

Article

A New Representation of the Generalized Krätzel Function

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Abstract: The confluence of distributions (generalized functions) with integral transforms has become a remarkably powerful tool to address important unsolved problems. The purpose of the present study is to investigate a distributional representation of the generalized Krätzel function. Hence, a new definition of these functions is formulated over a particular set of test functions. This is validated using the classical Fourier transform. The results lead to a novel extension of Krätzel functions by introducing distributions in terms of the delta function. A new version of the generalized Krätzel integral transform emerges as a natural consequence of this research. The relationship between the Krätzel function and the H -function is also explored to study new identities.

Keywords: generalized Krätzel function; H -function; Fourier transformation; slowly increasing test functions; generalized functions (distributions); delta function

1. Introduction

As a versatile integral in physics and astronomy [1], the Krätzel function is important in various branches of science. The basic Krätzel function [2], denoted as $Z_\rho^\nu(x)$, is defined as:

$$Z_\rho^\nu(x) = \int_0^\infty t^{\nu-1} \exp\left(-t^\rho - \frac{x}{t}\right) dt; (x > 0, \rho \in \mathbb{R}, \nu \in \mathbb{C}; \rho \leq 0, \Re(\nu) < 0). \quad (1)$$

where $\mathbb{R}$; $\mathbb{C}$, and $\Re$ denote the set of real numbers, complex numbers, and the real part of complex numbers, respectively. For $\rho = 1$; $x = \frac{t^2}{4}$, (1) can be written in terms of the McDonald function, denoted as $K_\nu(t)$, in accordance with [2] (Section 7.2.2),

$$Z_1^\nu\left(\frac{t^2}{4}\right) = \left(\frac{t}{2}\right)^\nu K_\nu(t). \quad (2)$$

The Krätzel integral transform, denoted by K_ρ^ν and defined by [3]

$$K_\rho^\nu(f(x)) = \int_0^\infty Z_\rho^\nu(xt) f(t) dt; (x > 0; \rho \geq 1), \quad (3)$$

contains the integral operators of Meijer and Laplace when $\rho = 1$ and $\rho = 1$; $\nu = \pm 1/2$, respectively. The Krätzel integral transform has been discussed in the literature by considering different aspects. For example, the Krätzel integral on a certain space of distributions was discussed in [4] by Rao and Debnath. Kilbas and Shlapakov [5] discussed these types of functions using fractional operators. Their results were further generalized by Trujillo, Kilbas, Rodríguez Rivero, Glaeske, and Bonilla [6,7]. The mentioned references [4–7] considered the Krätzel function (1) involving real positive variables x

and ρ . However, Kilbas, Saxena, and Trujillo [8] extended the results to complex variables using the following relationships between the Krätzel and H -functions [8,9]:

$$Z_{\rho}^{\nu}(s) = \frac{1}{\rho} H_{0,2}^{2,0} \left[s \left| \begin{matrix} (0,1) \left(\frac{\nu}{\rho}, \frac{1}{\rho} \right) \end{matrix} \right. \right]; (\rho > 0, \nu \in \mathbb{C}; s \in \mathbb{C}, s \neq 0) \quad (4)$$

when $\rho > 0$, while for $\rho < 0$,

$$Z_{\rho}^{\nu}(s) = \frac{1}{|\rho|} H_{1,1}^{1,1} \left[s \left| \begin{matrix} \left(1 - \frac{\nu}{\rho}, -\frac{1}{\rho} \right) \end{matrix} \right. \right]; (\rho < 0, \Re(\nu) < 0; s \in \mathbb{C}, s \neq 0). \quad (5)$$

The notations in Equations (1)–(5) for the Krätzel function, McDonald function, H -function, and Krätzel integral transform are standard. For example, they are used by Kilbas et al. [8]. The notation for the generalized Krätzel function $Z_{\sigma,\rho}^{a,b}(s)$ is adopted in accordance with the basic Krätzel function defined in (1). In this investigation, I will consider the generalized Krätzel function, defined as [10] (p. 2, Equation (2)):

$$Z_{\sigma,\rho}^{a,b}(s) = \int_0^{\infty} t^{s-1} \exp\left(-at^{\sigma} - \frac{b}{t^{\rho}}\right) dt; (a, b > 0; \sigma \in \mathbb{R}; \rho \in \mathbb{R}^+; s \in \mathbb{C}). \quad (6)$$

Equation (1) is a special case of (6), where $a = \rho = 1$ and replacing σ with ρ ; b with x . The reader should keep in mind the difference between the three terms throughout this manuscript: the basic Krätzel function (1); the Krätzel integral transform (3); and the generalized Krätzel function (6). Recurrence relations, the Mellin transform, fractional derivatives, and the integral of the function $Z_{\sigma,\rho}^{a,b}(s)$ are discussed in [11,12]. It is also shown that the function $Z_{\sigma,\rho}^{a,b}(s)$ and its special cases satisfy certain difference and differential equations [11,12], for example:

$$\nu Z_{\rho}^{\nu}(x) = \rho Z_{\rho}^{\nu+\rho}(x) - x Z_{\rho}^{\nu-1}(x). \quad (7)$$

Furthermore, the use of fractional differential equations has become vital for modeling and solving numerous physical problems that are otherwise unsolvable; see, for example [13–15]. It is worth mentioning that $Z_{\sigma,\rho}^{a,b}(s)$ is the solution of differential equations of fractional order [12]. Similarly, $Z_1^{\nu}(x)$ is a solution of the following differential equations [11,12]

$$xy'' + (\nu - 1)y' - y = 0; \quad (8)$$

$$x^2 y^{(IV)} + (2\nu - 4)xy''' + (\nu - 1)(\nu - 2)y'' + y = 0. \quad (9)$$

For further discussion of these functions and their applications, the interested reader is referred to [16,17] and references therein. Furthermore, recent investigations [18–27] are fundamental to achieving the goals of this paper.

Mathematical aspects of the generalized Krätzel function and integral operators are discussed in recent works, e.g., [1,10,16,17]. A new version of the generalized Krätzel–integral operators is discussed in [28] using the elements of distribution theory. Such functions are also considered in [29] using the Fréchet space of Boehmians. To the best of the author's knowledge, the Krätzel function has not been studied in the literature as a distribution in terms of the delta function. Motivated by the above discussion, the present study is focused on investigating a new representation of the generalized Krätzel function. By doing so, the domain of the generalized Krätzel function (7) can be extended from complex numbers to the space of complex test functions. Obviously, similar results will be valid for the H -function, considering the relationships (5) and (6).

The plan for this paper is as follows: essential preliminaries related to test function spaces are given in Section 1.1. The organization of the remaining parts is given as follows: Section 2.1 includes a new series representation of the generalized Krätzel function. Section 2.2 consists of the criteria

regarding the existence, as well as uses, of the novel series. Validation of these outcomes is given in Section 2.3. For completeness, distributional properties are stated in Section 2.4 in the form of a theorem. Some examples are discussed in Section 2.5. Section 3 highlights and concludes the present as well as future work.

