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Nonlinear Integro-Differential Equations Involving Mixed Right and Left Fractional Derivatives and Integrals with Nonlocal Boundary Data

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Abstract: In this paper, we study the existence of solutions for a new nonlocal boundary value problem of integro-differential equations involving mixed left and right Caputo and Riemann–Liouville fractional derivatives and Riemann–Liouville fractional integrals of different orders. Our results rely on the standard tools of functional analysis. Examples are constructed to demonstrate the application of the derived results.

Keywords: caputo-type fractional derivative; fractional integral; existence; fixed point

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1. Introduction

In the last few decades, fractional-order single-valued and multivalued boundary value problems containing different fractional derivatives such as Caputo, Riemann–Liouville, Hadamard, etc., and classical, nonlocal, integral boundary conditions have been extensively studied, for example, see the articles [1–12] and the references cited therein.

In the study of variational principles, fractional differential equations involving both left and right fractional derivatives give rise to a special class of Euler–Lagrange equations, for details, see [13] and the references cited therein. Let us consider some works on mixed fractional-order boundary value problems. In [14], the authors discussed the existence of an extremal solution to a nonlinear system involving the right-handed Riemann–Liouville fractional derivative. In [15], a two-point nonlinear higher order fractional boundary value problem involving left Riemann–Liouville and right Caputo fractional derivatives was investigated, while a problem in terms of left Caputo and right Riemann–Liouville fractional derivatives was studied in [16]. A nonlinear fractional oscillator equation containing left Riemann–Liouville and right Caputo fractional derivatives was investigated in [17]. In a recent paper [18], the authors proved some existence results for nonlocal boundary value problems of differential equations and inclusions containing both left Caputo and right Riemann–Liouville fractional derivatives.

Integro-differential equations appear in the mathematical modeling of several real world problems such as, heat transfer phenomena [19,20], forced-convective flow over a heat-conducting plate [21], etc. In [22], the authors studied the steady heat-transfer in fractal media via the local fractional nonlinear Volterra integro-differential equations. Electromagnetic waves in a variety of dielectric media with susceptibility following a fractional power-law are described by the fractional integro-differential equations [23].

Motivated by aforementioned applications of integro-differential equations and [18], we introduce a new kind of integro-differential equation involving right-Caputo and left-Riemann–Liouville fractional derivatives of different orders and right-left Riemann–Liouville fractional integrals and solve it subject to nonlocal boundary conditions. In precise terms, we prove existence and uniqueness of solutions for the problem given by

$$ {}^{C}D^{\alpha}_{1-}\,{}^{RL}D^{\beta}_{0+}y(t) + \lambda I^{p}_{1-}I^{q}_{0+}h(t,y(t)) = f(t,y(t)), \;\; t \in J := [0,1], \tag{1} $$

$$ y(0) = y(\xi) = 0, \quad y(1) = \delta y(\mu), \;\; 0 < \xi < \mu < 1, \tag{2} $$

where ${}^{C}D^{\alpha}_{1-}$ and ${}^{RL}D^{\beta}_{0+}$ denote the right Caputo fractional derivative of order $\alpha \in (1,2]$ and the left Riemann–Liouville fractional derivative of order $\beta \in (0,1]$, I^{p}_{1-} and I^{q}_{0+} denote the right and left Riemann–Liouville fractional integrals of orders $p,q > 0$ respectively, $f,h : [0,1] \times \mathbb{R} \to \mathbb{R}$ are given continuous functions and $\delta, \lambda \in \mathbb{R}$. It is imperative to notice that the integro-differential equation in (1) and (2) contains mixed type (integral and nonintegral) nonlinearities.

We organize the rest of the paper as follows. Section 2 contains some preliminary concepts related to our work. In Section 3, we prove an auxiliary lemma for the linear variant of the problem (1) and (2). Then we derive the existence results for the problem (1) and (2) by applying a fixed point theorem due to Krasnoselski and Leray–Schauder nonlinear alternative, while the uniqueness result is established via Banach contraction mapping principle. Examples illustrating the main results are also presented.

2. Preliminaries

In this section, we recall some related definitions of fractional calculus [1].

Definition 1. *The left and right Riemann–Liouville fractional integrals of order $\beta > 0$ for an integrable function $g : (0,\infty) \to \mathbb{R}$ are respectively defined by*

$$ I^{\beta}_{0+}g(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g(s)ds \;\; and \;\; I^{\beta}_{1-}g(t) = \int_t^1 \frac{(s-t)^{\beta-1}}{\Gamma(\beta)} g(s)ds. $$

Definition 2. *The left Riemann–Liouville fractional derivative and the right Caputo fractional derivative of order $\beta \in (n-1,n], n \in \mathbb{N}$ for a function $g : (0,\infty) \to \mathbb{R}$ with $g \in \mathbb{C}^n((0,\infty),\mathbb{R})$ are respectively given by*

$$ D^{\beta}_{0+}g(t) = \frac{d^n}{dt^n} \int_0^t \frac{(t-s)^{n-\beta-1}}{\Gamma(n-\beta)} g(s)ds \;\; and \;\; {}^{c}D^{\beta}_{1-}g(t) = (-1)^n \int_t^1 \frac{(s-t)^{n-\beta-1}}{\Gamma(n-\beta)} g^{(n)}(s)ds. $$

Lemma 1. *If $p > 0$ and $q > 0$, then the following relations hold almost everywhere on $[a,b]$:*

$$ I^{p}_{1-}I^{q}_{1-}f(x) = I^{p+q}_{1-}f(x), \quad I^{p}_{0+}I^{q}_{0+}f(x) = I^{p+q}_{0+}f(x). $$

3. Main Results

In the following lemma, we solve a linear variant of the problem (1) and (2).

