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The Relations between Residuated Frames and Residuated Connections

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Abstract: We introduce the notion of (dual) residuated frames as a viewpoint of relational semantics for a fuzzy logic. We investigate the relations between (dual) residuated frames and (dual) residuated connections as a topological viewpoint of fuzzy rough sets in a complete residuated lattice. As a result, we show that the Alexandrov topology induced by fuzzy posets is a fuzzy complete lattice with residuated connections. From this result, we obtain fuzzy rough sets on the Alexandrov topology. Moreover, as a generalization of the Dedekind–MacNeille completion, we introduce R-R (resp. DR-DR) embedding maps and R-R (resp. DR-DR) frame embedding maps.

Keywords: complete residuated lattice; (dual) residuated frames; (dual) residuated connections; R-R (resp. DR-DR) embedding maps

1. Introduction

Blyth and Janovitz [1] introduced the residuated connection as a pair (f, g) of maps from a partially ordered set $(X, \leq_X)$ to a partially ordered set $(Y, \leq_Y)$ such that for all $x \in X, y \in Y$, $f(x) \leq_Y y$ if and only if $x \leq_X g(y)$. Examples of maps which form residuated connections play an important role [2–4]. Orłowska and Rewitzky [5–7] introduced the residuated frame of logical relational systems for residuated connections.

Pawlak [8,9] introduced the rough set theory as a formal tool to deal with imprecision and uncertainty in the data analysis. Rough sets form residuated connections in the following sense: let R be an equivalence relation on X. For $A \subset X$ and $[x]_R = \{y \in X \mid (x, y) \in R\}$,

$$\overline{R}(A) = \{x \in X \mid [x]_R \cap A \neq \emptyset\},\ \underline{R}(A) = \{x \in X \mid [x]_R \subset A\}. \tag{1}$$

Let $P(X)$ be the class of all subsets of X and $(P(X), \subset)$ be a partially ordered set. A rough set $(\underline{R}, \overline{R})$ forms a residuated connection because for all $A, B \subset X$, $\overline{R}(A) \subset B$ if and only if $A \subset \underline{R}(B)$.

Ward et al. [10] introduced a complete residuated lattice L as an important algebraic structure for many valued logics [11–16]. For an extension of Pawlak's rough sets, many researchers have developed L-lower and L-upper approximation operators in algebraic structures L [17–25]. She and Wang [26] developed an L-fuzzy rough set (G, H) with L-lower approximation operator G and L-upper approximation operator F in complete residuated lattices as follows. Let (X, e_X) be an L-fuzzy partially ordered set. For $A, B \in L^X$,

$$F(A)(y) = \bigvee_{x \in X} (e_X(x, y) \odot A(x)),\ G(B)(x) = \bigwedge_{y \in X} (e_X(x, y) \to B(y)). \tag{2}$$

Moreover, fuzzy rough sets form residuated connections in the following sense: for all $A, B \subset X$,

$$e_{L^Y}(F(A), B) = \bigwedge_{y \in X} (F(A)(y) \to B(y)) = \bigwedge_{x \in X} (A(x) \to G(B)(x)) = e_{L^X}(A, G(B)). \tag{3}$$

Perfilieva [27–30] introduced the theory of fuzzy transform and inverse fuzzy transform in complete residuated lattices, which is similar to other well-known transform theories such as the Fourier, Laplace, Hilbert and wavelet transforms, as well as fuzzy various concept analysis and fuzzy relation equations [31–33]. Oh and Kim [34] interpreted Perfilieva's fuzzy transform as a residuated connection (e_{L^X}, F, G, e_{L^Y}) with fuzzy transform and inverse fuzzy transform G. By using the residuated connection, F is a fuzzy join preserving map and G is a fuzzy meet preserving map in a Kim's fuzzy complete lattice sense [20], as a generalization of a complete lattice [35–38]. If X and Y are solutions of fuzzy relation equations $F(X) = B$ and $G(Y) = A$, then $G(B)$ and $F(A)$ are solutions, respectively.

Discrete and stone dualities are dualities between algebras and logical relational systems such as Boolean algebras and classical propositional logics; MV-algebra and Lukasiewicz logic; and BL-algebra and basic fuzzy logics [3–6,39–41]. The duality leads in a natural way to relational semantics for a logic [39–41].

In this paper, as a duality between algebras and logical relational systems, we introduce the notion of residuated connections and residuated frames in fuzzy logics. In Theorems 3 and 4, we show that (dual) residuated frames induce (dual) residuated connections.

Let (X, e_X) be an L-fuzzy partially ordered set. As a generalization of the classic Tarski's fixed point theorem [42,43] for isotone maps, we show that $\tau_{e_X} = \{A \in L^X \mid A = F(A) = \bigvee_{x \in X}(e_X(x, y) \odot A(x))\}$ is an Alexandrov L-topology and $(\tau_{e_X}, \vee, \wedge, e_{\tau_{e_X}})$ is a fuzzy complete lattice [20].

If (e_X, R, S, e_Y) is a residuated frame, then we show that $F : \tau_{e_X} \to \tau_{e_Y}$ and $G : \tau_{e_Y} \to \tau_{e_X}$ are well-defined and $(e_{\tau_{e_X}}, F, G, e_{\tau_{e_Y}})$ is a residuated connection; $e_{\tau_{e_Y}}(F(A), B) = e_{\tau_{e_X}}(A, G(B))$ is defined by

$$F(A)(y) = \bigvee_{x \in X} (A(x) \odot R(x, y)), \;\; G(B)(x) = \bigwedge_{y \in Y} (S(y, x) \to B(y)) \tag{4}$$

where τ_{e_X} and τ_{e_Y} are Alexandrov L-topologies induced by fuzzy posets (X, e_X) and (Y, e_Y) in Theorem 1. Using this result, one can show that the pair $(F(A), G(A))$ is an fuzzy rough set for A on τ_{e_X} because $(e_X, R = e_X, S = e_X^{-1}, e_X)$ is a residuated frame. Moreover, we show the existence of fuzzy rough sets from residuated connections.

Similarly, by Theorem 4, dual residuated frames induce dual residuated connections. In Theorem 5 (resp. 9), (resp. dual) residuated connections induce (resp. dual) residuated frames. Under various relations, we investigate the (dual) residuated connections and frames on Alexandrov L-topologies.

As a generalization of the Dedekind–MacNeille completion [37], we prove the existence of R-R (resp. DR-DR) embedding maps and R-R (resp. DR-DR) frame embedding maps.

2. Preliminaries

Definition 1 ([10]). *An algebra $(L, \wedge, \vee, \odot, \to, \perp, \top)$ is called a complete residuated lattice if it satisfies the following conditions:*
(L1) $(L, \leq, \vee, \wedge, \perp, \top)$ is a complete lattice with the greatest element $\top$ and the least element $\perp$;
(L2) $(L, \odot, \top)$ is a commutative monoid;
(L3) $x \odot y \leq z$ if and only if $x \leq y \to z$ for $x, y, z \in L$.

In this paper, we always assume that $(L, \leq, \odot, \to, *)$ is a complete residuated lattice with $x^* = x \to \perp$ and $(x^*)^* = x$.

For $\alpha \in L, A \in L^X$, we denote $(\alpha \to A), (\alpha \odot A), \alpha_X \in L^X$ by $(\alpha \to A)(x) = \alpha \to A(x)$, $(\alpha \odot A)(x) = \alpha \odot A(x)$, $\alpha_X(x) = \alpha$.

Lemma 1 ([2]). *Let $x, y, z, x_i, y_i, w \in L$. Then the following hold:*
(1) $\top \to x = x$, $\perp \odot x = \perp$;
(2) *If* $y \leq z$, *then* $x \odot y \leq x \odot z$, $x \to y \leq x \to z$ *and* $z \to x \leq y \to x$;
(3) $x \leq y$ *if and only if* $x \to y = \top$;
(4) $x \to (\bigwedge_i y_i) = \bigwedge_i (x \to y_i)$;
(5) $(\bigvee_i x_i) \to y = \bigwedge_i (x_i \to y)$;
(6) $x \odot (\bigvee_i y_i) = \bigvee_i (x \odot y_i)$;
(7) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$;
(8) $(x \to y) \odot (z \to w) \leq (x \odot z) \to (y \odot w)$ *and* $x \to y \leq (x \odot z) \to (y \odot z)$;
(9) $(x \to y) \odot (y \to z) \leq x \to z$;
(10) $\bigvee_{i \in \Gamma} x_i \to \bigvee_{i \in \Gamma} y_i \geq \bigwedge_{i \in \Gamma} (x_i \to y_i)$ *and* $\bigwedge_{i \in \Gamma} x_i \to \bigwedge_{i \in \Gamma} y_i \geq \bigwedge_{i \in \Gamma} (x_i \to y_i)$;
(11) $x \to y \leq (y \to z) \to (x \to z)$ *and* $x \to y \leq (z \to x) \to (z \to y)$;
(12) $(x \odot y^*)^* = x \to y$ *and* $x \to y = y^* \to x^*$.

Definition 2 ([21]). *Let X be a set. A function $e_X : X \times X \to L$ is called:*
(E1) Reflexive if $e_X(x, x) = \top$ *for all* $x \in X$;
(E2) Transitive if $e_X(x, y) \odot e_X(y, z) \leq e_X(x, z)$, *for all* $x, y, z \in X$;
(E3) If $e_X(x, y) = e_X(y, x) = \top$, *then* $x = y$. *If e_X satisfies (E1) and (E2), then (X, e_X) is called a fuzzy preorder set. If e satisfies (E1), (E2) and (E3), then (X, e_X) is called a fuzzy partially order set (simply, fuzzy poset).*

Definition 3 ([18]). *(1) A subset $\tau_X \subset L^X$ is called an Alexandrov L-topology on X if it satisfies the following conditions:*
(O1) $\alpha_X \in \tau_X$;
(O2) If $A_i \in \tau_X$ *for all* $i \in I$, *then* $\bigvee_{i \in I} A_i, \bigwedge_{i \in I} A_i \in \tau_X$;
(O3) If $A \in \tau_X$ and $\alpha \in L$, *then* $\alpha \odot A, \alpha \to A \in \tau_X$. *The pair (X, τ_X) is called an Alexandrov L-topological space.*

Lemma 2. *Let $\tau_X \subset L^X$. Define $e_{\tau_X} : \tau_X \times \tau_X \to L$ by $e_{\tau_X}(A, B) = \bigwedge_{x \in X} (A(x) \to B(x))$. Then (τ_X, e_{τ_X}) is a fuzzy poset.*

Proof. (E1) For all $A \in \tau_X$, we have $e_{\tau_X}(A, A) = \bigwedge_{x \in X} (A(x) \to A(x)) = \top$.
(E2) Let $A, B, C \in \tau_X$. Then by Lemma 1(9), we have

$$\begin{aligned} e_{\tau_X}(A, B) \odot e_{\tau_X}(B, C) &= \bigwedge_{x \in X} (A(x) \to B(x)) \odot \bigwedge_{x \in X} (B(x) \to C(x)) \\ &\leq \bigwedge_{x \in X} ((A(x) \to B(x)) \odot (B(x) \to C(x))) \\ &\leq e_{\tau_X}(A, C). \end{aligned} \tag{5}$$

(E3) Let $e_{\tau_X}(A, B) = e_{\tau_X}(B, A) = \top$. Then by Lemma 1(3), $A = B$.
Hence (τ_X, e_{τ_X}) is a fuzzy poset. □

Theorem 1. ([18]) *Let (X, e_X) be a fuzzy poset. Define*

$$\tau_{e_X} = \{A \in L^X \mid A(x) \odot e_X(x, z) \leq A(z)\}. \tag{6}$$

Then τ_{e_X} is an Alexandrov L-topology on X.

Remark 1. *(1) Let $(X, \top_{\triangle_X})$ be a fuzzy poset where $\top_{\triangle_X}(x, x) = \top$ and $\top_{\triangle_X}(x, y) = \perp$ for $x \neq y \in X$. Then $\tau_{\top_{\triangle_X}} = L^X$ and $e_{\tau_{\top_{\triangle_X}}} = e_{L^X} : L^X \times L^X \to L$ as $e_{L^X}(A, B) = \bigwedge_{x \in X} (A(x) \to B(x))$.*

(2) Let $(X, \top_{X\times X})$ be a fuzzy poset where $\top_{X\times X}(x,y) = \top$ for each $x, y \in X$. Then $\tau_{\top_{X\times X}} = \{\alpha_X \in L^X \mid \alpha \in L\}$ and $e_{\tau_{\top_{X\times X}}} : \tau_{\top_{X\times X}} \times \tau_{\top_{X\times X}} \to L$ by $e_{\tau_{\top_{X\times X}}}(\alpha_X, \beta_X) = \alpha \to \beta$.