1.1. Distributions and Test Functions

Corresponding to each space of test functions, there is a dual space known as the space of distributions (or generalized functions). Consideration of such functions is vital because of their important property of representing singular functions. In this way, different calculus operations can be applied for such functions, as in the case of classical functions. The notations used in this subsection are standard and adopted from [30,31]. However, the notation φ is used for test functions throughout this manuscript. For the requirements of this investigation, the delta function, which is a commonly used singular function, needs to be mentioned. For any test function $\varphi(\omega) \in \mathcal{D}$, the delta function is defined by

$$\langle \delta(t - \omega), \varphi(t) \rangle = \varphi(\omega) \quad (\forall \varphi \in \mathcal{D}, \omega \in \mathbb{R}), \quad (10)$$

and

$$\delta(-t) = \delta(t); \delta(\omega t) = \frac{\delta(t)}{|\omega|}, \text{ where } \omega \neq 0. \quad (11)$$

An ample discussion and explanation of distributions (or generalized functions) was presented in five different volumes by Gelfand and Shilov [30]. Functions with compact support and that are infinitely differentiable, as well as quickly decaying, are commonly used as test functions. The spaces containing such functions are denoted by $\mathcal{D}$ and $\mathcal{S}$, respectively. Obviously, the corresponding duals are the spaces $\mathcal{D}'$ and $\mathcal{S}'$. A noteworthy fact about such spaces is that $\mathcal{D}$ and $\mathcal{D}'$ do not hold the closeness property with respect to the Fourier transform, but $\mathcal{S}$ and $\mathcal{S}'$ do. In this way, it is remarkable that the elements of $\mathcal{D}'$ have Fourier transforms that form distributions for the entire function space $\mathcal{Z}$ whose Fourier transforms belong to $\mathcal{D}$ [31]. Further to this explanation, it is notable that as the entire function is nonzero for a particular range $\omega_1 < s < \omega_2$, but zero otherwise, the following inclusion of the abovementioned spaces holds:

$$\mathcal{Z} \cap \mathcal{D} \equiv 0; \mathcal{Z} \subset \mathcal{S} \subset \mathcal{S}' \subset \mathcal{Z}'; \mathcal{D} \subset \mathcal{S} \subset \mathcal{S}' \subset \mathcal{D}' \quad (12)$$

More specifically, space $\mathcal{Z}$ comprises the entire and analytic functions sustaining the subsequent criteria

$$|s^q \varphi(s)| \leq C_q e^{\eta|q|}; \quad (q \in \mathbb{N} \setminus \{0\}). \quad (13)$$

Here and in the following, the numbers η and C_q are dependent on φ , and $\mathbb{N}$ denotes the set of natural numbers. The following identities ([30], Volume 1, p. 169, Equation (8)), ([31], (p. 159), Equation (4)), see also ([32], p. 201, Equation (9)) will be used in the proof of our main result, where $\mathcal{F}$ denotes the Fourier transform.

$$\mathcal{F}[e^{\alpha t}; \theta] = 2\pi\delta(\theta - i\alpha); \quad (14)$$

$$g(s + b) = \sum_{j=0}^{\infty} g^{(j)}(s) \frac{b^j}{j!}, \quad \forall g \in \mathcal{Z}'; \quad (15)$$

$$\delta(s + b) = \sum_{j=0}^{\infty} \delta^{(j)}(s) \frac{b^j}{j!}, \quad \text{where } \langle \delta^{(j)}(s), \varphi(s) \rangle = (-1)^j \varphi^{(j)}(0); \quad (16)$$

$$\delta(\omega_1 - s)\delta(s - \omega_2) = \delta(\omega_1 - \omega_2). \quad (17)$$

Some examples include $\sin(t)$, $\cos(t)$, $\sinh t$, and $\cosh t$, whose Fourier transforms are delta (singular) functions. Relevant detailed discussions about such spaces can be found in [30–33].

Throughout this paper, except if mentioned specifically, the conditions for the involved parameters are taken as stated in Section 1.

2. Results

2.1. New Representation of Generalized Krätzel Function

In this section, the results are computed as a series of complex delta functions, and discussion about its rigorous use as a generalized function over a space of test functions is provided in the next section.

Theorem 1. *The generalized Krätzel function has the following representation in terms of complex delta functions.*

$$Z_{\sigma,\rho}^{a;b}(s) = 2\pi \sum_{n,r=0}^{\infty} \frac{(-a)^n (-b)^r}{n!r!} \delta(\theta - i(\nu + \sigma n - \rho r)). \quad (18)$$

Proof. A replacement of $t = e^{\omega}$ and $s = \nu + i\theta$ in the integral representation of the generalized Krätzel function as given in (6) yields the following:

$$Z_{\sigma,\rho}^{a;b}(s) = \int_{-\infty}^{\infty} e^{\omega(\nu+i\theta)} \exp(-ae^{\sigma\omega}) \exp(-be^{-\rho\omega}) d\omega. \quad (19)$$

Then, the involved exponential function can be represented as

$$\exp(-ae^{\sigma\omega}) \exp(-be^{-\rho\omega}) = \sum_{n=0}^{\infty} \frac{(-ae^{\sigma\omega})^n}{n!} \sum_{r=0}^{\infty} \frac{(-be^{-\rho\omega})^r}{r!}. \quad (20)$$

Next, combining expressions (19) and (20) leads to the following:

$$Z_{\sigma,\rho}^{a;b}(s) = \int_{-\infty}^{\infty} e^{ix\theta} \sum_{n,r=0}^{\infty} \frac{(-a)^n (-b)^r}{n!r!} e^{(\nu+\sigma n-\rho r)\omega} d\omega, \quad (21)$$

which gives

$$Z_{\sigma,\rho}^{a;b}(s) = \sum_{n,r=0}^{\infty} \frac{(-a)^n (-b)^r}{n!r!} \int_{-\infty}^{\infty} e^{i\omega\theta} e^{(\nu+\sigma n-\rho r)\omega} d\omega. \quad (22)$$

The actions of summation and integration are exchangeable because the involved integral is uniformly convergent. An application of identity (14) produces the following:

$$\int_{-\infty}^{\infty} e^{i\theta x} e^{(\nu+\sigma n-\rho r)\omega} d\omega = \mathcal{F} [e^{(\nu+\sigma n-\rho r)\omega}; \theta] = 2\pi \delta(\theta - i(\nu + \sigma n - \rho r)). \quad (23)$$

A combination of Equations (22)–(23) yields the required result (18). $\square$

Corollary 1. *The generalized Krätzel function has the following series form.*

$$Z_{\sigma,\rho}^{a;b}(s) = 2\pi \sum_{n,r,p=0}^{\infty} \frac{(-a)^n (-b)^r (-i(\nu + \sigma n - \rho r))^p}{n!r!p!} \delta^{(p)}(\theta). \quad (24)$$

Proof. Equation (24) can be obtained by considering the following combination of Equation (16) and Equation (23):

$$\delta(\theta - i(\nu + \sigma n - \rho r)) = \sum_{p=0}^{\infty} \frac{(-i(\nu + \sigma n - \rho r))^p}{p!} \delta^{(p)}(\theta). \quad (25)$$

Next, making use of this relation in (18) leads to the required form. $\square$

Corollary 2. The generalized Krätzel function has the following series form.

$$Z_{\sigma, \rho}^{a, b}(s) = 2\pi \sum_{n, r=0}^{\infty} \frac{(-a)^n (-b)^r}{n! r!} \delta(s + \sigma n - \rho r). \quad (26)$$

Proof. Equation (23) can be rewritten as follows:

$$\begin{aligned} \int_{-\infty}^{\infty} e^{i\theta x} e^{(\nu + \sigma n - \rho r)\omega} d\omega &= \mathcal{F} \left[e^{(\nu + \sigma n - \rho r)\omega}; \theta \right] = 2\pi \delta(\theta - i(\nu + \sigma n - \rho r)) \\ &= 2\pi \delta \left[\frac{1}{i} (i\theta + (\nu + \sigma n - \rho r)) \right] \\ &= 2\pi i |\delta(\nu + i\theta + \sigma n - \rho r)| = 2\pi \delta(s + \sigma n - \rho r). \end{aligned} \quad (27)$$

