) = \frac{1}{\sqrt{M}} \sum_{b=1}^{M} \hat{f}(b)\varepsilon_b(-a) = \hat{\hat{f}}(-a). \tag{34}$$

Note that (34) is the expression of f with respect to the basis $\{\varepsilon_j\}$. An important relation reads

$$\chi(n) = \tau(\bar{\chi})^{-1} \sum_{j=1}^{M-1} \bar{\chi}(j)\varepsilon_j(n). \tag{35}$$

with the Gauss sum

$$\tau(\chi) = \sum_{a=1}^{M} \chi(a) e^{\frac{2\pi i a}{M}}. \tag{36}$$

Another instance is the following. Let χ_a be the characteristic function χ_a mod M ([22], p. 73).

$$\chi_a(n) = \begin{cases} 1 & n \equiv a \bmod M \\ 0 & n \not\equiv a \bmod M \end{cases}. \tag{37}$$

Then $\{\chi_a | 1 \le a \le M\}$ is a basis of $C(M)$ and

$$\sqrt{M}\hat{\chi}_a(n) = \sum_{j=1}^{M} \varepsilon_n(-j)\chi_a(j) = \varepsilon_a(-n). \tag{38}$$

Let

$$D(s, f) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}. \tag{39}$$

Since

$$\sum_{n=1}^{\infty} \frac{|f(n)|}{n^\sigma} \ll \zeta(\sigma),$$

the series in (39) is absolutely convergent for $\sigma > 1$. Let $D(M)$ denote the set of all Dirichlet series of the form (39). Then it forms a vector space of dimension M canonically isomorphic to $C(M)$. One of the bases of $D(M)$ is $\{\ell_s\left(\frac{a}{M}\right) | 1 \le a \le M\}$. Hence we have

$$D(s, f) = \frac{1}{\sqrt{M}} \sum_{a=1}^{M} \hat{f}(-a)\ell_s\left(\frac{a}{M}\right) = \frac{1}{\sqrt{M}} \sum_{a=1}^{M-1} \hat{f}(a)\ell_s\left(\frac{a}{M}\right) + \frac{\hat{f}(M)}{\sqrt{M}}\zeta(s). \tag{40}$$

It follows that $D(s, f)$ can be continued meromorphically over the whole plane and that it is entire if and only if $\hat{f}(M) = 0$ which is defined by

$$\hat{f}(M) = \frac{1}{\sqrt{M}} \sum_{a=1}^{M} f(a). \tag{41}$$

Another basis of $D(M)$ is $\{D(s,\chi_a) | 1 \le a \le M\}$, where

$$D(s,\chi_a) = \sum_{n=1}^{\infty} \frac{\chi_a(n)}{n^s} = \sum_{\substack{n=1 \\ n \equiv a \bmod M}}^{\infty} \frac{1}{n^s} = \zeta(s,a,M) = M^{-s}\zeta\left(s, \frac{a}{M}\right), \tag{42}$$

where $\zeta(s,a,M)$ indicates the partial zeta-function. Hence in parallel to (40), we have another expression

$$D(s,f) = \sum_{a=1}^{M} f(a)\zeta(s,a,M) = \frac{1}{M^s}\sum_{a=1}^{M} f(a)\zeta\left(s, \frac{a}{M}\right). \tag{43}$$

Proposition 1.

$$\begin{aligned} \frac{1}{M^s}\sum_{a=1}^{M} f(a)\zeta\left(s, \frac{a}{M}\right) &= \sum_{a=1}^{M} f(a)\zeta(s,a,M) = D(s,f) \\ &= \frac{1}{\sqrt{M}}\sum_{a=1}^{M} \hat{f}(-a)\ell_s\left(\frac{a}{M}\right) = \frac{1}{\sqrt{M}}\sum_{a=1}^{M-1} \hat{f}(a)\ell_s\left(\frac{a}{M}\right) + \frac{\hat{f}(M)}{\sqrt{M}}\zeta(s). \end{aligned} \tag{44}$$

Proof. The first equality follows from (43) while the second follows from (40). □

Theorem 2 (Dirichlet-Abel)**.** *Let M be a fixed modulus > 1. Let $R_n(x)$ denote a complex-valued function defined on $\mathrm{I} = [0,1]$ such that $R_n(x) = R_k(x)$ for $n \equiv k \pmod q$, and a fortiori, there are M different functions. Assume that each $R_k(x)$ is of Lipschitz α, $\alpha \ge 1$, $R_k(x) \in Lip\ \alpha$ and $\sum_{k=1}^{M} R_k(x) = 0$ for each $x \in \mathrm{I}$. Then the Dirichlet series*

$$F(s) = F(s,x) = \sum_{n=1}^{\infty} \frac{R_n(x^n)}{n^s} \tag{45}$$

is uniformly convergent in $\sigma > 0$ and $x \in \mathrm{I}$.

If, further, R_k are all continuous on I, then $F(s)$ is also continuous on I and

$$F(1,1) = \sum_{n=1}^{\infty} \frac{R_n(1)}{n} = -\frac{1}{M}\sum_{k=1}^{M} R_k(1)\psi\left(\frac{k}{M}\right) = \sum_{k=1}^{M} \hat{R}_k(1)\ell_1\left(\frac{k}{M}\right), \tag{46}$$

where ψ is the Euler digamma function (1) and $\ell_1(x)$ is the first polylogarithm function (21). For a periodic sequence $\{f(n)\}$ of period M, (46) reads

$$\begin{aligned} \gamma_0(f) &= \lim_{s \to 1}\left(D(s.f) - \frac{\frac{\hat{f}(M)}{\sqrt{M}}}{s-1}\right) \\ &= -\frac{1}{M}\sum_{k=1}^{M} f(k)\psi\left(\frac{k}{M}\right) = \sum_{k=1}^{M-1} \hat{f}(k)\ell_1\left(\frac{k}{M}\right) + \frac{\hat{f}(M)}{\sqrt{M}}\gamma, \end{aligned} \tag{47}$$

where $\hat{f}(M)$ is defined by (41) and $\gamma_0(f)$ by (76).

We note that if f is odd (see (49) below), then $\hat{f}(M) = \frac{1}{2}f_{\mathrm{odd}} + f_{\mathrm{odd}} = 0$, which explains why the odd parts are far easier to treat and the left-hand side of (47) reduces to $D(1,f)$. Enough to recall the case of the class number formula for an odd Dirichlet character (see (81)). For a more general result than (47), see Theorem 4 below.

