mathematics

MDPI

Article

A Unified Approach: Split Quaternions with Quaternion Coefficients and Quaternions with Dual Coefficients

Emel Karaca [1,*], Fatih Yılmaz [1] and Mustafa Çalışkan [2]

[1] Polatlı Science and Arts Faculty, Ankara Hacı Bayram Veli University, Ankara 06900, Turkey; fatih.yilmaz@hbv.edu.tr
[2] Faculty of Science, Gazi University, Ankara 06560, Turkey; mustafacaliskan@gazi.edu.tr
* Correspondence: emel.karaca@hbv.edu.tr

Received: 10 September 2020; Accepted: 29 October 2020; Published: 2 December 2020

check for updates

Abstract: This paper aims to present, in a unified manner, results which are valid on both split quaternions with quaternion coefficients and quaternions with dual coefficients, simultaneously, calling the attention to the main differences between these two quaternions. Taking into account some results obtained by Karaca, E. et al., 2020, each of these quaternions is studied and some important differences are remarked on.

Keywords: quaternions; split quaternions with quaternion coefficients; dual quaternions

1. Introduction

Quaternions, introduced in 1843 by the Irish mathematician Hamilton as a generalization of complex numbers, have become a useful tool for modeling and solving problems in classical fields of mathematics, engineering and physics [1]. Quaternion algebra sits at the intersection of many mathematical subjects. It captures the main features of noncommutative ring theory, group theory, geometric topology, representation theory, etc. After the discovery of quaternions, split quaternion algebra or coquaternion algebra was initially introduced by J. Cackle. Split quaternion algebra is especially beneficial to study because it often reflects some of general aspects for the mentioned subjects. Both quaternion and split quaternion algebras are associative and non-commutative 4-dimensional Clifford algebras. With this in mind, the properties and roots of quaternions and split quaternions are given in detail; see [2–5].

Like matrix representations of complex numbers, the quaternions are also given by matrix representation. It enables for calculating some algebraic properties in quaternion algebra. Hence, quaternions and matrices of quaternions were studied by many authors in the literature; see [6,7].

A brief introduction of the generalized quaternions is given in detail in [8]. Split Fibonacci quaternions, split Lucas quaternions and split generalized Fibonacci quaternions were defined in [9]. The relationships among these quaternions were given in the same study. Similarly, split Pell and split Pell–Lucas quaternions were defined in [10]. In that study, many identities between split Pell and split Pell–Lucas quaternions were mentioned.

Some algebraic concepts for complex quaternions and complex split quaternions were given in [11,12]. In these studies, a 4×4 quaternion coefficients matrix representation was used. Moreover, the correspondences between complex quaternions and complex split quaternions were discussed in detail.

Dual numbers were initially introduced by Clifford. Additionally, they were used as representing the dual angle which measures the relative positions of two skew lines in space by E. study. Using dual numbers, dual quaternions provide a set of tools to help solve problems in rigid transforms, robotics, etc. The generalization of Euler's and De Moivre's formulas for dual quaternions and matrix representations of basic algebraic concepts are studied in [13–15].

The main purpose of this paper is to present, based on quaternions with complex coefficients, results on split quaternions with quaternion coefficients and quaternions with dual coefficients.

The rest of the paper is organized as follows: Section 1 contains a mathematical summary of real quaternions and some concepts of dual numbers. Section 2 is dedicated to quaternions with dual coefficients and Section 3 shows some properties of split quaternions with quaternion coefficients. Finally, Section 4 contains the similarities and differences between quaternions with dual coefficients and split quaternions with quaternion coefficients.

2. Preliminaries

In this section, a brief summary of real quaternions is outlined and some properties of these quaternions are represented.

Definition 1. *A real quaternion is defined as*

$$q = q_0e_0 + q_1e_1 + q_2e_2 + q_3e_3, \tag{1}$$

where q_0, q_1, q_2 and q_3 are real numbers and e_0, e_1, e_2, e_3 of q are four basic vectors of a Cartesian set of coordinates which satisfy the non-commutative multiplication conditions:

$$\begin{aligned} e_1^2 &= e_2^2 = e_3^2 = e_1e_2e_3 = -1, \\ e_1e_2 &= e_3 = -e_2e_1, \;\; e_2e_3 = e_1 = -e_3e_2, \;\; e_3e_1 = e_2 = -e_1e_3. \end{aligned}$$

The set of quaternions can be represented as

$$H = \{q = q_0e_0 + q_1e_1 + q_2e_2 + q_3e_3 : q_0, q_1, q_2, q_3 \in \mathbb{R}\}, \tag{2}$$

where it is a 4-dimensional vector space on $\mathbb{R}$. A real quaternion is defined as a couple (S_q, V_q). That is, q consists of a scalar and a vector. Here $S_q = q_0e_0$ is scalar part and $V_q = q_1e_1 + q_2e_2 + q_3e_3$ is vector part of q, respectively.

For any given two quaternions p and q, the addition is

$$p + q = (S_p + S_q) + (V_p + V_q)$$

and the quaternion product is

$$pq = S_pS_q - \langle V_p, V_q\rangle + S_qV_q + S_qV_p + V_p \times V_q, \tag{3}$$

where $p = p_0e_0 + p_1e_1 + p_2e_2 + p_3e_3$ and $q = q_0e_0 + q_1e_1 + q_2e_2 + q_3e_3$ are real quaternions. Here "$\langle , \rangle$" is the inner product and "$\times$" is vector product in $\mathbb{R}^3$.

The scalar product of q is defined as

$$\lambda q = (\lambda q_0)e_0 + (\lambda q_1)e_1 + (\lambda q_2)e_2 + (\lambda q_3)e_3.$$

The conjugate of q is

$$\bar{q} = S_q - V_q.$$

Additionally, the norm of quaternion is given as

$$\|q\| = \sqrt{q\bar{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}. \tag{4}$$

If $\|q\| = 1$, then q is called unit real quaternion. The inverse of the real quaternion q is

$$q^{-1} = \frac{\bar{q}}{\|q\|^2}, \quad \|q\| \neq 0.$$

Theorem 1. *For $p, q \in H$, the following properties are satisfied, and for more details see [13]:*

(i) $\bar{\bar{q}} = q$,
(ii) $\overline{pq} = \bar{q}\bar{p}$,
(iii) $\|qp\| = \|q\|\|p\|$,
(iv) $\|q^{-1}\| = \frac{1}{\|q\|}$.