Next, making use of this relation in (18) leads to the required form. $\square$

Corollary 3. The generalized Krätzel function has the following series form.

$$Z_{\sigma, \rho}^{a, b}(s) = 2\pi \sum_{n, r, p=0}^{\infty} \frac{(-a)^n (-b)^r (\sigma n - \rho r)^p}{n! r! p!} \delta^{(p)}(s). \quad (28)$$

Proof. A suitable combination of Equations (16) and (26) gives

$$\delta(s + \sigma n - \rho r) = \sum_{p=0}^{\infty} \frac{(\sigma n - \rho r)^p}{p!} \delta^{(p)}(s), \quad (29)$$

which is a key to the required form. $\square$

Remark 1. The following results which represent the basic Krätzel function $Z_{\rho}^v(x)$ (1) are deduced from the above corollaries, by letting $a = \rho = 1$ and then replacing σ and b with ρ and x , respectively.

$$Z_{\rho}^s(x) = 2\pi \sum_{n, r=0}^{\infty} \frac{(-1)^n (-x)^r}{n! r!} \delta(\theta - i(\nu + \rho n - r)); \quad (30)$$

$$Z_{\rho}^s(x) = 2\pi \sum_{n, r, p=0}^{\infty} \frac{(-1)^n (-x)^r (-i(\nu + \rho n - r))^p}{n! r! p!} \delta^{(p)}(\theta); \quad (31)$$

$$Z_{\rho}^s(x) = 2\pi \sum_{n, r=0}^{\infty} \frac{(-1)^n (-x)^r}{n! r!} \delta(s + \rho n - r); \quad (32)$$

$$Z_{\rho}^s(x) = 2\pi \sum_{n, r, p=0}^{\infty} \frac{(-1)^n (-x)^r (\rho n - r)^p}{n! r! p!} \delta^{(p)}(s). \quad (33)$$

Now, letting $x = 0$ in (30–33), yields the following

$$Z_{\rho}^s(0) = 2\pi \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \delta(\theta - i(\nu + \rho n)) = 2\pi \sum_{n,p=0}^{\infty} \frac{(-1)^n (-i(\nu + \rho n))^p}{n!r!} \delta^{(p)}(\theta); \quad (34)$$

$$Z_{\rho}^s(0) = 2\pi \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \delta(s + \rho n); \quad (35)$$

$$Z_{\rho}^s(0) = 2\pi \sum_{n,p=0}^{\infty} \frac{(-1)^n (\rho n)^p}{n!r!} \delta^{(p)}(s). \quad (36)$$

It is notable that the above series representations are given in the form of delta functions. Such functions make sense only if defined as distributions (generalized functions) over a space of test functions, as discussed in Section 2.1. Consequently, it is essential to choose a suitable function for which this representation holds true. As an illustration, one can let $b = 0$ in identity (26) and multiply it by $\frac{1}{Z_{\sigma,\rho}^{a,0}(s)}$ to obtain the following:

$$1 = 2\pi \sum_{n=0}^{\infty} \frac{(-a)^n}{n!Z_{\sigma,\rho}^{a,0}(s)} \delta(s + \sigma n). \quad (37)$$

Therefore, singular points of delta function at $s = -\sigma n$ are canceled with the zeros of $Z_{\sigma,\rho}^{a,0}(s)$ in this expression, i.e., $\lim_{s \rightarrow -n} \frac{\delta(s + \sigma n)}{Z_{\sigma,\rho}^{a,0}(-n)} = \lim_{s \rightarrow -n} \frac{\frac{1}{s + \sigma n}}{\frac{1}{s + \sigma n}} = \lim_{s \rightarrow -n} \frac{s + \sigma n}{s + \sigma n} = 1$. Hence, making use of

$$\delta(t) = \begin{cases} \infty & (t = 0) \\ 0 & (t \neq 0), \end{cases} \quad (38)$$

in statement (37), the following can be obtained:

$$1 = \begin{cases} 2\pi \exp(-a); & s = -\sigma n \\ 0; & s \in \mathbb{C} \setminus \{-\sigma n\}, \end{cases} \quad (39)$$

which is false or inconsistent. At the same time, a consideration of the following special product,

$$\langle Z_{\sigma,\rho}^{a,0}(s), \frac{1}{Z_{\sigma,\rho}^{a,0}(s)} \rangle = 2\pi \sum_{n=0}^{\infty} \frac{(-a)^n}{n!} \langle \delta(s + \sigma n), \frac{1}{Z_{\sigma,\rho}^{a,0}(s)} \rangle, \quad (40)$$

gives the following:

$$\int_{s \in \mathbb{C}} 1 \, ds = 2\pi \sum_{n=0}^{\infty} \frac{(-a)^n}{n!Z_{\sigma,\rho}^{a,0}(-\sigma n)}. \quad (41)$$

Because $\frac{1}{Z_{\sigma,\rho}^{a,0}(-\sigma n)} = 0$ as a result of the poles of the special case of the Krätzel function, hence

$$\begin{aligned} \int_{s \in \mathbb{C}} 1 \, ds &= 0, \\ \int_{s \in \mathbb{C}} 1 \, ds &= \int_{-\infty}^{+\infty} 1 \, ds = 0, \\ &\implies \infty = 0 \end{aligned} \quad (42)$$

Therefore, one needs to be very careful in making a choice of function to analyze the behavior of the new series representation, which is discussed in the next subsection.

2.2. Analysis of the Behavior of the New Representation

The generalized Krätzel function $Z_{\sigma,\rho}^{a;b}(s)$ is expressed in a new form involving singular distributions, namely, the delta function. Therefore, it is proved in the subsequent theorem that this new form of $Z_{\sigma,\rho}^{a;b}(s)$ is a generalized function (distribution) over $\mathcal{Z}$ (the space of the entire test functions).

Theorem 2. Prove that the new representation of $Z_{\sigma,\rho}^{a;b}(s)$ is convergent in the sense of distributions over the space of the entire analytic functions, denoted by $\mathcal{Z}$.

Proof. For each $\varphi_1(s), \varphi_2(s) \in \mathcal{Z}$ and $c_1, c_2 \in \mathbb{C}$

$$\begin{aligned} & \langle Z_{\sigma,\rho}^{a;b}(s), c_1\varphi_1(s) + c_2\varphi_2(s) \rangle \\ &= \langle 2\pi \sum_{n,r=0}^{\infty} \frac{(-a)^n(-b)^r}{n!r!} \delta(s + \sigma n - \rho r), c_1\varphi_1(s) + c_2\varphi_2(s) \rangle. \end{aligned} \quad (43)$$

$$\implies \langle Z_{\sigma,\rho}^{a;b}(s), c_1\varphi_1(s) + c_2\varphi_2(s) \rangle = c_1 \langle Z_{\sigma,\rho}^{a;b}(s), \varphi_1(s) \rangle + c_2 \langle Z_{\sigma,\rho}^{a;b}(s), \varphi_2(s) \rangle. \quad (44)$$