Example 1. *The well-known Dirichlet's test for uniform convergence is a special case of Theorem 2. Theorem 1 follows from Theorem 2 by choosing essentially*

$$R_n(x) = \frac{4e^{\pi i \xi n}}{1 + e^{\pi i \xi n}}. \tag{48}$$

Example 2 ([8,43,44]). *We let*

$$\begin{aligned} f_{\text{odd}} &= \frac{1}{2}\left(f(n \bmod M) - f(-n \bmod M)\right), \\ f_{\text{even}} &= \frac{1}{2}\left(f(n \bmod M) + f(-n \bmod M)\right) \end{aligned} \tag{49}$$

be odd, resp. even part of f: $f = f_{\text{even}} + f_{\text{odd}}$. Then

$$D(1-s, f) = \left(\frac{\pi}{M}\right)^{\frac{1}{2}-s} \left(\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} D(s, \hat{f}_{\text{even}}) + \frac{\Gamma\left(\frac{1+s}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)} D(s, \hat{f}_{\text{odd}}) \right), \tag{50}$$

which amounts to (51) on clearing the denominators and multiplying by $\left(\frac{\pi}{M}\right)^{\frac{s-1}{2}}$.

The Dirichlet series $\phi(s)$, $\psi_j(s)$ as in [20] are said to satisfy the ramified functional equation, a special case of ([20], (7.18), p. 216) with $r = 1$ if

$$\begin{aligned} &\Gamma\left(\tfrac{s}{2}\right)\Gamma\left(\tfrac{1}{2}+\tfrac{s}{2}\right)\varphi(s) \\ &= \Gamma\left(\tfrac{1}{2}-\tfrac{s}{2}\right)\Gamma\left(\tfrac{1}{2}+\tfrac{s}{2}\right)\psi_1(1-s) + \Gamma\left(-\tfrac{s}{2}\right)\Gamma\left(1+\tfrac{s}{2}\right)\psi_2(1-s). \end{aligned} \tag{51}$$

Thsi will be developed in several directions starting from [45].

4. Generalized Euler Constants

The genesis of generalized Euler constants lies in the Laurent expansion of $\zeta(s)$ around $s = 1$:

$$\zeta(s) = \frac{1}{s-1} + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\gamma_k (s-1)^k, \tag{52}$$

where $\gamma = \gamma_0$ indicates the Euler constant.

Since one of the driving forces in number theory has been generalizing to an arithmetic progression, it is natural to consider the generalized Euler constants $\gamma_k(a, M)$ for an arithmetic progression a mod M. The kth one is defined in the first instance by

$$\gamma_k(a, M) = \lim_{x \to \infty} \left(\sum_{\substack{n \le x \\ n \equiv a \ (\mathrm{mod}\ M)}} \frac{\log^k n}{n} - \frac{\log^{k+1} x}{M(k+1)} \right). \tag{53}$$

A concise survey of literature about them may be found in [13]. One of two most proper ways of introducing them is through the Laurent expansion of the partial zeta-function defined in (42):

$$\zeta(s, a, M) = D(s, \chi_a) = \sum_{\substack{n=1 \\ n \equiv a \bmod M}}^{\infty} \frac{1}{n^s} = M^{-s}\zeta\left(s, \frac{a}{M}\right). \tag{54}$$

Here $\zeta(s,x)$ indicates the Hurwitz zeta-function (10) whose Laurent expansion were obtained by Wilton [14], Berndt [1], Balakrishnan [46], et al.

$$\zeta(s,a)=\frac{1}{s-1}+\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\gamma_k(a)(s-1)^k,\quad \gamma_0(a)=-\psi(a). \tag{55}$$

(54) is a genuine generating function for $\gamma_k(a,M)$:

Theorem 3 ([12])**.** *The Laurent expansion*

$$\zeta(s,a,M)=\frac{1}{M(s-1)}+\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\gamma_k(a,M)(s-1)^k. \tag{56}$$

holds and moreover,

$$\gamma_k(a,M)=\frac{1}{M}\sum_{j=0}^{k}\binom{k}{j}(\log M)^{k-j}\gamma_j\left(\frac{a}{M}\right). \tag{57}$$

The other method, as can be predicted e.g., from the Hurwitz formula (11), depends on the Lerch zeta-function and the method is called the Deninger-Meyer method as described below.

Since (38) reads for $f=\chi$

$$L(s,\chi)=M^{-s}\sum_{a=1}^{M}\chi(a)\zeta\left(s,\frac{a}{M}\right)=\sum_{a=1}^{M}\chi(a)\zeta\left(s,a,M\right), \tag{58}$$

it follows that

$$L^{(k)}(1,\chi)=(-1)^k\sum_{a=1}^{M}\chi(a)\gamma_k(a,M). \tag{59}$$

as is indicated by [47]. Hence finding the derivatives and the generalized Euler constants lead to the same thing.

As has been remarked in [9], the three types of quantities $L^{(k)}(1,\chi)$, $\gamma_k(a,M)$ and $L(k+1,\chi)$ may be treated in a unified way based on the expressions ($\sigma>0$ for χ non-trivial) which follow from (35):

$$L^{(k)}(s,\chi)=\frac{1}{\tau(\bar{\chi})}\sum_{a=1}^{M}\bar{\chi}(a)\frac{\partial^k}{\partial^k s}\ell_s\left(\frac{a}{M}\right), \tag{60}$$

$$L(s,\chi)=\frac{1}{\tau(\bar{\chi})}\sum_{a=1}^{M}\bar{\chi}(a)\ell_s\left(\frac{a}{M}\right), \tag{61}$$

where $\tau(\chi)$ is the normalized Gauss sum (36).

Generalized Euler constants have been studied in other two contexts. One is related to the Piltz divisor problem. References [48–52] all of which have been elucidated in [44]. The other is related to the estimate of them including [53–55], etc. For the Ableian theorem, refer to Section 7.