3. Quaternions with Dual Coefficients

Dual and complex numbers are significant two-dimensional number systems. Especially in the literature, many mathematicians dealt with the algebraic applications and interpretations of these numbers. Just as the algebra of complex numbers can be described with quaternions, the algebra of dual numbers can be described with quaternions. In this section, quaternions with dual coefficients (QDC) are introduced and some significant definitions and theorems are obtained.

A dual number is given by an expression of the form $a + \epsilon b$, where a and b are real numbers and $\epsilon^2 = 0$. Moreover, the set of dual numbers is given as

$$ID = \{z = a + \epsilon b : a, b \in \mathbb{R}, \ \ \epsilon^2 = 0\}. \tag{5}$$

Addition and multiplication of dual numbers are represented, respectively, as follows:

$$\begin{aligned} (x + \epsilon y) + (a + \epsilon b) &= (x + a) + \epsilon(y + b), \\ (x + \epsilon y).(a + \epsilon b) &= (xa) + \epsilon(xb + ya). \end{aligned}$$

The multiplication has commutatitve, associative property and distributes over addition. The conjugate of $z = a + \epsilon b$ is given as $\bar{z} = a - \epsilon b$. Additionally, the norm of z is

$$\sqrt{z\bar{z}} = a.$$

Let us define a quaternion with dual coefficients with the form

$$Q = Q_0E_0 + Q_1E_1 + Q_2E_2 + Q_3E_3, \tag{6}$$

where Q_0, Q_1, Q_2 and Q_3 are dual numbers and E_0, E_1, E_2, E_3 satisfy the following equalities:

$$\begin{aligned} E_1^2 &= E_2^2 = E_3^2 = E_1E_2E_3 = -E_0, \\ E_1E_2 &= E_3 = -E_2E_1, \ \ E_2E_3 = E_1 = -E_3E_2, \ \ E_3E_1 = E_2 = -E_1E_3. \end{aligned}$$

Furthermore, the quaternion with dual coefficients Q can be rewritten as

$$Q = \sum (a_m + \epsilon b_m)E_m, \tag{7}$$

where a_m and b_m are real numbers for $0 \leq m \leq 3$. Here i, j, k denote the quaternion units and commutes with e_1, e_2 and e_3, respectively. Additionally, $S_Q = Q_0E_0$ is the scalar part and $V_Q =$

$Q_1E_1 + Q_2E_2 + Q_3E_3$ is the vector part of Q. For any given two quaternions with quaternion coefficients Q and P, the addition is

$$Q + P = (S_P + S_Q) + (V_P + V_Q)$$

and the quaternion product is

$$\begin{aligned} QP &= S_A S_C - \langle V_A, V_C \rangle + S_A V_C + S_C V_A + (V_A \times V_C) \\ &+ \epsilon[(S_B S_C - \langle V_B, V_C \rangle + S_B V_C + S_C V_B + (V_B \times V_C)) \\ &+ (S_A S_D - \langle V_A, V_D \rangle + S_A V_D + S_D V_A + (V_A \times V_D))], \end{aligned}$$

where $Q = (a_0E_0 + a_1E_1 + a_2E_2 + a_3E_3) + \epsilon(b_0E_0 + b_1E_1 + b_2E_2 + b_3E_3)$ and $P = (c_0E_0 + c_1E_1 + c_2E_2 + c_3E_3) + \epsilon(d_0E_0 + d_1E_1 + d_2E_2 + d_3E_3)$ are quaternions with dual coefficients. The coefficients of P and Q can be given as follows:

$$\begin{aligned} A &= a_0E_0 + a_1E_1 + a_2E_2 + a_3E_3, \\ B &= b_0E_0 + b_1E_1 + b_2E_2 + b_3E_3, \\ C &= c_0E_0 + c_1E_1 + c_2E_2 + c_3E_3, \\ D &= d_0E_0 + d_1E_1 + d_2E_2 + d_3E_3 \end{aligned}$$

In other words, we can rewrite $Q = A + \epsilon B$ and $P = C + \epsilon D$.

The scalar product of Q is defined as

$$\mu Q = (\mu Q_0)E_0 + (\mu Q_1)E_1 + (\mu Q_2)E_2 + (\mu Q_3)E_3.$$

The conjugate of Q is

$$\begin{aligned} \bar{Q} &= S_Q - V_Q, \\ &= (a_0E_0 - a_1E_1 - a_2E_2 - a_3E_3) + \epsilon(b_0E_0 - b_1E_1 - b_2E_2 - b_3E_3), \\ &= \bar{A} + \epsilon\bar{B}. \end{aligned}$$

Example 1. *Let $Q = 3E_0 + (1 + 2\epsilon)E_1 + (1 - \epsilon)E_2 + E_3$ and $P = (6 + \epsilon)E_0 + (2 - \epsilon)E_1 + (3 + \epsilon)E_2 + \epsilon E_3$ be quaternions with dual coefficients.*

The addition of Q and P is

$$Q + P = (9 + \epsilon)E_0 + (3 + \epsilon)E_1 + 4E_2 + (1 + \epsilon)E_3.$$

Additionally, we can rewrite

$$Q = (3E_0 + E_1 + E_2 + E_3) + \epsilon(2E_1 - E_2)$$

and

$$P = (6E_0 + 2E_1 + 3E_2) + \epsilon(E_0 - E_1 + E_2 + E_3).$$

Moreover, the quaternion product of Q and P is

$$QP = Q = (13E_0 + 9E_1 + 17E_2 + 7E_3) + \epsilon(5E_0 + 10E_1 - 4E_2 + 14E_3).$$