Then, for any sequence $\{\varphi_\kappa\}_{\kappa=1}^{\infty}$ in $\mathcal{Z}$ converging to zero, one can assume that $\{\langle \delta(s + \sigma n - \rho r), \varphi_\kappa \rangle\}_{\kappa=1}^{\infty} \rightarrow 0$ because of the continuity of $\delta(s)$

$$\implies \left\{ \langle Z_{\sigma,\rho}^{a;b}(s), \varphi_\kappa(s) \rangle \right\}_{\kappa=1}^{\infty} = 2\pi \sum_{n,r=0}^{\infty} \frac{(-a)^n(-b)^r}{n!r!} \{ \langle \delta(s + \sigma n - \rho r), \varphi_\kappa(s) \rangle \}_{\kappa=1}^{\infty} \longrightarrow 0 \quad (45)$$

Hence, the generalized Krätzel function is a generalized function (distribution) over test function space $\mathcal{Z}$ because of the convergence of its new form (26), explored below:

$$\begin{aligned} \langle Z_{\sigma,\rho}^{a;b}(s), \varphi(s) \rangle &= 2\pi \sum_{n,r=0}^{\infty} \frac{(-a)^n(-b)^r}{n!r!} \langle \delta(s + \sigma n - \rho r), \varphi(s) \rangle \quad (\forall \varphi(s) \in \mathcal{Z}) \\ &= 2\pi \sum_{n,r=0}^{\infty} \frac{(-a)^n(-b)^r}{n!r!} \varphi(\rho r - \sigma n), \end{aligned} \quad (46)$$

where

$$\langle \delta(s + \sigma n - \rho r), \varphi(s) \rangle = \varphi(\rho r - \sigma n). \quad (47)$$

It can be seen that $\forall \varphi \in \mathcal{Z}; \varphi(\rho r - \sigma n)$ are functions of slow growth, and

$$\text{sum over the coefficients} = \sum_{n,r=0}^{\infty} \frac{(-a)^n(-b)^r}{n!r!} = \exp(-a-b) \quad (48)$$

exists and is rapidly decreasing. Consequently, for $\forall \varphi(s) \in \mathcal{Z}; \langle Z_{\sigma,\rho}^{a;b}(s), \varphi(s) \rangle$ as a product of the functions of slow growth and rapid decay is convergent. Similarly, other special and supplementary cases given in (30)–(36) are also meaningful in the sense of distributions. This fact is also obvious by making use of the basic Abel theorem. $\square$

Hence, the behavior of this new series is discussed for the functions of slow growth, but it is worth mentioning that this new series may converge for a larger class of functions. Consequently, new integrals of products of different functions in view of this new form of $Z_{\sigma,\rho}^{a;b}(s)$ are achieved here.

Consider a basic illustration, i.e., $\wp(s) = \tau^{s\xi}$ ($\xi > 0; s \in \mathbb{C}$). Therefore, by considering (26) and the shifting property of the delta function, the inner product $\langle Z_{\sigma,\rho}^{a,b}(s), \wp(s) \rangle$ yields

$$\begin{aligned} \int_{s \in \mathbb{C}} \tau^{s\xi} Z_{\sigma,\rho}^{a,b}(s) ds &= 2\pi \sum_{n,r=0}^{\infty} \frac{(-a)^n (-b)^r}{n!r!} \tau^{-n\sigma\xi + r\rho\xi} = 2\pi \sum_{n,r=0}^{\infty} \frac{(-a\tau^{-\sigma\xi})^n (-b\tau^{\rho\xi})^r}{n!r!} \\ &= 2\pi \exp(-a\tau^{-\sigma\xi} - b\tau^{\rho\xi}). \end{aligned} \quad (49)$$

The special case for the basic Krätzel function (1) can be obtained by taking $a = \rho = 1$; $\sigma = \rho$; $b = x$ in (49) as follows:

$$\int_{s \in \mathbb{C}} \tau^{s\xi} Z_{\rho}^s(x) ds = 2\pi \sum_{n,r=0}^{\infty} \frac{(-\tau^{-\rho\xi})^n (-x\tau^{\rho\xi})^r}{n!r!} = 2\pi \exp(-\tau^{-\rho\xi} - x\tau^{\rho\xi}). \quad (50)$$

Remark 2. Sequences as well as sums of delta function have significance in various engineering problems, for example, they are used as an electromotive force in electrical engineering. Thus, it is notable that if one multiplies $\{\delta(s + \sigma n - \rho r)\}_{n,r=0}^{\infty}$ with $2\pi \exp(-a - b)$, it will produce the distributional representation of the generalized Krätzel function. Furthermore, letting $a = 1$; $b = 0$, related outcomes hold for special cases as well. This discussion illustrates the possibility of further important identities. For instance, if one considers $\tau = e^{-1}$ in (49), then it can be used to compute the Laplace transform of $Z_{\sigma,\rho}^{a,b}(s)$. Therefore, it becomes important to check the validation of such results, as discussed in the following section.

2.3. Validation of the Results Obtained by the New Representation

Considering $t = e^{\omega}$ as well as $s = \nu + i\theta$ in (6), the generalized Krätzel function can be expressed as a Fourier transform, given below,

$$Z_{\sigma,\rho}^{a,b}(\nu + i\theta) = \sqrt{2\pi} \mathcal{F}[e^{\nu\omega} \exp(-ae^{\sigma\omega} - be^{-\rho\omega}); \theta]. \quad (51)$$

By letting $a = \rho = 1$ and then replacing σ and b with ρ and x , respectively, in (51), the basic Krätzel function (1) can be expressed as follows:

$$Z_{\rho}^{\nu+i\theta}(x) = \sqrt{2\pi} \mathcal{F}[e^{\nu\omega} \exp(-e^{\rho\omega} - xe^{-\omega}); \theta]. \quad (52)$$

The Fourier transform of an arbitrary function $u(t)$ satisfies the following:

$$\mathcal{F}[\sqrt{2\pi} \mathcal{F}[u(\omega); \theta]; \xi] = 2\pi u(-\xi). \quad (53)$$

Hence, applying this on identities (51)–(52) will lead to the following:

$$\begin{aligned} \mathcal{F}\{Z_{\sigma,\rho}^{a,b}(s); \xi\} &= \mathcal{F}[\sqrt{2\pi} \mathcal{F}[e^{\nu\omega} \exp(-ae^{\sigma\omega} - be^{-\rho\omega}); \theta]; \xi] \\ &= f(-\xi) = 2\pi e^{-\nu\xi} \exp(-ae^{-\sigma\xi} - be^{\rho\xi}), \end{aligned} \quad (54)$$

and equivalently,

$$\int_{-\infty}^{+\infty} e^{i\theta\xi} Z_{\sigma,\rho}^{a,b}(s) d\theta = 2\pi e^{-\nu\xi} \exp(-ae^{-\sigma\xi} - be^{\rho\xi}), \quad (55)$$

which is also obtainable as a specific case of our main result (49) by substituting $\tau = e$; $s = \nu + i\theta$. Furthermore, a substitution $\xi = 0$ in (55) leads to the following,

$$\int_{-\infty}^{+\infty} Z_{\sigma,\rho}^{a,b}(s) d\theta = 2\pi \exp(-a - b), \quad (56)$$

which is also attainable as a precise case of our main result (49). Hence, it is clear that the new representation of the generalized Krätzel function produces novel identities, which are unattainable by known techniques, but specific forms of new identities are trustworthy with known methods. The special case for the basic Krätzel function (1) can be obtained by taking $a = \rho = 1$ and then replacing σ and b with ρ and x , respectively, in (55–56) as follows:

$$\int_{-\infty}^{+\infty} e^{i\theta\xi} Z_{\rho}^{\nu+i\theta}(x) d\theta = 2\pi e^{-\nu\xi} \exp(-e^{-\rho\xi} - xe^{\xi}), \quad (57)$$

$$\int_{-\infty}^{+\infty} Z_{\rho}^{\nu+i\theta}(0) d\theta = \frac{2\pi}{e}. \quad (58)$$

Remark 3. It is notable that the new obtained integrals contribute only the sum over residues as a result of the existing poles or singular points in the integrand, which is consistent with the basic result of complex analysis.

