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Intrinsic Discontinuities in Solutions of Evolution Equations Involving Fractional Caputo–Fabrizio and Atangana–Baleanu Operators

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Abstract: There has been considerable recent interest in certain integral transform operators with non-singular kernels and their ability to be considered as fractional derivatives. Two such operators are the Caputo–Fabrizio operator and the Atangana–Baleanu operator. Here we present solutions to simple initial value problems involving these two operators and show that, apart from some special cases, the solutions have an intrinsic discontinuity at the origin. The intrinsic nature of the discontinuity in the solution raises concerns about using such operators in modelling. Solutions to initial value problems involving the traditional Caputo operator, which has a singularity inits kernel, do not have these intrinsic discontinuities.

Keywords: Caputo–Fabrizio operator; Atangana–Baleanu operator; fractional falculus

MSC: 23A33

1. Introduction

In mathematical models based on evolution equations it is standard to restrict consideration to models whose solutions exist in a space of continuous functions. This is also true in cases of evolution equations with fractional order differential operators, such as the Riemann–Liouville operator [1] and the Caputo operator [2]. Both of these operators are based on integrals with power law kernels that are singular at the origin. In recent years, there has been a great deal of attention focussed on fractional differential operators based on integrals with non-singular kernels. Included in this are the Caputo–Fabrizio (CF) operator [3] and the Atangana–Baleanu in the sense of a Caputo (ABC) operator [4].

The CF and ABC operators have since been employed in numerous modelling applications, including applications to phase transitions [5], fluid and ground water flow [6], cancer treatment [7], and epidemiology [8], among others [9–12]. Many of the works in this field introduce the model evolution equations by the ad hoc "fractionalisation" of simply replacing integer order derivatives in traditional models with CF or ABC derivatives, without further phenomenological consideration.

On the theoretical side, there has been considerable effort devoted to understanding the interpretations of these differential operators. In a sequence of studies, Tarasov [13–15], Ortigueira and Machado [16], and Giusti [17] have shown that the CF differential operator should not be regarded as a fractional order operator and the ABC operator does not extend beyond the Caputo operator.

Nevertheless, modelling applications still persist [8], as do numerical studies [18–20] and algebraic methods of solution [21]. We also note that, as presented by Hilfer and Luchko [22], there is no absolute agreement on a well defined set of properties to which a fractional derivative must adhere and it is not the aim of this work to interrogate such properties. A greater concern, expressed in the present work, is that CF and ABC operators are not generally suitable for modelling, whenever a solution is sought in a space of continuous functions. In particular, we show that we can construct well formed solutions to initial values problems (IVPs) with CF operators but the solutions have discontinuities at the origin. Many of the works discussed above make extensive use of integral transforms, and more specifically, the Laplace transform. A key point of the present contribution is highlighting that the Laplace transform is not bijective for the class of functions which solve IVPs under the CF operator and careful treatment of the solution near the initial condition is paramount.

The discontinuities in the solution of the IVPs necessitate the more general consideration of the derivatives in the definitions of both the CF and ABC operators. Such generalisations to distributional derivatives are well defined [23]. There is a large body of work concerned with the analysis of such derivatives, both for the case of the CF operator [24] and for the Riemann–Liouville, Caputo, and other fractional derivatives [25–27].

The remainder of this paper is organised as follows. We first construct solutions of CF IVPs and show that such solutions must, in general, feature a discontinuity at the origin. We then briefly discuss the impacts of these results on numerical methods for the solution of IVPs involving CF operators and point out a seemingly overlooked simple approach to the numerical evaluation of such equations. Next we repeat this treatment for the ABC operator and again show that solutions, in general, will have a discontinuity at the origin. Finally we consider a more traditional fractional derivative, the Caputo derivative, and show that solutions to Caputo IVPs can not feature such discontinuities.

2. Caputo–Fabrizio Operator

Definition 1 (The Caputo–Fabrizio operator)**.** *The Caputo–Fabrizio (CF) operator is defined as [3],*

$$ {}^{\mathrm{CF}}_{0}\mathcal{D}^{\alpha}_{t}u(t) = \frac{M(\alpha)}{1-\alpha}\int_0^t u'(\tau)\exp\left(-\frac{\alpha(t-\tau)}{1-\alpha}\right)d\tau, \tag{1} $$

for $0 \leq \alpha < 1$. Here $M(\alpha)$ is a weighting function such that $M(0) = M(1) = 1$.

It is typical and sufficient to take $M(\alpha) = 1$. The CF operator has been purported to be a fractional derivative when $0 < \alpha < 1$ which limits to an integer order derivative as $\alpha \to 1^-$ [3,28]. It should also be noted that the derivative in the integral may be considered in the distributional sense; for more information, see [24].

We will consider CF equations in the form of an IVP with

Definition 2 (A Caputo–Fabrizio Initial Value Problem)**.** *A CF IVP is given by both a CF equation of the form*

$$ {}^{\mathrm{CF}}_{0}\mathcal{D}^{\alpha}_{t}u(t) = F(t), \tag{2} $$

with $F(t)$ a continuous function for $t \geq 0$ and an initial value,

$$ u(0) = u_0, \tag{3} $$

with $u_0 \in \mathbb{R}$.

For now, it will suffice to say that we will consider solutions, $u(t)$, defined over the interval $t \in [0, \infty)$ without being overly concerned with the smoothness of such solutions other than asking for the derivative, at least in the distributional sense, to be well defined.