For $\mu = 2$, the scalar product of Q is

$$2Q = 6E_0 + (2 + 4\epsilon)E_1 + (2 - 2\epsilon)E_2 + 2E_3.$$

Moreover, the conjugate of Q is

$$\bar{Q} = (3E_0 - E_1 - E_2 - E_3) + \epsilon(-2E_1 + E_2).$$

Moreover, the norm of Q is given as

$$\|Q\| = \sqrt{Q\bar{Q}} = \sqrt{A\bar{A}}. \tag{8}$$

If $\|Q\| = 1$, then Q is called unit quaternion with dual coefficients. The inverse of Q is

$$Q^{-1} = \frac{\bar{Q}}{\|Q\|^2}, \ \ \|Q\| \neq 0.$$

Example 2. *Let* $Q = \sqrt{5}E_0 + (1+\epsilon)E_1 + (2-\epsilon)E_2 + E_3$ *be a quaternion with dual coefficients. The inverse of Q is*

$$Q^{-1} = \frac{\sqrt{5}E_0 + (1-\epsilon)E_1 + (2+\epsilon)E_2 + E_3}{\sqrt{11}}.$$

The *conjugate* and *dual conjugate* are defined, respectively, as follows:

$$\begin{aligned} \bar{Q} &= Q_0E_0 - Q_1E_1 - Q_2E_2 - Q_3E_3, \\ \widetilde{Q} &= \bar{Q}_0E_0 + \bar{Q}_1E_1 + \bar{Q}_2E_2 + \bar{Q}_3E_3. \end{aligned}$$

Furthermore, above equations can be written as $\bar{Q} = \bar{A} + \epsilon\bar{B}$ and $\widetilde{Q} = A - \epsilon B$, respectively. Hence, we get the following equations:

$$Q\widetilde{Q} = A^2 - 2\epsilon(V_A \times V_B), \tag{9}$$

$$\widetilde{Q}Q = A^2 + 2\epsilon(V_A \times V_B). \tag{10}$$

Therefore, the product $Q\widetilde{Q}$ is not commutative.

Basis elements of 4×4 matrices are given as follows:

$$E_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, E_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$E_2 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, E_3 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}.$$

Here, the multiplication of the matrices E_0, E_1, E_2, E_3 satisfies the equalities given in the definition of the basis elements e_0, e_1, e_2, e_3.

The algebra of the matrix representation for a quaternion with dual coefficients, denoted by H_D^Q, is defined as

$$H_D^Q = \{Q = \begin{pmatrix} Q_0 + iQ_2 & 0 & 0 & -Q_1 + iQ_3 \\ 0 & Q_0 - iQ_2 & Q_1 + iQ_3 & 0 \\ 0 & -Q_1 + iQ_3 & Q_0 + iQ_2 & 0 \\ Q_1 + iQ_3 & 0 & 0 & Q_0 - iQ_2 \end{pmatrix} : Q_0, Q_1, Q_2, Q_3 \in D\}.$$

In other words, the matrix representation of Q, where $Q = Q_0E_0 + Q_1E_1 + Q_2E_2 + Q_3E_3$ is a quaternion with dual coefficients, is

$$\begin{pmatrix} x+\epsilon y & 0 & 0 & -(\bar{z}+\epsilon\bar{t}) \\ 0 & \bar{x}+\epsilon\bar{y} & z+\epsilon t & 0 \\ 0 & -(\bar{z}+\epsilon\bar{t}) & x+\epsilon y & 0 \\ z+\epsilon t & 0 & 0 & \bar{x}+\epsilon\bar{y} \end{pmatrix},$$

where $x = a_0 + ia_2, y = b_0 + ib_2, z = a_1 + ia_3$ and $t = b_1 + ib_3$ are complex numbers.

The transpose and the adjoint matrix of Q, denoted by Q^T and $AdjQ$ respectively, are obtained as

$$Q^T = \begin{pmatrix} x+\epsilon y & 0 & 0 & z+\epsilon\, t \\ 0 & \bar{x}+\epsilon\bar{y} & -(\bar{z}+\epsilon\bar{t}) & 0 \\ 0 & z+\epsilon t & x+\epsilon y & 0 \\ -(\bar{z}+\epsilon\bar{t}) & 0 & 0 & \bar{x}+\epsilon\bar{y} \end{pmatrix},$$

$$AdjQ = \begin{pmatrix} \bar{A}(A\bar{A}+B\bar{B}) & 0 & 0 & -B(A\bar{A}+B\bar{B}) \\ 0 & A(A\bar{A}+B\bar{B}) & \bar{B}(A\bar{A}+B\bar{B}) & 0 \\ 0 & -B(A\bar{A}+B\bar{B}) & \bar{A}(A\bar{A}+B\bar{B}) & 0 \\ \bar{B}(A\bar{A}+B\bar{B}) & 0 & 0 & A(A\bar{A}+B\bar{B}) \end{pmatrix},$$

where $A = Q_0 + iQ_2$ and $B = Q_1 + iQ_3$ are considered as coefficients. Here $\bar{A}$ and $\bar{B}$ are the complex conjugates of A and B, respectively. From above matrices, we can write

$$AdjQ = (A\bar{A} + B\bar{B})\bar{Q}. \tag{11}$$

If $A\bar{A} + B\bar{B} = 1$, then we get Equation (11) as below:

$$AdjQ = \bar{Q}.$$

If off-diagonal entries of Q are 0 then Q is called a diagonal matrix and Q is in form of $Q = Q_0E_0 + Q_2E_2$.

If $Q^T = Q$ then Q is called a symmetric matrix and Q is in form of $Q = Q_0E_0 + Q_2E_2 + Q_3E_3$.

If $Q^T = Q$ then Q is called orthogonal matrix and Q is in form of $Q = Q_0E_0 + Q_2E_2$.

If $(\bar{Q})^T = Q$ then Q is called Hermitian matrix and Q is in form of $Q = Q_0E_0 + Q_2E_2$.

If $(\bar{Q})^T = Q^{-1}$ then Q is called a unitary matrix and Q is in form of $Q = Q_0E_0 + Q_1E_1 + Q_3E_3$ and $detQ = 1$.

Definition 2. *A determinant of $Q \in H_D^Q$ is defined as*

$$detQ = Q_0^2 detE_0 + Q_1^2 detE_1 + Q_2^2 detE_2 + Q_3^2 detE_3. \tag{12}$$

Moreover, from the definition of determinant, we can write

$$detQ = Q_0^2 + Q_1^2 + Q_2^2 + Q_3^2. \tag{13}$$

Here, we point out that the definition of detQ in Equation (13) is different from the determinant for the matrix representation of Q. Namely, the determinant for the matrix representation of Q is calculated as $(Q_0^2 + Q_1^2 + Q_2^2 + Q_3^2)^2$.