3. Fuzzy Residuated Frames and Fuzzy Residuated Connections on Alexandrov *L*-topologies

Definition 4. *Let (X, e_X) and (Y, e_Y) be fuzzy posets. Let $f : X \to Y$* and *$g : Y \to X$ be maps.*
(1) *(e_X, f, g, e_Y) is a residuated connection if $e_Y(f(x), y) = e_X(x, g(y))$ for all $x \in X, y \in Y$;*
(2) *(e_X, f, g, e_Y) is a dual residuated connection if $e_Y(y, f(x)) = e_X(g(y), x)$ for all $x \in X, y \in Y$;*
(3) *f is an isotone map if $e_Y(f(x_1), f(x_2)) \geq e_X(x_1, x_2)$ for all $x_1, x_2 \in X$;*
(4) *f is an antitone map if $e_Y(f(x_1), f(x_2)) \geq e_X(x_2, x_1)$ for all $x_1, x_2 \in X$;*
(5) *f is an embedding map* if *$e_Y(f(x_1), f(x_2)) = e_X(x_1, x_2)$ for all $x_1, x_2 \in X$.*

Theorem 2. *Let (X, e_X) and (Y, e_Y) be fuzzy posets. Let $f : X \to Y$ and $g : Y \to X$ be maps.*
(1) *(e_X, f, g, e_Y) is a residuated connection if and only if f, g are isotone maps and $e_Y(f(g(y)), y) = e_X(x, g(f(x))) = \top$ for all $x, y \in X$;*
(2) *(e_X, f, g, e_Y) is a dual residuated connection if and only if f, g are isotone maps and $e_Y(y, f(g(y))) = e_X(g(f(x)), x) = \top$ for all $x, y \in X$.*

Proof. (1) Let (f, g) be a residuated connection. Since $e_Y(f(x), y) = e_X(x, g(y))$, we have $\top = e_Y(f(x), f(x)) = e_X(x, g(f(x)))$ and $e_Y(f(g(y)), y) = e_X(g(y), g(y)) = \top$. Furthermore,

$$e_Y(f(x_1), f(x_2)) = e_X(x_1, g(f(x_2))) \geq e_X(x_1, x_2) \odot e_X(x_2, g(f(x_2))) = e_X(x_1, x_2).$$

Conversely,

$$e_Y(f(x), y) \geq e_Y(f(g(y)), y) \odot e_Y(f(x), f(g(y))) = e_Y(f(x), f(g(y))) \geq e_X(x, g(y)).$$

Similarly, $e_Y(f(x), y) \leq e_X(x, g(y))$.
(2) Since $e_Y(f(x), y) = e_X(g(y), x)$, we have $\top = e_Y(f(x), f(x)) = e_X(g(f(x)), x)$ and $e_Y(f(g(y)), y) = e_X(g(y), g(y)) = \top$. Furthermore,

$$e_Y(f(x_1), f(x_2)) = e_X(g(f(x_2)), x_1) \geq e_X(x_2, x_1) \odot e_X(g(f(x_2)), x_2) = e_X(x_2, x_1).$$

□

For $R_1 \in L^{X\times Y}$ and $R_2 \in L^{Y\times Z}$, define

$$R_1 \circ R_2(x, z) = \bigvee_y (R_1(x, y) \odot R_2(y, z)), \quad R_1^{-1}(y, x) = R_1(x, y). \tag{7}$$

Lemma 3. *Let (X, e_X) and (X, e_Y) be fuzzy posets. Let $R \in L^{X\times Y}$. Then the following hold:*
(1) *$(e_X \circ R)^{-1} = R^{-1} \circ e_X^{-1}$ and $(R \circ e_X)^{-1} = e_X^{-1} \circ R^{-1}$;*
(2) *$e_X \circ R \leq R$ if and only if $e_X^{-1} \circ R^* \leq R^*$;*
(3) *$R \circ e_X^{-1} \leq R$ if and only if $R^* \circ e_X \leq R^*$;*
(4) *$e_X \circ R \circ e_Y \leq R$ if and only if $e_X \circ R \leq R$ and $R \circ e_Y \leq R$;*
(5) *$e_X^{-1} \circ R \circ e_Y^{-1} \leq R$ if and only if $e_X^{-1} \circ R \leq R$ and $R \circ e_Y^{-1} \leq R$;*
(6) *$e_X^{-1} \circ R \circ e_Y^{-1} \leq R$ if and only if $e_X \circ R^* \circ e_Y \leq R^*$.*

Proof. (1) $(e_X \circ R)^{-1}(y, x) = e_X \circ R(x, y) = \bigvee_{z\in X}(e_X(x, z) \odot R(z, y)) = \bigvee_{z\in X}(e_X^{-1}(z, x) \odot R^{-1}(y, z)) = R^{-1} \circ e_X^{-1}(y, x)$. Similarly, $(R \circ e_X)^{-1} = e_X^{-1} \circ R^{-1}$.
(2) $e_X(x, z) \odot R(z, y) \leq R(x, y)$ if and only if $R(z, y) \leq e_X(x, z) \to R(x, y)$ if and only if $e_X(x, z) \odot R^*(x, y) \leq R^*(z, y)$ if and only if $e_X^{-1}(z, x) \odot R^*(x, y) \leq R^*(z, y)$.

(3) $R(w,y) \odot e_X^{-1}(y,x) \leq R(w,x)$ if and only if $R(w,y) \odot e_X(x,y) \leq R(w,x)$ if and only if $e_X(x,y) \to R^*(w,y) \geq R^*(w,x)$ if and only if $e_X(x,y) \odot R^*(w,x) \leq R^*(w,y)$.
(4) $e_X \circ R \circ e_Y(x,y) = \bigvee_{y_1 \in Y}(e_X \circ R)(x,y_1) \odot e_Y(y_1,y) \geq (e_X \circ R)(x,y) \odot e_Y(y,y) = (e_X \circ R)(x,y)$. Similarly, $R \circ e_Y \leq R$. The converse part can be proved easily.
(5) and (6) can be proved easily by using (2)–(4). □

Definition 5. *Let (X,e_X) and (Y,e_Y) be fuzzy posets. Let $R \in L^{X\times Y}$ and $S \in L^{Y\times X}$. A structure (e_X,R,S,e_Y) is called:*
(1) *A residuated frame if $S = R^{-1}$ and $e_X \circ R \circ e_Y \leq R$;*
(2) *A dual residuated frame* if *$S = R^{-1}$ and $e_X^{-1} \circ R \circ e_Y^{-1} \leq R$.*

Lemma 4. *Let (X,e_X) and (Y,e_Y) be fuzzy posets. Then the following hold:*
(1) *Let (e_X,f,g,e_Y) be a residuated connection. Define maps $R : X \times Y \to L$ and $S : Y \times X \to L$ by*

$$R(x,y) = e_X(x,g(y)) = e_Y(f(x),y), \quad S(y,x) = R(x,y). \tag{8}$$

Then (e_X,R,S,e_Y) is a residuated frame;
(2) *Let (e_X,f,g,e_Y) be a dual residuated connection. Define maps $R : X \times Y \to L$ and $S : Y \times X \to L$ by*

$$R(x,y) = e_X(g(y),x) = e_Y(y,f(x)), \quad S(y,x) = R(x,y). \tag{9}$$

Then (e_X,R,S,e_Y) is a dual residuated frame;
(3) *If g is isotone and $R_1(x,y) = e_X(x,g(y))$ (resp. $R_2(x,y) = e_X(g(y),x)$), then $e_X \circ R_1 \circ e_Y \leq R_1$ (resp. $e_X^{-1} \circ R_2 \circ e_Y^{-1} \leq R_2$);*
(4) *If f is isotone and $R_1(x,y) = e_Y(y,f(x))$ (resp. $R_2(x,y) = e_Y(f(x),y)$), then $e_X^{-1} \circ R_1 \circ e_Y^{-1} \leq R_1$ (resp. $e_X \circ R_2 \circ e_Y \leq R_2$).*

Proof. (1) For all $x,x_1 \in X$ and $y,y_1 \in Y$,

$$\begin{aligned} e_X(x,x_1) \odot R(x_1,y_1) \odot e_Y(y_1,y) &= e_X(x,x_1) \odot e_X(x_1,g(y_1)) \odot e_Y(y_1,y) \\ &\leq e_X(x,g(y_1)) \odot e_X(y_1,y) \\ &= e_Y(f(x),y_1) \odot e_Y(y_1,y) \\ &\leq e_Y(f(x),y) = R(x,y). \end{aligned} \tag{10}$$

Hence $e_X \circ R \circ e_Y \leq R$.
(3) For all $x,x_1 \in X$ and $y,y_1 \in Y$,

$$\begin{aligned} e_X(x,x_1) \odot R_1(x_1,y_1) \odot e_Y(y_1,y) &= e_X(x,x_1) \odot e_X(x_1,g(y_1)) \odot e_Y(y_1,y) \\ &\leq e_X(x,x_1) \odot e_X(x_1,g(y_1)) \odot e_X(g(y_1),g(y)) \\ &\leq e_X(x,x_1) \odot e_X(x_1,g(y)) \\ &\leq e_X(x,g(y)) = R(x,y). \end{aligned} \tag{11}$$

Hence $e_X \circ R_1 \circ e_Y \leq R_1$.
(2) and (4) can be proved similarly. □

Theorem 3. *Let (e_X,R,S,e_Y) be a residuated frame. Let τ_{e_X} and τ_{e_Y} be Alexandrov L-topologies. Then the following hold:*
(1) *$(e_{\tau_{e_X}},F,G,e_{\tau_{e_Y}})$ is a residuated connection where*

$$F(A)(y) = \bigvee_{x\in X}(A(x) \odot R(x,y)), \quad G(B)(x) = \bigwedge_{y\in Y}(S(y,x) \to B(y)); \tag{12}$$

(2) $(e_{\tau_{e_X}}, F, G, e_{\tau_{e_Y}})$ *is an dual residuated connection where*

$$F(A)(y) = \bigwedge_{x\in X}(R^*(x,y) \to A(x)), \quad G(B)(x) = \bigvee_{y\in Y}(R^*(x,y) \odot B(y)). \tag{13}$$

Proof. (1) Since $R \circ e_Y \le R$ and $e_X \circ R \le R$ by Lemma 3(4), we have $F(A) \in \tau_{e_Y}$ and $G(B) \in \tau_{e_X}$ from:

$$F(A)(y) \odot e_Y(y,w) = \bigvee_{x\in X}(A(x) \odot R(x,y) \odot e_Y(y,w)) \le \bigvee_{x\in X}(A(x) \odot R(x,w)) = F(A)(w), \tag{14}$$

and

$$\begin{aligned} G(B)(x) \odot e_X(x,z) \odot R(z,y) \le \bigwedge_{y\in Y}((R(x,y) \to B(y)) \odot R(x,y)) \le B(y) \\ \Leftrightarrow G(B)(x) \odot e_X(x,z) \le G(B)(z). \end{aligned} \tag{15}$$

Moreover, for all $A \in \tau_{e_X}$ and $B \in \tau_{e_Y}$,

$$\begin{aligned} e_{\tau_{e_Y}}(F(A), B) &= \bigwedge_{y\in Y}(F(A)(y) \to B(y)) = \bigwedge_{y\in Y}\Big(\bigvee_{x\in Y}(R(x,y) \odot A(x)) \to B(y)\Big) \\ &= \bigwedge_{x\in X}\bigwedge_{y\in Y}\Big(A(x) \to (R(x,y) \to B(y))\Big) = \bigwedge_{x\in X}\Big(A(x) \to \bigwedge_{x\in X}(R(x,y) \to B(y))\Big) \\ &= \bigwedge_{x\in X}\Big(A(x) \to G(B)(x)\Big) = e_{\tau_{e_X}}(A, G(B)). \end{aligned} \tag{16}$$

(2) Since $R^* \circ e_Y^{-1} \le R^*$ and $e_X^{-1} \circ R^* \le R^*$ by Lemma 3 (5)–(6), we have

$$\begin{aligned} F(A)(y) \odot e_Y(y,w) \odot R^*(x,w) &= (\bigwedge_{x\in X}(R^*(x,y) \to A(x))) \odot e_Y(y,w) \odot R^*(x,w) \\ &\le \bigwedge_{x\in X}(R^*(x,y) \to A(x) \odot R^*(x,y)) \le A(x), \\ G(B)(x) \odot e_X(x,z) &\le \bigvee_{y\in Y}((R^*(x,y) \odot B(y)) \odot e_X(x,z)) \\ &\le \bigvee_{y\in Y}(R^*(z,y) \odot B(y)) = G(B)(z). \end{aligned} \tag{17}$$

Thus $F(A) \in \tau_{e_Y}$ and $G(B) \in \tau_{e_X}$.
Moreover, for all $A \in \tau_{e_X}$ and $B \in \tau_{e_Y}$,

$$\begin{aligned} e_{\tau_{e_X}}(G(B), A) &= \bigwedge_{x\in X}(G(B)(x) \to A(x)) = \bigwedge_{x\in X}\Big(\bigvee_{y\in Y}(R^*(x,y) \odot B(y)) \to A(x)\Big) \\ &= \bigwedge_{x\in X}\bigwedge_{y\in Y}\Big(B(y) \to (R^*(x,y) \to A(x))\Big) = \bigwedge_{y\in Y}\Big(B(y) \to \bigwedge_{x\in X}(R^*(x,y) \to A(x))\Big) \\ &= \bigwedge_{y\in Y}\Big(B(y) \to F(A)(y)\Big) = e_{\tau_{e_Y}}(B, F(A)). \end{aligned} \tag{18}$$

□

Remark 2. *Since* $(\top_{\triangle_X}, e_X, e_X^{-1}, \top_{\triangle_X})$ *is a residuated frame where* e_X *is a fuzzy poset and* $\tau_{\top_{\triangle_X}} = L^X$ *by Remark 1(1),* (e_{L^X}, F, G, e_{L^X}) *is a residuated connection where*

$$F(A)(y) = \bigvee_{x\in X}(A(x) \odot e_X(x,y)), \quad G(B)(x) = \bigwedge_{y\in X}(e_X(x,y) \to B(y)). \tag{19}$$

The pair (G, F) is a fuzzy rough set ([26]).