It should be noted that, in general, continuous solutions of this IVP do not exist [29]. Here we will consider a more general form of a solution by taking an ansatz such that

$$u'(t) = u_c'(t) + a\delta(t - 0^+), \tag{4}$$

where u_c is a continuous function, δ is a Dirac delta, and a is an unknown constant. Note that this form of a solution still permits a purely continuous form with $a = 0$. The form of this Dirac delta is chosen so that we have

$$\int_0^t \delta(\tau - 0^+) d\tau = H(t) = \begin{cases} 0 & t \leq 0, \\ 1 & t > 0. \end{cases} \tag{5}$$

With the ansatz in Equation (4) it follows that the solution will be of the form

$$u(t) = u(0) + u_c(t) - u_c(0) + aH(t). \tag{6}$$

We will further simplify this by taking $u_c(0) = u(0)$, to give

$$u(t) = u_c(t) + aH(t). \tag{7}$$

Care has to be taken here due to the fact that the Laplace transform is not bijective, i.e.,

$$\mathcal{L}^{-1}\left\{\mathcal{L}\left\{u(t)\right\}\right\} = u_c(t) + a \neq u(t). \tag{8}$$

As such, we can not rely on Laplace transform techniques to find solutions of this form.

Theorem 1. *For a CF IVP (Definition 2) assume that a solution in the form of the ansatz (Equation (7)) exists. Then such a solution is given by*

$$u(t) = u(0) + \frac{1-\alpha}{M(\alpha)}\left(F(t) - F(0)\right) + \frac{\alpha}{M(\alpha)}\int_0^t F(\tau) d\tau + \frac{(1-\alpha)F(0)}{M(\alpha)} H(t). \tag{9}$$

Proof. The solution is found by assuming that a solution in the form of the ansatz exists, substituting it into the IVP, and showing that the result is then consistent. To find the value of the unknown constant a from the ansatz we first substitute the Equation (4) into Equation (2), which gives

$$\frac{M(\alpha)}{1-\alpha}\int_0^t u_c'(\tau)\exp\left(-\frac{\alpha(t-\tau)}{1-\alpha}\right) d\tau + \frac{aM(\alpha)}{1-\alpha}\exp\left(-\frac{\alpha t}{1-\alpha}\right) = F(t) \tag{10}$$

for $t > 0$. Next we take the limit as t approaches 0 from above to give

$$\lim_{t\to 0^+}\frac{M(\alpha)}{1-\alpha}\int_0^t u_c'(\tau)\exp\left(-\frac{\alpha(t-\tau)}{1-\alpha}\right) d\tau + \lim_{t\to 0^+}\frac{aM(\alpha)}{1-\alpha}\exp\left(-\frac{\alpha t}{1-\alpha}\right) = \lim_{t\to 0^+} F(t) \tag{11}$$

and rearrange to find

$$a = \frac{(1-\alpha)F(0^+)}{M(\alpha)} = \frac{(1-\alpha)F(0)}{M(\alpha)} \tag{12}$$

as F is continuous.

To find the continuous part of this solution, we may differentiate the IVP, Equation (10), with respect to t:

$$\frac{M(\alpha)}{1-\alpha}u_c'(t) - \frac{\alpha M(\alpha)}{(1-\alpha)^2}\int_0^t u_c'(\tau)\exp\left(-\frac{\alpha(t-\tau)}{1-\alpha}\right) d\tau - \frac{a\alpha M(\alpha)}{(1-\alpha)^2}\exp\left(-\frac{\alpha t}{1-\alpha}\right) = F'(t). \tag{13}$$

From Equation (10) we also have

$$\int_0^t u_c'(\tau)\exp\left(-\frac{\alpha(t-\tau)}{1-\alpha}\right)d\tau = \frac{1-\alpha}{M(\alpha)}F(t) - a\exp\left(-\frac{\alpha t}{1-\alpha}\right). \tag{14}$$

Combining these two expressions gives an integer order differential equation for u_c,

$$u_c'(t) = \frac{\alpha}{M(\alpha)}F(t) + \frac{1-\alpha}{M(\alpha)}F'(t). \tag{15}$$

The integral form of this equation is

$$u_c(t) = u_c(0) + \frac{1-\alpha}{M(\alpha)}\left(F(t)-F(0)\right) + \frac{\alpha}{M(\alpha)}\int_0^t F(\tau)d\tau. \tag{16}$$

Hence the general form for the solution of the IVP is

$$u(t) = u(0) + \frac{1-\alpha}{M(\alpha)}\left(F(t)-F(0)\right) + \frac{\alpha}{M(\alpha)}\int_0^t F(\tau)d\tau + \frac{(1-\alpha)F(0)}{M(\alpha)}H(t). \tag{17}$$

□

From Equation (12) we see that the CF IVP only permits a continuous solution in the case where $F(0) = 0$. This requirement on the existence of continuous solutions has been noted in [29], where the result was obtained via Laplace transforms, and is the case considered in [30]. In all other cases the solution will involve both a continuous component and a step discontinuity at the origin.

This solution of the CF IVP can easily be alternatively verified by taking the CF operator of the solution to recover the original IVP,

$$\begin{aligned}
&\frac{M(\alpha)}{1-\alpha}\int_0^t u'(\tau)\exp\left(-\frac{\alpha(t-\tau)}{1-\alpha}\right)d\tau \\
&= \int_0^t F'(\tau)\exp\left(-\frac{\alpha(t-\tau)}{1-\alpha}\right)d\tau \\
&+ \frac{\alpha}{1-\alpha}\int_0^t F(\tau)\exp\left(-\frac{\alpha(t-\tau)}{1-\alpha}\right)d\tau + F(0)\exp\left(-\frac{\alpha t}{1-\alpha}\right) \\
&= F(t).
\end{aligned} \tag{18}$$

Here we have simply applied integration by parts.

This general solution may alternatively be written as

$$u(t) = \begin{cases} u(0) & t = 0, \\ u(0) + \frac{1-\alpha}{M(\alpha)}F(t) + \frac{\alpha}{M(\alpha)}\int_0^t F(\tau)d\tau & t > 0. \end{cases} \tag{19}$$

It should be noted that this solution differs from the solution given in other papers, such as [28,31], although it is in agreement with the solution given in [17] for $t > 0$.