Example 3. *Let $Q = (1+\epsilon)E_0 + (3+2\epsilon)E_1 + E_2 + (1-\epsilon)E_3$ be a quaternion with dual coefficients. The determinant of Q is given as*

$$detQ = 12.$$

Theorem 2. *For any $Q, P \in H_D^Q$ and $\lambda \in H$, the following properties are satisfied:*

(i) $detQ = detQ^T$;
(ii) $det(\lambda Q) = \lambda^2 detQ$;
(iii) $det(QP) = detQdetP$;
(iv) $det(\tilde{Q}) = det(Q)$.

Proof. (i) For any $Q = Q_0E_0 + Q_1E_1 + Q_2E_2 + Q_3E_3 \in H_D^Q$, from the given matrices, it can be easily seen that

$$det(Q) = ((x + \epsilon y)(\bar{x} + \epsilon \bar{y}) + (z + \epsilon t)(\bar{z} + \epsilon \bar{t}))^2 = det(Q^T). \tag{14}$$

(ii) For $\lambda \in H$, we get $\lambda Q = (\lambda Q_0)E_0 + (\lambda Q_1)E_1 + (\lambda Q_2)E_2 + (\lambda Q_3)E_3$. Therefore,

$$\begin{aligned} det(\lambda Q) &= \lambda^2 Q_0^2 + \lambda^2 Q_1^2 + \lambda^2 Q_2^2 + \lambda^2 Q_3^2 \\ &= \lambda^2 (Q_0^2 + Q_1^2 + Q_2^2 + Q_3^2) \\ &= \lambda^2 detQ. \end{aligned}$$

(iii) For $Q = Q_0E_0 + Q_1E_1 + Q_2E_2 + Q_3E_3$ and $P = P_0E_0 + P_1E_1 + P_2E_2 + P_3E_3$, the product QP is denoted as

$$\begin{aligned} QP &= E_0(Q_0P_0 - (Q_1P_1 + Q_2P_2 + Q_3P_3)) \\ &+ Q_0(P_1E_1 + P_2E_2 + P_3E_3) \\ &+ P_0(Q_1E_1 + Q_2E_2 + Q_3E_3) \\ &+ E_1(Q_2P_3 - Q_3P_2) \\ &+ E_2(Q_3P_1 - Q_1P_3) \\ &+ E_3(Q_1P_2 - Q_2P_1) \end{aligned}$$

and from the definition of determinant, we have

$$\begin{aligned} det(QP) &= Q_0^2P_0^2 + Q_0^2P_1^2 + Q_0^2P_2^2 + Q_0^2P_3^2 + Q_1^2P_0^2 + Q_1^2P_1^2 + Q_1^2P_2^2 + Q_1^2P_3^2 + Q_2^2P_0^2 \\ &+ Q_2^2P_1^2 + Q_2^2P_2^2 + Q_2^2P_3^2 + Q_3^2P_0^2 + Q_3^2P_1^2 + Q_3^2P_2^2 + Q_3^2P_3^2. \end{aligned}$$

In addition, the determinants of Q and P are $Q_0^2 + Q_1^2 + Q_2^2 + Q_3^2$ and $P_0^2 + P_1^2 + P_2^2 + P_3^2$, respectively. Then,

$$\begin{aligned} detQdetP &= Q_0^2P_0^2 + Q_0^2P_1^2 + Q_0^2P_2^2 + Q_0^2P_3^2 + Q_1^2P_0^2 + Q_1^2P_1^2 + Q_1^2P_2^2 + Q_1^2P_3^2 + Q_2^2P_0^2 \\ &+ Q_2^2P_1^2 + Q_2^2P_2^2 + Q_2^2P_3^2 + Q_3^2P_0^2 + Q_3^2P_1^2 + Q_3^2P_2^2 + Q_3^2P_3^2. \end{aligned}$$

Therefore, we have

$$det(QP) = detQdetP.$$

(iv) From the definitions of the conjugate and dual conjugate, it can be proved easily. □

On the other hand, we obtain the following result for the determinants of the product $Q^T\tilde{Q}$ and Q :

$$det(Q^T\tilde{Q}) \neq (detQ)^2, \tag{15}$$

where Q is any non-zero quaternion.

If $detQ \neq 0$, then the inverse of the quaternion with dual coefficients is

$$\begin{aligned} Q^{-1} &= \bar{Q}\frac{1}{detQ} \\ &= \frac{1}{Q_0^2+Q_1^2+Q_2^2+Q_3^2}(Q_0E_0 - Q_1E_1 - Q_2E_2 - Q_3E_3). \end{aligned}$$

Theorem 3. *Quaternion with dual coefficients matrices satisfy the following properties:*

(i) $E_1C \neq CE_1, E_2C \neq CE_2, E_3C \neq CE_3,$
(ii) $Q^2 = S_Q^2 - det(V_Q)E_0 + 2S_QV_Q$

where C is any non-zero quaternion.

Proof. (i) By using multiplication, it can be seen easily.
(ii) For $Q = Q_0E_0 + Q_1E_1 + Q_2E_2 + Q_3E_3 \in H_D^Q$,

$$Q^2 = (Q_0^2 - Q_1^2 - Q_2^2 - Q_3^2)E_0 + 2Q_0(Q_1E_1 + Q_2E_2 + Q_3E_3).$$

Thus, we get

$$Q^2 = S_Q^2 - det(V_Q)E_0 + 2S_QV_Q.$$

□

Here we would like to bring to your attention that this result is different from Theorem 2, in [1].