Next, an application of Parseval's identity of the Fourier transform in (54) leads to the following new results regarding the generalized Krätzel function, $Z_{\sigma,\rho}^{a,b}(\nu + i\theta)$:

$$\int_{-\infty}^{+\infty} Z_{\sigma,\rho}^{a,b}(\nu + i\theta) \overline{Z_{\sigma,\rho}^{a,b}(\mu + i\theta)} d\theta = 2\pi \int_0^{\infty} t^{\nu+\mu-1} e^{-2a\sigma t - 2bt^{-\rho}} dt = 2\pi Z_{\sigma,\rho}^{2a,2b}(\nu + \mu). \quad (59)$$

A substitution of $a = \rho = 1$ and then replacement of σ and b with ρ and x , respectively, in (59) leads to the following identity for the basic Krätzel function (1):

$$\int_{-\infty}^{+\infty} Z_{\rho}^{\nu+i\theta}(x) \overline{Z_{\rho}^{\mu+i\theta}(x)} d\theta = 2\pi \int_0^{\infty} t^{\nu+\mu-1} e^{-2\rho t - 2xt^{-1}} dt = 2\pi Z_{\rho}^{\nu+\mu}(2x), \quad (60)$$

and for $\nu = \mu$, one can obtain

$$\int_{-\infty}^{+\infty} |Z_{\sigma,\rho}^{a,b}(\nu + i\theta)|^2 d\tau = \int_0^{\infty} t^{2\nu-1} e^{-2a\sigma t - 2bt^{-\rho}} dt = 2\pi Z_{\sigma,\rho}^{2a,2b}(2\nu). \quad (61)$$

2.4. Further Properties of the Generalized Krätzel Function as a Distribution

For completeness, a list of basic properties are stated and proved here in the sense of distributions by following the methodology of [31], Chapter 7.

Theorem 3. The generalized Krätzel function holds the subsequent properties as a distribution

- (i) $\langle Z_{\sigma,\rho}^{a,b}(s), \varphi_1(s) + \varphi_2(s) \rangle = \langle Z_{\sigma,\rho}^{a,b}(s), \varphi_1(s) \rangle + \langle Z_{\sigma,\rho}^{a,b}(s), \varphi_2(s) \rangle; \forall \varphi(s) \in \mathcal{Z}$
- (ii) $\langle c_1 Z_{\sigma,\rho}^{a,b}(s), \varphi(s) \rangle = \langle Z_{\sigma,\rho}^{a,b}(s), c_1 \varphi(s) \rangle; \forall \varphi(s) \in \mathcal{Z}$
- (iii) $\langle Z_{\sigma,\rho}^{a,b}(s - \gamma), \varphi(s) \rangle = \langle Z_{\sigma,\rho}^{a,b}(s), \varphi(s + \gamma) \rangle; \forall \varphi(s) \in \mathcal{Z}$
- (iv) $\langle Z_{\sigma,\rho}^{a,b}(c_1 s), \varphi(s) \rangle = \langle Z_{\sigma,\rho}^{a,b}(s), \frac{1}{c_1} \varphi\left(\frac{s}{c_1}\right) \rangle; \forall \varphi(s) \in \mathcal{Z}$
- (v) $\langle Z_{\sigma,\rho}^{a,b}(c_1 s - \gamma), \varphi(s) \rangle = \langle Z_{\sigma,\rho}^{a,b}(s), \frac{1}{c_1} \varphi\left(\frac{s}{c_1} + \gamma\right) \rangle; \forall \varphi(s) \in \mathcal{Z}$
- (vi) $\psi(s) Z_{\sigma,\rho}^{a,b}(s) \in \mathcal{Z}$ is a distribution over $\mathcal{Z}$ for any regular distribution $\psi(z)$.
- (vii) For $b = 0$, $Z_{\sigma,\rho}^{a,0}(s) = s Z_{\sigma,\rho}^{a,0}(s)$ iff $\varphi(s - 1) = s\varphi(s)$, where $\varphi \in \mathcal{Z}$
- (viii) $\langle Z_{\sigma,\rho}^{a,b(m)}(s), \varphi(s) \rangle = \sum_{n,r=0}^{\infty} \frac{(-a)^n (-b)^r}{n!r!} (-1)^m \varphi^m(-\sigma n + \rho r); m = 0, 1, 2, \dots; \forall \varphi(s) \in \mathcal{Z}$
- (ix) $Z_{\sigma,\rho}^{a,b}(\omega_1 - s) Z_{\sigma,\rho}^{a,b}(s - \omega_2) = (2\pi \exp(-a - b))^2 \delta(\omega_1 - \omega_2); \forall \varphi(s) \in \mathcal{Z}$
- (x) $\langle \mathcal{F}[Z_{\sigma,\rho}^{a,b}(s)], \varphi(s) \rangle = \langle Z_{\sigma,\rho}^{a,b}(s), \mathcal{F}[\varphi](s) \rangle; \forall \varphi(s) \in \mathcal{Z}$
- (xi) $\langle \mathcal{F}[Z_{\sigma,\rho}^{a,b}(s)], \mathcal{F}[\varphi(s)] \rangle = 2\pi \langle Z_{\sigma,\rho}^{a,b}(\nu), \varphi(-\nu) \rangle, \nu = \Re(s); \forall \varphi(s) \in \mathcal{Z}$

- (xii) $\langle \mathcal{F}[Z_{\sigma,\rho}^{a;b}(s)], \mathcal{F}[\wp(s)] \rangle = 2\pi \langle Z_{\sigma,\rho}^{a;b}(\nu), \wp^T(\nu) \rangle$, where $\overline{\wp(-s)} = \wp^T(s)$; $\forall \wp(s) \in \mathcal{Z}$
 (xiii) $\langle \mathcal{F}[Z_{\sigma,\rho}^{a;b}(s)], \overline{\mathcal{F}[\wp(s)]} \rangle = 2\pi \langle Z_{\sigma,\rho}^{a;b}(\nu), \wp^T(\nu) \rangle$; $\forall \wp(s) \in \mathcal{Z}$
 (xiv) $\langle \mathcal{F}[Z_{\sigma,\rho}^{a;b}(s)], \overline{\mathcal{F}[\wp(s)]} \rangle = 2\pi \langle [Z_{\sigma,\rho}^{a;b}(\nu)], [\wp(\nu)] \rangle$; $\forall \wp(s) \in \mathcal{Z}$
 (xv) $\mathcal{F}[Z_{\sigma,\rho}^{a;b(m)}(s)] = [(-it)^m Z_{\sigma,\rho}^{a;b}(s)]$; $m = 0, 1, 2, \dots$; $\forall \wp(s) \in \mathcal{Z}$
 (xvi) $Z_{\sigma,\rho}^{a;b}(s + c_1) = \sum_{n=0}^{\infty} \frac{(c_1)^n}{n!} Z_{\sigma,\rho}^{a;b(n)}(s)$; $\forall \wp(s) \in \mathcal{Z}$

where c_1 , γ , and c_2 are arbitrary real or complex constants.