Theorem 4. *Let (e_X, R, S, e_Y) be a dual residuated frame. Let τ_{e_X} and τ_{e_Y} be Alexandrov L-topologies. Then the following hold:*
(1) *$(e_{\tau_{e_X}}, F, G, e_{\tau_{e_Y}})$ is a dual residuated connection where*

$$F(A)(y) = \bigwedge_{x\in X}(R(x,y)\to A(x)),\ \ G(B)(x) = \bigvee_{y\in Y}(R(x,y)\odot B(y)); \tag{20}$$

(2) *$(e_{\tau_{e_X}}, F, G, e_{\tau_{e_Y}})$ is a residuated connection where*

$$F(A)(y) = \bigvee_{x\in X}(A(x)\odot R^*(x,y)),\ \ G(B)(x) = \bigwedge_{y\in Y}(R^*(x,y)\to B(y)). \tag{21}$$

Proof. (1) Since $R\circ e_Y^{-1}\le R$ and $e_X^{-1}\circ R\le R$ by Lemma 3(5), we have

$$\begin{aligned} F(A)(y)\odot e_Y(y,w)\odot R(x,w) &= (\bigwedge_{x\in X}(R(x,y)\to A(x)))\odot e_Y^{-1}(w,y)\odot R(x,w)\\ &\le \bigwedge_{x\in X}(R(x,y)\to A(x)\odot R(x,y))\le A(x),\\ G(B)(x)\odot e_X(x,z) &\le \bigvee_{y\in Y}(R(x,y)\odot B(y)\odot\odot e_X(x,z))\le G(B)(z). \end{aligned} \tag{22}$$

Moreover, for all $A\in\tau_{e_X}$ and $B\in\tau_{e_Y}$,

$$\begin{aligned} e_{\tau_{e_X}}(G(B),A) &= \bigwedge_{x\in X}(G(B)(x)\to A(x)) = \bigwedge_{x\in X}\Big(\bigvee_{y\in Y}(R(x,y)\odot B(y))\to A(x)\Big)\\ &= \bigwedge_{x\in X}\bigwedge_{y\in Y}\Big(B(y)\to(R(x,y)\to A(x))\Big) = \bigwedge_{y\in Y}\Big(B(y)\to\bigwedge_{x\in X}(R(x,y)\to A(x))\Big)\\ &= \bigwedge_{y\in Y}\Big(B(y)\to F(A)(y)\Big) = e_{\tau_{e_Y}}(B,F(A)). \end{aligned} \tag{23}$$

Thus $F(A)\in\tau_{e_Y}$ and $G(B)\in\tau_{e_X}$.
(2) Since $R^*\circ e_Y\le R^*$ and $e_X\circ R^*\le R^*$ by Lemma 3(2–3), we have

$$\begin{aligned} F(A)(y)\odot e_Y(y,w) &= \bigvee_{x\in X}(A(x)\odot R^*(x,y)\odot e_Y(y,w))\\ &\le \bigvee_{x\in X}(A(x)\odot R^*(x,w)) = F(A)(w),\\ G(B)(x)\odot e_X(x,z)\odot R^*(z,y) &\le \bigwedge_{y\in Y}((R^*(x,y)\to B(y))\odot R^*(x,y))\le B(y). \end{aligned} \tag{24}$$

Thus $F(A)\in\tau_{e_Y}$ and $G(B)\in\tau_{e_X}$.
Moreover, for all $A\in\tau_{e_X}$, and $B\in\tau_{e_Y}$,

$$\begin{aligned} e_{\tau_{e_Y}}(F(A),B) &= \bigwedge_{y\in Y}(F(A)(y)\to B(y)) = \bigwedge_{y\in Y}\Big(\bigvee_{x\in Y}(R^*(x,y)\odot A(x))\to B(y)\Big)\\ &= \bigwedge_{x\in X}\bigwedge_{y\in Y}\Big(A(x)\to(R^*(x,y)\to B(y))\Big) = \bigwedge_{x\in X}\Big(A(x)\to\bigwedge_{x\in X}(R^*(x,y)\to B(y))\Big)\\ &= \bigwedge_{x\in X}\Big(A(x)\to G(B)(x)\Big) = e_{\tau_{e_X}}(A,G(B)). \end{aligned} \tag{25}$$

□

Remark 3. *Since $(\top_{\triangle_X}, e_X, e_X^{-1}, \top_{\triangle_X})$ is a dual residuated frame where e_X is a fuzzy poset and $\tau_{\top_{\triangle_X}} = L^X$ by Remark 1(1), (e_{L^X}, F, G, e_{L^X}) is a dual residuated connection where*

$$F(A)(y) = \bigwedge_{x \in X} (e_X(x,y) \to A(x)), \quad G(B)(x) = \bigvee_{y \in X} (e_X(x,y) \odot B(y)). \tag{26}$$

Example 1. *Let (X, e_X) and (Y, e_Y) be fuzzy posets. Let $f : X \to Y$ and $g : Y \to X$ be maps. Let τ_{e_X} and τ_{e_Y} be Alexandrov L-topologies.*
(1) *Let g be isotone and $R(x,y) = e_X(x, g(y))$. By Lemma 4(3), $(e_X, R, S = R^{-1}, e_Y)$ is a residuated frame. By Theorem 3(1), $(e_{\tau_{e_X}}, F, G, e_{\tau_{e_Y}})$ is a residuated connection with*

$$F(A)(y) = \bigvee_{x \in X} (A(x) \odot e_X(x, g(y))), \quad G(B)(x) = \bigwedge_{y \in Y} (e_X(x, g(y)) \to B(y)). \tag{27}$$

(2) *Let g be isotone and $R(x,y) = e_X(g(y), x)$. By Lemma 4(3), $(e_X, R, S = R^{-1}, e_Y)$ is a dual residuated frame. By Theorem 4(1), $(e_{\tau_{e_X}}, F, G, e_{\tau_{e_Y}})$ is a dual residuated connection where*

$$F(A)(y) = \bigwedge_{y \in Y} (e_X(g(y), x) \to A(x)), \quad G(B)(x) = \bigvee_{y \in Y} (B(y) \odot e_X(g(y), x)). \tag{28}$$

(3) *Let f be isotone and $R(x,y) = e_Y(y, f(x))$. By Lemma 4(4), $(e_X, R, S = R^{-1}, e_Y)$ is a dual residuated frame. By Theorem 4(1), $(e_{\tau_{e_X}}, F, G, e_{\tau_{e_X}})$ is a dual residuated connection where*

$$F(A)(y) = \bigwedge_{x \in X} (e_Y(y, f(x)) \to A(x)), \quad G(B)(y) = \bigvee_{y \in Y} (B(y) \odot e_Y(y, f(x))). \tag{29}$$

(4) *Let f be isotone and $R(x,y) = e_Y(f(x), y)$. By Lemma 4(4), $(e_X, R, S = R^{-1}, e_Y)$ is a residuated frame. By Theorem 3(1), $(e_{\tau_{e_X}}, F, G, e_{\tau_{e_Y}})$ is a residuated connection where*

$$F(A)(y) = \bigvee_{x \in X} (e_Y(f(x), y) \odot A(x)), \quad G(B)(y) = \bigwedge_{y \in Y} (e_Y(f(x), y) \to B(y)). \tag{30}$$

Theorem 5. *Let (X, e_X) and (Y, e_Y) be fuzzy posets. Let τ_{e_X} and τ_{e_Y} be Alexandrov L-topologies. Then the following hold:*
(1) *(e_X, f, g, e_Y) is a residuated connection. That is, $e_Y(f(x), y) = e_X(x, g(y))$ for all $x, y \in X$ if and only if there exist relations $R : \tau_{e_X} \times \tau_{e_Y} \to L$ and $S : \tau_{e_Y} \times \tau_{e_X} \to L$ by*

$$R(A, B) = \bigwedge_{x \in X} (A(x) \to B(f(x))), \quad S(B, A) = \bigwedge_{y \in Y} (A(g(y)) \to B(y)) \tag{31}$$

with isotone maps $f : X \to Y$, $g : Y \to X$ such that $(e_{\tau_{e_X}}, R, S, e_{\tau_{e_Y}})$ is a residuated frame.
(2) *In* (1),

$$R(A, B) = e_{\tau_{e_X}}(A, f^{\leftarrow}(B)) = e_{\tau_{e_Y}}(F(A), B) = e_{\tau_{e_X}}(A, G(B)) \tag{32}$$

where $F(A)(y) = \bigvee_{z \in X} (e_Y(f(z), y) \odot A(z))$ and $G(B) = \bigwedge_{y \in Y} (e_Y(f(z), y) \to B(y)))$.

$$S(B, A) = e_{\tau_{e_Y}}(g^{\leftarrow}(A), B) = e_{\tau_{e_Y}}(F_1(A), B) = e_{\tau_{e_X}}(A, G_1(B)) \tag{33}$$

where $F_1(A)(w) = \bigvee_{z \in X} (e_Y(z, g(w)) \odot A(z))$ and $G_1(B)(z) = \bigwedge_{w \in Y} (e_Y(z, g(w)) \to B(w))$.

Proof. (1) ($\Rightarrow$) Let $A \in \tau_{e_X}$ and $B \in \tau_{e_Y}$. Since $B(f(g(y))) \odot e_Y(f(g(y)), y) \le B(y)$, $e_Y(f(g(y)), y) = \top$, $A(x) \odot e_X(x, g(f(x))) \le A(g(f(x)))$ and $e_X(x, g(f(x))) = \top$,

$$\begin{aligned} R(A,B) &= \bigwedge_{x \in X} (A(x) \to B(f(x))) \le \bigwedge_{y \in Y} (A(g(y)) \to B(f(g(y))) \odot e_Y(f(g(y)), y)) \\ &\le \bigwedge_{y \in Y} (A(g(y)) \to B(y)) = S(B,A) \end{aligned} \tag{34}$$

and

$$\begin{aligned} S(B,A) &= \bigwedge_{x \in X} (A(g(y)) \to B(y)) \le \bigwedge_{x \in X} (A(g(f(x))) \to B(f(x))) \\ &\le \bigwedge_{y \in X} (A(x) \odot e_X(x, g(f(x))) \to B(f(x))) = R(A,B). \end{aligned} \tag{35}$$

Thus we have $R(A,B) = S(B,A)$. For all $A, A_1 \in \tau_{e_X}, B, B_1 \in \tau_{e_Y}$, we have

$$\begin{aligned} & e_{\tau_{e_X}}(A, A_1) \odot R(A_1, B_1) \odot e_{\tau_{e_Y}}(B_1, B) \\ &= e_{\tau_{e_X}}(A, A_1) \odot \bigwedge_{x \in X} (A_1(x) \to B_1(f(x))) \odot \bigwedge_{x \in X} (B_1(f(x)) \to B(f(x))) \\ &\le \bigwedge_{x \in X} (A(x) \to B(f(x))) = R(A,B). \end{aligned} \tag{36}$$

Thus $e_{\tau_{e_X}} \circ R \circ e_{\tau_{e_Y}} \le R$.