Lemma 1. *For any $Q, P \in H_D^Q$, the following properties are satisfied:*

$$\begin{aligned} (i) \quad & Q = [\widetilde{(\bar{Q})^T}], \\ (ii) \quad & Q^T = \tilde{\bar{Q}}. \end{aligned}$$

Proof. (i) For any $Q \in H_D^Q$, we write

$$Q = \begin{pmatrix} W & 0 & 0 & -\bar{Z} \\ 0 & \bar{W} & Z & 0 \\ 0 & -\bar{Z} & W & 0 \\ Z & 0 & 0 & \bar{W} \end{pmatrix},$$

where $W = Q_0 + iQ_2$ and $Z = Q_1 + iQ_3$ coefficients. From the conjugate of Q, we obtain

$$\bar{Q} = \begin{pmatrix} \bar{W} & 0 & 0 & -Z \\ 0 & W & \bar{Z} & 0 \\ 0 & -Z & \bar{W} & 0 \\ \bar{Z} & 0 & 0 & W \end{pmatrix}.$$

Using the definition of transpose, we get

$$\bar{Q}^T = \begin{pmatrix} \bar{W} & 0 & 0 & \bar{Z} \\ 0 & W & -Z & 0 \\ 0 & \bar{Z} & \bar{W} & 0 \\ -Z & 0 & 0 & W \end{pmatrix}.$$

As we use the definition of dual conjugate to the matrix $\bar{Q}^T$, i.e., applying the conjugate for every coefficients, we acquire Q.

(ii) If Q is considered as (i), the transpose and conjugate are given as

$$\bar{Q}^T = \begin{pmatrix} \bar{W} & 0 & 0 & \bar{Z} \\ 0 & W & -Z & 0 \\ 0 & \bar{Z} & \bar{W} & 0 \\ -Z & 0 & 0 & W \end{pmatrix},$$

$$\bar{Q} = \begin{pmatrix} \bar{W} & 0 & 0 & -Z \\ 0 & W & \bar{Z} & 0 \\ 0 & -Z & \bar{W} & 0 \\ \bar{Z} & 0 & 0 & W \end{pmatrix}.$$

As we apply the dual conjugate for every coefficients of the conjugate of Q, we acquire that

$$\tilde{\bar{Q}} = \begin{pmatrix} \bar{W} & 0 & 0 & \bar{Z} \\ 0 & W & -Z & 0 \\ 0 & \bar{Z} & \bar{W} & 0 \\ -Z & 0 & 0 & W \end{pmatrix}.$$

□

Theorem 4. *If $Q, P \in H_D^Q$ are invertible, then the following properties are satisfied:*

(i) $(\widetilde{Q})^{-1} \neq (\widetilde{Q^{-1}})$ *in general,*
(ii) $(\bar{Q})^{-1} = (\bar{Q})^{-1}$,
(iii) $\widetilde{QP} = \widetilde{Q}\widetilde{P}$,
(iv) $(QP)^{-1} = P^{-1}Q^{-1}$.

Proof. (i) For a given invertible $\widetilde{Q} = \bar{Q}_0 E_0 + \bar{Q}_1 E_1 + \bar{Q}_2 E_2 + \bar{Q}_3 E_3$, we can write

$$\begin{aligned} (\widetilde{Q})^{-1} &= \frac{1}{det(\widetilde{Q})}\tilde{\bar{Q}} \\ &= \frac{1}{(\bar{Q}_0)^2 + (\bar{Q}_1)^2 + (\bar{Q}_2)^2 + (\bar{Q}_3)^2}(Q_0 E_0 + Q_1 E_1 + Q_2 E_2 + Q_3 E_3). \end{aligned}$$

On the other hand, we can calculate

$$(\widetilde{Q^{-1}}) = \frac{1}{Q_0^2 + Q_1^2 + Q_2^2 + Q_3^2}(\bar{Q}_0 E_0 - \bar{Q}_1 E_1 - \bar{Q}_2 E_2 - \bar{Q}_3 E_3).$$

Thus, we obtain $(\widetilde{Q})^{-1} \neq (\widetilde{Q^{-1}})$.
By using the inverse definition and multiplication definition, (ii) and (iii) can be seen, easily.
(iv) For $Q = Q_0 E_0 + Q_1 E_1 + Q_2 E_2 + Q_3 E_3$ and $P = P_0 E_0 + P_1 E_1 + P_2 E_2 + P_3 E_3$, we obtain QP as

$$QP = Q = KE_0 + LE_1 + ME_2 + NE_3,$$

where

$$\begin{aligned} K &= Q_0 P_0 - Q_1 P_1 - Q_2 P_2 - Q_3 P_3, \\ L &= Q_0 P_1 + Q_1 P_0 + Q_2 P_3 - Q_3 P_2, \\ M &= Q_0 P_2 - Q_1 P_3 + Q_2 P_0 + Q_3 P_1, \\ N &= Q_0 P_3 + Q_1 P_2 - Q_2 P_1 + Q_3 P_0, \end{aligned}$$

are coefficients, respectively. From the definition of determinant, we write

$$det(QP) = K^2 + L^2 + M^2 + N^2.$$

Thus, we can find

$$(QP)^{-1} = \frac{\overline{QP}}{det(QP)} = \frac{KE_0 - LE_1 - ME_2 - NE_3}{K^2 + L^2 + M^2 + N^2}.$$

On the other hand, the inverses of P and Q are obtained as

$$\begin{aligned} P^{-1} &= \frac{P_0E_0 - P_1E_1 - P_2E_2 - P_3E_3}{P_0^2 + P_1^2 + P_2^2 + P_3^2}, \\ Q^{-1} &= \frac{Q_0E_0 - Q_1E_1 - Q_2E_2 - Q_3E_3}{Q_0^2 + Q_1^2 + Q_2^2 + Q_3^2}. \end{aligned}$$

Additionally, their inner product is

$$P^{-1}Q^{-1} = \frac{KE_0 - LE_1 - ME_2 - NE_3}{K^2 + L^2 + M^2 + N^2}.$$

Hence,

$$(QP)^{-1} = P^{-1}Q^{-1}.$$

□

Example 4. *Let $Q = (2-\epsilon)E_0 + (1+\epsilon)E_1 + (2-\epsilon)E_2 + (3+2\epsilon)E_3$ be a quaternion with dual coefficients. We obtain $\tilde{Q} = (2+\epsilon)E_0 + (1-\epsilon)E_1 + (2+\epsilon)E_2 + (3-2\epsilon)E_3$. Therefore, we can write*

$$\begin{aligned} (\widetilde{Q})^{-1} &= \frac{1}{det(\widetilde{Q})}\tilde{\tilde{Q}} \\ &= \frac{1}{18}((2+\epsilon)E_0 + (1-\epsilon)E_1 + (2+\epsilon)E_2 + (3-2\epsilon)E_3). \end{aligned}$$

On the other hand, we can calculate

$$(\widetilde{Q^{-1}}) = \frac{1}{18}(2-\epsilon)E_0 - (1+\epsilon)E_1 + -(2-\epsilon)E_2 + -(3+2\epsilon)E_3).$$

From these calculations, we observe that $(\widetilde{Q})^{-1} \neq (\widetilde{Q^{-1}})$ in general. However, if we take Q_0 as pure-real and $Q_1 = Q_2 = Q_3 = 0$, then equation (i) provides an equality.