2.1. Weakening Continuity Requirements for $F(t)$

In the above we assumed that $F(t)$ was a continuous function. This is a little restrictive, as it excludes most of the interesting cases where $F(t)$ depends on the function $u(t)$, as $u(t)$ is not a continuous function. To accommodate a discontinuity at the origin we may remove the requirement that F is continuous and assume that we can write

$$F(t) = F_c(t) + bH(t), \tag{20}$$

where F_c is a continuous function and $b \in \mathbb{R}$. By following the same methodology as above, we have

$$\frac{M(\alpha)}{1-\alpha}\int_0^t u_c'(\tau)\exp\left(-\frac{\alpha(t-\tau)}{1-\alpha}\right)d\tau + \frac{aM(\alpha)}{1-\alpha}\exp\left(-\frac{\alpha t}{1-\alpha}\right) = F_c(t) + b \tag{21}$$

for $t > 0$. By taking the limit from above as t tends to 0, we can find the unknown coefficient a.

$$a = \frac{(1-\alpha)(F_c(0)+b)}{M(\alpha)}. \tag{22}$$

The continuous part of the solution will again be reduced to the solution of an ODE.

$$u_c'(t) = \frac{\alpha(F_c(t)+b)}{M(\alpha)} + \frac{1-\alpha}{M(\alpha)}F_c'(t). \tag{23}$$

The general solution with a discontinuity at $t = 0$ for $F(t)$ is thus

$$u(t) = u(0) + \frac{1-\alpha}{M(\alpha)}\left(F_c(t) - F_c(0)\right) + \frac{\alpha}{M(\alpha)}\int_0^t (F_c(\tau)+b)d\tau + \frac{(1-\alpha)(F_c(0)+b)}{M(\alpha)}H(t). \tag{24}$$

This solution is completely equivalent to the general solution given above in Equation (17). From this we see that the weakening of the continuity requirement did not effect the solution and we can attempt to solve IVPs of the form

$${}^{\mathrm{CF}}_{0}\mathcal{D}^{\alpha}_{t}u(t) = F(u(t),t), \tag{25}$$

with $u(0) = u_0$. In the case of IVPs of this form the given solution may only exist in the case that a, found via Equation (22), is real valued. For example with $F(u(t),t) = \frac{M(\alpha)}{1-\alpha}u(t)$, the relation in Equation (22) does not hold with $u_o \neq 0$, and hence there is no solution of the ansatz form. Furthermore, for a non-linear equation, the resulting ODE for the continuous part of the solution, Equation (23), may not have solutions. Hence we must deal with each non-linear case individually.

2.2. Example Solutions for CF Initial Value Problems

We will construct some solutions of simple IVPs to illustrate the forms given above. In each case the validity of the solution can easily be seen by a direct substitution into the original equation.

2.2.1. Example CF IVP with $F(t) = 1$

Consider the CF IVP with

$${}^{\mathrm{CF}}_{0}\mathcal{D}^{\alpha}_{t}u(t) = 1, \tag{26}$$

and

$$u(0) = u_0. \tag{27}$$

The solution of this IVP can be found directly via Equation (17) and is

$$u(t) = u_0 + \frac{\alpha}{M(\alpha)}t + \frac{1-\alpha}{M(\alpha)}H(t). \tag{28}$$

Notice that the definition of $H(t)$ ensures that $u(0) = u_0$, but the solution has a step at $t = 0$. This is an illustrative example as it is simple to check against the definition of the CF operator.

2.2.2. Example CF IVP with $F(u(t), t) = -u_c(t)$

It is instructive to consider the differences induced by a discontinuity in F at $t = 0$. Here we present an example where F is taken to be the continuous part of the solution whilst in the next example we will show the case for F being the full solution.

Consider the CF IVP with

$$ {}^{\mathrm{CF}}_{0}\mathcal{D}^{\alpha}_{t} u(t) = -u_c(t), \tag{29} $$

and

$$ u(0) = u_0. \tag{30} $$

This solution can be found first by solving the ODE for u_c, Equation (15), which can be rearranged to obtain,

$$ u_c'(t) = -\frac{\alpha}{M(\alpha) + 1 - \alpha} u_c(t), \tag{31} $$

subject to the initial condition $u_c(0) = u_0$. The continuous part of the solution is thus

$$ u_c(t) = u_0 e^{-\frac{\alpha}{M(\alpha)+1-\alpha} t}. \tag{32} $$

The discontinuous part of the solution is readily found from Equation (12) and combining the two will give the above solution. Thus the CF IVP has a solution of,

$$ u(t) = u_0 e^{-\frac{\alpha}{M(\alpha)+1-\alpha} t} - \frac{(1-\alpha)u_0}{M(\alpha)} H(t). \tag{33} $$

We can see that as $t \to \infty$ this solution changes sign and asymptotes to $-\frac{(1-\alpha)u_0}{M(\alpha)}$.

2.2.3. Example CF IVP with $F(u(t), t) = -u(t)$

Using the weakened form of the continuity requirement we can consider the IVP of the form

$$ {}^{\mathrm{CF}}_{0}\mathcal{D}^{\alpha}_{t} u(t) = -u(t), \tag{34} $$

and

$$ u(0) = u_0. \tag{35} $$

As the right-hand side of this equation is dependent on the solution, we will first find the unknown coefficient from the ansatz via the relation given in Equation (22), with $b = a$ and $F_c(0) = u_c(0) = u_0$. This gives,

$$ a = -\frac{(1-\alpha)u_0}{M(\alpha) + 1 - \alpha}. \tag{36} $$

Following the same procedure as the previous example, we obtain the following ODE for the continuous part of the solution:

$$ u_c'(t) = -\frac{\alpha}{M(\alpha) + 1 - \alpha} (u_c(t) + a). \tag{37} $$

This can be solved with the initial condition $u_c(0) = u_0$ to give

$$ u_c(t) = \frac{M(\alpha)u_0}{M(\alpha) + 1 - \alpha} e^{-\frac{\alpha}{M(\alpha)+1-\alpha} t} + \frac{(1-\alpha)u_0}{M(\alpha) + 1 - \alpha}. \tag{38} $$

Again, combining the continuous and discontinuous parts of the solution will give the full solution,

$$u(t) = \frac{M(\alpha)u_0}{M(\alpha)+1-\alpha}e^{-\frac{\alpha}{M(\alpha)+1-\alpha}t} + \frac{(1-\alpha)u_0}{M(\alpha)+1-\alpha}(1-H(t)). \tag{39}$$

In contrast to the second example, this solution will remain positive, for $u_0 > 0$, and will asymptote to 0 as $t \to \infty$.