In Deninger [31] (see also Meyer [56], pp. 526–557) it has been shown that the best and most proper approach to the determination of Laurent coefficients of a zeta-function is to use the reciprocal of the Hurwitz formula, i.e., to appeal to the functional equation (12) ([34], pp. 48–51), expanding the analytic function on the right-hand side into power series around $s=0$, which involves higher derivatives of the Hurwitz zeta-function. We call this the Deninger-Meyer method. The Hurwitz zeta-function and its derivatives can be characterized as a principal solution to the difference Equation (91) (see Section 5 below). This establishes the closed formula for the derivative of the Dirichlet L-function $L(s,\chi)$ at $s=1$ leading to the generalized Lerch-Chowla-Selberg formula for a real quadratic field beyond that of an imaginary quadratic field. Thus, this gives the evaluation of $L^{(k)}(1,\chi)$ for $k=0,1$. Higher order derivatives are computed in [9] and then more extensively in [57]. See ([34], pp. 177–191).

This procedure is a reminiscence of Stark's method of evaluating the value at $s = 0$ rather at $s = 1$ of L-functions, [58–60] etc.

Let

$$D = \frac{\mathrm{d}}{\mathrm{d}t}$$

([61], p. 192, Problem 5) gives

$$D^n A(t) = \sum_{k=0}^{n} S(n,k) e^{kt} f_k, \quad f_k = D_u^k f(u)|_{u=e^t}, \tag{62}$$

where $S(n,k)$ are Stirling numbers of the second kind defined by

$$S(n,k) = \frac{1}{k!}\Delta^k 0^n = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} j^n = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{j}\binom{k}{j}(k-j)^n. \tag{63}$$

Here Δ indicates the difference operator ([62], p. 13), ([61], p. 202) defined by

$$\Delta f(x) = f(x+1) - f(x) = (E - I)f(x), \quad Ef(x) = f(x+1), \quad If(x) = f(x) \tag{64}$$

and in general

$$\Delta^k f(x) = \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} f(x+j). \tag{65}$$

This clarifies the second equality in (63). For a general form, see (100).

Comtet ([62], pp. 224–229) is the most informative reference book on Stirling numbers.

Proposition 2 ([30])**.** *For $\tau \in \mathcal{H}$ and $n \in \mathbb{N}$*

$$\begin{aligned} \ell_{-n}(\tau) &= \frac{1}{(2\pi i)^n}\frac{\mathrm{d}^n}{\mathrm{d}^n\tau}\frac{e^{2\pi i\tau}}{1-e^{2\pi i\tau}} \\ \ell_{n+1}(\tau) &= 2\pi i \int_0^\tau \ell_n(w)\,\mathrm{d}w + \zeta(k+1), \end{aligned} \tag{66}$$

where the integral is taken along a line joining 0 *and* τ*. The boundary functions are*

$$\ell_{-n}(x) = \sum_{k=1}^{n} k! S(n,k) \frac{e^{2\pi i n x}}{(1-e^{2\pi i n x})^{k+1}} \tag{67}$$

whose special case being

$$\ell_0(x) = \frac{e^{2\pi i x}}{1-e^{2\pi i x}} = \frac{1}{2}(-1 + i\cot \pi x) \tag{68}$$

for $x \in \mathbb{R} - \mathbb{Z}$ while for $0 < x < 1$

$$\ell_\varkappa(x) = \frac{(2\pi i)^{\varkappa-1}}{\varkappa!}\left(A_\varkappa(x) - \pi i B_\varkappa(x)\right), \tag{69}$$

where $\bar{B}_\varkappa(x)$ is the $\varkappa$-th periodic Bernoulli polynomial in (17) and $A_\varkappa(x)$ is the counterpart, the $\varkappa$-th Clausen function. The first Clausen function is given by (22) and higher order ones are obtained by integration:

$$A_\varkappa(x) = \frac{\varkappa!}{2(2\pi i)^{\varkappa-1}}\sum_{n=1}^{\infty}\frac{e^{2\pi i n x} + (-1)^{\varkappa-1}e^{-2\pi i n x}}{n^\varkappa} = \frac{\varkappa!}{2(2\pi i)^{\varkappa-1}}\sum_{n=-\infty}^{\infty}\frac{\mathrm{sgn}(n) e^{2\pi i n x}}{n^\varkappa}, \tag{70}$$

where the sign function $\mathrm{sgn}(n)$ is 1 for $n > 0$, -1 for $n < 0$ and 0 for $n = 0$.

Proof. It suffices to prove (67), which is an immediate consequence of (62). □

However, using the analytic function $\ell_s(x)$, the situation is much more transparent. Indeed, from (69) we have natural expressions:

$$\bar{B}_\varkappa(x) = -\frac{\varkappa!}{(2\pi i)^\varkappa}\left(\ell_\varkappa(x) - (-1)^{\varkappa-1}\ell_\varkappa(1-x)\right) \tag{71}$$

and

$$A_\varkappa(x) = \frac{\varkappa!}{2(2\pi i)^{\varkappa-1}}\left(\ell_\varkappa(x) + (-1)^{\varkappa-1}\ell_\varkappa(1-x)\right) \tag{72}$$

which are in conformity with the functional equations (parity relation)

$$A_\varkappa(x) = (-1)^{\varkappa-1}A_\varkappa(1-x), \quad \bar{B}_\varkappa(x) = (-1)^\varkappa \bar{B}_\varkappa(1-x). \tag{73}$$

Since we have the formula

$$\zeta(-l,x) = -\frac{1}{l+1}\bar{B}_{l+1}(x), \tag{74}$$

the parity relation (73) entails

$$\zeta(-l,x) + (-1)^l\zeta(-l,1-x) = 0, \quad \zeta(0,x) + \zeta(0,1-x) = 0. \tag{75}$$

This is a generalization of trivial zeros of the Riemann zeta-function.

We turn to the case of Dirichlet series $D(s,f)$ with periodic coefficients $f(n)$ described toward the end of Section 3.

Let

$$D(s,f) = \frac{\frac{\hat{f}(M)}{\sqrt{M}}}{s-1} + \frac{1}{\sqrt{M}}\sum_{k=0}^\infty \frac{\gamma_k(f)}{k!}(s-1)^k \tag{76}$$

be the Laurent expansion of $D(s.f)$ around $s=1$. Then we have following one of our main theorems generalizing ([8], Theorem 2) and ([12], Theorem).