Lemma 2. *Let $H, F \in C[0,1] \cap L(0,1)$. Then the linear problem*

$$ \begin{cases} {}^{C}D^{\alpha}_{1-}\,{}^{RL}D^{\beta}_{0+}y(t) + \lambda I^{p}_{1-}I^{q}_{0+}H(t) = F(t), & t \in J := [0,1], \\ y(0) = y(\xi) = 0, \quad y(1) = \delta y(\mu), \end{cases} \tag{3} $$

is equivalent to the fractional integral equation:

$$
\begin{aligned}
y(t) = &\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\Big[I_{1-}^{\alpha}F(s) - \lambda I_{1-}^{\alpha+p} I_{0+}^{q} H(s)\Big]ds \\
&+ a_1(t)\Big\{\delta\int_0^{\mu} \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\Big[I_{1-}^{\alpha}F(s) - \lambda I_{1-}^{\alpha+p} I_{0+}^{q} H(s)\Big]ds \\
&- \int_0^{1} \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\Big[I_{1-}^{\alpha}F(s) - \lambda I_{1-}^{\alpha+p} I_{0+}^{q} H(s)\Big]ds\Big\} \\
&+ a_2(t)\int_0^{\xi} \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)}\Big[I_{1-}^{\alpha}F(s) - \lambda I_{1-}^{\alpha+p} I_{0+}^{q} H(s)\Big]ds,
\end{aligned} \tag{4}
$$

where

$$
a_1(t) = \frac{1}{\Lambda}\Big[\xi^{\beta+1}t^{\beta} - \xi^{\beta}t^{\beta+1}\Big],\ a_2(t) = \frac{1}{\Lambda}\Big[t^{\beta}(1-\delta\mu^{\beta+1}) - t^{\beta+1}(1-\delta\mu^{\beta})\Big], \tag{5}
$$

and it is assumed that

$$
\Lambda = \xi^{\beta+1}(1-\delta\mu^{\beta}) - \xi^{\beta}(1-\delta\mu^{\beta+1}) \neq 0. \tag{6}
$$

Proof. Applying the left and right fractional integrals I_{1-}^{α} and I_{0+}^{β} successively to the integro-differential equation in (3), and then using Lemma 1, we get

$$
y(t) = I_{0+}^{\beta} I_{1-}^{\alpha}F(t) - \lambda I_{0+}^{\beta} I_{1-}^{\alpha+p} I_{0+}^{q} H(t) + c_0\frac{t^{\beta}}{\Gamma(\beta+1)} + c_1\frac{t^{\beta+1}}{\Gamma(\beta+2)} + c_2 t^{\beta-1}, \tag{7}
$$

where c_0, c_1 and c_2 are unknown arbitrary constants.

In view of the condition $y(0) = 0$, it follows from (7) that $c_2 = 0$. Inserting $c_2 = 0$ in (7) and then using the nonlocal boundary conditions $y(\xi) = 0, y(1) = \delta y(\mu)$ in the resulting equation, we obtain a system of equations in c_0 and c_1 given by

$$
\begin{aligned}
c_0\Big(\frac{1-\delta\mu^{\beta}}{\Gamma(\beta+1)}\Big) + c_1\Big(\frac{1-\delta\mu^{\beta+1}}{\Gamma(\beta+2)}\Big) &= \delta A_1 - A_2, \\
c_0\Big(\frac{\xi^{\beta}}{\Gamma(\beta+1)}\Big) + c_1\Big(\frac{\xi^{\beta+1}}{\Gamma(\beta+2)}\Big) &= -A_3,
\end{aligned} \tag{8}
$$

where

$$
\begin{gathered}
A_1 = I_{0+}^{\beta} I_{1-}^{\alpha}F(\mu) - \lambda I_{0+}^{\beta} I_{1-}^{\alpha+p} I_{0+}^{q} H(\mu),\ \ A_2 = I_{0+}^{\beta} I_{1-}^{\alpha}F(1) - \lambda I_{0+}^{\beta} I_{1-}^{\alpha+p} I_{0+}^{q} H(1), \\
A_3 = I_{0+}^{\beta} I_{1-}^{\alpha}F(\xi) - \lambda I_{0+}^{\beta} I_{1-}^{\alpha+p} I_{0+}^{q} H(\xi).
\end{gathered}
$$

Solving the system (8), we find that

$$
\begin{aligned}
c_0 &= \frac{\Gamma(\beta+1)}{\Lambda}\Big[\xi^{\beta+1}(\delta A_1 - A_2) + (1-\delta\mu^{\beta+1})A_3\Big], \\
c_1 &= \frac{-\Gamma(\beta+2)}{\Lambda}\Big[\xi^{\beta}(\delta A_1 - A_2) + (1-\delta\mu^{\beta})A_3\Big],
\end{aligned}
$$

where Λ is defined by (6). Substituting the values of c_0 and c_1 together with the notations (5) in (7), we obtain the solution (4). The converse follows by direct computation. This completes the proof. □