($\Leftarrow$) Since $e_Y(z,w) \odot e_Y(w,y) \le e_Y(z,y)$ if and only if $(e_Y)_y^{-1*}(z) \odot e_Y(z,w) \le (e_Y)_y^{-1*}(w)$, we have $(e_Y)_y^{-1*} \in \tau_{e_Y}$. For all $(e_X)_x \in \tau_{e_X}$ and $(e_Y)_y^{-1*} \in \tau_{e_Y}$,

$$\begin{aligned} R((e_X)_x, (e_Y)_y^{-1*}) &= \bigwedge_{z \in X} ((e_X)_x(z) \to (e_Y)_y^{-1*}(f(z))) \\ &\le (e_X)_x(x) \to (e_Y)_y^{-1*}(f(x)) = e_Y(f(x), y)^*. \end{aligned} \tag{37}$$

Since $e_X(x,z) \odot e_Y(f(z),y) \le e_Y(f(x), f(z)) \odot e_Y(f(z), y) \le e_Y(f(x), y)$, we have $e_X(x,z) \to e_Y^*(f(z), y) \ge e_Y^*(f(x), y)$. Hence $R((e_X)_x, (e_Y)_y^{-1*}) = e_Y^*(f(x), y)$. Moreover,

$$\begin{aligned} S((e_Y)_y^{-1*}, (e_X)_x) &= \bigwedge_{z \in X} ((e_X)_x(g(z)) \to (e_Y)_y^{-1*}(z)) \\ &\le (e_X)_x(g(y)) \to (e_Y)_y^{-1*}(y) = e_X(x, g(y))^*. \end{aligned} \tag{38}$$

Since $e_X(x, g(z)) \odot e_Y(z,y) \le e_X(x, g(z)) \odot e_X(g(z), g(y)) \le e_X(x, g(y))$, we have $e_X(x, g(z)) \to e_Y^*(z,y) \ge e_X^*(x, g(y))$. Hence $S((e_Y)_y^{-1*}, (e_X)_x) = e_X^*(x, g(y))$. Now, from

$$R((e_X)_x, (e_Y)_y^{-1*}) = e_Y^*(f(x), y) = S((e_Y)_y^{-1*}, (e_X)_x) = e_X^*(x, g(y)), \tag{39}$$

we have $e_Y(f(x), y) = e_X(x, g(y))$ for all $x, y \in X$.

(2) Let $A \in \tau_{e_X}$ and $B \in \tau_{e_Y}$. Since $A = \bigvee_{z\in X}(A(z) \odot e_X(z,-))$ and $B = \bigwedge_{y\in Y}(B^*(y) \to e^*_Y(-,y))$, we have

$$\begin{aligned}
R(A,B) &= \bigwedge_{x\in X}(A(x) \to B(f(x))) = \bigwedge_{x\in X}(\bigvee_{z\in X}(A(z) \odot e_X(z,x)) \to \bigwedge_{y\in Y}(B^*(y) \to e^*_Y(f(x),y))) \\
&= \bigwedge_{x,z\in X}\bigvee_{y\in Y}(A(z) \odot B^*(y) \to (e_X(z,x) \to e^*_Y(f(x),y))) \\
&= \bigwedge_{z\in X}\bigvee_{y\in Y}(A(z) \odot B^*(y) \to \bigwedge_{x\in X}(e_X(z,x) \to e^*_Y(f(x),y))) \\
&= \bigwedge_{z\in X}\bigvee_{y\in Y}(A(z) \odot B^*(y) \to e^*_Y(f(z),y)) \\
&= \bigwedge_{y\in Y}(\bigvee_{z\in X}(e_Y(f(z),y) \odot A(z)) \to B(y)) = e_{\tau_{e_Y}}(F(A),B) \\
&= \bigwedge_{z\in X}(A(z) \to \bigwedge_{y\in Y}(e_Y(f(z),y) \to B(y))) = e_{\tau_{e_X}}(A,G(B))
\end{aligned} \tag{40}$$

and

$$\begin{aligned}
S(B,A) &= \bigwedge_{y\in Y}(A(g(y)) \to B(y)) = \bigwedge_{y\in Y}(\bigvee_{z\in X}(A(z) \odot e_X(z,g(y))) \to \bigwedge_{w\in Y}(B^*(w) \to e^*_Y(y,w))) \\
&= \bigwedge_{y,w\in Y}\bigvee_{z\in X}(A(z) \odot B^*(w) \to (e_X(z,g(y)) \to e^*_Y(y,w))) \\
&= \bigwedge_{w\in Y}\bigvee_{z\in X}(A(z) \odot B^*(w) \to \bigwedge_{y\in Y}(e_X(z,g(y)) \to e^*_Y(y,w))) \\
&= \bigwedge_{w\in Y}\bigvee_{z\in X}(A(z) \odot B^*(w) \to e^*_Y(z,g(w))) \\
&= \bigwedge_{w\in Y}(\bigvee_{z\in X}(e_Y(z,g(w)) \odot A(z)) \to B(w)) = e_{\tau_{e_Y}}(F_1(A),B) \\
&= \bigwedge_{z\in X}(A(z) \to \bigwedge_{w\in Y}(e_Y(z,g(w)) \to B(w))) = e_{\tau_{e_X}}(A,G_1(B)).
\end{aligned} \tag{41}$$

□

Example 2. *Let (L^X, F, G, L^Y) be a residuated connection where for $R \in L^{X\times Y}$,*

$$F(A)(y) = \bigvee_{x\in X}(R(x,y) \odot A(x)), \quad G(B)(x) = \bigwedge_{y\in Y}(R(x,y) \to B(y)). \tag{42}$$

Let $\tau_{e_{L^X}} = \{\alpha \in L^{L^X} \mid \alpha(A) \odot e_{L^X}(A,B) \le \alpha(B)\}$ and $\tau_{e_{L^Y}} = \{\beta \in L^{L^Y} \mid \beta(A) \odot e_{L^Y}(A,B) \le \beta(B)\}$. Define two maps $T_1, S_1^{-1} : \tau_{e_{L^X}} \times \tau_{e_{L^Y}} \to L$ by

$$T_1(\alpha,\beta) = \bigwedge_{A\in L^X}(\alpha(A) \to \beta(F(A))), \quad S_1(\beta,\alpha) = \bigwedge_{B\in L^X}(\alpha(G(B)) \to \beta(B)). \tag{43}$$

Then $(e_{\tau_{e_{L^X}}}, T_1, S_1, e_{\tau_{e_{L^Y}}})$ is a residuated frame.

Theorem 6. *Let (X, e_X) be a fuzzy poset. Let τ_{e_X} be an Alexandrov L-topology. Let $\tau_{e_{\tau_{e_X}}} = \{\alpha \in L^{\tau_{e_X}} \mid \alpha(A) \odot e_{\tau_{e_X}}(A,B) \le \alpha(B)\}$. Define a map $h : X \to \tau_{e_{\tau_{e_X}}}$ by $h(x)(A) = \hat{x}(A) = A(x)$. Then $h : (X, e_X) \to (\tau_{e_{\tau_{e_X}}}, e_{\tau_{e_{\tau_{e_X}}}})$ is an embedding map.*

Proof. Assume that $h(x)(A) = h(y)(A)$ for all $A \in \tau_{e_X}$. Then $h(x)((e_X)_x) = h(y)((e_X)_x) = e_X(x,y) = \top$ for $(e_X)_x \in \tau_{e_X}$, and $h(x)((e_X)_y) = h(y)((e_X)_y) = e_X(y,x) = \top$ for $(e_X)_y \in \tau_{e_X}$. Thus $x = y$. Hence h is injective.
Since

$$\hat{x}(A) \odot e_{\tau_{e_X}}(A,B) = \hat{x}(A) \odot \bigwedge_{y \in X}(A(y) \to B(y)) \le A(x) \odot (A(x) \to B(x)) \le B(x) = \hat{x}(B), \tag{44}$$

we have $h(x) = \hat{x} \in \tau_{e_{\tau_{e_X}}}$. Let $A \in \tau_{e_X}$. Since $A(x) = \bigwedge_{y \in Y}(e_X(x,y) \to A(y))$, we have

$$e_X(x,y) \le \bigwedge_{A \in \tau_{e_X}} (A(x) \to A(y)) = \bigwedge_{A \in \tau_{e_X}} (\hat{x}(A) \to \hat{y}(A)) = e_{\tau_{e_{\tau_{e_X}}}}(\hat{x},\hat{y}). \tag{45}$$

Let $(e_X)_z(x) = e_X(z,x)$. Since $(e_X)_z(x) \odot e_X(x,y) \le (e_X)_z(y)$, we have $(e_X)_z \in \tau_{e_X}$ for all $z \in X$. Note that

$$\begin{aligned} e_{\tau_{e_{\tau_{e_X}}}}(\hat{x},\hat{y}) &= \bigwedge_{A \in \tau_{e_X}} (A(x) \to A(y)) \le \bigwedge_{(e_X)_z \in \tau_{e_X}} ((e_X)_z(x) \to (e_X)_z(y)) \\ &= \bigwedge_{z \in X} (e_X(z,x) \to e_X(z,y)) = e_X(x,y). \end{aligned} \tag{46}$$

Hence $e_{\tau_{e_{\tau_{e_X}}}}(\hat{x},\hat{y}) = e_X(x,y)$. $\square$

Definition 6. *Let* (e_X, f, g, e_X) and $(e_Z, \tilde{f}, \tilde{g}, e_Z)$ *be residuated connections. An injective function* $k : (e_X, f, g, e_X) \to (e_Z, \tilde{f}, \tilde{g}, e_Z)$ is an *R-R embedding if*

$$e_X(x,y) = e_Z(k(x),k(y)),\ e_X(f(x),y) = e_Z(\tilde{f}(k(x)),k(y)),\ e_X(x,g(y)) = e_Z(k(x),\tilde{g}(k(y))). \tag{47}$$

If k is a bijective R-R embedding map, then k is called an R-R isomorphism.

Theorem 7. *Let* (e_X, f, g, e_X) *be a residuated connection,* τ_{e_X} *be an Alexandrov L-topology and* $\tau_{e_{\tau_{e_X}}} = \{\alpha \in L^{\tau_{e_X}} \mid \alpha(A) \odot e_{\tau_{e_X}}(A,B) \le \alpha(B)\}$. *Define a map* $h : X \to \tau_{e_{\tau_{e_X}}}$ *by* $h(x)(A) = \hat{x}(A) = A(x)$. *Then the map* $h : (e_X, f, g, e_X) \to (e_{\tau_{e_{\tau_{e_X}}}}, F, G, e_{\tau_{e_{\tau_{e_X}}}})$ *is an R-R embedding map with*

$$e_X(x,y) = e_{\tau_{e_{\tau_{e_X}}}}(\hat{x},\hat{y}), F(h(x))(B) = F(\hat{x})(B) = \widehat{f(x)}(B) \tag{48}$$

for all $B \in \tau_{e_X}$ *and* $G(h(y))(A) = G(\hat{y})(A) = \hat{g(y)}(A)$ *for all* $A \in \tau_{e_X}$ *where*

$$R(A,B) = \bigwedge_{x \in X}(A(x) \to B(f(x))), \quad S(B,A) = \bigwedge_{y \in X}(A(g(y)) \to B(y)), \tag{49}$$

$$F(\hat{x})(B) = \bigvee_{A \in \tau_{e_X}} (R(A,B) \odot \hat{x}(A)), \quad G(\hat{y})(A) = \bigwedge_{B \in \tau_{e_X}} (S(B,A) \to \hat{y}(B)). \tag{50}$$

Moreover, $e_{\tau_{e_{\tau_{e_X}}}}(F(\hat{x}),\hat{y}) = e_{\tau_{e_{\tau_{e_X}}}}(\hat{x},G(\hat{y}))$.

Proof. By Theorem 6, $h : (X, e_X) \to (\tau_{e_{\tau_{e_X}}}, e_{\tau_{e_{\tau_{e_X}}}})$ is an embedding map. By Theorem 5(1), $(e_{\tau_{e_X}}, R, S, e_{\tau_{e_X}})$ is a residuated frame where

$$R(A,B) = \bigwedge_{x \in X}(A(x) \to B(f(x))), \quad S(B,A) = \bigwedge_{y \in X}(A(g(y)) \to B(y)). \tag{51}$$

By Theorem 3(1), $(e_{\tau_{e_{\tau_{e_X}}}}, F, G, e_{\tau_{e_{\tau_{e_X}}}})$ is a residuated connection where

$$F(\alpha)(B) = \bigvee_{A\in\tau_{e_X}} (R(A,B)\odot\alpha(A)) = \bigvee_{A\in\tau_{e_X}} \Big(\bigwedge_{z\in X}(A(z)\to B(f(z)))\odot\alpha(A)\Big), \tag{52}$$

$$G(\alpha)(A) = \bigwedge_{B\in\tau_{e_X}} (S(B,A)\to\alpha(B)) = \bigwedge_{B\in\tau_{e_X}} \Big(\bigwedge_{z\in X}(A(g(z))\to B(z))\to\alpha(B)\Big). \tag{53}$$

Moreover,

$$\begin{aligned} F(\hat{x})(B) = \bigvee_{A\in\tau_{e_X}} (R(A,B)\odot\hat{x}(A)) &= \bigvee_{A\in\tau_{e_X}} \Big(\bigwedge_{z\in X}(A(z)\to B(f(z)))\odot A(x)\Big) \\ &\le B(f(x)) = \widehat{f(x)}(B). \end{aligned} \tag{54}$$

Since f is isotone and $B\in\tau_{e_X}$, we have $B(f(x))\odot e_X(x,y)\le B(f(x))\odot e_X(f(x),f(y))\le B(f(y))$. Hence $f^{\leftarrow}(B)\in\tau_{e_X}$.