Here, we would like to point out that (i) is only satisfied when Q_0 are considered as pure-real and $Q_1 = Q_2 = Q_3 = 0$.

4. Split Quaternions with Quaternion Coefficients

In [11], Karaca et. al. introduced the split quaternions with quaternion coefficients (SQC) and obtained some significant properties. Moreover, they gave some definitions and theorems about split quaternions with quaternion coefficients.

A split quaternion with quaternion coefficients is the form

$$P = P_0E_0 + P_1E_1 + P_2E_2 + P_3E_3, \tag{16}$$

where P_0, P_1, P_2 and P_3 are quaternions and the split quaternion matrix basis $\{E_0, E_1, E_2, E_3\}$ of P satisfies the following equalities:

$$\begin{aligned} E_1^2 &= -E_0, \ E_2^2 = E_3^2 = E_0, \\ E_1E_2 &= E_3 = -E_2E_1, \ E_2E_3 = E_1 = -E_3E_2, \ E_3E_1 = E_2 = -E_1E_3. \end{aligned}$$

Additionally, the quaternion with quaternion coefficients P can be rewritten as

$$P = \sum(a_m + b_m i + c_m j + d_m k)E_m, \tag{17}$$

where a_m, b_m, c_m, d_m are real numbers for $0 \leq m \leq 3$. Here i, j, k denote the quaternion units and commutes with e_1, e_2 and e_3, respectively. Furthermore, $S_P = P_0E_0$ is the scalar part and $V_P = P_1E_1 + P_2E_2 + P_3E_3$ is vector part of P in [11]. The set of split quaternions with quaternion coefficients are denoted by H_S^Q in [11]. Basis elements of 4x4 matrices are given as follows:

$$E_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, E_1 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix},$$

$$E_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, E_3 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}.$$

The conjugate, quaternionic conjugate and total conjugate are defined, respectively, as follows [11]:

$$\begin{aligned} \bar{P} &= P_0E_0 - P_1E_1 - P_2E_2 - P_3E_3, \\ \widetilde{P} &= \bar{P}_0E_0 + \bar{P}_1E_1 + \bar{P}_2E_2 + \bar{P}_3E_3, \\ \widetilde{\bar{P}} &= \bar{P}_0E_0 - \bar{P}_1E_1 - \bar{P}_2E_2 - \bar{P}_3E_3. \end{aligned}$$

In [11], the determinant of P is defined as

$$detP = P_0^2 detE_0 + P_1^2 detE_1 + P_2^2 detE_2 + P_3^2 detE_3. \tag{18}$$

Simply, we can write

$$detP = P_0^2 + P_1^2 + P_2^2 + P_3^2.$$

Additionally, the norm of P is given as

$$\begin{aligned} \|P\| = \sqrt{P\bar{P}} &= P_0^2 + P_1^2 - P_2^2 - P_3^2, \\ &= (P_0^2 + P_1^2) - (P_2^2 + P_3^2), \\ &= \sqrt{||p_{11}|^2 - |p_{12}|^2|}, \end{aligned}$$

where $p_{11} = P_0 + iP_1$ and $p_{12} = P_2 + iP_3$ are considered for calculations. If $\|P\| = 1$, then P is called unit split quaternion with quaternion coefficients in [11].

Theorem 5. *For any non-zero $Q, P \in H_S^Q$ and $\lambda \in H$, the following properties are satisfied:*

(i) $detQ = det\bar{Q} = detQ^T$,

(ii) $det(\lambda Q) = \lambda^2 detQ$,

(iii) $det(QP) \neq detQdetP$ *in general.*

Proof. (i) For any $Q = Q_0E_0 + Q_1E_1 + Q_2E_2 + Q_3E_3 \in H_S^Q$, from the equations of transpose and conjugate

$$detQ = det\bar{Q} = detQ^T = Q_0^2 + Q_1^2 + Q_2^2 + Q_3^2.$$

(ii) For $\lambda \in H$, we get $\lambda Q = (\lambda Q_0)E_0 + (\lambda Q_1)E_1 + (\lambda Q_2)E_2 + (\lambda Q_3)E_3$. Therefore, we obtain

$$\begin{aligned} det(\lambda Q) &= \lambda^2 Q_0^2 + \lambda^2 Q_1^2 + \lambda^2 Q_2^2 + \lambda^2 Q_3^2 \\ &= \lambda^2 (Q_0^2 + Q_1^2 + Q_2^2 + Q_3^2) \\ &= \lambda^2 detQ. \end{aligned}$$

(iii) For $Q = Q_0E_0 + Q_1E_1 + Q_2E_2 + Q_3E_3$ and $P = P_0E_0 + P_1E_1 + P_2E_2 + P_3E_3$, the product QP is denoted as

$$\begin{aligned} det(QP) &= (Q_0P_0 - (Q_1P_1 + Q_2P_2 + Q_3P_3))^2 \\ &+ (Q_0P_1 + Q_1P_0 + Q_2P_3 - Q_3P_2)^2 \\ &+ (Q_0P_2 + Q_2P_0 + Q_3P_1 - Q_1P_3)^2 \\ &+ (Q_0P_3 + Q_3P_0 + Q_1P_2 - Q_2P_1)^2. \end{aligned}$$

