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A Notion of Convergence in Fuzzy Partially Ordered Sets

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Abstract: The notion of sequential convergence in fuzzy partially ordered sets, under the name o_F-convergence, is well known. Our aim in this paper is to introduce and study a notion of net convergence, with respect to the fuzzy order relation, named o-convergence, which generalizes the former notion and is also closer to our sense of the classic concept of "convergence". The main result of this article is that the two notions of convergence are identical in the area of complete F-lattices.

Keywords: fuzzy order relation; o_F-convergence; o-convergence

MSC: 54A20; 06D72

1. Introduction

Zadeh, in his seminal paper [1] in 1971, introduced and studied the concept of fuzzy relation. In particular, the notion of fuzzy order relation was initiated by generalizing the notions of reflexivity, antisymmetry and transitivity. Since then, many authors have studied fuzzy orders and relations by adopting different approaches [2–8]. Fuzzy orders have a wider range of utility when compared to the classic orders by allowing the expression not only the preference for one alternative over another, in a set of alternatives, but also the "power" of that preference. Generally, fuzzy relations are important because of their applications in fuzzy modeling, fuzzy diagnosis, and fuzzy control (see for example [9]).

Using a notion of fuzzy order, the authors in [3] defined and studied a notion of convergence for sequences, in the sense of Birkhoff [10] which was further investigated in the context of fuzzy Riesz spaces in [11,12], where is considered as net convergence. Moreover, this notion was redefined to unbounded fuzzy order convergence in [13]. Motivated by the previous works, we provide, in the general context of fuzzy posets, a notion of convergence for nets, in the sense of McShane [14]. Particularly, Section 2 contains preliminaries. In Section 3 we introduce and study o-convergence which is a generalization of o_F-convergence, considered in [3]. In Section 4, we prove that, in the setting of complete F-lattices, both notions of convergence are equivalent with the equality of limit inferior and limit superior, with respect to the fuzzy order relation and, therefore, coincide. Finally, in Section 5, we add some concluding remarks for possible future study in this field.

2. Preliminaries

This section contains preliminary material that will be needed in the sequel.

Let X be a nonempty set. A fuzzy set α on X (due to Zadeh [15]) is a membership function $\mu_\alpha : X \to [0,1]$ with the value of $\mu_\alpha(x)$ at x representing the "grade of membership" of x in α. When α is an ordinary set its membership function μ_α reduces to its characteristic function and α is called a crisp set on X.

In what follows, we recall the basic notions and results from [3,7,16].

Definition 1 ([3])**.** *Let X be a nonempty set. A fuzzy order on X is a fuzzy set on $X \times X$ whose membership function μ satisfies the following properties:*

(1) (reflexivity) for all $x \in X$, $\mu(x,x) = 1$;
(2) (antisymmetry) for all $x, y \in X$, $\mu(x,y) + \mu(y,x) > 1$ implies $x = y$; and,
(3) (transitivity) for all $x, z \in X$, $\mu(x,z) \geqslant \bigvee_{y \in X} [\mu(x,y) \wedge \mu(y,z)]$, where $\vee$ and $\wedge$ denote the supremum and the infimum, with respect to the usual order on the unit interval, respectively.

A set with a fuzzy order defined on it is called a fuzzy ordered set (or foset for short.)

Notation 1 ([7])**.** *Let X be a foset and $x \in X$. With $\uparrow x$ we will denote the fuzzy set on X defined by $\uparrow x(y) = \mu(x,y)$, for all $y \in X$. Dually, with $\downarrow x$ we will denote the fuzzy set on X defined by $\downarrow x(y) = \mu(y,x)$, for all $y \in X$. If M is a subset of X, $\uparrow M = \bigvee_{x \in M} \uparrow x$ and $\downarrow M = \bigvee_{x \in M} \downarrow x$.*

Definition 2 ([7])**.** *Let M be a subset of a foset X. The upper bound $U(M)$ of M is the fuzzy set on X, defined as follows:*

$$U(M)(y) = \begin{cases} 0, & \text{if } \uparrow x(y) \leqslant 1/2 \text{ for some } x \in M \\ \left(\bigwedge_{x \in M} \uparrow x \right)(y), & \text{otherwise.} \end{cases}$$

Dually, the lower bound $L(M)$ of M is the fuzzy set on X, defined as follows:

$$L(M)(y) = \begin{cases} 0, & \text{if } \downarrow x(y) \leqslant 1/2 \text{ for some } x \in M \\ \left(\bigwedge_{x \in M} \downarrow x \right)(y), & \text{otherwise.} \end{cases}$$