Let $A = f^{\leftarrow}(B)$. Note that

$$\begin{aligned} F(\hat{x})(B) = \bigvee_{A\in\tau_{e_X}} (R(A,B)\odot\hat{x}(A)) &\ge \Big(\bigwedge_{z\in X}(f^{\leftarrow}(B)(z)\to B(f(z)))\odot f^{\leftarrow}(B)(x)\Big) \\ &= B(f(x)) = \widehat{f(x)}(B). \end{aligned} \tag{55}$$

Hence $F(\hat{x}) = \widehat{f(x)}$. Note that

$$\begin{aligned} G(\hat{y})(A) = \bigwedge_{B\in\tau_{e_X}} (S(B,A)\to\hat{y}(B)) &= \bigwedge_{B\in\tau_{e_X}} \Big(\bigwedge_{z\in X}(A(g(z))\to B(z))\to B(y)\Big) \\ \ge \bigwedge_{B\in\tau_{e_X}} \Big(A(g(y))\to B(y))\to B(y)\Big) &\ge A(g(y)) = \widehat{g(y)}(A). \end{aligned} \tag{56}$$

Since g is isotone, we have $g^{\leftarrow}(A)\in\tau_{e_X}$. Thus $G(\hat{y})\le\widehat{g(y)}$. Moreover,

$$e_{\tau_{e_{\tau_{e_X}}}}(F(\hat{x}),\hat{y}) = e_{\tau_{e_{\tau_{e_X}}}}(\widehat{f(x)},\hat{y}) = e_X(f(x),y) = e_X(x,g(y)) = e_{\tau_{e_{\tau_{e_X}}}}(\hat{x},\widehat{g(y)}) = e_{\tau_{e_{\tau_{e_X}}}}(\hat{x},G(\hat{y})). \tag{57}$$

□

Definition 7. *Let (e_X, R, S, e_X) and $(e_Z, \tilde{R}, \tilde{S}, e_Z)$ be residuated frames. An injective map $k : (e_X, R, S, e_X) \to (e_Z, \tilde{R}, \tilde{S}, e_Y)$ is an R-R frame embedding if*

$$e_X(x,y) = e_Z(k(x),k(y)),\ R(x,y) = \tilde{R}(k(x),k(y)),\ S(x,y) = \tilde{S}(k(x),k(y)). \tag{58}$$

If k is a bijective R-R embedding map, then k is called an R-R frame isomorphism.

Theorem 8. *Let (e_X, R, S, e_X) be a residual frame, τ_{e_X} be an Alexandrov L-topology and $\tau_{e_{\tau_{e_X}}} = \{\alpha\in L^{\tau_{e_X}} \mid \alpha(A)\odot e_{\tau_{e_X}}(A,B)\le\alpha(B)\}$. Define a map $k : X\to\tau_{e_{\tau_{e_X}}}$ by $k(x)(A) = \hat{x}(A) = A(x)$. Then the map $k : (e_X, R, S, e_X) \to (e_{\tau_{e_{\tau_{e_X}}}}, \hat{R}, \hat{S}, e_{\tau_{e_{\tau_{e_X}}}})$ is an R-R frame embedding map with $e(x,y) = e_{\tau_{e_{\tau_{e_X}}}}(k(x),k(y))$, $R(x,y) = \hat{R}(k(x),k(y)) = \hat{R}(\hat{x},\hat{y})$ and $S(x,y) = \hat{S}(k(x),k(y)) = \hat{S}(\hat{x},\hat{y})$ where*

$$F(A)(y) = \bigvee_{x\in X}(R(x,y)\odot A(x)),\quad G(B)(x) = \bigwedge_{y\in X}(R(x,y)\to B(y)), \tag{59}$$

$$\hat{R}(\alpha,\beta) = \bigwedge_{A\in\tau_{e_X}} (\alpha(A) \to \beta(F(A))), \quad \hat{S}(\beta,\alpha) = \bigwedge_{B\in\tau_{e_X}} (\alpha(G(B)) \to \beta(B)). \tag{60}$$

Proof. By Theorem 6, $k : (X, e_X) \to (\tau_{e_{\tau_{e_X}}}, e_{\tau_{e_{\tau_{e_X}}}})$ is an embedding map. Hence $e_X(x,y) = e_{\tau_{e_{\tau_{e_X}}}}(\hat{x},\hat{y})$. By Theorem 3(1), $(e_{\tau_{e_X}}, F, G, e_{\tau_{e_X}})$ is a residuated connection where

$$F(A)(y) = \bigvee_{x\in X} (R(x,y) \odot A(x)), \quad G(B)(x) = \bigwedge_{y\in X} (R(x,y) \to B(y)). \tag{61}$$

By Theorem 5(1), $(e_{\tau_{e_{\tau_{e_X}}}}, \hat{R}, \hat{S}, e_{\tau_{e_{\tau_{e_X}}}})$ is a residuated frame where

$$\hat{R}(\alpha,\beta) = \bigwedge_{A\in\tau_{e_X}} (\alpha(A) \to \beta(F(A))), \quad \hat{S}(\beta,\alpha) = \bigwedge_{B\in\tau_{e_X}} (\alpha(G(B)) \to \beta(B)). \tag{62}$$

Note that for all $\hat{x},\hat{y} \in \tau_{e_{\tau_{e_X}}}$,

$$\begin{aligned}
\hat{R}(\hat{x},\hat{y}) &= \bigwedge_{A\in\tau_{e_X}} (\hat{x}(A) \to \hat{y}(F(A))) = \bigwedge_{A\in\tau_{e_X}} (A(x) \to F(A)(y)) \\
&= \bigwedge_{A\in\tau_{e_X}} \Big(A(x) \to \bigvee_{z\in X} (R(z,y) \odot A(z))\Big) \geq \bigwedge_{A\in\tau_{e_X}} \Big(A(x) \to (R(x,y) \odot A(x))\Big) \\
&\geq R(x,y).
\end{aligned} \tag{63}$$

Let $(e_X)_x(z) = e_X(x,z)$. Then $(e_X)_x \in \tau_{e_X}$. Since $e_X \circ R \circ e_X \leq R$, we have $e_X \circ R \leq R$. Thus

$$\begin{aligned}
\hat{R}(\hat{x},\hat{y}) &= \bigwedge_{A\in\tau_{e_X}} \Big(A(x) \to \bigvee_{z\in X} (R(z,y) \odot A(z))\Big) \leq \Big((e_X)_x(x) \to \bigvee_{z\in X} (R(z,y) \odot (e_X)_x(z))\Big) \\
&= R(x,y)
\end{aligned} \tag{64}$$

and

$$\begin{aligned}
\hat{S}(\hat{y},\hat{x}) &= \bigwedge_{B\in\tau_{e_X}} (\hat{x}(G(B)) \to \hat{y}(B)) = \bigwedge_{B\in\tau_{e_X}} (G(B)(x) \to B(y)) \\
&= \bigwedge_{B\in\tau_{e_X}} \Big(\bigwedge_{z\in X} (R(x,z) \to B(z)) \to B(y)\Big) \geq \bigwedge_{B\in\tau_{e_X}} \Big((R(x,y) \to B(y)) \to B(y)\Big) \\
&\geq R(x,y) = S(y,x).
\end{aligned} \tag{65}$$

Since $R \circ e_X \leq e_X \circ R \circ e_X \leq R$, we have $R(x,y) \odot e_X(y,w) \leq R(x,w)$. Thus $R_x = R(x,-) \in \tau_{e_X}$. Hence

$$\begin{aligned}
\hat{S}(\hat{y},\hat{x}) &= \bigwedge_{B\in\tau_{e_X}} \Big(\bigwedge_{z\in X} (R(x,z) \to B(z)) \to B(y)\Big) \leq \Big(\bigwedge_{z\in X} (R(x,z) \to R_x(z)) \to R_x(y)\Big) \\
&= R(x,y) = S(y,x).
\end{aligned} \tag{66}$$

□

Corollary 1. *Let $(e_X, R = e_X, S = e_X^{-1}, e_X)$ be a residual frame and $\tau_{e_{\tau_{e_X}}} = \{\alpha \in L^{\tau_{e_X}} \mid \alpha(A) \odot e_{\tau_{e_X}}(A,B) \leq \alpha(B)\}$. Define a map $k : X \to \tau_{e_{\tau_{e_X}}}$ by $k(x)(A) = \hat{x}(A) = A(x)$. Then the map*

$$k : (e_X, R = e_X, S = e_X^{-1}, e_X) \to (e_{\tau_{e_{\tau_{e_X}}}}, \hat{R} = \widehat{e_X}, \hat{S} = \widehat{e_X^{-1}}, e_{\tau_{e_{\tau_{e_X}}}}) \tag{67}$$

is an embedding map with $e_X(x,y) = e_{\tau_{e_{\tau_{e_X}}}}(k(x),k(y))$, $e_X(x,y) = \widehat{e_X}(\hat{x},\hat{y})$ *and* $e_X^{-1}(x,y) = \widehat{e_X^{-1}}(\hat{x},\hat{y})$ *where*

$$\begin{aligned} \widehat{e_X}(\hat{x},\hat{y}) &= \bigwedge_{A\in\tau_{e_X}} (\hat{x}(A) \to \hat{y}(F(A))) = \bigwedge_{A\in\tau_{e_X}} (A(x) \to \bigvee_{z\in X} (e_X(z,y) \odot A(z))) = e_X(x,y), \\ \widehat{e_X^{-1}}(\hat{y},\hat{x}) &= \bigwedge_{A\in\tau_{e_X}} (\hat{x}(G(B)) \to \hat{y}(B)) = \bigwedge_{A\in\tau_{e_X}} (\bigwedge_{z\in X} (e_X(x,z) \to B(z)) \to B(y)) = e_X^{-1}(y,x). \end{aligned} \tag{68}$$

Example 3. *Let* $X = \{a,b,c\}$ *be a set. Let* $f : X \to X$ *be a map by* $f(a) = b, f(b) = a, f(c) = c$ *and* $f = f^{-1}$. *Define a binary operation* $\odot$ *on* $L = [0,1]$ *by*

$$x \odot y = \max\{0, x+y-1\},\ x \to y = \min\{1-x+y, 1\}. \tag{69}$$

(1) *Let* $(X = \{a,b,c\}, e_X)$ *be a fuzzy poset where*

$$e_X = \begin{pmatrix} 1 & 0.6 & 0.5 \\ 0.6 & 1 & 0.5 \\ 0.7 & 0.7 & 1 \end{pmatrix}. \tag{70}$$

Since $e_X(x,y) = e_X(f(x),f(y)), e_X(x,f(f(x))) = e_X(f(f(x)),x) = 1$, *we have that* (e_X,f,f,e_X) *are both residuated and dual residuated connections.* Since (e_X,f,f,e_X) is a residuated connection, we have that $e_X(f(x),y) = e_X(x,f(y))$ for all $x,y \in X$ if and only if there the exist relations $R : \tau_{e_X} \times \tau_{e_X} \to L$ and $S : \tau_{e_X} \times \tau_{e_X} \to L$ by

$$R(A,B) = \bigwedge_{x\in X} (A(x) \to B(f(x))), \quad S(B,A) = \bigwedge_{y\in Y} (A(f(y)) \to B(y)) \tag{71}$$

with an isotone map $f : X \to Y$ *such that* $(e_{\tau_{e_X}}, R, S, e_{\tau_{e_X}})$ *is a residuated frame.*

Let $(e_X)_z(x) = e(z,x)$ *for all* $z \in X$. *Then* $(e_X)_z \in \tau_{e_X}$. *Now, we have*

$$\begin{aligned} &R((e_X)_a,(e_X)_b) = \bigwedge_{x\in X} (e_X(a,x) \to e_X(b,f(x))) = 1, \\ &R((e_X)_b,(e_X)_a) = 1,\ R((e_X)_a,(e_X)_a) = R((e_X)_b,(e_X)_b) = 0.6,\ R((e_X)_c,(e_X)_c) = 1, \\ &R((e_X)_a,(e_X)_c) = 0.7, R((e_X)_c,(e_X)_a) = 0.5, R((e_X)_b,(e_X)_c) = 0.7, R((e_X)_c,(e_X)_b) = 0.5. \\ &S((e_X)_x,(e_X)_y) = R((e_X)_y,(e_X)_x) \quad \text{for all } x,y \in X. \end{aligned} \tag{72}$$