In addition, the determinants of Q and P are $Q_0^2 + Q_1^2 + Q_2^2 + Q_3^2$ and $P_0^2 + P_1^2 + P_2^2 + P_3^2$, respectively. Then,

$$\begin{aligned} detQdetP &= Q_0^2P_0^2 + Q_0^2P_1^2 + Q_0^2P_2^2 + Q_0^2P_3^2 + Q_1^2P_1^2 + Q_1^2P_2^2 + Q_1^2P_3^2 + Q_2^2P_0^2 \\ &+ Q_2^2P_1^2 + Q_2^2P_2^2 + Q_2^2P_3^2 + Q_3^2P_0^2 + Q_3^2P_1^2 + Q_3^2P_2^2 + Q_3^2P_3^2. \end{aligned}$$

Therefore, we have

$$det(QP) \neq detQdetP.$$

□

Theorem 6. *Split quaternions with quaternion coefficients matrices satisfy the following properties:*

(i) $E_1C \neq CE_1, E_2C \neq CE_2, E_3C \neq CE_3,$

(ii) $Q^2 = S_Q^2 + (V_Q \times V_Q) + 2Q_0(Q_1E_1 + Q_2E_2 + Q_3E_3)$ *in general,*

where C is any non-zero quaternion.

Proof. (i) For any non-zero quaternion C, from the quaternion product, it can be seen easily.

(ii) For Q, we acquire that

$$\begin{aligned} Q^2 &= Q_0^2E_0 + Q_0Q_1E_1 + Q_0Q_2E_2 + Q_0Q_3E_3 \\ &+ Q_1^2E_0 + Q_1Q_0E_1 - Q_1Q_3E_2 + Q_1Q_2E_3 \\ &+ Q_2^2E_0 + Q_2Q_3E_1 + Q_2Q_0E_2 - Q_2Q_1E_3 \\ &+ Q_3^2E_0 - Q_3Q_2E_1 + Q_3Q_0E_2 + Q_3Q_0E_3. \end{aligned}$$

Thus, we can write $Q^2 = S_Q^2 + (V_Q \times V_Q) + 2Q_0(Q_1E_1 + Q_2E_2 + Q_3E_3)$. □

Definition 3. *Every split quaternion with quaternion coefficients* $Q = Q_0E_0 + Q_1E_1 + Q_2E_2 + Q_3E_3$ *can be written in the polar form*

$$e^Q = e^{Q_0}(\cos\phi + \frac{V_Q}{\phi}\sin\phi), \tag{19}$$

where $V_Q = Q_1E_1 + Q_2E_2 + Q_3E_3$ *and* $\phi = \|V_Q\| = \sqrt{Q_1^2 + Q_2^2 + Q_3^2}$, *respectively.*

Example 5. *Let $Q = 2E_0 + (i+j)E_1 + kE_2 + E_3$ be split quaternion with quaternion coefficients. The polar form of Q is obtained as*

$$e^Q = e^2(\cos(\sqrt{2+2k}) + \frac{(i+j)E_1 + kE_2 + E_3}{(\sqrt{2+2k})}\sin(\sqrt{2+2k})),$$

where $V_Q = (i+j)E_1 + kE_2 + E_3$ and $\phi = 2 + 2k$, respectively.

Additionally, every split quaternion with quaternion coefficients P can be uniquely written as $P = (P_0 + P_1E_1) + E_2(P_2 + P_3E_1)$.

Theorem 7. *For any non-zero $P, Q \in H_S^Q$, the following properties are satisfied:*

(i) $detP \neq \|P\|^2 \ \ in \ \ general,$
(ii) $If \ P \ is \ invertible, \ then \ \overline{(P^{-1})} = \bar{P}^{-1},$
(iii) $If \ P \ is \ invertible, \ then \ (\tilde{P})^{-1} = \widetilde{P^{-1}},$
(iv) $If \ P \ is \ invertible, \ then \ [(\bar{P})^T]^{-1} = [\overline{(P^{-1})}]^T,$
(v) $\overline{QP} \neq \bar{Q}\bar{P} \ \ in \ \ general,$
(vi) $If \ P \ and \ Q \ are \ invertible, \ then \ (PQ)^{-1} = Q^{-1}P^{-1}.$

Proof. Let $P = P_0 + P_1E_1 + E_2(P_2 + P_3E_1$ be a split quaternion with quaternion coefficients.

(i) From the definition of the determinant, we obtain

$$detP = P_0^2 + P_1^2 + P_2^2 + P_3^2$$

and

$$\|P\|^2 = P\bar{P} = (P_0^2 + P_1^2 - P_2^2 - P_3^2)^2.$$

(ii) Let P be invertible. Thus, we can write

$$\overline{P^{-1}} = \frac{P_0E_0 + P_1E_1 + P_2E_2 + P_3E_3}{P_0^2 + P_1^2 + P_2^2 + P_3^2}.$$

As we consider $\overline{P} = P_0 - P_1E_1 - P_2E_2 - P_3E_1$, we acquire

$$\overline{P}^{-1} = \frac{P_0E_0 + P_1E_1 + P_2E_2 + P_3E_3}{P_0^2 + P_1^2 + P_2^2 + P_3^2}.$$

(iii) Let P be invertible. Using quaternionic conjugate, i.e., $\tilde{P} = \bar{P}_0E_0 + \bar{P}_1E_1 + \bar{P}_2E_2 + \bar{P}_3E_3$, we obtain

$$\tilde{P}^{-1} = \frac{\bar{P}_0E_0 - \bar{P}_1E_1 - \bar{P}_2E_2 - \bar{P}_3E_3}{\bar{P}_0^2 + \bar{P}_1^2 + \bar{P}_2^2 + \bar{P}_3^2} = \widetilde{P^{-1}}.$$

(iv) Let P be invertible. Considering transpose and conjugate, we get

$$[(\bar{P})^T]^{-1} = \frac{P_0E_0 + P_1E_1 + P_2E_2 - P_3E_3}{P_0^2 + P_1^2 + P_2^2 + P_3^2} = [\overline{(P^{-1})}]^T.$$