2.2.4. Example CF IVP with $F(u(t),t) = -\left(\frac{(1-\alpha)u_0^2}{M(\alpha)}\right)^2 - \frac{2(1-\alpha)u_0^2 u(t)}{M(\alpha)} - u(t)^2$

Again, using the weakened form of the continuity requirement, we can consider the IVP of the form

$${}_0^{\mathrm{CF}}\mathcal{D}_t^{\alpha} u(t) = -\left(\frac{(1-\alpha)u_0^2}{M(\alpha)}\right)^2 - \frac{2(1-\alpha)u_0^2 u(t)}{M(\alpha)} - u(t)^2, \tag{40}$$

and

$$u(0) = u_0, \ \text{ with } \ u_0 < 0. \tag{41}$$

In this case we see that

$$F_c(t) + bH(t) = -\left(\frac{(1-\alpha)u_0^2}{M(\alpha)}\right)^2 - \frac{2(1-\alpha)u_0^2 u(t)}{M(\alpha)} - u(t)^2. \tag{42}$$

From the ansatz we also have

$$u(t) = u_c(t) + aH(t), \tag{43}$$

hence,

$$F_c(t) = -\left(\frac{(1-\alpha)u_0^2}{M(\alpha)}\right)^2 - \frac{2(1-\alpha)u_0^2 u_c(t)}{M(\alpha)} - u_c(t)^2 - 2a(u_c(t) - u_0), \tag{44}$$

and

$$b = -\frac{2(1-\alpha)u_0^2 a}{M(\alpha)} - a^2 - 2au_0. \tag{45}$$

From Equation (22) we then have the relation

$$a = \frac{(1-\alpha)}{M(\alpha)}\left(-\left(\frac{(1-\alpha)u_0^2}{M(\alpha)}\right)^2 - \frac{2(1-\alpha)u_0^3}{M(\alpha)} - u_0^2 - \frac{2(1-\alpha)u_0^2 a}{M(\alpha)} - a^2 - 2au_0\right). \tag{46}$$

This has solutions

$$a = -\frac{(1-\alpha)u_0^2}{M(\alpha)}. \tag{47}$$

From Equation (23) we have an ODE for the continuous part of the solution,

$$u_c'(t) = -\frac{\alpha(u_c(t)^2)}{M(\alpha)} + \frac{2(1-\alpha)}{M(\alpha)}u_c(t)u_c'(t). \tag{48}$$

This ODE has a solution,

$$u_c(t) = -\frac{M(\alpha)}{(1-\alpha)W_0\left(-\frac{M(\alpha)\exp\left(\frac{\alpha}{1-\alpha}t - \frac{M(\alpha)}{(1-\alpha)u_0}\right)}{(1-\alpha)u_0}\right)}, \tag{49}$$

where W_0 is a Lambert W function [32]. The full solution to the IVP is thus

$$u(t) = -\frac{M(\alpha)}{(1-\alpha)W_0\left(-\frac{M(\alpha)\exp\left(\frac{\alpha}{1-\alpha}t - \frac{M(\alpha)}{(1-\alpha)u_0}\right)}{(1-\alpha)u_0}\right)} - \frac{(1-\alpha)u_0^2}{M(\alpha)}H(t). \tag{50}$$

2.3. Numerical Considerations for Equations Involving the CF Operator

Equation (1) purports a memory effect by way of a convolution through time. Discretising the CF operator in this form leads to the unnecessary computation of memory terms. As is shown above, the solution to the IVP (2) may be obtained through the solution of the auxiliary ordinary differential Equation (15) (or in integral form Equation (17)).

The numerics contained within the recent literature are largely restricted to low order numerical methods. From the formulation presented in this work, we suggest that any numerical method appropriate for ODEs may be used to accurately solve IVPs with CF operators, and as such many efficient, highly accurate methods are available to these equations. To the best of the authors' knowledge no preceding work has proposed a numerical method which recovers discontinuous solutions to CF equations. As shown above these equations do not exhibit nontrivial continuous solutions, and as such require numerical methods tailored to recover the discontinuous dynamics.

3. The Atangana–Baleanu Operator

In a similar manner to the CF operator we will consider another non-singular kernel operator, the Atangana–Baleanu, in the sense of Caputo, (ABC) operator [4].

Definition 3 (The Atangana–Baleanu, in the sense of Caputo, Operator)**.** *The ABC operator for* $0 \leq \alpha < 1$ *is defined as*

$${}^{\mathrm{ABC}}_{0}\mathcal{D}^{\alpha}_{t}u(t) = \frac{B(\alpha)}{1-\alpha}\int_0^t u'(\tau)E_\alpha\left(-\frac{\alpha(t-\tau)^\alpha}{1-\alpha}\right)d\tau, \tag{51}$$

where E_α *is the Mittag-Leffler function, defined by:*

$$E_\alpha(x) = \sum_{n=0}^{\infty}\frac{x^n}{\Gamma(\alpha n+1)}. \tag{52}$$

Here $B(\alpha)$ *is a normalisation constant, that must obey* $B(0) = B(1) = 1$.