Let $\mathcal{X} = C([0,1], \mathbb{R})$ denote the Banach space of all continuous functions from $[0,1] \to \mathbb{R}$ equipped with the norm $\|y\| = \sup\{|y(t)| : t \in [0,1]\}$. By Lemma 2, we define an operator $\mathcal{G} : \mathcal{X} \to \mathcal{X}$ associated with the problem (1) and (2) as

$$\begin{aligned}\mathcal{G}y(t) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\Big[I_{1-}^{\alpha} f(s,y(s)) - \lambda I_{1-}^{\alpha+p} I_{0+}^{q} h(s,y(s))\Big] ds \\ &\quad + a_1(t)\Bigg[\delta \int_0^{\mu} \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\Big[I_{1-}^{\alpha} f(s,y(s)) - \lambda I_{1-}^{\alpha+p} I_{0+}^{q} h(s,y(s))\Big] ds \\ &\quad - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\Big[I_{1-}^{\alpha} f(s,y(s)) - \lambda I_{1-}^{\alpha+p} I_{0+}^{q} h(s,y(s))\Big] ds\Bigg] \\ &\quad + a_2(t)\int_0^{\xi} \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)}\Big[I_{1-}^{\alpha} f(s,y(s)) - \lambda I_{1-}^{\alpha+p} I_{0+}^{q} h(s,y(s))\Big] ds.\end{aligned}$$

Notice that the fixed points of the operator $\mathcal{G}$ are solutions of the problem (1) and (2).

In the forthcoming analysis, we use the following estimates:

$$\begin{aligned}\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha+p} I_{0+}^{q} ds &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha+p-1}}{\Gamma(\alpha+p)} \int_0^u \frac{(u-r)^{q-1}}{\Gamma(q)} dr\, du\, ds \\ &\le \frac{t^{\beta}}{\Gamma(\beta+1)\Gamma(\alpha+p+1)\Gamma(q+1)}, \\ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} ds &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} du\, ds \le \frac{t^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)},\end{aligned}$$

where we have used $u^q \le 1,\ (1-s)^{\alpha+p} < 1;\ p,q > 0, 1 < \alpha \le 2$.

In the sequel, we set

$$\Omega_1 = \frac{\Delta}{\Gamma(\alpha+1)}, \quad \Omega_2 = \frac{|\lambda|\Delta}{\Gamma(\alpha+p+1)\Gamma(q+1)}, \tag{9}$$

where

$$\begin{aligned}\Delta &= \frac{1}{\Gamma(\beta+1)}\Big[1 + \bar{a}_1(|\delta|\mu^{\beta} + 1) + \bar{a}_2\xi^{\beta}\Big], \\ \bar{a}_1 &= \max_{t\in[0,1]} |a_1(t)|, \quad \bar{a}_2 = \max_{t\in[0,1]} |a_2(t)|.\end{aligned}$$

3.1. Existence Results

In the following, we prove our first existence result for the problem (1) and (2), which relies on Krasnoselskii's fixed point theorem [24].

Theorem 1. *Assumed that:*

(B_1) *There exist* $L > 0$ *such that* $|f(t,x) - f(t,y)| \le L|x-y|$, $\forall t \in [0,1]$, $x, y \in \mathbb{R}$;
(B_2) *There exist* $K > 0$ *such that* $|h(t,x) - h(t,y)| \le K|x-y|$, $\forall t \in [0,1]$, $x, y \in \mathbb{R}$;
(B_3) $|f(t,y)| \le \sigma(t)$ *and* $|h(t,y)| \le \rho(t)$, *where* $\sigma, \rho \in C([0,1], \mathbb{R}^+)$.

Then the problem (1) and (2) has at least one solution on $[0,1]$ *if* $L\gamma_1 + K\gamma_2 < 1$, *where*

$$\gamma_1 = \frac{1}{\Gamma(\beta+1)\Gamma(\alpha+1)}, \quad \gamma_2 = \frac{|\lambda|}{\Gamma(\beta+1)\Gamma(\alpha+p+1)\Gamma(q+1)}. \tag{10}$$

Proof. Introduce the ball $B_\theta = \{y \in \mathcal{X} : \|y\| \leq \theta\}$, where $\|\sigma\| = \sup_{t\in[0,1]} |\sigma(t)|$, $\|\rho\| = \sup_{t\in[0,1]} |\rho(t)|$ and

$$\theta \geq \|\sigma\|\Omega_1 + \|\rho\|\Omega_2. \tag{11}$$

Let us split the operator $\mathcal{G} : \mathcal{X} \to \mathcal{X}$ on B_θ as $\mathcal{G} = \mathcal{G}_1 + \mathcal{G}_2$, where

$$\begin{aligned}
\mathcal{G}_1 y(t) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} f(s,y(s))ds - \lambda \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha+p} I_{0+}^{q} h(s,y(s))ds, \\
\mathcal{G}_2 y(t) &= a_1(t)\Bigg[\delta\bigg(\int_0^\mu \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} f(s,y(s))ds - \lambda \int_0^\mu \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha+p} I_{0+}^{q} h(s,y(s))ds\bigg) \\
&\quad -\bigg(\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} f(s,y(s))ds - \lambda \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha+p} I_{0+}^{q} h(s,y(s))ds\bigg)\Bigg] \\
&\quad + a_2(t)\Bigg[\int_0^\xi \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} f(s,y(s))ds - \lambda \int_0^\xi \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha+p} I_{0+}^{q} h(s,y(s))ds\Bigg].
\end{aligned}$$

Now, we verify that the operators $\mathcal{G}_1$ and $\mathcal{G}_2$ satisfy the hypothesis of Krasnoselskii's theorem [24] in three steps.