(v) For any $P, Q \in H_S^Q$, we can write

$$\begin{aligned}
\bar{Q}\bar{P} \ = \ & Q_0P_0E_0 - Q_0P_1E_1 - Q_0P_2E_2 - Q_0P_3E_3 \\
- \ & Q_1P_0E_1 - Q_1P_1E_0 + Q_1P_2E_3 - Q_1P_3E_2 \\
- \ & Q_2P_0E_2 - Q_2P_1E_3 + Q_2P_2E_0 + Q_2P_3E_1 \\
- \ & Q_3P_0E_3 + Q_3P_1E_2 - Q_3P_2E_1 + Q_3P_3E_0
\end{aligned}$$

and

$$\begin{aligned}\overline{QP} &= Q_0P_0E_0 - Q_0P_1E_1 - Q_0P_2E_2 - Q_0P_3E_3 \\ &- Q_1P_0E_1 - Q_1P_1E_0 - Q_1P_2E_3 + Q_1P_3E_2 \\ &- Q_2P_0E_2 + Q_2P_1E_3 + Q_2P_2E_0 - Q_2P_3E_1 \\ &- Q_3P_0E_3 - Q_3P_1E_2 + Q_3P_2E_1 + Q_3P_3E_0.\end{aligned}$$

Thus, $\bar{Q}\bar{P} \neq \overline{QP}$.
(vi) From the definition of conjugate, it can be proved easily. □

Let us to exemplify this theorem.

Example 6. *Let $Q = E_0 + iE_1 + (j+k)E_3$ and $P = E_0 + E_2 + jE_3$ be split quaternions with quaternion coefficients. Then:*
(i) $detP = 1 = \|P\|^2$.
(ii) *Let P be invertible. It can be found that*

$$\overline{(P^{-1})} = E_0 + E_2 + jE_3 = \bar{P}^{-1}.$$

(iii) *Let P be invertible. It can be easily seen that*

$$(\tilde{P})^{-1} = E_0 - E_2 + jE_3 = \widetilde{P^{-1}}.$$

(v) *By exploiting the multiplication definition of quaternions and their properties, it is seen that*

$$\begin{aligned}\overline{QP} &= 2E_0 + 2E_1 + E_2 - E_3, \\ \bar{Q}\bar{P} &= 2E_0 - 2E_1 - 3E_2 + E_3.\end{aligned}$$

5. Conclusions

In this article, we defined quaternions with dual coefficients (QDC). Then we got some algebraic properties for QDC. Moreover, we gave some important theorems for split quaternions with quaternion coefficients (SQC).

One of the important differences between QDC and SQC is the definition of the inner product. Another difference between them is that the product of conjugate and dual conjugate is different from the product of the conjugate and quaternionic conjugate. Thirdly, we found that the expression of the norm is different in each case. Finally, the determinant of multiplication of two SQC is equal to the multiplication of their determinants. However, this property does not hold for QDC. In other words, the determinant of multiplication of two QDC is not equal to the multiplication of their determinants.

Author Contributions: Conceptualization, F.Y. and E.K.; methodology, E.K.; software, F.Y. and E.K.; validation, F.Y., M.Ç. and E.K.; formal analysis, F.Y. and E.K.; investigation, F.Y. and E.K.; resources, F.Y. and E.K.; data curation, E.K.; writing—original draft preparation, F.Y. and E.K.; writing—review and editing, F.Y. and E.K.; visualization, F.Y. and E.K.; supervision, F.Y. and M.Ç.; project administration, F.Y.; funding acquisition, TUBITAK. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Malonek, H.R. Quaternions in applied sciences—A historical perspective of a mathematical concept. In *International Kolloquium Applications of Computer Science Additionally, Mathematics in Architecture Additionally, Building Industry*; IKM: Sola, Norway, 2003; Volume 16.
2. Ward, J.P. *Quaternions and Cayley Numbers: Algebra and Applications*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 1997. [CrossRef]
3. Kula, L.; Yaylı, Y. Split quaternions and rotations in semi-Euclidean space E_2^4. *J. Korean Math. Soc.* **2007**, *44*, 1313–1327. [CrossRef]
4. Özdemir, M. The roots of a split quaternion. *Appl. Math. Lett.* **2009**, *22*, 258–263. [CrossRef]
5. Karaca, E.; Yılmaz, F.; Çalışkan, M. Some characterizations for split quaternions. In Proceedings of the International Conference on Mathematics and Its Applications in Science and Engineering (ICMASE 2020), Ankara, Turkey, 9–11 July 2020; pp. 61–65.
6. Alagöz, Y.; Oral, K.H.; Yüce, S. Split quaternion matrices. *Miskolc Math. Notes* **2012**, *13*, 223–232. [CrossRef]
7. Akyiğit, M.; Köksal, H.H.; Tosun, M. Split Fibonacci quaternions. *Adv. Appl. Clifford Algebr.* **2013**, *23*, 535–545. [CrossRef]
8. Tokeşer, Ü.; Ünal, Z.; Bilici, G. Split Pell and Pell-Lucas quaternions. *Adv. Appl. Clifford Algebr.* **2017**, *27*, 1881–1993. [CrossRef]
9. Zhang, F. Quaternions and matrices of quaternions. *Linear Algebra Appl.* **1997**, *251*, 21–57. [CrossRef]
10. Jafari, M.; Yaylı, Y. Generalized quaternions and their algebraic properties. *Commun. Fac. Sci. Univ. Ank. Ser. A1* **2015**, *64*, 15–27.
11. Alagöz, Y.; Özyurt, G. Some properties of complex quaternion and complex split quaternion. *Miskolc Math. Notes* **2019**, *20*, 45–58. [CrossRef]
12. Erdoğdu, M.; Özdemir, M. On complex split quaternion matrices. *Adv. Appl. Clifford Algebr.* **2013**, *23*, 625–638. [CrossRef]
13. Ercan, Z.; Yüce, S. On properties of the dual quaternions. *Appl. Math. Lett.* **2009**, *22*, 142–146.
14. Dağdeviren, A.; Yüce, S. Dual quaternions and dual quaternionic curves. *Filomat* **2009**, *33*, 1037–1046. [CrossRef]
15. Thomas, F. Approaching dual quaternions from matrix algebra. *IEEE Trans. Robot.* **2014**, *30*, 1037–1048. [CrossRef]