It is sufficient to take $B(\alpha) = 1$. The ABC operator is again often seen to be a fractional derivative as we recover an integer order derivative in the case $\alpha \to 1^-$. The use of the ABC operator as a fractional derivative is less contentious than the CF operator as it is non-local in time. As we are considering discontinuous solutions it is again necessary to interpret the derivative in the definition of the ABC operator in a distributional sense.

Again we will consider simple IVPs arising from this operator.

Definition 4 (An ABC Initial Value Problem)**.** *An ABC IVP is given by both an ABC equation,*

$${}^{\mathrm{ABC}}_{0}\mathcal{D}^{\alpha}_{t}u(t) = F(t) \tag{53}$$

with $F(t)$ *a continuous function in* t*, and an initial condition* $u(0) = u_0$ *for some* $u_0 \in \mathbb{R}$.

To find a solution we consider the ansatz

$$u'(t) = u'_c(t) + a\delta(t-0^+) \tag{54}$$

with the integral form of the solution

$$u(t) = u(0) + u_c(t) - u_c(0) + aH(t). \tag{55}$$

We will further simplify this by taking $u_c(0) = u(0)$, to give

$$u(t) = u_c(t) + aH(t). \tag{56}$$

Theorem 2. *For an ABC IVP (Definition 4) assume that a solution in the form of the ansatz (Equation (56)) exists. Then such a solution is given by*

$$u(t) = u(0) + \frac{1-\alpha}{B(\alpha)}(F(t) - F(0)) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t F(\tau)(t-\tau)^{\alpha-1}d\tau + \frac{(1-\alpha)F(0)}{B(\alpha)}H(t). \tag{57}$$

Proof. Combining Equation (53) and Equation (54) gives

$$\frac{B(\alpha)}{1-\alpha}\int_0^t u_c'(\tau)E_\alpha\left(-\frac{\alpha(t-\tau)^\alpha}{1-\alpha}\right)d\tau + \frac{aB(\alpha)}{1-\alpha}E_\alpha\left(-\frac{\alpha t^\alpha}{1-\alpha}\right) = F(t) \tag{58}$$

for $t > 0$. Taking the limit as t approaches 0 from above, we obtain

$$\lim_{t\to 0^+}\frac{B(\alpha)}{1-\alpha}\int_0^t u_c'(\tau)E_\alpha\left(-\frac{\alpha(t-\tau)^\alpha}{1-\alpha}\right)d\tau + \lim_{t\to 0^+}\frac{aB(\alpha)}{1-\alpha}E_\alpha\left(-\frac{\alpha t^\alpha}{1-\alpha}\right) = \lim_{t\to 0^+}F(t). \tag{59}$$

Thus the coefficient a is given by

$$a = \frac{(1-\alpha)F(0^+)}{B(\alpha)} = \frac{(1-\alpha)F(0)}{B(\alpha)}, \tag{60}$$

provided $F(x)$ is continuous.

As $u_c(t)$ is a continuous function and the Laplace transform is bijective over the space of continuous functions, Laplace transform techniques can be applied. Therefore we take the Laplace transform of both sides of Equation (58) from t to s domain to obtain

$$\frac{B(\alpha)}{1-\alpha}\frac{s^{\alpha-1}(s\mathcal{L}\{u_c(t)\} - u_c(0))}{s^\alpha + \frac{\alpha}{1-\alpha}} + \frac{aB(\alpha)}{1-\alpha}\frac{s^{\alpha-1}}{s^\alpha + \frac{\alpha}{1-\alpha}} = \mathcal{L}\{F(t)\}, \tag{61}$$

having used the result

$$\mathcal{L}\{t^{\alpha k+\beta-1}E_{\alpha,\beta}^{(k)}(yt^\alpha)\} = \frac{k!s^{\alpha-\beta}}{(s^\alpha - y)^{k+1}}, \qquad \Re(s) > |y|^{1/\alpha}, \tag{62}$$

from [1]. By rearranging the equation above, we see that

$$s\mathcal{L}\{u_c(t)\} - u_c(0) = \frac{1-\alpha}{B(\alpha)}(s\mathcal{L}\{F(t)\} - F(0)) + \frac{\alpha}{B(\alpha)}s^{1-\alpha}\mathcal{L}\{F(t)\}. \tag{63}$$

To deal with the $s^{1-\alpha}\mathcal{L}\{F(t)\}$ on the RHS of Equation (63) we will utilise some results from fractional calculus. The Riemann–Liouville fractional derivative [1] of order $1-\alpha$ with $0 \le \alpha \le 1$ is defined by

$${}^{\mathrm{RL}}_{0}\mathcal{D}_t^{1-\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\frac{d}{dt}\int_0^t f(\tau)(t-\tau)^{\alpha-1}d\tau. \tag{64}$$

The Laplace transform of the Riemann–Liouville derivative is given by [33],

$$\mathcal{L}\{{}^{\mathrm{RL}}_{0}\mathcal{D}_t^{1-\alpha}f(t)\} = s^{1-\alpha}\mathcal{L}\{f(t)\} + \lim_{t\to 0^+}{}^{\mathrm{RL}}_{0}\mathcal{D}_t^{-\alpha}f(t), \tag{65}$$

where ${}^{\mathrm{RL}}_{0}\mathcal{D}_t^{-\alpha} f(t)$ is a Riemann–Liouville fractional integral of order α. Furthermore, provided the limits exist, we also have [33,34],

$$\lim_{t\to 0^+} {}^{\mathrm{RL}}_{0}\mathcal{D}_t^{-\alpha} f(t) = \lim_{t\to 0^+} \Gamma(1-\alpha) t^{\alpha} f(t), \tag{66}$$

and hence provided that $\lim_{t\to 0^+} F(t)$ exists we have

$$s^{1-\alpha}\mathcal{L}\{F(t)\} = \mathcal{L}\{{}^{\mathrm{RL}}_{0}\mathcal{D}_t^{1-\alpha} F(t)\}. \tag{67}$$