Moreover,

$$R((e_X)_a,(e_X)_b^{-1*}) = \bigwedge_{x\in X} (e_X(a,x) \to e_X^*(f(x),b)) = e_X^*(f(a),b). \tag{73}$$

Since f *is isotone and* $R(x,y) = e_X(x,f(y)) = e_X(f(x),y)$, *we have by Example 1(4) that* $(e_{\tau_{e_X}}, F, G, e_{\tau_{e_Y}})$ *is a residuated connection with*

$$F(A)(y) = \bigvee_{x\in X} (A(x) \odot e_X(x,f(y))), \quad G(B)(x) = \bigwedge_{y\in X} (e_X(x,f(y)) \to B(y)). \tag{74}$$

Since f *is isotone and* $R(x,y) = e_X(f(y),x) = e_X(y,f(x))$, *we have by Example 1(3) that* $(e_{\tau_{e_X}}, F, G, e_{\tau_{e_Y}})$ *is a dual residuated connection with*

$$F(A)(y) = \bigwedge_{x\in X} (e_X(f(y),x) \to A(x)), \quad G(B)(x) = \bigvee_{y\in X} (B(y) \odot e_X(f(y),x)). \tag{75}$$

Since $(e_{\tau_{e_X}}, R, S, e_{\tau_{e_X}})$ is a residuated frame, we have by Theorem 7 that $(e_{\tau_{e_{\tau_{e_X}}}}, F, G, e_{\tau_{e_{\tau_{e_X}}}})$ is a residuated connection where

$$F(\alpha)(B) = \bigvee_{A\in\tau_{e_X}} \Big(\bigwedge_{z\in X} (A(z) \to B(f(z))) \odot \alpha(A)\Big), G(\alpha)(B) = \bigwedge_{C\in\tau_{e_X}} \Big(\bigwedge_{z\in X} (B(f(z)) \to C(z)) \to \alpha(C)\Big). \quad (76)$$

Since

$$(A(z) \to B(f(z))) \odot (B(f(z)) \to A(z)) \odot \alpha(A) \le \alpha(A), \quad (77)$$

we have

$$(A(z) \to B(f(z))) \odot \alpha(A) \le (B(f(z)) \to A(z)) \to \alpha(A). \quad (78)$$

Hence $F(\alpha)(B) \le G(\alpha)(B)$. Since f is isotone, we have that $f^{\leftarrow}(B) \in \tau_{e_X}$ for all $B \in \tau_{e_X}$, and so

$$G(\alpha)(B) \le (B(f(z)) \to B(f(z))) \to \alpha(f^{\leftarrow}(B)) = (B(f(z)) \to B(f(z))) \odot \alpha(f^{\leftarrow}(B)) \le F(\alpha)(B). \quad (79)$$

Hence the map $h : (e_X, f, f, e_X) \to (e_{\tau_{e_{\tau_{e_X}}}}, F, F, e_{\tau_{e_{\tau_{e_X}}}})$ is an R-R embedding map.
(2) *Let $(X = \{a, b, c\}, e_X)$ be a fuzzy poset where*

$$e_X = \begin{pmatrix} 1 & 0.6 & 0.5 \\ 0.6 & 1 & 0.7 \\ 0.7 & 0.5 & 1 \end{pmatrix}. \quad (80)$$

Since

$$0.7 = e_X(c, a) \not\le e_X(f(c), f(a)) = e_X(c, b) = 0.5,$$

f is not an isotone map. Hence (e_X, f, f, e_X) are neither residuated nor dual residuated connections. Let $R(x, y) = e_X(x, f(y))$. Then $(e_{\tau_{e_X}}, F, G, e_{\tau_{e_Y}})$ is not a residuated connection with

$$F(A)(y) = \bigvee_{x\in X} (A(x) \odot e_X(x, f(y)), \quad G(B)(x) = \bigwedge_{y\in X} (e_X(x, f(y)) \to B(y)), \quad (81)$$

because $F((e_X)_c) \notin \tau_{e_X}$ for $(e_X)_c \in \tau_{e_X}$ from $F((e_X)_c)(c) \odot e_X(c, a) = 0.7 \not\le F((e_X)_c)(a) = 0.5$ where

$$\begin{aligned} F((e_X)_c)(c) &= \bigvee_{x\in X} ((e_X)_c(x) \odot e_X(x, f(c)) = e_X(c, c) = 1, \\ F((e_X)_c)(a) &= \bigvee_{x\in X} ((e_X)_c(x) \odot e_X(x, f(a)) = e_X(c, b) = 0.5. \end{aligned} \quad (82)$$

Let $R(x, y) = e_X(f(y), x)$. Then $(e_{\tau_{e_X}}, F, G, e_{\tau_{e_Y}})$ is not a dual residuated connection with

$$F(A)(y) = \bigwedge_{y\in X} (e_X(f(y), x) \to A(x)), \quad G(B)(x) = \bigvee_{y\in X} (B(y) \odot e_X(f(y), x)), \quad (83)$$

because $F((e_X^{-1*})_c) \notin \tau_{e_X}$ for $(e_X^{-1*})_c \in \tau_{e_X}$ from $F((e_X^{-1*})_c)(b) \odot e_X(b, c) = 0.2 \not\le F((e_X^{-1*})_c)(c) = 0$ *where*

$$\begin{aligned} F((e_X^{-1*})_c)(b) &= \bigwedge_{y\in X} (e_X(f(b), x) \to (e_X^{-1*})_c(x)) = e_X^*(f(b), c) = 0.5, \\ F((e_X^{-1*})_c)(c) &= \bigwedge_{y\in X} (e_X(f(c), x) \to (e_X^{-1*})_c(x)) = e_X^*(f(c), c) = 0. \end{aligned} \quad (84)$$

(3) *Let* $(X = \{a, b, c\}, e_X)$ *be a fuzzy poset where*

$$e_X = \begin{pmatrix} 1 & 1 & 0.7 \\ 0.6 & 1 & 0.7 \\ 0.7 & 0.7 & 1 \end{pmatrix}. \quad (85)$$

Let $g, h : X \to X$ *be maps by*

$$g(a) = g(b) = a, g(c) = c \quad \text{and} \quad h(a) = h(b) = b, h(c) = c. \quad (86)$$

Since

$$\begin{aligned} &e_X(x, y) \leq e_X(g(x), g(y)), \quad e_X(x, y) \leq e_X(h(x), h(y)), \quad g(h(a)) = g(h(b)) = a, \\ &g(h(c)) = c, \quad h(g(a)) = h(g(b)) = b, \quad g(h(c)) = c, \end{aligned} \quad (87)$$

we have

$$e_X(g(h(x)), x) = e_X(x, h(g(x))) = 1, \quad e_X(h(g(a)), a) = e_X(b, g(h(b))) = 0.6. \quad (88)$$

Hence (e_X, g, h, e_X) *is a residuated connection, but not a dual residuated connection. Since* (e_X, g, h, e_X) *is a residuated connection, we have by Theorem 5 that* $(e_{\tau_{e_X}}, R, S, e_{\tau_{e_X}})$ *is a residuated frame where*

$$R(A, B) = \bigwedge_{x \in X} (A(x) \to B(g(x))), \quad S(B, A) = \bigwedge_{y \in Y} (A(h(y)) \to B(y)). \quad (89)$$

Since $(e_{\tau_{e_X}}, R, S, e_{\tau_{e_X}})$ *is a residuated frame, we have by Theorem 7 that* $(e_{\tau_{e_{\tau_{e_X}}}}, F, G, e_{\tau_{e_{\tau_{e_X}}}})$ *is a residuated connection where*

$$F(\alpha)(B) = \bigvee_{A \in \tau_{e_X}} (R(A, B) \odot \alpha(A)) = \bigvee_{A \in \tau_{e_X}} \Big(\bigwedge_{z \in X} (A(z) \to B(g(z))) \odot \alpha(A) \Big), \quad (90)$$

$$G(\alpha)(A) = \bigwedge_{B \in \tau_{e_X}} (S(B, A) \to \alpha(B)) = \bigwedge_{B \in \tau_{e_X}} \Big(\bigwedge_{z \in X} (A(h(z)) \to B(z)) \to \alpha(B) \Big). \quad (91)$$

4. Fuzzy Dual Residuated Connections on Alexandrov *L*-Topologies

Theorem 9. *Let* (X, e_X) *and* (Y, e_Y) *be fuzzy posets. Let* τ_{e_X} *and* τ_{e_Y} *be Alexandrov L-topologies. Then the following hold:*
(1) (e_X, f, g, e_Y) *is a dual residuated connection. That is,* $e_Y(y, f(x)) = e_X(g(y), x)$ *for all* $x, y \in X$ *if and only if there exist maps* $R : \tau_{e_X} \times \tau_{e_Y} \to L$ *and* $S : \tau_{e_Y} \times \tau_{e_X} \to L$ *by*

$$R(A, B) = \bigwedge_{x \in X} (B(f(x)) \to A(x)), \quad S(B, A) = \bigwedge_{y \in Y} (B(y) \to A(g(y))) \quad (92)$$

with isotone maps $f : X \to Y$, $g : Y \to X$ *such that* $(e_{\tau_{e_X}}, R, S, e_{\tau_{e_X}})$ *is a dual residuated frame.*
(2) *In* (1),

$$R(A, B) = e_{\tau_{e_X}}(f^{\leftarrow}(B), A) = e_{\tau_{e_Y}}(B, F(A)) = e_{\tau_{e_X}}(G(B), A) \quad (93)$$

where $F(A)(y) = \bigwedge_{z \in X}(e_Y(y, f(z)) \to A(z))$ *and* $G(B) = \bigvee_{y \in Y}(e_Y(y, f(z)) \odot B(y))$.

$$S(B, A) = e_{\tau_{e_Y}}(B, g^{\leftarrow}(A)) = e_{\tau_{e_Y}}(B, F_1(A)) = e_{\tau_{e_X}}(G_1(B), A) \quad (94)$$

where $F_1(A)(w) = \bigwedge_{z \in X}(e_Y(g(w), z) \to A(z))$ *and* $G_1(B)(z) = \bigvee_{w \in Y}(e_Y(g(w), z) \odot B(w))$.

Proof. (1) ($\Rightarrow$) Let $A \in \tau_{e_X}$. Since $A(g(f(x))) \odot e_X(g(f(x)), x) \leq A(x)$ and $B(y) \odot e_Y(y, f(g(y))) \leq B(f(g(y)))$ and $e_X(g(f(x)), x)) = e_Y(y, f(g(y))) = \top$ by Theorem 2, we have

$$\begin{aligned} S(B,A) &= \bigwedge_{y \in X} (B(y) \to A(g(y))) \leq \bigwedge_{x \in X} (B(f(x)) \to A(g(f(x))) \odot e_X(g(f(x)), x)) \\ &\leq \bigwedge_{x \in X} (B(f(x)) \to A(x)) = R(A,B) \end{aligned} \tag{95}$$

and

$$\begin{aligned} R(A,B) &= \bigwedge_{x \in X} (B(f(x)) \to A(x)) \leq \bigwedge_{y \in X} (B(f(g(y))) \to A(g(y))) \\ &\leq \bigwedge_{y \in X} (B(y) \odot e_Y(y, f(g(y))) \to A(g(y))) \\ &= S(B,A). \end{aligned} \tag{96}$$

Thus $S = R^{-1}$. For all $A, A_1 \in \tau_{e_X}$ and $B, B_1 \in \tau_{e_Y}$, we have

$$\begin{aligned} & e^{-1}_{\tau_{e_X}}(A, A_1) \odot R(A_1, B_1) \odot e^{-1}_{\tau_{e_Y}}(B_1, B) \\ & \leq e_{\tau_{e_X}}(A_1, A) \odot \bigwedge_{x \in X} (B_1(f(x)) \to A_1(x)) \odot \bigwedge_{x \in X} (B(f(x)) \to B_1(f(x))) \\ & \leq \bigwedge_{x \in X} (B(f(x)) \to A(x)) = R(A,B). \end{aligned} \tag{97}$$

($\Leftarrow$) For all $(e_X)_x^{-1*} \in \tau_{e_X}$ and $(e_Y)_y \in \tau_{e_Y}$, we have

$$\begin{aligned} R((e_X)_x^{-1*}, (e_Y)_y) &= \bigwedge_{z \in X} ((e_Y)_y(f(z)) \to (e_X)_x^{-1*}(z)) \leq (e_Y)_y(f(x)) \to (e_X)_x^{-1*}(x) \\ &= e_Y(y, f(x))^*. \end{aligned} \tag{98}$$