Theorem 4 (Base change relation)**.**

$$\begin{aligned}
&\frac{\frac{\hat{f}(M)}{\sqrt{M}}}{s-1} + \sum_{k=0}^\infty \frac{(-1)^k}{k!}\left(\sum_{a=1}^M f(a)\gamma_k(a,M)\right)(s-1)^k = D(s,f) \\
&= \frac{\frac{\hat{f}(M)}{\sqrt{M}}}{s-1} + \frac{1}{\sqrt{M}}\sum_{k=0}^\infty \frac{1}{k!}\left(\sum_{a=1}^{M-1}\hat{f}(a)(-1)^k\sigma_k^a + \hat{f}(M)\gamma_k\right)(s-1)^k
\end{aligned} \tag{77}$$

where

$$\sigma_k^a = \sum_{n=1}^\infty \frac{e^{2\pi i n\frac{a}{M}}}{n}\log^k n = (-1)^k\frac{\partial^k}{\partial^k s}\ell_1\left(\frac{a}{M}\right), \tag{78}$$

i.e., $\ell_1\left(\frac{a}{M}\right)$ *is a generating function of* σ_k^a *and a fortiori,*

$$\begin{aligned}
&\sum_{a=1}^M f(a)\sum_{j=0}^k \binom{k}{j}(\log M)^{k-j}\gamma_j\left(\frac{a}{M}\right) = \sum_{a=1}^M f(a)\gamma_k(a,M) \\
&= \gamma_k(f) = \sum_{a=1}^{M-1}\hat{f}(a)\frac{\partial^k}{\partial^k s}\ell_1\left(\frac{a}{M}\right) + \frac{\hat{f}(M)}{\sqrt{M}}\gamma_k,
\end{aligned} \tag{79}$$

the case $k=0$ *leading to* (47).

Proof. Equation (77) follows from Proposition 1.

Equation (79) follows from (77) save for the first equality which is a consequence of (57). □

Corollary 1. *Gauss' formula for the digamma function at rational arguments ([11], p. 135)*

$$\psi\left(\frac{a}{M}\right) = -\gamma - \log\frac{M}{2} - \frac{\pi}{2}\cot\frac{a}{M} + 2\sum_{0<j<M/2}\cos\frac{2\pi aj}{M}\log\sin\frac{\pi j}{M}. \tag{80}$$

is a consequence of (79)*. Moreover, the odd and even parts of* (79) *is a slight generalization of the Dirichlet class number formula in finite form, see [18,63]. See also Section 8 below.*

Proof. Equations (47) or (79) with $k = 0$ and $f(n) = 1$ amounts to (80). □

This is a prototype of the equivalence theorem in [63] to the effect that the finite expressions for the Dirichlet class number formula is equivalent to (80). See also ([18], Chapter 8). For χ odd, we have the finite expression

$$L(1,\chi) = \frac{\pi}{2M}\sum_{a=1}^{M-1}\chi(a)\cot\frac{\pi a}{M}. \tag{81}$$

which leads to the finite form of the class number formula and it is here that (2) appears as the odd part.

Funakura [7] deduces as his main theorem, the Kronecker limit formula for $D(s,f)$ which is compactly represented as Corollary 2 below. His proof depends on the first equality of Proposition 1 and therein he uses Gauss' formula to transform the result into the required form. For its elucidation we refer to [13].

Corollary 2 ([7], Theorem 5)**.**

$$\frac{\gamma_0(f)}{\sqrt{M}} = \lim_{s\to 1}\left(D(s,f) - \frac{\frac{\hat{f}(M)}{\sqrt{M}}}{s-1}\right) = \frac{1}{\sqrt{M}}\sum_{a=1}^{M-1}\hat{f}(a)\ell_1\left(\frac{a}{M}\right) + \frac{\hat{f}(M)}{\sqrt{M}}\gamma, \tag{82}$$

which may be expressed as

$$= \frac{1}{\sqrt{M}}\sum_{a=1}^{M-1}\hat{f}(a)A_1\left(\frac{a}{M}\right) - \frac{\pi i}{\sqrt{M}}\sum_{a=1}^{M-1}\hat{f}(a)\bar{B}_1\left(\frac{a}{M}\right) + \frac{\hat{f}(M)}{\sqrt{M}}\gamma \tag{83}$$

and as

$$= -\frac{1}{\sqrt{M}}\sum_{a=1}^{M-1}\hat{f}(a)\log 2\sin\frac{\pi a}{M} + \frac{\pi}{2M}\sum_{a=1}^{M-1}f(a)\cot\frac{\pi a}{M} + \frac{\hat{f}(M)}{\sqrt{M}}\gamma. \tag{84}$$

Equations (83) and (84) follow from the results stated toward the end of Section 2.

The pioneer results of Briggs [21] and Lehmer [11] are immediate and natural consequences of (47) or (79) with $k = 0$ as well as a corollary to Funakura's theorem, Corollary 2:

Example 3 ([11], (6))**.**

$$M\gamma_0(a,M) = \sum_{j=1}^{M-1}\varepsilon_a(-j)\ell_1\left(j\frac{a}{M}\right) + \gamma. \tag{85}$$

Example 4 ([21])**.**

$$M\gamma_0(a,M) = -\psi\left(\frac{a}{M}\right) - \log M. \tag{86}$$

This follows from (55).

Theorem 4 shows ubiquity of $\ell_1(x)$. However, the main formula (79) is in its first stage. For the Deninger-Meyer method involves its second stage of expressing the derivatives of the Lerch zeta-function by those of the Hurwitz zeta-function.

We may compute the higher order Laurent coefficients of $D(s, f)$ around $s = 1$ by (40) by applying the Deninger-Meyer method, i.e., by finding the coefficient $(-1)^k \frac{\partial^k}{\partial^k s} \ell_1(x)$ of s^k of $\ell_k(1-s)$ in (12).

$$\begin{aligned}&(-1)^k \frac{\partial^k}{\partial^k s} \ell_1(x) \\ &= \sum_{a+b+c+d=k} \frac{(-\log 2\pi)^a}{a!} \frac{\Gamma^{(b)}(1)}{b!} \frac{\left(\frac{\pi}{2}\right)^c}{c!d!} \{\zeta^{(d)}(0,x) + (-1)^c \zeta^{(d)}(0,1-x)\},\end{aligned} \tag{87}$$

where a, b, c, d run through integers ≥ 0 whose sum is $k+1$. In actual calculation, one can omit those terms which correspond to $d = 0, 2|c$ by (75).