The inverse Laplace transform of Equation (63) then gives the following ordinary integro-differential equation for $u_c(t)$:

$$\frac{d}{dt}u_c(t) = \frac{1-\alpha}{B(\alpha)}\frac{d}{dt}F(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\frac{d}{dt}\int_0^t F(\tau)(t-\tau)^{\alpha-1}d\tau. \tag{68}$$

The integral form of the solution is obtained immediately from the equation above, the result is

$$u_c(t) = u_c(0) + \frac{1-\alpha}{B(\alpha)}(F(t)-F(0)) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t F(\tau)(t-\tau)^{\alpha-1}d\tau. \tag{69}$$

Hence the general form for the solution of the IVP is

$$u(t) = u(0) + \frac{1-\alpha}{B(\alpha)}(F(t)-F(0)) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t F(\tau)(t-\tau)^{\alpha-1}d\tau + \frac{(1-\alpha)F(0)}{B(\alpha)}H(t). \tag{70}$$

□

Again, we can alternatively verify this this solution by substituting the solution back to the ABC operator,

$$\begin{aligned} &\frac{B(\alpha)}{1-\alpha}\int_0^t u'(\tau)E_\alpha\left(-\frac{\alpha(t-\tau)^\alpha}{1-\alpha}\right)d\tau \\ &= F(0)E_\alpha\left(-\frac{\alpha t^\alpha}{1-\alpha}\right) + \int_0^t F'(\tau)E_\alpha\left(-\frac{\alpha(t-\tau)^\alpha}{1-\alpha}\right)d\tau \\ &+ \frac{\alpha}{1-\alpha}\int_0^t \left({}^{\mathrm{RL}}_{0}\mathcal{D}_t^{1-\alpha}F(t)\right)E_\alpha\left(-\frac{\alpha(t-\tau)^\alpha}{1-\alpha}\right)d\tau. \end{aligned} \tag{71}$$

Note that both sides of Equation (71) are continuous; thus, the corresponding equation in Laplace space reads

$$\begin{aligned} &\mathcal{L}\left\{\frac{B(\alpha)}{1-\alpha}\int_0^t u'(\tau)E_\alpha\left(-\frac{\alpha(t-\tau)^\alpha}{1-\alpha}\right)d\tau\right\} \\ &= \frac{s^{\alpha-1}F(0)}{s^\alpha + \frac{\alpha}{1-\alpha}} + \frac{s^{\alpha-1}(s\mathcal{L}\{F(t)\}-F(0))}{s^\alpha+\frac{\alpha}{1-\alpha}} + \frac{\alpha}{1-\alpha}\frac{s^{\alpha-1}}{s^\alpha+\frac{\alpha}{1-\alpha}}\left(s^{1-\alpha}\mathcal{L}\{F(t)\}\right) \\ &= \mathcal{L}\{F(t)\}. \end{aligned} \tag{72}$$

We see that inverse Laplace transform of the equation above recovers the IVP:

$${}^{\mathrm{ABC}}_{0}\mathcal{D}_t^{\alpha} u(t) = F(t). \tag{73}$$

We can note that the continuity requirement on $F(t)$ can be eased in the same manner as the CF operator by considering a discontinuous $F(t)$ such that $F(t) = F_c(t) + bH(t)$, where $F_c(t)$ is a continuous function. As such, we can attempt to consider IVPs of the form

$$ {}^{\mathrm{ABC}}_{0}\mathcal{D}^{\alpha}_{t} u(t) = F(u(t), t), \tag{74} $$

with $u(0) = u_0$. Again a solution of the ansatz form will only exist if a real valued constant a can be found such that the following relation holds,

$$ a = \frac{(1-\alpha)(F_c(u_0, 0) + b)}{B(\alpha)}, \tag{75} $$

where F_c, a continuous function, and b, a real valued constant, are found from $F(u(t), t) = F_c(u(t), t) + bH(t)$.

3.1. Example ABC Initial Value Problems

3.1.1. Example ABC IVP with $F(t) = 1$

Consider the ABC IVP with

$$ {}^{\mathrm{ABC}}_{0}\mathcal{D}^{\alpha}_{t} u(t) = 1, \tag{76} $$

and

$$ u(0) = u_0. \tag{77} $$

The solution of this IVP follows immediately from the Equation (70) and is

$$ u(t) = u_0 + \frac{1}{B(\alpha)\Gamma(\alpha)} t^{\alpha} + \frac{1-\alpha}{B(\alpha)} H(t). \tag{78} $$

3.1.2. Example ABC IVP with $F(t) = u_c(t)$

Consider the ABC IVP with

$$ {}^{\mathrm{ABC}}_{0}\mathcal{D}^{\alpha}_{t} u(t) = u_c(t), \tag{79} $$

and

$$ u(0) = u_0. \tag{80} $$

The solution can be found by independently calculating the continuous and discontinuous parts of the solution. The continuous part of the solution is found by first considering the equation for u_c in Laplace space, Equation (63), which can be rearranged to obtain

$$ \mathcal{L}\{u_c(t)\} = \frac{s^{\alpha-1}}{s^{\alpha} - \frac{\alpha}{B(\alpha)-1+\alpha}} u_c(0). \tag{81} $$

The inverse Laplace transform from s to t domain then gives

$$ u_c(t) = u_0 E_{\alpha}\left(\frac{\alpha}{B(\alpha) - 1 + \alpha} t^{\alpha} \right) \tag{82} $$

subject to the initial condition $u_c(0) = u_0$. The discontinuous part of the solution is found by calculating the coefficient a from Equation (60), giving

$$ a = \frac{(1-\alpha)u_0}{B(\alpha)}. \tag{83} $$

The solution of the IVP is then as follows:

$$ u(t) = u_0 E_{\alpha}\left(\frac{\alpha}{B(\alpha) - 1 + \alpha} t^{\alpha} \right) + \frac{(1-\alpha)u_0}{B(\alpha)} H(t). \tag{84} $$