(i) For $y, x \in B_\theta$, we have

$$\begin{aligned}
&\|\mathcal{G}_1 y + \mathcal{G}_2 x\| \\
\leq\ & \sup_{t\in[0,1]} \Bigg\{ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} |f(s,y(s))|ds + |\lambda| \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha+p} I_{0+}^{q} |h(s,y(s))|ds \\
& + |a_1(t)|\bigg\{|\delta|\bigg(\int_0^\mu \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} |f(s,x(s))|ds + |\lambda| \int_0^\mu \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha+p} I_{0+}^{q} |h(s,x(s))|ds\bigg) \\
& + \bigg(\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} |f(s,x(s))|ds + |\lambda| \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha+p} I_{0+}^{q} |h(s,x(s))|ds\bigg)\bigg\} \\
& + |a_2(t)|\bigg\{\int_0^\xi \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} |f(s,x(s))|ds + |\lambda| \int_0^\xi \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha+p} I_{0+}^{q} |h(s,x(s))|ds\bigg\}\Bigg\} \\
\leq\ & \|\sigma\| \sup_{t\in[0,1]} \bigg\{ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} ds + |a_1(t)|\bigg[|\delta| \int_0^\mu \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} ds \\
& + \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} ds\bigg] + |a_2(t)| \int_0^\xi \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} ds\bigg\} \\
& + \|\rho\||\lambda| \sup_{t\in[0,1]} \bigg\{ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha+p} I_{0+}^{q} ds + |a_1(t)|\bigg[|\delta| \int_0^\mu \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha+p} I_{0+}^{q} ds \\
& + \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha+p} I_{0+}^{q} ds\bigg] + |a_2(t)| \int_0^\xi \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha+p} I_{0+}^{q} ds\bigg\} \\
\leq\ & \bigg\{\frac{\|\sigma\|}{\Gamma(\alpha+1)} + \frac{\|\rho\||\lambda|}{\Gamma(\alpha+p+1)\Gamma(q+1)}\bigg\}\Delta \\
=\ & \|\sigma\|\Omega_1 + \|\rho\|\Omega_2 < \theta,
\end{aligned}$$

where we used (11). Thus $\mathcal{G}_1 y + \mathcal{G}_2 x \in B_\theta$.

(ii) Using (B_1) and (B_2), it is easy to show that

$$\begin{aligned}\|\mathcal{G}_1 y-\mathcal{G}_1 x\| &\leq \sup_{t\in[0,1]}\left\{\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha}|f(s,y(s))-f(s,x(s))|ds\right.\\ &\quad\left.+|\lambda|\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha+p} I_{0+}^{q}|h(s,y(s))-h(s,x(s))|ds\right\}\\ &\leq (L\gamma_1+K\gamma_2)\|y-x\|,\end{aligned}$$

which, in view of the condition: $L\gamma_1+K\gamma_2<1$, implies that the operator $\mathcal{G}_1$ is a contraction.

(iii) Continuity of the functions f,h implies that the operator $\mathcal{G}_2$ is continuous. In addition, $\mathcal{G}_2$ is uniformly bounded on B_θ as

$$\begin{aligned}\|\mathcal{G}_2 y\| &\leq \sup_{t\in[0,1]}\left\{|a_1(t)|\left\{|\delta|\int_0^\mu \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\left[I_{1-}^{\alpha}|f(s,y(s))|+|\lambda|I_{1-}^{\alpha+p}I_{0+}^{q}|h(s,y(s))|\right]ds\right.\right.\\ &\quad\left.+\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\left[I_{1-}^{\alpha}|f(s,y(s))|+|\lambda|I_{1-}^{\alpha+p}I_{0+}^{q}|h(s,y(s))|\right]ds\right\}\\ &\quad\left.+|a_2(t)|\int_0^\xi \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)}\left[I_{1-}^{\alpha}|f(s,y(s))|+|\lambda|I_{1-}^{\alpha+p}I_{0+}^{q}|h(s,y(s))|\right]ds\right\}\\ &\leq \|\sigma\|\sup_{t\in[0,1]}\left\{|a_1(t)|\left[|\delta|\int_0^\mu \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}I_{1-}^{\alpha}ds+\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}I_{1-}^{\alpha}ds\right]\right.\\ &\quad\left.+|a_2(t)|\int_0^\xi \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)}I_{1-}^{\alpha}ds\right\}\\ &\quad+\|\rho\||\lambda|\sup_{t\in[0,1]}\left\{|a_1(t)|\left[|\delta|\int_0^\mu \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}I_{1-}^{\alpha+p}I_{0+}^{q}ds+\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}I_{1-}^{\alpha+p}I_{0+}^{q}ds\right]\right.\\ &\quad\left.+|a_2(t)|\int_0^\xi \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)}I_{1-}^{\alpha+p}I_{0+}^{q}ds\right\}\\ &\leq \|\sigma\|(\Omega_1-\gamma_1)+\|\rho\|(\Omega_2-\gamma_2),\end{aligned}$$

where Ω_i, and γ_i $(i=1,2)$ are defined by (9) and (10) respectively.