Corresponding to (87), we have

$$\begin{aligned}&\frac{\partial^k}{\partial^k s} \zeta(0,x) \\ &= \sum_{a+b+c+d=k} \frac{(-\log 2\pi)^a}{a!} \frac{\Gamma^{(b)}(1)}{b!} \frac{\left(\frac{\pi i}{2}\right)^c}{c!d!} i \left((-1)^{c+1} \ell_1^{(d)}(x) + \ell_1^{(d)}(1-x)\right).\end{aligned} \tag{88}$$

Hence

$$\begin{aligned}\zeta'(0,x) = &-(-\log 2\pi + \Gamma'(1))\,(\ell_1(x) - \ell_1(1-x)) + \frac{\pi i}{2} i(\ell_1(x) + \ell_1(1-x)) \\ &- i(\ell_1'(x) - \ell_1'(1-x))\end{aligned} \tag{89}$$

Substituting Equations (71), (72),

$$\ell_1'(x) = -\sum_{n=1}^{\infty} \frac{\log n}{n} e^{2\pi i n x},$$

and invoking (97), we deduce Kummer's Fourier series

$$\log \frac{\Gamma(x)}{\sqrt{2\pi}} = \sum_{n=1}^{\infty} \left(\frac{1}{2n} \cos 2\pi n x + \frac{\gamma + \log 2\pi n}{\pi n} \sin 2\pi n x \right). \tag{90}$$

It can be shown that Kummer's Fourier series is, in the long run, equivalent to the functional equation for the Riemann zeta-function ([18], p. 108; [64], pp. 168–175).

5. Difference Equations

According to ([31], p. 173) there is a solution to the difference equation

$$f(x+1) - f(x) = \log^k x, \tag{91}$$

given by the Gaussian representation

$$R_k(x) = \lim_{n\to\infty} \left(\lambda + x \log^k n - \log^k x - \sum_{\nu=1}^{n-1} \left(\log^k(x+\nu) - \log^k \nu \right) \right), \tag{92}$$

such that $R_k(x)$ is convex for large argument $> A > 0$ and

$$\lambda = R_k(1). \tag{93}$$

Theorem 5 ([31], Theorem 2.2)**.**

$$(-1)^{k+1}\left(\frac{\partial^k}{\partial^k s}\zeta(0,x)-\zeta^{(k)}(0)\right) \tag{94}$$

is the uniquely determined solution to (91) *having the value* 0 *at* $x=1$ *and a fortiori*

$$R_k(x)=(-1)^{k+1}\frac{\partial^k}{\partial^k s}\zeta(0,x) \tag{95}$$

on choosing

$$\lambda=(-1)^{k+1}\zeta^{(k)}(0). \tag{96}$$

The well-known Lerch formula

$$\zeta'(0,x)=\log\frac{\Gamma(x)}{\sqrt{2\pi}} \tag{97}$$

is a special case of (95).

The R-function or higher order derivatives of the Hurwitz zeta-function is introduced using the Dufresnoy-Pisot theorem which is a generalization of the Bohr-Mollerop theorem [65] while in [9] Nörlund's principal solution [66] is used which is also used in [4] to introduce a generalized gamma function to express the generalized Euler constants. Its logarithmic derivative appears in Ramanujan's second notebook [67]. Higher-order derivatives of Dirichlet L-function has been studied in [22], ([34], §8.1) using the Deninger R_k-function. In a similar setting, Vignéras [68] uses the Dufresnoy-Pisot theorem to introduce the Barnes multiple gamma function (see [16], pp. 49–50).

After [11,21] it was Dilcher [4,69] and Kanemitsu [9] who developed the theory of generalized Euler constants. Most of these results have been summarized and elucidated in [12]. There are further generalizations and we refer to [3] and references there given. We may use the Deninger-Meyer method to find Laurent coefficients of a more general class of Dirichlet series including, for example, References [5,47] in terms of principal solutions to a difference equation, which will be conducted elsewhere.

As an analogy to $\log\Gamma(x)$, Dilcher ([4], (4.3)) introduces the kth generalized gamma function $\log\Gamma_k(x)$ as a principal solution to the DE

$$f(x+1)-f(x)=\frac{1}{k+1}\log^{k+1}x, \quad \log\Gamma_k(1)=0. \tag{98}$$

By Theorem 5, we have apparently

$$\log\Gamma_k(x)=\frac{1}{k+1}R_{k+1}(x)=(-1)^k\frac{\partial^{k+1}}{\partial^{k+1}s}\zeta(0,x), \tag{99}$$

so that we could build a theory of $\log\Gamma_k(x)$ on that of the Hurwitz zeta-function (for the case of the theory of gamma function on the Riemann zeta-function, see [18], Chapter 5).

The general formula for the difference operator of order $\alpha\in\mathbb{N}$ with difference $y\geq 0$ is given by

$$\Delta_y^\alpha f(x)=\sum_{\nu=0}^{\alpha}(-1)^{\alpha-\nu}\binom{\alpha}{\nu}f(x+\nu y) \tag{100}$$

If f has the α-th derivative $f^{(\alpha)}$, then

$$\Delta_y^\alpha f(x)=\int_x^{x+y}\mathrm{d}t_1\int_{t_1}^{t_1+y}\mathrm{d}t_2\cdots\int_{t_{\alpha-1}}^{t_{\alpha-1}+y}f^{(\alpha)}\left(t_\alpha\right)\,\mathrm{d}t_\alpha. \tag{101}$$

This has been most prominently applied by Landau [70] and has been constantly used by later authors in the context of Riesz means.

6. Abel-Tauber Process

Our aim in this section is to formulate a theorem which improves the asymptotic formulas in [71] and generalizes [72] to the case of a general Dirichlet series $f(s)$ satisfying a functional equation, where we let

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^s} \tag{102}$$

be a Dirichlet series absolutely convergent in some half-plane where $\{a_n\}$ is a complex sequence and $\{\lambda_n\}$ is an increasing sequence with $\lambda_1 > 0$. Instead of studying the behavior of each value of a_n, it is customary and more effective to consider the summatory function

$$A(x) = \sum_{\lambda_n \le x} a_n \tag{103}$$

with empty sum being 0. Asymptotic formulas for the summatory function (103) may be obtained by various methods. The most typical way is the use of the Cauchy residue theorem and application of the functional equation, if any, satisfied by $f(s)$. Historically, Voronoǐ's expression for the summatory function of the divisor function led to the estimate $O(x^{\frac{1}{3}+\varepsilon})$ [73]. One of the most comprehensive treatment has been given by [25]. Its generalization to the class of Dirichlet series with periodic coefficients has been done in [26].