Proof. It can be verified that the methodology to prove (i–vi) can be achieved using the properties of the delta function. Therefore, begin by proving (vii):

$$\begin{aligned} \langle Z_{\sigma,\rho}^{a;0}(s+1), \wp(s) \rangle &= \langle Z_{\sigma,\rho}^{a;0}(s), \wp(s-1) \rangle, \\ \Leftrightarrow \langle s Z_{\sigma,\rho}^{a;0}(s), \wp(s) \rangle &= \langle Z_{\sigma,\rho}^{a;0}(s), \wp(s-1) \rangle, \\ \Leftrightarrow \langle Z_{\sigma,\rho}^{a;0}(s), s \wp(s) \rangle &= \langle Z_{\sigma,\rho}^{a;0}(s), \wp(s-1) \rangle, \end{aligned}$$

as required. Next, result (viii) is proved by making use of Equation (16) (see Section 2.1),

$$\langle Z_{\sigma,\rho}^{a;b(m)}(s), \wp(s) \rangle = \sum_{n,r=0}^{\infty} \frac{(-a)^n (-b)^r}{n!r!} (-1)^m \wp^m(-\sigma n + \rho r), m = 0, 1, 2, \dots$$

That is meaningful and finite as a product of rapidly decaying, as well as slow growth, functions. Result (ix) is proved here in view of relation (17) (see Section 2.1),

$$\begin{aligned} \langle Z_{\sigma,\rho}^{a;b}(\omega_1 - s) Z_{\sigma,\rho}^{a;b}(s - \omega_2), \wp(s) \rangle &= \left(2\pi \sum_{n,r=0}^{\infty} \frac{(-a)^n (-b)^r}{n!r!} \right)^2 \langle \delta(\omega_1 - \omega_2), \wp(s) \rangle \\ &= (2\pi \exp(-a - b))^2 \langle \delta(\omega_1 - \omega_2), \wp(s) \rangle. \end{aligned}$$

Identities (x)–(xv) can also be proved in view of various properties of the delta function. Let us start by proving (x):

$$\begin{aligned} \langle \mathcal{F}[Z_{\sigma,\rho}^{a;b}(s)], \wp(s) \rangle &= 2\pi \sum_{n,r=0}^{\infty} \frac{(-a)^n (-b)^r}{n!r!} \langle \mathcal{F}[\delta(s + \sigma n - \rho r)], \wp(s) \rangle \\ &= 2\pi \sum_{n,r=0}^{\infty} \frac{(-a)^n (-b)^r}{n!r!} \langle \delta(s + \sigma n - \rho r), \mathcal{F}[\wp(s)] \rangle = \langle Z_{\sigma,\rho}^{a;b}(s), \mathcal{F}[\wp(s)] \rangle. \end{aligned}$$

Next, the results (xi–xii) are proved as follows:

$$\begin{aligned} \langle \mathcal{F}[Z_{\sigma,\rho}^{a;b}(s)], \mathcal{F}[\wp(s)] \rangle &= 2\pi \langle Z_{\sigma,\rho}^{a;b}(s), \mathcal{F}[\mathcal{F}[\wp(s)]] \rangle = 2\pi \langle Z_{\sigma,\rho}^{a;b}(\nu), \wp(-\nu) \rangle, \\ \langle \overline{\mathcal{F}[Z_{\sigma,\rho}^{a;b}(s)]}, \mathcal{F}[\wp(s)] \rangle &= 2\pi \langle Z_{\sigma,\rho}^{a;b}(s), \overline{\mathcal{F}[\mathcal{F}[\wp(s)]]} \rangle = 2\pi \langle Z_{\sigma,\rho}^{a;b}(\nu), \overline{\wp(-\nu)} \rangle \\ &= 2\pi \langle Z_{\sigma,\rho}^{a;b}(\nu), \wp^T(\nu) \rangle, \end{aligned}$$

where the transpose of $\wp$ is denoted by $\wp^T$. Proof of the results (xiii)–(xiv) are as follows:

$$\begin{aligned} \langle \mathcal{F}[Z_{\sigma,\rho}^{a;b}(s)], \overline{\mathcal{F}[\wp(s)]} \rangle &= 2\pi \langle Z_{\sigma,\rho}^{a;b}(s), \overline{\wp(-s)} \rangle = 2\pi \langle Z_{\sigma,\rho}^{a;b}(\nu), \wp^T(\nu) \rangle, \langle \overline{\mathcal{F}[Z_{\sigma,\rho}^{a;b}(s)]}, \mathcal{F}[\wp(s)] \rangle \\ &= 2\pi \langle \mathcal{F}[Z_{\sigma,\rho}^{a;b}(s)], \overline{\mathcal{F}[\wp(s)]} \rangle = 2\pi \langle \mathcal{F}[Z_{\sigma,\rho}^{a;b}(s)], \mathcal{F}[\wp(s)] \rangle \\ &= 2\pi \langle Z_{\sigma,\rho}^{a;b}(\nu), \wp(-\nu) \rangle, \end{aligned}$$

where the above lines also follow by considering Parseval's formula of the Fourier transform. The proof of (xv) is as follows:

$$\begin{aligned}\langle \mathcal{F}[Z_{\sigma,\rho}^{a;b}(s)], \varphi^{(1)}(s) \rangle &= \langle \mathcal{F}[Z_{\sigma,\rho}^{a;b(1)}(s)], \varphi(s) \rangle = \langle Z_{\sigma,\rho}^{a;b}(s), \mathcal{F}[\varphi^{(1)}(s)] \rangle \\ \langle \mathcal{F}[Z_{\sigma,\rho}^{a;b(1)}(s)], \varphi(s) \rangle &= \langle Z_{\sigma,\rho}^{a;b}(s), (-it)\varphi(s)e^{-ist} \rangle \\ \langle Z_{\sigma,\rho}^{a;b(1)}(s), \varphi(s) \rangle &= \langle (-it)\mathcal{F}[Z_{\sigma,\rho}^{a;b}(s)], \varphi(s) \rangle \\ \langle \mathcal{F}[Z_{\sigma,\rho}^{a;b(1)}(s)], \varphi(s) \rangle &= \langle (-it)\mathcal{F}[Z_{\sigma,\rho}^{a;b}(s)], \varphi(s) \rangle\end{aligned}$$

and so on; one can finally obtain

$$\langle \mathcal{F}[Z_{\sigma,\rho}^{a;b(m)}(s)], \varphi(s) \rangle = \langle (-it)^m \mathcal{F}[Z_{\sigma,\rho}^{a;b}(s)], \varphi(s) \rangle,$$

as the requirement for (xv). The last result (xvi) is true in view of the statement mentioned in [31] p. 201, "Suppose $f \in \mathcal{Z}'$ and Δ is a complex constant then the translation of the function f by the quantity $-\Delta$ is represented by $f(z + \Delta) = \sum_{n=0}^{\infty} \frac{(\Delta)^n}{n!} f^{(n)}(z)$ ". Consequently, it yields

$$\begin{aligned}\langle Z_{\sigma,\rho}^{a;b}(s + c_1; a), \varphi(s) \rangle &= \langle Z_{\sigma,\rho}^{a;b}(s), \varphi(s - c_1) \rangle = \lim_{\nu \rightarrow \infty} \langle Z_{\sigma,\rho}^{a;b}(s), \sum_{n=0}^{\nu} \frac{(-c_1)^n}{n!} \varphi^{(n)}(s) \rangle \\ &= \lim_{\nu \rightarrow \infty} \langle \sum_{n=0}^{\nu} \frac{(c_1)^n}{n!} Z_{\sigma,\rho}^{a;b(n)}(s), \varphi(s) \rangle,\end{aligned}$$

as required. $\square$

Remark 4. The space of generalized functions denoted by $\mathcal{D}'$ is mapped onto $\mathcal{Z}'$ with the help of the Fourier transform; similarly, this mapping can be inverted from $\mathcal{Z}'$ onto $\mathcal{D}'$ [31] p. 203. Both ways, it is a continuous linear mapping. Therefore, (54) explores that $2\pi e^{-v\xi} \exp(-ae^{\sigma\xi} - be^{-\rho\xi}) \in \mathcal{D}'$. In the same way, if one considers (55) and inverts it by the Fourier transform, then $\mathcal{F}\{Z_{\sigma,\rho}^{a;b}(s)\} \in \mathcal{D}'$.