To show the compactness of $\mathcal{G}_2$, we fix $\sup_{(t,y)\in[0,1]\times B_\theta}|f(t,y)|=\overline{f}$, $\sup_{(t,y)\in[0,1]\times B_\theta}|h(t,y)|=\overline{h}$. Then, for $0<t_1<t_2<1$, we have

$$\begin{aligned}&|(\mathcal{G}_2 y)(t_2)-(\mathcal{G}_2 y)(t_1)|\\ \leq\ & |a_1(t_2)-a_1(t_1)|\left\{|\delta|\int_0^\mu \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)}\left[I_{1-}^{\alpha}|f(s,y(s))|+|\lambda|I_{1-}^{\alpha+p}I_{0+}^{q}|h(s,y(s))|\right]ds\right.\\ &\left.+\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\left[I_{1-}^{\alpha}|f(s,y(s))|+|\lambda|I_{1-}^{\alpha+p}I_{0+}^{q}|h(s,y(s))|\right]ds\right\}\\ &+|a_2(t_2)-a_2(t_1)|\left\{\int_0^\xi \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)}\left[I_{1-}^{\alpha}|f(s,y(s))|+|\lambda|I_{1-}^{\alpha+p}I_{0+}^{q}|h(s,y(s))|\right]ds\right\}\\ \leq\ & (\gamma_1\overline{f}+\gamma_2\overline{h})\left\{\left(\xi^{\beta+1}|t_2^\beta-t_1^\beta|+\xi^\beta|t_1^{\beta+1}-t_2^{\beta+1}|\right)\frac{(|\delta|\mu^\beta+1)}{|\Lambda|}\right.\\ &\left.+\left(|1-\delta\mu^{\beta+1}||t_2^\beta-t_1^\beta|+|1-\delta\mu^\beta||t_1^{\beta+1}-t_2^{\beta+1}|\right)\frac{\xi^\beta}{|\Lambda|}\right\},\end{aligned}$$

which tends to zero independent of y as $t_2\to t_1$. This shows that $\mathcal{G}_2$ is equicontinuous. It is clear from the foregoing arguments that the operator $\mathcal{G}_2$ is relatively compact on B_θ. Hence, by the Arzelá-Ascoli theorem, $\mathcal{G}_2$ is compact on B_θ.

In view of the foregoing arguments (*i*)-(*iii*), the hypothesis of the Krasnoselskii's fixed point theorem [24] holds true. Thus, the operator $\mathcal{G}_1 + \mathcal{G}_2 = \mathcal{G}$ has a fixed point, which implies that the problem (1) and (2) has at least one solution on $[0,1]$. The proof is finished. □

Remark 1. *If we interchange the roles of the operators $\mathcal{G}_1$ and $\mathcal{G}_2$ in the previous result, the condition $L\gamma_1 + K\gamma_2 < 1$, is replaced with the following one:*

$$L(\Omega_1 - \gamma_1) + K(\Omega_2 - \gamma_2) < 1,$$

where Ω_1, Ω_2 and γ_1, γ_2 are defined by (9), (10) respectively.

The following existence result relies on Leray–Schauder nonlinear alternative [25].

Theorem 2. *Suppose that the following conditions hold:*

(B_4) *There exist continuous nondecreasing functions $\phi, \psi : [0,\infty) \to (0,\infty)$ such that $\forall (t,y) \in [0,1] \times \mathbb{R}$, $|f(t,y)| \le \omega_1(t)\phi(\|y\|)$ and $|h(t,y)| \le \omega_2(t)\psi(\|y\|)$, where $\omega_1, \omega_2 \in C([0,T], \mathbb{R}^+)$.*
(B_5) *There exist a constant $M > 0$ such that*

$$\frac{M}{\|\omega_1\|\phi(M)\Omega_1 + \|\omega_2\|\psi(M)\Omega_2} > 1,$$

Then, the problem (1) and (2) has at least one solution on $[0,1]$.

Proof. First we show that the operator $\mathcal{G}$ is completely continuous. This will be established in several steps.

(*i*) $\mathcal{G}$ maps bounded sets into bounded sets in $\mathcal{X}$.

Let $y \in \mathcal{B}_r = \{y \in \mathcal{X} : \|y\| \le r\}$, where r is a fixed number. Then, using the strategy employed in the proof of Theorem 1, we obtain

$$\begin{aligned} \|\mathcal{G}y(t)\| &\le \left\{ \frac{\|\omega_1\|\phi(r)}{\Gamma(\alpha+1)} + \frac{\|\omega_2\|\psi(r)|\lambda|}{\Gamma(\alpha+p+1)\Gamma(q+1)} \right\}\Delta \\ &= \|\omega_1\|\phi(r)\Omega_1 + \|\omega_2\|\psi(r)\Omega_2 < \infty. \end{aligned}$$

(ii) $\mathcal{G}$ maps bounded sets into equicontinuous sets.