3.1.3. Example ABC IVP with $F(u(t), t) = u(t)$

Using the weakened form of the continuity requirement we can consider the IVP of the form

$$ {}^{\mathrm{ABC}}_{\ 0}\mathcal{D}_t^{\alpha} u(t) = u(t), \tag{85} $$

and

$$ u(0) = u_0. \tag{86} $$

Again the solution is found by considering the continuous and discontinuous parts separately. From Equation (60), we obtain

$$ a = \frac{u_0(1-\alpha)}{B(\alpha)+\alpha-1}. \tag{87} $$

Subject to the initial condition $u_c(0) = u_0$. The continuous part of the solution can be found via its Laplace transform. Substituting the value for a into Equation (63) gives

$$ \mathcal{L}(u_c(t)) = \frac{u_0 B(\alpha)}{B(\alpha)+\alpha-1}\frac{s^{\alpha-1}}{s^{\alpha}-\frac{\alpha}{B(\alpha)+\alpha-1}} - \frac{u_0(1-\alpha)}{B(\alpha)+\alpha-1}s^{-1}. \tag{88} $$

Inverting the Laplace transform then gives the continuous part of the solution,

$$ u_c(t) = \frac{u_0 B(\alpha)}{B(\alpha)+\alpha-1}E_{\alpha}\left(\frac{\alpha}{B(\alpha)-1+\alpha}t^{\alpha}\right) - \frac{u_0(1-\alpha)}{B(\alpha)+\alpha-1}. \tag{89} $$

Combining the continuous and discontinuous parts will then give the full solution,

$$ u(t) = \frac{u_0 B(\alpha)}{B(\alpha)+\alpha-1}E_{\alpha}\left(\frac{\alpha}{B(\alpha)-1+\alpha}t^{\alpha}\right) + \frac{u_0(1-\alpha)}{B(\alpha)+\alpha-1}(H(t)-1). \tag{90} $$

4. Singular Kernel Operator Example: Caputo Derivative

Here we attempt to apply the same technique to a fractional derivative with a singular kernel, the Caputo derivative.

Definition 5 (Caputo Derivative)**.** *The Caputo derivative of order α with $0 < \alpha < 1$, is defined as [2],*

$$ {}^{\mathrm{C}}_{0}\mathcal{D}_t^{\alpha} u(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t u'(\tau)(t-\tau)^{-\alpha}d\tau. \tag{91} $$

In order to explore the possibility of discontinuous solutions we will need to allow for the derivative in the Caputo definition to be interpreted in a distributional sense; see [25] for a more detailed exposition of the use of distributional derivatives in Caputo derivatives. We again consider a simple IVP.

Definition 6 (Caputo IVP)**.** *A Caputo IVP is given by a Caputo equation,*

$$ {}^{\mathrm{C}}_{0}\mathcal{D}_t^{\alpha} u(t) = F(t), \tag{92} $$

with an initial condition, $u(0) = u_0$. We will assume that $F(t)$ a continuous function in t, and $u_0 \in \mathbb{R}$.

In a similar manner as above we consider an ansatz and look for solutions of the form

$$ u(t) = u_c(t) + aH(t), \tag{93} $$

with $u_c(t)$ a continuous function, a a constant, and $H(t)$ as defined in the previous sections.

Theorem 3. *For an Caputo IVP (Definition 6) assume that a solution in the form of the ansatz (Equation (93)) exists. Then such a solution does not possess a step discontinuity at $t = 0$.*

Proof. The proof follows from assuming a solution exists in the form of the ansatz and then showing that the only permitted value of the parameter a is zero. From the ansatz we have

$$u'(t) = u_c'(t) + a\delta(t - 0^+) \tag{94}$$

with $u_c'(t)$ being the derivative of a continuous function, δ a Dirac delta, and a some unknown constant. To obtain the value of the unknown constant we will substitute this into the IVP to give

$$\frac{1}{\Gamma(1-\alpha)} \int_0^t u_c'(\tau)(t-\tau)^{-\alpha} d\tau + \frac{a}{\Gamma(1-\alpha)} t^{-\alpha} = F(t). \tag{95}$$

Next we take the limit as $t \to 0^+$,

$$\lim_{t \to 0^+} \frac{1}{\Gamma(1-\alpha)} \int_0^t u_c'(\tau)(t-\tau)^{-\alpha} d\tau + \lim_{t \to 0^+} \frac{a}{\Gamma(1-\alpha)} t^{-\alpha} = F(0^+). \tag{96}$$

The left-hand side of this expression only exists if $a = 0$, whilst the right-hand side is well defined. As such solutions with a step discontinuity at the origin do not exist for the Caputo derivative. □

5. Conclusions

We have shown that the solutions of initial value problems using both the CF and ABC operators feature, in almost all cases, a discontinuity at the origin. The occurrence of the discontinuity is problematic for the application of the CF and ABC operators in modelling. Very few physical processes are well described by discontinuous functions, and fewer still with the discontinuity at the origin.

This discontinuity also raises issues with the use of these operators as fractional derivatives. Whilst both operators are generalisations of derivatives, in the sense that as $\alpha \to 1^-$ we recover an integer order derivative, the lack of smooth solutions is problematic. Many proponents of the use of these operators claim that the non-singular kernel is desirable, but as we have shown here in order for the solution of the IVP to exist the derivative of the solution must be singular, and thus the integrand is still singular.

In the Appendix A we have considered a more generalized ansatz for the solution of CF operator equations. This generalization does not give solutions to IVPs, but we can find solutions that diverge at the origin.

Traditional numerical methods are based on approximating the solution, and derivatives thereof, by their respective discrete counterparts. In the case where the solution exhibits a discontinuity, these approaches attempt to capture an infinite gradient in the same manner as the gradient of a smooth function. As such, resulting schemes are inadequate for capturing the dynamics of solutions admitted by the present IVPs. By decomposing the solution into discontinuous and continuous parts, traditional methods can be modified to approximate the continuous dynamics only.