2.5. New Version of the Krätzel Integral Transform: A Case Study of New Representation

As a case study of new representation, a new version of the generalized Krätzel integral transform is introduced and defined over $\mathcal{Z}$ as follows:

$$K_{\sigma,\rho}^{a;b}(f(s)) = \int_{s \in \mathbb{C}} Z_{\sigma,\rho}^{a;b}(s) f(s) ds; \forall \varphi(s) \in \mathcal{Z}. \quad (62)$$

By letting $a = \rho = 1$ and then replacing σ and b with ρ and x , respectively, in (62), one can obtain the following new version of the basic Krätzel integral transform (3):

$$K_{\rho}^s(f(x)) = \int_{s \in \mathbb{C}} Z_{\rho}^s(x) f(s) ds; \forall \varphi(s) \in \mathcal{Z}. \quad (63)$$

As a singular generalized function, the delta function is a linear mapping that maps every function to its value at zero. Because of this property, this new representation has the power to calculate integrals that cannot be computed in the usual sense. For example, the Riemann zeta function (see, for example, [34–39] Equation (26)) is considered here:

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \left(\frac{1}{e^t - 1} - \frac{1}{t} \right) t^{s-1} dt; (0 < \Re(s) < 1).$$

Example 1. Let $f(s) = \zeta(s)$; then, its generalized Krätzel transform is

$$\begin{aligned} \int_{s \in \mathbb{C}} Z_{\sigma, \rho}^{a, b}(s) \zeta(s) ds &= 2\pi \sum_{n, r=0}^{\infty} \frac{(-a)^n (-b)^r}{n! r!} \int_{s \in \mathbb{C}} \delta(s + \sigma n - \rho r) \zeta(s) ds \\ &= 2\pi \sum_{n, r=0}^{\infty} \frac{(-a)^n (-b)^r}{n! r!} \zeta(-\sigma n + \rho r), \end{aligned} \quad (64)$$

By letting $a = \rho = 1$ and then replacing σ and b with ρ and x , respectively, in (64), one can obtain the basic Krätzel integral transform of the zeta function

$$\begin{aligned} \int_{s \in \mathbb{C}} Z_{\rho}^s(x) \zeta(s) ds &= 2\pi \sum_{n, r=0}^{\infty} \frac{(-1)^n (-x)^r}{n! r!} \int_{s \in \mathbb{C}} \delta(s + \rho n - r) \zeta(s) ds \\ &= 2\pi \sum_{n, r=0}^{\infty} \frac{(-1)^n (-x)^r}{n! r!} \zeta(-\rho n + r). \end{aligned} \quad (65)$$

For $x = 0$, $\rho = 1$ (65) yields the following:

$$\int_{s \in \mathbb{C}} Z_1^s(0) \zeta(s) ds = 2\pi \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \zeta(-n). \quad (66)$$

Because the zeta function vanishes at even negative integers (see, for example, [40] p. 330), this sum can further be evaluated using the following zeta function relation with Bernoulli numbers, (see, for example, [40,41]),

$$\zeta(1 - 2n) = -\frac{B_{2n}}{2n}; \quad (n \in \mathbb{N} \setminus \{0\}).$$

where the Bernoulli numbers B_n are defined by their relation with the Bernoulli polynomial

$$B_n := B_n(0) = (-1)^n B_n(1); \quad (n \in \mathbb{N} \setminus \{0\}),$$

and the classical Bernoulli polynomials $B_n(x)$ of degree n in x are defined by the following generating functions [40–42]:

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad (|t| < 2\pi).$$

In view of the above details, the following approximation can be obtained:

$$\int_{s \in \mathbb{C}} Z_1^s(0) \zeta(s) ds = 2\pi \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \zeta(-n) = 2\pi \left[\zeta(0) + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} \right] \cong -2.63$$

For further related studies about such computations, the interested reader is referred to [40–45].

Example 2. For any constant Ω , let $f(s) = e^{\Omega s}$; then, its generalized Krätzel transform is

$$\begin{aligned} \int_{s \in \mathbb{C}} Z_{\sigma, \rho}^{a, b}(s) e^{\Omega s} ds &= 2\pi \sum_{n, r=0}^{\infty} \frac{(-a)^n (-b)^r}{n! r!} \langle \delta(s + \sigma n - \rho r), e^{\Omega s} \rangle \\ &= 2\pi \sum_{n, r=0}^{\infty} \frac{(-a)^n (-b)^r}{n! r!} e^{-\Omega \sigma n + \Omega \rho r} \\ &= 2\pi \exp(-a e^{-\Omega \sigma} - b e^{\Omega \rho}). \end{aligned} \quad (67)$$

Example 3. For any constant Ω , let $f(s) = \sin \Omega s$; then, its generalized Krätzel transform is

$$\begin{aligned} \int_{s \in \mathbb{C}} Z_{\sigma, \rho}^{a; b}(s) \sin \Omega s ds &= 2\pi \sum_{n, r=0}^{\infty} \frac{(-a)^n (-b)^r}{n! r!} \langle \delta(s + \sigma n - \rho r), \sin \Omega s \rangle \\ &= 2\pi \sum_{n, r=0}^{\infty} \frac{(-a)^n (-b)^r \sin \Omega(\rho r - \sigma n)}{n! r!} \\ &= \text{IMG} \left(2\pi \sum_{n, r=0}^{\infty} \frac{(-a)^n (-b)^r e^{i(\Omega(\rho r - \sigma n))}}{n! r!} \right) \\ &= \text{IMG} \left(2\pi \sum_{n, r=0}^{\infty} \frac{(-a)^n (-b)^r (e^{i\Omega})^{\rho r} e^{-i\Omega \sigma n}}{n! r!} \right) \\ &= \text{IMG} \left(2\pi \sum_{n, r=0}^{\infty} \frac{(-ae^{-i\Omega \sigma})^n (-be^{i\Omega \rho})^r}{n! r!} \right) \\ &= \text{IMG} \left(2\pi \exp(-ae^{-i\Omega \sigma} - be^{i\Omega \rho}) \right). \end{aligned} \quad (68)$$

Similarly, $f(s) = \cos \Omega s$,

$$\int_{s \in \mathbb{C}} Z_{\sigma, \rho}^{a; b}(s) \cos \Omega s ds = \Re \left(2\pi \exp(-ae^{-i\Omega \sigma} - be^{i\Omega \rho}) \right). \quad (69)$$

In the limiting case, $\cos \Omega s \rightarrow \pm 1$; hence,

$$\int_{s \in \mathbb{C}} Z_{\sigma, \rho}^{a; b}(s) \cos \Omega s ds \rightarrow \pm 2\pi \exp(-a - b). \quad (70)$$

Example 4. Let $f(s) = \tan^{-1} s$; then, its generalized Krätzel transform is

$$\int_{s \in \mathbb{C}} Z_{\sigma, \rho}^{a; b}(s) \tan^{-1} s ds = 2\pi \sum_{n, r=0}^{\infty} \frac{(-a)^n (-b)^r \tan^{-1}(\rho r - \sigma n)}{n! r!}. \quad (71)$$