Lemma 1 ([74], Satz 1.4, p. 371)**.** *For $\{a_n\}$ and $A(x)$ as above suppose $g(t)$ is of class C^1 on $[\lambda_1, \infty)$. then the formula for integration by parts*

$$\sum_{\lambda_1 \le \lambda_k \le x} a_k g(\lambda_k) = \int_{\lambda_1}^{x} g(t)\, \mathrm{d}A(t) + a_1 g(\lambda_1) = A(x)g(x) - \int_{\lambda_1}^{x} A(t) g'(t)\, \mathrm{d}t \tag{104}$$

holds true.

In the following lemma we use the notation. $\varphi(s) = f(s)$ as in (102) and $\psi(s) = \sum_{n=1}^{\infty} \frac{b_n}{\mu_n^s}$ and they satisfy the functional equation (107) with the multiple gamma factor

$$\Delta(s) = \prod_{\nu=1}^{\mathrm{N}} \Gamma\left(\alpha_\nu s + \beta_\nu\right), \quad \alpha_\nu > 0 \tag{105}$$

and

$$\mathrm{A} = \sum_{\nu=1}^{\mathrm{N}} \alpha_\nu \ge 1. \tag{106}$$

$A_\lambda^0(x)$ below is essentially $A(x)$ in (103) and $Q_0(x)$ is essentially the sum of residues of φ which gives rise to the main term $M(x)$ in Theorem 6.

Lemma 2 ([25], Theorem 4.1)**.** *Suppose that the functional equation*

$$\Delta(s)\varphi(s) = \Delta(r-s)\psi(r-s) \tag{107}$$

is satisfied with $r \ge 0$ and that the only singularities of the function $\varphi(s)$ are poles. Then we have

$$A_\lambda^0(x) = Q_0(x) + E(x), \tag{108}$$

where

$$E(x) = O\left(x^{\frac{r}{2}-\frac{1}{4A}+2A\eta u}\right) + O\left(x^{q-\frac{1}{2A}-\eta}\log^{l-1} x\right) + O\left(\sum_{x<\lambda_n\le y} |a_n|\right) \tag{109}$$

for every $\eta \ge 0$, *and where* $y = x + O\left(x^{1-\eta-\frac{1}{2A}}\right)$, $q =$ *maximum of the real parts of singularities of* φ, $l =$ *maximum of order of a pole with real part* q, *and* $u = \beta - \frac{r}{2} - \frac{1}{4A}$. *Herein,* β *is such that* $\sum_{n=1}^{\infty} \frac{\mu_n}{\mu_n^{\beta}} < \infty$.

If in addition, $a_n \ge 0$, *then the last term in* (109) *can be suppressed.*

Theorem 6 (Abel-Tauber theorem)**.** *Suppose*

$$\begin{aligned} A(x) &= M(x) + E(x), \quad M(x) = xP_l(\log x) = cx\log^l x(1+o(1)), \\ E(x) &= O(x^u), \quad 0 < u < 1 \end{aligned} \tag{110}$$

P_l *denoting a polynomial of degree* l, *as a result of the functional equation (Lemma 2) or otherwise. Then for* $0 \le b < 1 - u (< 1)$ *and* k *a non-negative integer*

$$S_b(x) = \sum_{\lambda_n \le x} \lambda_n^{b-1} a_n \log^k \lambda_n = \int_{\lambda_1}^{x} t^{b-1}\log^k t M'(t)\,\mathrm{d}t + C_k + O\left(x^{b-1+u}\log^l x\right) \tag{111}$$

where

$$C_k = C_k(b) = \lim_{x\to\infty}\left(\sum_{\lambda_n\le x} n^{b-1} a_n \log^k \lambda_n - \int_{\lambda_1}^{x} E(t)\,\mathrm{d}(t^{b-1}\log^k t)\right), \tag{112}$$

whence

$$f(s) = \frac{cl!s}{(s-1)^{l+1}} + \cdots + \frac{C_{-1}}{s-1} + \sum_{k=0}^{\infty} \frac{(-1)^k C_k}{k!}(s-1)^k. \tag{113}$$

Proof. By the first equality in (104),

$$S_b(x) = \sum_{\lambda_n \le x} \lambda_n^{b-1} a_n \log^k \lambda_n = \int_{\lambda_1}^{x} t^{b-1}\log^k t\,\mathrm{d}A(t) + a_1\lambda_1^{b-1}\log^k \lambda_1. \tag{114}$$

Substituting the first equality in (110) and applying integration by parts to the integral involving $E(t)$, we have

$$\begin{aligned} \int_{\lambda_1}^{x} t^{b-1}\log^k t\,\mathrm{d}E(t) &= \lambda_1^{b-1}\log^k \lambda_1 E(\lambda_1) - \int_{\lambda_1}^{x} E(t)\,\mathrm{d}(t^{b-1}\log^k t) + O(x^{b-1+u}\log^k x) \\ &= C_k' + O(x^{b-1+u}\log^K x), \quad K = \max\{k, l\}, \end{aligned}$$

where $C_k' = \lambda_1^{b-1}\log^k \lambda_1 E(\lambda_1) - \int_{\lambda_1}^{\infty} E(t)((b-1)t^{b-2}\log^k t + kt^{b-1}\log^{k-1} t)\,\mathrm{d}t$ and $C_k = C_k' + a_1\lambda_1^{b-1}\log^k \lambda_1$. Hence (112) follows from which (111) follows. To prove (112), we note that

$$f(s) = \int_{\lambda_1}^{\infty} t^{-s} A(t)\,\mathrm{d}t + a_1\lambda_1^{-s} = s\int_{\lambda_1}^{\infty} t^{-s-1}A(t)\,\mathrm{d}t. \tag{115}$$

Hence for $\sigma > 1$

$$(-1)^k f^{(k)}(s) = \sum_{n=1}^{\infty} \frac{\lambda_n^{-1} a_n \log^k \lambda_n}{\lambda_n^{1-s}} = (s-1)\int_{\lambda_1}^{\infty} t^{-s} S_b(t)\,\mathrm{d}t. \tag{116}$$