In the case of the CF operator, the continuous part of the solution follows from a relatively simple integer order differential equation and the vast literature of methods are available to find efficient high order solutions. The numerical solution of the ABC operator equations is more complicated, as the operator does involve a history dependence.

In this work we have concentrated on the often explored case of $0 < \alpha < 1$, although extensions to the case $\alpha > 1$ are possible. Generalisations of the CF and ABC operators for larger values of α exist. To investigate the occurrence of discontinuities in such systems alternate forms of our ansatz would need to be taken.

We have also shown that this type of discontinuity at the origin can not occur in Caputo derivatives. The ansatz approach that we use is applicable to cases where we have derivatives appearing in integrands, such as the CF, ABC, and Caputo operators. This approach would need further modifications to be applicable to Riemann–Liouville type operators where the derivative occurs outside of the integral.

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Appendix A. Higher Order Singularities in the the Solution

We could consider a more generalised ansatz, with a higher order singularities,

$$u'(t) = u_c'(t) + \sum_{i=0}^{\infty} a_i \frac{d^i}{dt^i}\delta(t-0^+), \tag{A1}$$

The solution in this case will then be of the form

$$u(t) = u_c(t) + a_0 H(t) + \sum_{i=0}^{\infty} a_{i+1} \frac{d^i}{dt^i}\delta(t-0^+), \tag{A2}$$

where $\frac{d^i}{dt^i}\delta(t)$ denotes i-th derivative of the Dirac delta $\delta(t)$ with corresponding unknown constant a_i. Inserting this generalised ansatz into Equation (2), one finds

$$\frac{M(\alpha)}{1-\alpha}\int_0^t u_c'(\tau)\exp\left(-\frac{\alpha(t-\tau)}{1-\alpha}\right)d\tau + \frac{M(\alpha)}{1-\alpha}\sum_{i=0}^{\infty} a_i\left(\frac{\alpha}{1-\alpha}\right)^i \exp\left(-\frac{\alpha t}{1-\alpha}\right) = F_c(t)+b \tag{A3}$$

for $t>0$. In the limit as $t \to 0$ from above, we have

$$\sum_{i=0}^{\infty} a_i\left(\frac{\alpha}{1-\alpha}\right)^i = \frac{(1-\alpha)(F_c(0^+)+b)}{M(\alpha)} = \frac{(1-\alpha)(F_c(0)+b)}{M(\alpha)}, \tag{A4}$$

since F_c is continous. Again, we make use of Leibniz's rule and differentiate Equation (A3) with respect to t; this gives

$$\tfrac{M(\alpha)}{1-\alpha}u_c'(t) - \tfrac{\alpha M(\alpha)}{(1-\alpha)^2}\int_0^t u_c'(\tau)\exp\left(-\tfrac{\alpha(t-\tau)}{1-\alpha}\right)d\tau - \tfrac{\alpha M(\alpha)}{(1-\alpha)^2}\sum_{i=0}^{\infty} a_i\left(\tfrac{\alpha}{1-\alpha}\right)^i \exp\left(-\tfrac{\alpha t}{1-\alpha}\right) = F_c'(t). \tag{A5}$$

From Equation (A3), one finds

$$\int_0^t u_c'(\tau)\exp\left(-\frac{\alpha(t-\tau)}{1-\alpha}\right)d\tau = \frac{(1-\alpha)(F_c(t)+b)}{M(\alpha)} - \sum_{i=0}^{\infty} a_i\left(\frac{\alpha}{1-\alpha}\right)^i \exp\left(-\frac{\alpha t}{1-\alpha}\right). \tag{A6}$$

Replacing the integral in Equation (A5) with RHS of Equation (A6) and rearranging yields

$$u_c'(t) = \frac{\alpha(F_c(t)+b)}{M(\alpha)} + \frac{1-\alpha}{M(\alpha)}F_c'(t). \tag{A7}$$

The integral solution of Equation (A7) is given as

$$u_c(t) = u_c(0) + \frac{1-\alpha}{M(\alpha)}(F_c(t)-F_c(0)) + \frac{\alpha}{M(\alpha)}\int_0^t (F_c(\tau)+b)d\tau. \tag{A8}$$

Hence the general form of the unbounded solution takes the following form

$$u(t) = u_c(0) + \frac{1-\alpha}{M(\alpha)}\left(F_c(t) - F_c(0)\right) + \frac{\alpha}{M(\alpha)}\int_0^t (F_c(\tau)+b)d\tau + a_0 H(t) + \sum_{i=0}^{\infty} a_{i+1}\frac{d^i}{dt^i}\delta(t-0^+). \quad \text{(A9)}$$

Therefore, we see that the unbounded solution is non-unique with the only condition for the coefficients $a_i's$ given by Equation (A4). Note that generally the initial value $u(0) = u_0$ can not be imposed for solution of this form, as it involves a delta function and its distributional derivatives at the origin, unless we force the unknown coefficients $a_i = 0$ for $i \in \mathbb{Z}^+$. If we set $\{a_1, a_2, ...\} = 0$, from Equation (A4), it is then required that

$$a_0 = \frac{(1-\alpha)(F_c(0)+b)}{M(\alpha)}. \quad \text{(A10)}$$

This allows us to impose the initial condition $u(0) = u_0$, and the solution for the IVP in this case reads

$$u(t) = u(0) + \frac{1-\alpha}{M(\alpha)}\left(F_c(t) - F_c(0)\right) + \frac{\alpha}{M(\alpha)}\int_0^t (F_c(\tau)+b)d\tau + \frac{(1-\alpha)(F_c(0)+b)}{M(\alpha)}H(t). \quad \text{(A11)}$$

From this we see that higher order discontinuities at the origin can still produce solutions to the equation, but do not provide solutions for an IVP.

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