Since

$$e_Y(y, f(z)) \odot e_X(z, x) \leq e_Y(y, f(z)) \odot e_Y(f(z), f(x)) \leq e_Y(y, f(x)), \tag{99}$$

we have $e_X(x,z) \to e^*_Y(y, f(z)) \geq e^*_Y(y, f(x))$. Hence $R((e_X)_x^{-1*}, (e_Y)_y) = e^*_Y(y, f(x))$. Additionally,

$$\begin{aligned} S((e_Y)_y, (e_X)_x^{-1*}) &= \bigwedge_{z \in X} ((e_Y)_y(z) \to (e_X)_x^{-1*}(g(z)) \\ &\leq (e_Y)_y(y) \to (e_Y)_x^{-1*}(g(y)) = e_X(g(y), x)^*. \end{aligned} \tag{100}$$

Since

$$e_X(g(z), x) \odot e_Y(y, z) \leq e_X(g(z), x) \odot e_X(g(y), g(z)) \leq e_X(g(y), x), \tag{101}$$

we have $e_Y(y,z) \to e^*_X(g(z), x) \geq e^*_X(g(y), x)$. Hence $S((e_Y)_y, (e_X)_x^{-1*}) = e^*_X(g(y), x)$. Since

$$e^*_Y(y, f(x)) = R((e_X)_x^{-1*}, (e_Y)_y) = S((e_Y)_y, (e_X)_x^{-1*}) = e^*_X(g(y), x), \tag{102}$$

we have that (e_X, f, g, e_Y) is a dual residuated connection. $\square$

Example 4. *Let (e_{L^X}, F, G, e_{L^Y}) be a dual residuated connection for $R \in L^{X \times Y}$ defined by*

$$F(A)(y) = \bigwedge_{x \in X} (R(x,y) \to A(x)), \quad G(B)(x) = \bigvee_{y \in Y} (R(x,y) \odot B(y)), \tag{103}$$

and $\tau_{e_{L^X}} = \{\alpha \in L^{L^X} \mid \alpha(A) \odot e_{L^X}(A,B) \leq \alpha(B)\}$ *and* $\tau_{e_{L^Y}} = \{\beta \in L^{L^Y} \mid \beta(A) \odot e_{L^Y}(A,B) \leq \beta(B)\}$. *Two maps* $T_1, S_1 : \tau_{e_{L^X}} \times \tau_{e_{L^Y}} \to L$ *are defined by*

$$T_1(\alpha,\beta) = \bigwedge_{A \in L^X} (\beta(F(A)) \to \alpha(A)), \quad S_1(\beta,\alpha) = \bigwedge_{B \in L^X} (\beta(B) \to \alpha(G(B))). \tag{104}$$

Then $(e_{\tau_{e_{L^X}}}, T_1, S_1, e_{\tau_{e_{L^Y}}})$ *is a dual residuated frame.*

Definition 8. *Let* (e_X, f, g, e_X) *and* $(e_Z, \tilde{f}, \tilde{g}, e_Z)$ *be dual residuated connections. An injective function* $k : (e_X, f, g, e_X) \to (e_Z, \tilde{f}, \tilde{g}, e_Z)$ *is a DR-DR embedding if*

$$e_X(x,y) = e_Z(k(x),k(y)),\ e_X(y,f(x)) = e_Z(k(y),\tilde{f}(k(x))),\ e_X(g(y),x) = e_Z(\tilde{g}(k(y)),k(x)). \tag{105}$$

If k is a bijective DR-DR embedding map, then k is called a DR-DR isomorphism.

Theorem 10. *Let* (e_X, f, g, e_X) *be a dual residuated connection,* τ_{e_X} *be an Alexandrov L-topology and* $\tau_{e_{\tau_{e_X}}} = \{\alpha \in L^{\tau_{e_X}} \mid \alpha(A) \odot e_{\tau_{e_X}}(A,B) \leq \alpha(B)\}$. *Define a map* $h : X \to \tau_{e_{\tau_{e_X}}}$ *by* $h(x)(A) = \hat{x}(A) = A(x)$. *Then* $h : (e_X, f, g, e_X) \to (e_{\tau_{e_{\tau_{e_X}}}}, F, G, e_{\tau_{e_{\tau_{e_X}}}})$ *is a DR-DR embedding map with* $e_X(x,y) = e_{\tau_{e_{\tau_{e_X}}}}(\hat{x},\hat{y})$, $F(h(x))(B) = F(\hat{x})(B) = \widehat{f(x)}(B)$ *and* $G(h(y))(A) = G(\hat{y})(A) = \widehat{g(y)}(A)$ *for all* $A \in \tau_{e_X}$ *where*

$$\begin{aligned} R(A,B) &= \bigwedge_{x \in X} (B(f(x)) \to A(x)), \quad S(B,A) = \bigwedge_{y \in X} (B(y) \to A(g(y))), \\ F(\alpha)(B) &= \bigwedge_{A \in \tau_{e_X}} (R(A,B) \to \alpha(A)), \quad G(\alpha)(A) = \bigvee_{B \in \tau_{e_X}} (S(B,A) \odot \alpha(B)). \end{aligned} \tag{106}$$

Moreover, $e_{\tau_{e_{\tau_{e_X}}}}(\hat{y}, F(\hat{x})) = e_{\tau_{e_{\tau_{e_X}}}}(G(\hat{y}), \hat{x})$.

Proof. By Theorem 9, $(e_{\tau_{e_X}}, R, S, e_{\tau_{e_X}})$ is a dual residuated frame where

$$R(A,B) = \bigwedge_{x \in X} (B(f(x)) \to A(x)), \quad S(B,A) = \bigwedge_{y \in X} (B(y) \to A(g(y))). \tag{107}$$

By Theorem 4(1), $(e_{\tau_{e_{\tau_{e_X}}}}, F, G, e_{\tau_{e_{\tau_{e_X}}}})$ is a dual residuated connection where

$$\begin{aligned} F(\alpha)(B) &= \bigwedge_{A \in \tau_{e_X}} (R(A,B) \to \alpha(A)) = \bigwedge_{A \in \tau_{e_X}} \Big(\bigwedge_{z \in X} (B(f(z)) \to A(z)) \to \alpha(A) \Big), \\ G(\alpha)(A) &= \bigvee_{B \in \tau_{e_X}} (S(B,A) \odot \alpha(B)) = \bigvee_{B \in \tau_{e_X}} \Big(\bigwedge_{z \in X} (B(z) \to A(g(z))) \odot \alpha(B) \Big). \end{aligned} \tag{108}$$

By Theorem 6, a map $h : X \to \tau_{e_{\tau_{e_X}}}$ by $h(x)(A) = \hat{x}(A) = A(x)$ is embedding. That is, $e_X(x,y) = e_{\tau_{e_{\tau_{e_X}}}}(\hat{x},\hat{y})$. For all $B \in \tau_{e_X}$, we have

$$\begin{aligned} F(\hat{x})(B) &= \bigwedge_{A \in \tau_{e_X}} (R(A,B) \to \hat{x}(A)) = \bigwedge_{A \in \tau_{e_X}} \Big(\bigwedge_{z \in X} (B(f(z)) \to A(z)) \to A(x) \Big) \\ &\geq \bigwedge_{A \in \tau_{e_X}} \Big((B(f(x)) \to A(x)) \to A(x) \Big) \geq B(f(x)) = \widehat{f(x)}(B). \end{aligned} \tag{109}$$

Since f is isotone and $B \in \tau_{e_X}$, we have

$$B(f(x)) \odot e_X(x,y) \leq B(f(x)) \odot e_X(f(x), f(y)) \leq B(f(y)). \tag{110}$$

Hence $f^{\leftarrow}(B) \in \tau_{e_X}$.
Let $A = f^{\leftarrow}(B)$. For all $A, B \in \tau_{e_X}$,

$$\begin{aligned} F(\hat{x})(B) &= \bigwedge_{A\in\tau_{e_X}} \Big(\bigwedge_{z\in X} (B(f(z)) \to A(z)) \to A(x) \Big) \leq \bigwedge_{z\in X} (B(f(z)) \to B(f(z)) \to B(f(x))) \\ &= \top \to B(f(x)) = B(f(x)) = \widehat{f(x)}(B) \end{aligned} \tag{111}$$

and

$$\begin{aligned} G(\hat{y})(A) &= \bigvee_{B\in\tau_{e_X}} (S(B, A) \odot \hat{y}(B)) = \bigvee_{B\in\tau_{e_X}} \Big(\bigwedge_{z\in X} (B(z) \to A(g(z))) \odot B(y) \Big) \\ &\leq \bigwedge_{B\in\tau_{e_X}} \Big(B(y) \to A(g(y))) \odot B(y) \Big) \leq A(g(y)) = \widehat{g(y)}(A). \end{aligned} \tag{112}$$

Let $B(y) = g^{\leftarrow}(A)(y) = A(g(y))$ for all $y \in X$. Since

$$g^{\leftarrow}(A)(y) \odot e_Y(y, w) \leq A(g(y)) \odot e_X(g(y), g(w)) \leq A(g(w)), \tag{113}$$

we have $g^{\leftarrow}(A) \in \tau_{e_X}$. Moreover,

$$\begin{aligned} G(\hat{y})(B) &= \bigvee_{A\in\tau_{e_X}} \Big(\bigwedge_{z\in X} (B(z) \to A(g(z))) \odot B(y) \Big) \geq \bigwedge_{z\in X} (A(g(z)) \to A(g(z))) \odot A(g(y)) \Big) \\ &= \top \odot A(g(y)) = A(g(y)) = \widehat{g(y)}(A). \end{aligned} \tag{114}$$

Moreover,

$$e_{\tau_{e_{\tau_{e_X}}}}(\hat{y}, F(\hat{x})) = e_{\tau_{e_{\tau_{e_X}}}}(\hat{y}, \widehat{f(x)}) = e_X(y, f(x)) = e_X(g(y), x) = e_{\tau_{e_{\tau_{e_X}}}}(\widehat{g(y)}, \hat{x}) = e_{\tau_{e_{\tau_{e_X}}}}(G(\hat{y}), \hat{x}). \tag{115}$$

□

Definition 9. *Let (e_X, R, S, e_X) and $(e_Z, \tilde{R}, \tilde{S}, e_Z)$ be dual residuated frames. An injective map $k : (e_X, R, S, e_X) \to (e_Z, \tilde{R}, \tilde{S}, e_Z)$ is a DR-DR frame embedding if*

$$e_X(x, y) = e_Z(k(x), k(y)),\ R(x, y) = \tilde{R}(k(x), k(y)),\ S(x, y) = \tilde{S}(k(x), k(y)). \tag{116}$$

If k is a bijective DR-DR frame embedding map, then k is called a DR-DR frame isomorphism*.*

Theorem 11. *Let (e_X, R, S, e_X) be a dual residual frame, τ_{e_X} be an Alexandrov L-topology and $\tau_{e_{\tau_{e_X}}} = \{\alpha \in L^{\tau_{e_X}} \mid \alpha(A) \odot e_{\tau_{e_X}}(A, B) \leq \alpha(B)\}$. Define a map $k : X \to \tau_{e_{\tau_{e_X}}}$ by $k(x)(A) = \hat{x}(A) = A(x)$. Then the map $k : (e_X, R, S, e_X) \to (e_{\tau_{e_{\tau_{e_X}}}}, \hat{R}, \hat{S}, e_{\tau_{e_{\tau_{e_X}}}})$ is a DR-DR frame embedding map with $e_X(x, y) = e_{\tau_{e_{\tau_{e_X}}}}(k(x), k(y))$, $R(x, y) = \hat{R}(\hat{x}, \hat{y})$ and $S(x, y) = \hat{S}(\hat{x}, \hat{y})$ where*

$$\begin{aligned} F(A)(y) &= \bigwedge_{x\in X} (R(x, y) \to A(x)), \quad G(B)(x) = \bigvee_{x\in X} (S(y, x) \odot B(y)), \\ \hat{R}(\alpha, \beta) &= \bigwedge_{A\in\tau_{e_X}} (\beta(F(A)) \to \alpha(A)), \quad \hat{S}(\beta, \alpha) = \bigwedge_{B\in\tau_{e_X}} (\beta(B) \to \alpha(G(B))). \end{aligned} \tag{117}$$

Proof. By Theorem 4(1), $(\tau_{e_X}, F, G, \tau_{e_X})$ is a dual residuated connection where

$$F(A)(y) = \bigwedge_{x\in X} (R(x, y) \to A(x)), \quad G(B)(x) = \bigvee_{y\in Y} (R(x, y) \odot B(y)). \tag{118}$$

By Theorem 9, $(e_{\tau_{e_{\tau_{e_X}}}}, \hat{R}, \hat{S}, e_{\tau_{e_{\tau_{e_X}}}})$ is a dual residuated frame where

$$\hat{R}(\alpha,\beta) = \bigwedge_{A\in\tau_{e_X}} (\beta(F(A)) \to \alpha(A)), \quad \hat{S}(\beta,\alpha) = \bigwedge_{B\in\tau_{e_X}} (\beta(B) \to \alpha(G(B))). \tag{119}$$