For simplicity we take $b = 0$ and substitute (111). It suffices to consider the case $M(t) = ct\log^l t$ in which case $M'(t) = c(\log^l t + l\log^{l-1} t)$ and so its contribution to the main term in (111) is

$$\int_{\lambda_1}^{x} (t^{b-1}\log^{k+l} t + t^{b-1}\log^{k+l-1})\,\mathrm{d}t.$$

We use the case $b = 0$. Then this simply becomes the same as in ([2], p. 41) and so

$$(-1)^k f^{(k)}(s) = (-1)^k \left(\frac{\mathrm{d}}{\mathrm{d}s}\right)\left(\frac{cl!s}{(s-1)^{l+1}}\right), \tag{117}$$

which shows (113) almost verbatim to that of Theorem 1 of [2]. □

Corollary 3 ([2], Theorem 1). *If $\lambda_n = n$ and*

$$A(x) = cx\log^l x + O(x^u), \quad 0 < u < 1 \tag{118}$$

then

$$\sum_{n \le x} n^{-1} a_n \log^k n = \frac{c}{k+l+1}(\log x)^{k+l+1} + \frac{cl}{k+l}(\log x)^{k+l} + C_k + O\left(x^{u-1}\log^l x\right) \tag{119}$$

where

$$C_k = \lim_{x\to\infty}\left(\sum_{n\le x} n^{-1}a_n \log^l n - \frac{c}{k+l+1}(\log x)^{k+l+1} - \frac{cl}{k+l}(\log x)^{k+l}\right), \tag{120}$$

whence

$$f(s) = \frac{cl!}{(s-1)^{l+1}} + \sum_{r=0}^{\infty}\frac{(-1)^k C_k}{k!}(s-1)^k. \tag{121}$$

Remark 1. *As remarked by [2], the result is true for $A(x) = x^a P(\log x) + O(x^u)$, $a-1 < u < a$.* (111) *gives asymptotic formulas for a large class of arithmetic functions including those studied by [71]. The results of Theorem 6 are still true if* (110) *is replaced by the PNT type estimate, which corresponds to the case* $\mathrm{A} < 0$ *in ([25], p. 112). Theorem 6 generalized in this way may serve to unify the unorganized tables of asymptotic formulas given in some collections of formulas in number theory.*

The proof of Lemma 2 depends on the Riesz sums and then the differencing (100)*. In order to cover the case of periodic Dirichlet series, we need to incorporate the situation of ramified functional equations as in Smith [26], which will be conducted elsewhere.*

7. Quellenangaben

Some authors are concerned with analogues of the generalized Euler constants in the Gaussian field $\mathbb{Q}(i)$ with the ring $\mathbb{Z}(i)$ of Gaussian integers.

Ref. [6] is concerned with a bi-dimensional analogue of the Euler constant defined by

$$\lim_{n\to\infty}\left(\sum_{2\le k\le n}\frac{1}{\pi r_k^2} - \log n\right),$$

where

$$r_k = \min\{r > 0 | \sharp(\mathbb{Z}[i] \cap \bar{D}(z,r)) \ge k \text{ for } \exists z \in \mathbb{C}\}$$

and in the spirit of [75].

Ref. [5] is concerned with the Laurent expansion

$$\zeta_{\mathbb{Q}(i)} = \frac{\frac{\pi}{4}}{s-1} + \frac{1}{4}\gamma_{\mathbb{Q}(i)} + O(s-1)$$

and he deduced from the Kronecker limit formula

$$\gamma_{\mathbb{Q}(i)} = 2\pi(\gamma - \log 2 - \log|\eta(i)|^2), \tag{122}$$

where $\eta(\tau)$ indicates the Dedekind eta-function whose 24th power is (a constant multiple of) the discriminant function

$$\Delta(\tau) = g_2^3(\tau) - 27g_3^2 = (2\pi)^{12}\eta^{24}(\tau) \tag{123}$$

see ([76], p. 95), ([77], Theorem 4.4). Then Elstrodt applies the pre-Lerch-Chowla-Selberg formula

$$\Delta(i) = \frac{1}{2^6(2\pi)^6}\Gamma\left(\frac{1}{4}\right)^{24} \tag{124}$$

which follows from Hurwitz's result [78]

$$\sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(mi+n)^4} = \frac{2^4}{4!}\frac{K_0^4}{10}, \tag{125}$$

where the prime on the summation sign means that the pair $(m, n) = (0, 0)$ is excluded and

$$K_0 = 2\int_0^1 \frac{1}{\sqrt{1-t^4}}\,\mathrm{d}t = \sqrt{2}\varpi \tag{126}$$

is a special case of the lemniscate function ([79], p. 524) and

$$K_0 = \frac{1}{4\sqrt{\pi}}\Gamma\left(\frac{1}{4}\right)^2 \tag{127}$$

first proved by Legendre. The same result is stated on ([10], p. 176). For the celebrated Lerch-Chowla-Selberg formula we refer to Section 8.1 below. It gives a relation between two independent special functions—elliptic and gamma functions, and is a consequence of the combination of the Kronecker limit formula (which is a consequence of the Fourier-Bessel expansion) and the decomposition of the Dedekind zeta-function. Equation (125) also follows from the modula relation, and so Legendre's result (127) might, in the long run, correspond to a pre-class field theory for lemniscate functions. We will come to this elsewhere.

According to Wintner ([29], p. 634), by the theory of elliptic modular functions, all expansions considered by Jacobi in his §40 must be identical consequences of the corresponding expansions of the (principal) logarithm of the fundamental invariant (123), that is, of the expansion

$$-\frac{1}{24}\log\frac{\Delta}{\pi} = -\sum_{n=1}^{\infty}\log(1-\tau^n) = \sum_{n=1}^{\infty}\frac{1}{n}\frac{\tau^n}{1-\tau^n}. \tag{128}$$

Wintner ([29], p. 628) also mentions "Riemann's posthumous fragment, based on \$ 40 of Jacobi's Fundament Nova, consists of two parts, I and II, $\cdots$" These imply that all the results in Section 3 are to be limit values of the discriminant function (or Dedekind eta-function). Thus, it seems probable that in Definition 1, one expresses the Lambert series in terms of Lerch zeta-functions and then apply the Dirichlet-Abel theorem to deduce all the results at a stretch. We might also obtain pre-Lerch-Chowla-Selberg type formulas in the spirit of (124) if we consider the even part. We will study this elsewhere.