In the limiting case, $\tan^{-1}(\rho r - \sigma n) \rightarrow \left(\pm \frac{\pi}{2}\right)$; hence,

$$\int_{s \in \mathbb{C}} Z_{\sigma, \rho}^{a; b}(s) \tan^{-1} s ds \rightarrow \pm \pi^2 \exp(-a - b). \quad (72)$$

Example 5. Let us next consider the Mittag–Leffler function [46] (p. 9, Equation (1.67)) (a generalization of the exponential function), defined by $f(s) = E_{\alpha}(s) = \sum_{m=0}^{\infty} \frac{s^m}{\Gamma(\alpha m + 1)}$; its generalized Krätzel transform is

$$\begin{aligned} \int_{s \in \mathbb{C}} Z_{\sigma, \rho}^{a; b}(s) E_{\alpha}(s) ds &= \int_{s \in \mathbb{C}} Z_{\sigma, \rho}^{a; b}(s) \sum_{m=0}^{\infty} \frac{s^m}{\Gamma(\alpha m + 1)} ds \\ \int_{s \in \mathbb{C}} Z_{\sigma, \rho}^{a; b}(s) E_{\alpha}(s) ds &= \sum_{n, r, m=0}^{\infty} \frac{(-a)^n (-b)^r}{n! r! \Gamma(\alpha m + 1)} \int_{s \in \mathbb{C}} \delta(s + \sigma n - \rho r) s^m ds \\ &= \sum_{n, r, m=0}^{\infty} \frac{(-a)^n (-b)^r}{n! r! \Gamma(\alpha m + 1)} (\rho r - \sigma n)^m \\ &= \sum_{n, r=0}^{\infty} \frac{(-a)^n (-b)^r}{n! r!} E_{\alpha}(\rho r - \sigma n) \end{aligned} \quad (73)$$

Taking $\alpha = 1$, one can obtain the following:

$$\int_{s \in \mathbb{C}} Z_{\sigma, \rho}^{a; b}(s) e^s ds = 2\pi \exp(-ae^{-\sigma} - be^{\rho}). \quad (74)$$

Example 6. Let us next consider the McDonald function [45] $K_v(s)$, also defined in Equation (2); its generalized Krätzel transform is

$$\begin{aligned}\int_{s \in \mathbb{C}} Z_{\sigma, \rho}^{a, b}(s) K_v(s) ds &= \sum_{n, r=0}^{\infty} \frac{(-a)^n (-b)^r}{n! r!} \int_{s \in \mathbb{C}} \delta(s + \sigma n - \rho r) K_v(s) ds \\ &= \sum_{n, r=0}^{\infty} \frac{(-a)^n (-b)^r}{n! r!} K_v(\rho r - \sigma n).\end{aligned}\quad (75)$$

Certain special cases of the above results may also be of interest. For example, letting $b = 0$ in (73) and (75) produces the following:

$$\int_{s \in \mathbb{C}} Z_{\sigma, \rho}^{a, 0}(s) E_\alpha(s) ds = \sum_{n=0}^{\infty} \frac{(-a)^n}{n!} \int_{s \in \mathbb{C}} \delta(s + \sigma n) E_\alpha(s) ds = \sum_{n=0}^{\infty} \frac{(-a)^n}{n!} E_\alpha(-\sigma n), \quad (76)$$

and

$$\int_{s \in \mathbb{C}} Z_{\sigma, \rho}^{a, 0}(s) K_v(s) ds = \sum_{n=0}^{\infty} \frac{(-a)^n}{n!} \int_{s \in \mathbb{C}} \delta(s + \sigma n) K_v(s) ds = \sum_{n=0}^{\infty} \frac{(-a)^n}{n!} K_v(-\sigma n). \quad (77)$$

Next, it can be observed that all the results that hold for the Laplace transform of delta functions similarly hold for the generalized Krätzel function, for example,

$$L\{\delta^{(r)}(s); \omega\} = \omega^r.$$

Therefore,

$$\begin{aligned}L(Z_{\sigma, \rho}^{a, b}(s); \omega) &= L\left(2\pi \sum_{n, r, p=0}^{\infty} \frac{(-a)^n (-b)^r (\sigma n - \rho r)^p}{n! r! p!} \delta^{(p)}(s); \omega\right) \\ L(Z_{\sigma, \rho}^{a, b}(s); \omega) &= 2\pi \sum_{n, r, p=0}^{\infty} \frac{(-a)^n (-b)^r (\sigma n - \rho r)^p}{n! r! p!} L\{\delta^{(p)}(s); \omega\} \\ &= 2\pi \sum_{n, r, p=0}^{\infty} \frac{(-a)^n (-b)^r (\sigma n - \rho r)^p}{n! r! p!} \omega^p = 2\pi \exp(-ae^{\sigma\omega} - be^{-\rho\omega}).\end{aligned}\quad (78)$$

By letting $a = \rho = 1$ and then replacing σ and b with ρ and x , respectively, in (78), the Laplace transform of the basic Krätzel function can be obtained as follows:

$$L(Z_\rho^s(x); \omega) = 2\pi \exp(-e^{\rho\omega} - xe^{-\omega}) \quad (79)$$

Similarly,

$$\left. \begin{aligned}L(Z_{\sigma, \rho}^{a, b}(s - c); \omega) &= 2\pi e^{-\omega c} \exp(-ae^{\sigma\omega} - be^{-\rho\omega}) \\ L(Z_\rho^{s-c}(x); \omega) &= 2\pi e^{-\omega c} \exp(-e^{\rho\omega} - xe^{-\omega})\end{aligned} \right\} \quad (80)$$

3. Summary and Forthcoming Directions

On the one hand, generalization of a function by extending its domain of definition has remained a novel and non-trivial problem. On the other hand, the Krätzel function and its generalizations have fundamental applications in various disciplines. In light of the details presented in Section 1, this function is widely studied as a function of real variables, and its investigation with respect to complex variables only became possible in 2006 following Kilbas et al. [8]. In this article, it is generalized from complex numbers to complex functions. Hence, one can continue the Krätzel function beyond its original domain of definition in terms of the delta function. Consequently, the problem of physically interpreting its values does not arise, as is the case for classical representation. Similarly, the use of differentiation and Fourier and Laplace transforms has become a continuous operation for the Krätzel function in view of this generalized representation. While the Laplace transform of this function is known in the literature with respect to the variable $b = x$, it is now obtained with respect to variable s

in this study. This is because the identities valid for delta functions can now be successfully applied to the Krätzel function using distributional representation.

The existence of a new version of the generalized Krätzel integral transform is a natural consequence of this research. The Krätzel integral transform is known only over the domain of real numbers with respect to the variable x . In this study, a new version of the Krätzel integral transform is established for variable s . It provides a computational technique to evaluate the integrals of products of the Krätzel function with other functions. Using the generalized representation, integrals of products of functions are converted into a sum over natural numbers. The relationship of the Krätzel function with the H -function sheds fresh light on new explorations of this important and newly emerging function. The method of computing the new identities involves the desired simplicity. The success and beauty of this new version is in the fact that the sum over the coefficients of the distributional representation is rapidly decreasing. This discussion provides insights for further, new results. For example, the generalized Krätzel function and its various special cases satisfy the differential equation of fractional order [10,11]. Hence, the solution of these fractional differential equations can be discussed over the space of complex test functions in future research. These facts may be significant for applying the function beyond the problems it was originally defined to address.

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