Let $0 < t_1 < t_2 < 1$ and $y \in \mathcal{B}_r$, where $\mathcal{B}_r$ is bounded set in $\mathcal{X}$. Then we obtain

$$
\begin{aligned}
& |\mathcal{G}y(t_2) - \mathcal{G}y(t_1)| \\
\leq \ & \left| \int_0^{t_1} \frac{(t_2-s)^{\beta-1} - (t_1-s)^{\beta-1}}{\Gamma(\beta)} \left[I_{1-}^{\alpha} |f(s,y(s))| + |\lambda| I_{1-}^{\alpha+p} I_{0+}^{q} |h(s,y(s))| \right] ds \right| \\
& + \left| \int_{t_1}^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} \left[I_{1-}^{\alpha} |f(s,y(s))| + |\lambda| I_{1-}^{\alpha+p} I_{0+}^{q} |h(s,y(s))| \right] ds \right| \\
& + |a_1(t_2) - a_1(t_1)| \left\{ |\delta| \left(\int_0^{\mu} \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)} \left[I_{1-}^{\alpha} |f(s,y(s))| + |\lambda| I_{1-}^{\alpha+p} I_{0+}^{q} |h(s,y(s))| \right] ds \right) \right. \\
& \left. + \left(\int_0^{1} \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \left[I_{1-}^{\alpha} |f(s,y(s))| + |\lambda| I_{1-}^{\alpha+p} I_{0+}^{q} |h(s,y(s))| \right] ds \right) \right\} \\
& + |a_2(t_2) - a_2(t_1)| \left\{ \int_0^{\xi} \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} \left[I_{1-}^{\alpha} |f(s,y(s))| + |\lambda| I_{1-}^{\alpha+p} I_{0+}^{q} |h(s,y(s))| \right] ds \right\} \\
\leq \ & \left[\frac{\|\omega_1\| \phi(r)}{\Gamma(\beta+1)\Gamma(\alpha+1)} + \frac{\|\omega_2\| \psi(r) |\lambda|}{\Gamma(\beta+1)\Gamma(\alpha+p+1)\Gamma(q+1)} \right] \\
& \times \left\{ 2(t_2-t_1)^{\beta} + |t_2{}^{\beta} - t_1{}^{\beta}| + \frac{(|\delta|\mu^{\beta}+1)}{|\Lambda|} \left(\xi^{\beta+1} |t_2^{\beta} - t_1^{\beta}| + \xi^{\beta} |t_2^{\beta+1} - t_1^{\beta+1}| \right) \right. \\
& \left. + \frac{\xi^{\beta}}{|\Lambda|} \left(|1 - \delta\mu^{\beta+1}| |t_2^{\beta} - t_1^{\beta}| + |1 - \delta\mu^{\beta}| |t_2^{\beta+1} - t_1^{\beta+1}| \right) \right\}.
\end{aligned}
$$

Notice that the right-hand side of the above inequality tends to 0 as $t_2 \to t_1$, independent of $y \in \mathcal{B}_r$. In view of the foregoing arguments, it follows by the Arzelá–Ascoli theorem that $\mathcal{G} : \mathcal{X} \to \mathcal{X}$ is completely continuous.

The conclusion of the Leray–Schauder nonlinear alternative [25] will be applicable once it is shown that there exists an open set $U \subset C([0,1],\mathbb{R})$ with $y \neq \nu\mathcal{G}y$ for $\nu \in (0,1)$ and $y \in \partial U$. Let $y \in C([0,1],\mathbb{R})$ such that $y = \nu\mathcal{G}y$ for $\nu \in (0,1)$. As argued in proving that the operator $\mathcal{G}$ is bounded, one can obtain that

$$|y(t)| = |\nu\mathcal{G}y(t)| \quad \leq |\omega_1(t)|\phi(\|y\|)\Omega_1 + |\omega_2(t)|\psi(\|y\|)\Omega_2,$$

which can be written as

$$\frac{\|y\|}{\|\omega_1\|\phi(\|y\|)\Omega_1 + \|\omega_2\|\psi(\|y\|)\Omega_2} \leq 1.$$

On the other hand, we can find a positive number M such that $\|y\| \neq M$ by assumption (B_5). Let us set

$$U = \{y \in \mathcal{X} : \|y\| < M\}.$$

Clearly, ∂U contains a solution only when $\|y\| = M$. In other words, there is no solution $y \in \partial U$ such that $y = \nu\mathcal{G}y$ for some $\nu \in (0,1)$. Therefore, $\mathcal{G}$ has a fixed point $y \in \overline{U}$ which is a solution of the problem (1) and (2). The proof is finished. □

3.2. Uniqueness Result

Here we prove a uniqueness result for the problem (1) and (2) with the aid of Banach contraction mapping principle.

Theorem 3. *If the conditions (B_1) and (B_2) hold, then the problem (1) and (2) has a unique solution on $[0,1]$ if*

$$L\Omega_1 + K\Omega_2 < 1, \tag{12}$$

where Ω_1 and Ω_2 are defined by (9).

Proof. In the first step, we show that $\mathcal{G}\mathcal{B}_r \subset \mathcal{B}_r$, where $\mathcal{B}_r = \{y \in \mathcal{X} : \|y\| \leq r\}$ with

$$r \geq \frac{f_0\Omega_1 + h_0\Omega_2}{1-(L\Omega_1 + K\Omega_2)}, \; f_0 = \sup_{t\in[0,1]} |f(t,0)|, \; h_0 = \sup_{t\in[0,1]} |h(t,0)|.$$

For $y \in \mathcal{B}_r$ and using the condition (B_1), we have

$$\begin{aligned} |f(t,y)| &= |f(t,y) - f(t,0) + f(t,0)| \leq |f(t,y) - f(t,0)| + |f(t,0)| \\ &\leq L\|y\| + f_0 \leq Lr + f_0. \end{aligned} \tag{13}$$