By Theorem 6, $e_X(x,y) = e_{\tau_{e_{\tau_{e_X}}}}(\hat{x},\hat{y})$. Moreover,

$$\begin{aligned}\hat{R}(\hat{x},\hat{y}) &= \bigwedge_{A\in\tau_{e_X}} (\hat{y}(F(A)) \to \hat{x}(A)) = \bigwedge_{A\in\tau_{e_X}} (\bigwedge_{z\in X} (R(z,y) \to A(z)) \to A(x)) \\ &\geq \bigwedge_{A\in\tau_{e_X}} ((R(x,y) \to A(x)) \to A(x)) \geq R(x,y).\end{aligned} \tag{120}$$

Let $R_y^{-1}(z) = R(z,y)$. Since $e_X^{-1} \circ R \leq e_X^{-1} \circ R \circ e_X^{-1} \leq R$, we have

$$R_y^{-1}(x) \odot e_X(x,z) = e_X^{-1}(z,x) \odot R(x,y) \leq R_y^{-1}(z). \tag{121}$$

Thus $R_y^{-1} \in \tau_{e_X}$, and so

$$\begin{aligned}\hat{R}(\hat{x},\hat{y}) &= \bigwedge_{A\in\tau_{e_X}} (\bigwedge_{z\in X} (R(z,y) \to A(z)) \to A(x)) \\ &\leq \bigwedge_{z\in X} ((R(z,y) \to R_y^{-1}(z)) \to R_y^{-1}(x)) = R(x,y),\end{aligned} \tag{122}$$

$$\begin{aligned}\hat{S}(\hat{y},\hat{x}) &= \bigwedge_{B\in\tau_{e_X}} (\hat{y}(B) \to \hat{x}(G(B))) = \bigwedge_{B\in\tau_{e_X}} (B(y) \to \bigvee_{z\in X} (S(z,x) \odot B(z)) \\ &\geq \bigwedge_{B\in\tau_{e_X}} (B(y) \to (S(y,x) \odot B(y)) \geq S(y,x).\end{aligned} \tag{123}$$

For all $R_y^{-1} \in \tau_{e_X}$,

$$\begin{aligned}\hat{S}(\hat{y},\hat{x}) &= \bigwedge_{B\in\tau_{e_X}} (\hat{y}(B) \to \hat{x}(G(B))) \leq (R_y^{-1}(y) \to \bigvee_{z\in X} (R(x,z) \odot R_y^{-1}(z)) \\ &\leq \top \to R(x,y) = R(x,y) = S(y,x).\end{aligned} \tag{124}$$

Hence $k : (e_X, R, S, e_X) \to (e_{\tau_{e_{\tau_{e_X}}}}, \hat{R}, \hat{S}, e_{\tau_{e_{\tau_{e_X}}}})$ is a DR-DR frame embedding map. □

Example 5. *Let $X = \{a,b,c\}$ be a set. Let $f : X \to X$ a map and $([0,1], \odot)$ defined as in Example 3.*
(1) Let $(X = \{a,b,c\}, e_X)$ be a fuzzy poset defined as in Example 3(1). Since (e_X, f, f, e_X) is a dual residuated connection, that is, $e_X(f(x),y) = e_X(x,f(y))$ for all $x,y \in X$, there exist maps $R : \tau_{e_X} \times \tau_{e_X} \to L$ and $S : \tau_{e_X} \times \tau_{e_X} \to L$ by

$$R(A,B) = \bigwedge_{x\in X} (B(f(x)) \to A(x)), \quad S(B,A) = \bigwedge_{y\in Y} (B(y) \to A(g(y))) \tag{125}$$

with an isotone map $f : X \to Y$ such that $(e_{\tau_{e_X}}, R, S, e_{\tau_{e_X}})$ is a dual residuated frame. For all $(e_X)_a, (e_X)_b \in \tau_{e_X}$,

$$\begin{aligned}&R((e_X)_a,(e_X)_b) = \bigwedge_{x\in X} (e_X(b,f(x)) \to e_X(a,x)) = 1,\ R((e_X)_b,(e_X)_a) = 1, \\ &R((e_X)_a,(e_X)_a) = R((e_X)_b,(e_X)_b) = 0.6,\ R((e_X)_c,(e_X)_c) = 1,\ R((e_X)_a,(e_X)_c) = 0.5, \\ &R((e_X)_c,(e_X)_a) = 0.7,\ R((e_X)_b,(e_X)_c) = 0.5,\ R((e_X)_c,(e_X)_b) = 0.7, \\ &S((e_X)_x,(e_X)_y) = R((e_X)_y,(e_X)_x) \quad \text{for all} \quad x,y \in X.\end{aligned} \tag{126}$$

Moreover,

$$R((e_X)_a^{-1*}, (e_X)_b) = \bigwedge_{x \in X} ((e_X)_b(f(x)) \to e_X)_a^{-1*}(x)) = e_X^*(b, f(a)). \tag{127}$$

By Theorem 4(1), $(e_{\tau_{e_{\tau_{e_X}}}}, F, G, e_{\tau_{e_{\tau_{e_X}}}})$ *is a dual residuated connection where*

$$\begin{aligned} F(\alpha)(B) &= \bigwedge_{A \in \tau_{e_X}} (R(A, B) \to \alpha(A)) = \bigwedge_{A \in \tau_{e_X}} \Big(\bigwedge_{z \in X} (B(f(z)) \to A(z)) \to \alpha(A) \Big), \\ G(\alpha)(A) &= \bigvee_{B \in \tau_{e_X}} (S(B, A) \odot \alpha(B)) = \bigvee_{B \in \tau_{e_X}} \Big(\bigwedge_{z \in X} (B(z) \to A(g(z))) \odot \alpha(B) \Big). \end{aligned} \tag{128}$$

By a similar method used in Example 3, one can see that $F = G$.

(2) *Let* $(X = \{a, b, c\}, e_X)$ *be a fuzzy poset and* $g, h : X \to X$ *defined as in Example 3(3). Since* (e_X, h, g, e_X) *is a dual residuated connection, that is,* $e_X(h(x), y) = e_X(x, g(y))$ *for all* $x, y \in X$*, there exist relations* $R : \tau_{e_X} \times \tau_{e_X} \to L$ *and* $S : \tau_{e_X} \times \tau_{e_X} \to L$ *by*

$$R(A, B) = \bigwedge_{x \in X} (B(h(x)) \to A(x)), \quad S(B, A) = \bigwedge_{y \in Y} (B(y) \to A(g(y))) \tag{129}$$

such that $(e_{\tau_{e_X}}, R, S, e_{\tau_{e_X}})$ *is a dual residuated frame. By Theorem 4(1),* $(e_{\tau_{e_{\tau_{e_X}}}}, F, G, e_{\tau_{e_{\tau_{e_X}}}})$ *is a dual residuated connection where*

$$\begin{aligned} F(\alpha)(B) &= \bigwedge_{A \in \tau_{e_X}} (R(A, B) \to \alpha(A)) = \bigwedge_{A \in \tau_{e_X}} \Big(\bigwedge_{z \in X} (B(h(z)) \to A(z)) \to \alpha(A) \Big), \\ G(\alpha)(A) &= \bigvee_{B \in \tau_{e_X}} (S(B, A) \odot \alpha(B)) = \bigvee_{B \in \tau_{e_X}} \Big(\bigwedge_{z \in X} (B(z) \to A(g(z))) \odot \alpha(B) \Big). \end{aligned} \tag{130}$$

Example 6. (1) *Let* $(X = \{a, b, c\}, e_X)$ *be a fuzzy poset where*

$$e_X = \begin{pmatrix} 1 & 0.6 & 0.5 \\ 0.6 & 1 & 0.7 \\ 0.5 & 0.7 & 1 \end{pmatrix}. \tag{131}$$

Define a binary operation $\odot$ on $[0, 1]$ by

$$x \odot y = \max\{0, x + y - 1\}, \ x \to y = \min\{1 - x + y, 1\}. \tag{132}$$

Then $(L = [0, 1], \odot, \to, 0, 1)$ *is a complete residuated lattice. Let*

$$R = \begin{pmatrix} 0.7 & 0.4 & 0.3 \\ 0.6 & 0.8 & 0.5 \\ 0.3 & 0.5 & 0.8 \end{pmatrix}. \tag{133}$$

Since (e_X, R, S, e_X) *is a residuated frame, we have by Theorem 3(1) that* $(e_{\tau_{e_X}}, F, G, e_{\tau_{e_X}})$ *is a residuated connection where*

$$F(A)(y) = \bigvee_{x \in X} (R(x, y) \odot A(x)), \quad G(B)(x) = \bigwedge_{y \in X} (R(x, y) \to B(y)). \tag{134}$$

By Theorem 11, $(e_{\tau_{e_{\tau_{e_X}}}}, \hat{R}, \hat{S}, e_{\tau_{e_{\tau_{e_X}}}})$ *is a residuated frame where*

$$\hat{R}(\alpha, \beta) = \bigwedge_{A \in \tau_{e_X}} (\alpha(A) \to \beta(F(A))), \quad \hat{S}(\beta, \alpha) = \bigwedge_{B \in \tau_{e_X}} (\alpha(G(B)) \to \beta(B)). \tag{135}$$

Since (e_X, R, S, e_X) *is a dual residuated frame, we have by Theorem 4(1) that* $(\tau_{e_X}, F, G, \tau_{e_X})$ *is a dual residuated connection where*

$$F(A)(y) = \bigwedge_{x\in X} (R(x,y) \to A(x)), \quad G(B)(x) = \bigvee_{y\in Y} (R(x,y) \odot B(y)). \tag{136}$$

By Theorem 11, $(e_{\tau_{e_{\tau_{e_X}}}}, \hat{R}, \hat{S}, e_{\tau_{e_{\tau_{e_X}}}})$ *is a dual residuated connection where*

$$\hat{R}(\alpha,\beta) = \bigwedge_{A\in\tau_{e_X}} (\beta(F(A)) \to \alpha(A)), \quad \hat{S}(\beta,\alpha) = \bigwedge_{B\in\tau_{e_X}} (\beta(B) \to \alpha(G(B))). \tag{137}$$

(2) *Let*

$$e_X = \begin{pmatrix} 1 & 0.7 & 0.5 \\ 0.4 & 1 & 0.3 \\ 0.3 & 0.5 & 1 \end{pmatrix}. \tag{138}$$

Then

$$e_X \circ R \circ e_X = \begin{pmatrix} 0.7 & 0.5 & 0.3 \\ 0.6 & 0.8 & 0.5 \\ 0.3 & 0.5 & 0.8 \end{pmatrix}, \tag{139}$$

and so $R < e_X \circ R \circ e_X$*. Hence* (e_X, R, S, e_X) *is not residuated frame. Since* $G((e_X)_b^{-1*})(a) \odot e_X(a,b) = R^*(a,b) \odot e_X(a,b) = 0.6 \odot 0.7 = 0.3 \not\leq 0.2 = R^*(b,b) = G((e_X)_b^{-1*})(b)$*, we have* $G((e_X)_b^{-1*}) \notin \tau_{e_X}$*. However, since* $R = e_X^{-1} \circ R \circ e_X^{-1}$*, we have that* $(e_{\tau_{e_X}}, F, G, e_{\tau_{e_X}})$ *is a dual residuated connection defined by*

$$F(A)(y) = \bigwedge_{x\in X} (R(x,y) \to A(x)), \quad G(B)(x) = \bigvee_{y\in Y} (R(x,y) \odot B(y)). \tag{140}$$

By Theorem 11, $(e_{\tau_{e_{\tau_{e_X}}}}, \hat{R}, \hat{S}, e_{\tau_{e_{\tau_{e_X}}}})$ *is a dual residuated frame where*

$$\hat{R}(\alpha,\beta) = \bigwedge_{A\in\tau_{e_X}} (\beta(F(A)) \to \alpha(A)), \quad \hat{S}(\beta,\alpha) = \bigwedge_{B\in\tau_{e_X}} (\beta(B) \to \alpha(G(B))). \tag{141}$$

5. Conclusions

As an extension of residuated frames for classical relational semantics, we have introduced (dual) residuated frames for fuzzy logics. As a generalization of the classical Tarski's fixed point theorem, we have shown that an Alexandrov L-topology is a fuzzy complete lattice with residuated connections. By using residuated connections, we have constructed fuzzy rough sets and have solved fuzzy relation equations on the Alexandrov L-topology. Moreover, as a generalization of the Dedekind–MacNeille completion, we have introduced R-R (resp. DR-DR) embedding maps and R-R (resp. DR-DR) frame embedding maps.

In the future, by using the concepts of (dual) residuated connections and frames, we plan to investigate fuzzy contexts, information systems and decision rules on Alexandrov L-topologies.

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