References [2,10,47] give Abelian theorems to the effect that if an asymptotic formula for the summatory function (103) is known, then the one for $\sum_{\lambda_n \le x} n^{b-1} a_n \log^k n$ can be derived where $k \ge 0$ is an integer, which then implies the Laurent expansion for $f(s)$ in (102). Ref. [47] uses the Laplace transform and so does [71]. The asymptotic formula is of the from $A(x) = cx^a + O(x^u \log^K x)$, $u < a$. Briggs [2] enhances [24] and arrive at an Abelian theorem with $A(x) = CxP(\log x) + O(x^u)$ with P a polynomial of degree l or $A(x) = cx + O\left(\frac{x}{\log^K x}\right)$ with K as large as we please. The latter is a weak form of the PNT and this setting has been fully generalized in [23] in which the asymptotic formula for $A(x)$ is derived from analytic properties of the generating Dirichlet series (102). It also includes the case where (102) has logarithmic singularities, covering the best known form of the PNT. References [80,81] are similar to [2].

8. Elucidation of Some Identities

We clarify the underlying reason for identities (2) and (3).

Proof of (2) depends on (55), (11) and (68). Indeed, substituting (2) with x replaced by $1-x$ from (2), we obtain after transformation

$$\zeta(1-s,x) - \zeta(1-s,1-x) = \frac{\Gamma(s+1)}{(2\pi)^s} \frac{\pi}{2} \frac{\sin\frac{\pi}{2}s}{\frac{\pi}{2}} (-2i)(\ell_s(x) - \ell_s(1-x)). \tag{129}$$

Now taking the limit $s \to 0$ on using (11) and (68), we conclude (2).

Remarkably enough, this in conjunction with (47) leads to a generalization of the reverse Eisenstein formula [82,83], ([84], pp. 318–319):

$$\sum_{a=1}^{M-1} \varepsilon_b(-a)\bar{B}_1\left(\frac{a}{M}\right) = -\frac{1}{2i}\cot\frac{\pi b}{M}. \tag{130}$$

This is not coincidental and clarifies the reason why the cotangent function appears in the Dedekind sums and the Dirichlet class number formula, ([34], pp. 72–74). According to [33], the cotangent function is an odd basis of the Kubert space $\mathcal{K}_0$ and $\cot \pi\frac{a}{M}$, $1 \le a \le M/2$ form a basis of a vector space of odd functions in $C(M)$. Ref. ([85], Lemma 2.1) which plays a crucial role in the proof of the reciprocity laws for Dedekind-like sums is a special case of (47).

We must recall the warning in [86] that formal use of (66) for linear independence over $\mathbb{Q}$ leads to triviality since what appear are only derivatives of cotangents.

Elucidation of the genesis of the partial fraction expansion (3) for the cotangent function has been done in many literature and we refer to ([18], pp. 87–104), ([20], pp. 10–11, Chapter 4), etc. It is shown that (3) is a special case of the Fourier-Bessel series equivalent to the functional equation of Riemann's type with a single gamma factor.

8.1. Concluding Remarks

We shall give a rather short but high-brow description of the Lerch-Chowla-Selberg formula according to ([20], pp. 12–15), see also ([64], pp. 119–124), [31,87,88], etc., for a related account. We omit most of the details and assume basic facts.

Suppose $\Omega = \mathbb{Q}(\sqrt{\Delta})$ is the imaginary quadratic field of discriminant $\Delta < 0$ and let $\zeta_\Omega(s)$ be the Dedekind zeta-functio of Ω. It can be decomposed into the sum of h class zeta-functions $\zeta_A(s)$ which corresponds to the partial zeta-functions (see (42)): $\zeta(s) = \sum_{A \in I/P} \zeta_A(s)$, where I/P is the ideal class group of Ω which is of order h—the class number of Ω. The class zeta-function can be viewed as an Epstein zeta-function of a positive definite binary quadratic form Q: $\zeta_A(s) \sim \zeta_Q(s)$. It has a simple pole at $s = 1$ with residue $\lambda = \frac{2\pi}{w\sqrt{|\Delta|}}$, w indicating the number of roots of 1 in Ω and so the residue of $\zeta_\Omega(s)$ at $s = 1$ is $h\lambda$ as found by Dirichlet. Cf. (81) for the case of an odd character.

The Laurent constant c_0 of $\zeta_Q(s)$ was found by Kronecker in terms of the Dedekind eta-function. This part arises from the Fourier-Bessel series which is equivalent to the functional equation (of Hecke type) for the Epstein zeta-function ([20], Chapter 4).

On the other hand, we have the product decomposition

$$\zeta_\Omega(s) = \zeta(s) L(s, \chi), \tag{131}$$

where $L(s, \chi)$ is the Dirichlet L-function associated to the primitive Dirichlet character modulo $|\Delta|$.

Hence the residue at $s = 1$ is also expressed as $L(1, \chi)$, whence the Dirichlet class number formula

$$h\lambda = L(1, \chi). \tag{132}$$

Since h is finite, it is desirable to express the infinite series $L(1, \chi)$ in finite form, which is called the class number deformation. One of the most famous expressions involves the generalized Bernoulli number.

For the Laurent constant $c_0 = L'(1, \chi) + \gamma L(1, \chi)$, where γ is the Euler constant, one has to express $L'(1, \chi)$ in finite terms by appealing to Kummer's Fourier series for $\log \Gamma(x)$, see [64]. As in [13], this part also depends on the functional equation for the Dirichlet L-function.

Then equating two expressions, one obtains the equality between finite sums of the gamma values and eta-values, which was first found by Lerch and re-discovered by Chowla and Selberg [89].

Thus the Lerch-Chowla-Selberg formula is a consequence of the functional equations for the Epstein zeta-function and Dirichlet L-function plus class field theory since the decomposition (131) in the very long run, a consequence of the theory. This suggests the following problem.

Problem. Suppose there are two zeta-functions $f(s), g(s)$ and that both $f(s)$ and the product $Z(s) = f(s)g(s)$ have a simple pole at $s = 1$, say and satisfy the functional equation (of Hecke type, say). Then equate the Laurent constants of $f(s)$ and $Z(s)$ to deduce a generalized Lerch-Chowla-Selberg formula. This should include the case of a cyclotomic field. These are more or less concerned with odd aspect. Naturally, the next step will be to study the case of even aspect of real quadratic fields etc. in which case even the Kronecker limit formula is not in preferable form [88].

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