Similarly, using (B_2), we get

$$|h(t,y)| \leq Kr + h_0. \tag{14}$$

In view of (13) and (14), we obtain

$$\begin{aligned} \|\mathcal{G}y\| &= \sup_{t\in[0,1]} |\mathcal{G}y(t)| \\ &\leq \sup_{t\in[0,1]} \left\{ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \Big[I_{1-}^{\alpha} |f(s,y(s))| + |\lambda| I_{1-}^{\alpha+p} I_{0+}^{q} |h(s,y(s))| \Big] ds \right. \\ &\quad + |a_1(t)| \Bigg\{ |\delta| \int_0^{\mu} \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)} \Big[I_{1-}^{\alpha} |f(s,y(s))| + |\lambda| I_{1-}^{\alpha+p} I_{0+}^{q} |h(s,y(s))| \Big] ds \\ &\quad + \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \Big[I_{1-}^{\alpha} |f(s,y(s))| + |\lambda| I_{1-}^{\alpha+p} I_{0+}^{q} |h(s,y(s))| \Big] ds \Bigg\} \\ &\quad \left. + |a_2(t)| \int_0^{\xi} \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} \Big[I_{1-}^{\alpha} |f(s,y(s))| + |\lambda| I_{1-}^{\alpha+p} I_{0+}^{q} |h(s,y(s))| \Big] ds \right\} \\ &\leq (Lr + f_0) \sup_{t\in[0,1]} \left\{ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} ds + |a_1(t)| \Bigg[|\delta| \int_0^{\mu} \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} ds \right. \\ &\quad \left. + \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} ds \Bigg] + |a_2(t)| \int_0^{\xi} \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} ds \right\} \\ &\quad + (Kr + h_0)|\lambda| \sup_{t\in[0,1]} \left\{ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha+p} I_{0+}^{q} ds \right. \\ &\quad + |a_1(t)| \Bigg[|\delta| \int_0^{\mu} \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha+p} I_{0+}^{q} ds + \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha+p} I_{0+}^{q} ds \Bigg] \\ &\quad \left. + |a_2(t)| \int_0^{\xi} \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha+p} I_{0+}^{q} ds \right\} \\ &\leq \left\{ \frac{(Lr + f_0)}{\Gamma(\alpha+1)} + \frac{(Kr + h_0)|\lambda|}{\Gamma(\alpha+p+1)\Gamma(q+1)} \right\} \Delta \\ &= (Lr + f_0)\Omega_1 + (Kr + h_0)\Omega_2 < r, \end{aligned}$$

which implies that $\mathcal{G}y \in \mathcal{B}_r$, for any $y \in \mathcal{B}_r$. Therefore, $\mathcal{G}\mathcal{B}_r \subset \mathcal{B}_r$. Next, we prove that $\mathcal{G}$ is a contraction. For that, let $x, y \in \mathcal{X}$ and $t \in [0,1]$. Then, by the conditions (B_1) and (B_2), we obtain

$$
\begin{aligned}
\|\mathcal{G}y-\mathcal{G}x\| &= \sup_{t\in[0,1]}\left|(\mathcal{G}y)(t)-(\mathcal{G}x)(t)\right| \\
&\le \sup_{t\in[0,1]}\left\{\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha}|f(s,y(s))-f(s,x(s))|ds\right. \\
&\quad +|\lambda|\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha+p} I_{0+}^{q}|h(s,y(s))-h(s,x(s))|ds \\
&\quad +|a_1(t)|\left[|\delta|\left(\int_0^\mu \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha}|f(s,y(s))-f(s,x(s))|ds\right.\right. \\
&\quad \left. +|\lambda|\int_0^\mu \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha+p} I_{0+}^{q}|h(s,y(s))-h(s,x(s))|ds\right) \\
&\quad +\left(\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha}|f(s,y(s))-f(s,x(s))|ds\right. \\
&\quad \left.\left. +|\lambda|\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha+p} I_{0+}^{q}|h(s,y(s))-h(s,x(s))|ds\right)\right] \\
&\quad +|a_2(t)|\left[\int_0^\xi \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha}|f(s,y(s))-f(s,x(s))|ds\right. \\
&\quad \left.\left. +|\lambda|\int_0^\xi \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha+p} I_{0+}^{q}|h(s,y(s))-h(s,x(s))|dr\,du\,ds\right]\right\} \\
&\le L\|y-x\|\sup_{t\in[0,1]}\left\{\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha}ds\right. \\
&\quad +|a_1(t)|\left[|\delta|\int_0^\mu \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha}ds+\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha}ds\right] \\
&\quad \left. +|a_2(t)|\int_0^\xi \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha}ds\right\} \\
&\quad +K\|y-x\||\lambda|\sup_{t\in[0,1]}\left\{\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha+p} I_{0+}^{q}ds\right. \\
&\quad +|a_1(t)|\left[|\delta|\int_0^\mu \frac{(\mu-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha+p} I_{0+}^{q}ds+\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha+p} I_{0+}^{q}ds\right] \\
&\quad \left. +|a_2(t)|\int_0^\xi \frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha+p} I_{0+}^{q}ds\right\} \\
&\le \left\{\frac{L\Delta}{\Gamma(\alpha+1)}+\frac{K|\lambda|\Delta}{\Gamma(\alpha+p+1)\Gamma(q+1)}\right\}\|y-x\| \\
&= (L\Omega_1+K\Omega_2)\|y-x\|.
\end{aligned}
$$

From the above inequality, it follows by the assumption $(L\Omega_1+K\Omega_2)<1$ that $\mathcal{G}$ is a contraction. Therefore, we deduce by Banach contraction mapping principle that there exists a unique fixed point for the operator $\mathcal{G}$, which corresponds to a unique solution for the problem (1) and (2) on $[0,1]$. The proof is completed. □

3.3. Examples

In this subsection, we construct examples to illustrate the existence and uniqueness results obtained in the last two subsections. Let us consider the following problem:

$$
\begin{cases}
D_{1-}^{3/2}D_{0+}^{1/2}y(t)+2I_{1-}^{4/3}I_{0+}^{5/4}h(t,y(t))=f(t,y(t)),\ t\in J:=[0,1], \\
y(0)=y(2/3)=0,\ \ y(1)=\frac{1}{2}y(3/4).
\end{cases}
\tag{15}
$$

Here $\alpha = 3/2, \beta = 1/2, \lambda = 2, p = 4/3, q = 5/4, \mu = 3/4, \delta = 1/2, \xi = 2/3$, and

$$f(t,y) = \frac{1}{(t^2+8)}\big(\tan^{-1} y + e^{-t}\big), \quad h(t,y) = \frac{1}{2\sqrt{t^2+9}}\big(\frac{|y|}{1+|y|} + e^{-t}\big). \tag{16}$$

Using the given data, it is found that $L = 1/8, K = 1/6$,

$$\bar{a}_1 = \max_{t\in[0,1]} |a_1(t)| = |a_1(t)|_{t=1} \approx 1.121394517474712,$$
$$\bar{a}_2 = \max_{t\in[0,1]} |a_2(t)| = |a_2(t)|_{t=t_{a_2}} \approx 1.168623082364286,$$

where

$$t_{a_2} = \frac{\beta(1-\delta\mu^{\beta+1})}{(1-\delta\mu^{\beta})(\beta+1)} \approx 0.396975661732535.$$

In consequence, we get

$$\Omega_1 \approx 3.022797441671726, \Omega_2 \approx 1.451691300771574, |\Lambda| \approx 0.242702744426469,$$

where Ω_1,Ω_2 are defined by (9) and Λ is given by (6).
(i) For illustrating Theorem 1, we have

$$|f(t,y)| \le \sigma(t) = \frac{e^{-t} + (\pi/2)}{t^2+8}, \quad |h(t,y)| \le \rho(t) = \frac{e^{-t}+1}{2\sqrt{t^2+9}},$$

and that

$$L\gamma_1 + K\gamma_2 \approx 0.174044436618777 < 1,$$

where $\gamma_1 \approx 0.848826363156775$ and $\gamma_2 \approx 0.407646847345084$. Clearly, the hypothesis of Theorem 1 is satisfied and consequently its conclusion applies to the problem (15).
(ii) In order to explain Theorem 2, we take the following values (instead of (16)) in the problem (15):

$$f(t,y) = \frac{1}{\sqrt{t^2+25}}\big(y\cos y + \pi/2\big), \quad h(t,y) = \frac{1}{5\sqrt{t^2+4}}\big(\sin y + 1/4\big), \tag{17}$$

and note that $\omega_1(t) = \frac{1}{\sqrt{t^2+25}}, \|\omega_1\| = 1/5, \omega_2(t) = \frac{1}{5\sqrt{t^2+4}}, \|\omega_2\| = 1/10, \phi(\|y\|) = \|y\| + \pi/2$ and $\psi(\|y\|) = \|y\| + 1/4$. By the condition (B_5), we find that $M > 3.939452045479877$. Thus, all the conditions of Theorem 2 are satisfied and, hence the problem (15) with $f(t,y)$ and $h(t,y)$ given by (17) has at least one solution on $[0,1]$.
(iii) It is easy to show that $f(t,y)$ and $h(t,y)$ satisfy the conditions (B_1) and (B_2) respectively with $L = 1/8$ and $K = 1/6$ and that $L\Omega_1 + K\Omega_2 \approx 0.619798230337561 < 1$. Thus, all the assumptions of Theorem 3 hold true and hence the problem (15) has a unique solution on $[0,1]$.

4. Conclusions

We considered a fractional differential equation involving left Caputo and right Riemann–Liouville fractional derivatives of different orders and a pair of nonlinearities: $I^{p}_{1-}I^{q}_{0+}h(t,y(t)) = \int_t^1 \frac{(s-t)^{p-1}}{\Gamma(p)} \int_0^s \frac{(s-v)^{q-1}}{\Gamma(q)} h(v,y(v))dvds$ (integral type) and $f(t,y(t))$, equipped with four-point nonlocal boundary conditions. Different criteria ensuring the existence of solutions for the given problem are presented in Theorems 1 and 2, while the uniqueness of solutions is shown in Theorem 3. An interesting and scientific feature of the fractional differential Equation (1) is that the integral type of nonlinearity can describe composition of a physical quantity (like density) over two different arbitrary subsegments of the given domain. In the case of $p = q = 1$, this composition takes the form

$\int_t^1 \int_0^s h(v, y(v))dvds$. As pointed out in the introduction, fractional differential equations containing mixed (left Caputo and right Riemann–Liouville) fractional derivatives appear as Euler–Lagrange equations in the study of variational principles. So, such equations in the presence of the integral type of nonlinearity of the form introduced in (1) enhances the scope of Euler–Lagrange equations studied in [26]. Moreover, the fractional integro-differential Equation (1) can improve the description of the electromagnetic waves in dielectric media considered in [23]. As a special case, our results correspond to a three-point nonlocal mixed fractional order boundary value problem by letting $\delta = 0$, which is indeed new in the given configuration.

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