If $U(M)(x) > 0$, for some $x \in X$, we write $x \in U(M)$; in such case, we say that M is bounded from above and we call x an upper bound of M. Similarly, if $L(M)(x) > 0$, then we write $x \in L(M)$; in such case we say that M is bounded from below and we call x a lower bound of M. If M is both bounded from above and bounded from below, then M is said to be bounded.
An element $z \in X$ is said to be the supremum of M (written $z = \sup M$) if

(1) $z \in U(M)$ and
(2) $y \in U(M)$ implies $y \in U(z)$.

Similarly, $z \in X$ is said to be the infimum of M (written $z = \inf M$) if

(3) $z \in L(M)$ and
(4) $y \in L(M)$ implies $y \in L(z)$.

Theorem 1 ([7])**.** *Let M be a subset of a foset X. Subsequently,*

(1) $\inf M$, if it exists, is unique;
(2) $\sup M$, if it exists, is unique.

If M is a subset of a foset X, then we will adopt from [3] the notations $\vee M$ and $\wedge M$ for $\sup M$ and $\inf M$, respectively. In the case that M is an indexed set i.e., $M = \{m_i : i \in I\}$ we will use alternatively, when it is more convenient, the abbreviated symbols $\bigvee_{i \in I} m_i$ and $\bigwedge_{i \in I} m_i$ for $\vee\{m_i : i \in I\}$ and $\wedge\{m_i : i \in I\}$, respectively.

Notation 2. *$x \vee y = \sup\{x, y\}$ and $x \wedge y = \inf\{x, y\}$.*

Theorem 2 ([7]). *Let X be a foset. Then the following identities hold, whenever the expressions referred exist.*

(1) $x \wedge x = x$ *and* $x \vee x = x$.
(2) $x \wedge y = y \wedge x$ *and* $x \vee y = y \vee x$.
(3) $x \wedge (x \vee y) = x \vee (x \wedge y) = x$.
(4) $\mu(x, y) > 1/2$ *if and only if* $x \wedge y = x$.
(5) $\mu(x, y) > 1/2$ *if and only if* $x \vee y = y$.

Definition 3 ([7]). *A foset X is called a fuzzy lattice (or F-lattice for short) if all finite subsets of X have suprema and infima. A fuzzy lattice is said to be complete if every subset of X has a supremum and an infimum.*

Definition 4 ([16]). *Let D be a subset of a foset X.*

(1) *D is said to be directed to the right if for every finite subset E of D, $D \cap U(E) \neq \emptyset$.*
(2) *D is said to be directed to the left if for every finite subset E of D, $D \cap L(E) \neq \emptyset$.*
(3) *D is said to be directed if it is both directed to the right and directed to the left.*

In our terminology for nets, we follow Kelley [17] i.e., a net in a set X is an arbitrary function $s : A \to X$, where A is a nonempty directed set. If $s(a) = s_a$, for all $a \in A$, then the net s will be denoted by the symbol $(s_a)_{a \in A}$.

Definition 5 ([12]). *Suppose that (X, μ) is a foset. A net $(s_a)_{a\in A}$, of elements in X, is said to be increasing if $a \leqslant b$ implies $\mu(s_a, s_b) > 1/2$, in which case we shall write $(s_a)_{a\in A}\uparrow$. Moreover, if $x = \vee\{s_a : a \in A\}$, then we write $(s_a)_{a\in A}\uparrow x$. The definition of a decreasing net and the symbols $(s_a)_{a\in A}\downarrow$, $(s_a)_{a\in A}\downarrow x$ are dual.*

The following notion of convergence in X, for the case of sequences, was introduced by I. Beg and M. Islam [3] (the primary version for posets is due, in essence, to Birkhoff [10]). Below, we summarize some basic notions and results from [3], where we refer the reader for more details.

Definition 6 ([3]). *Let X be a foset. We say that a net $(s_a)_{a\in A}$ in X is order-converging or (o_F)-converging to a point $x \in X$ and we write $(s_a)_{a\in A} \xrightarrow{o_F} x$ if there exists a pair of nets $(u_a)_{a\in A}$ and $(v_a)_{a\in A}$, in X, such that*

(1) $(u_a)_{a\in A}\uparrow x$, $(v_a)_{a\in A}\downarrow x$ *and*
(2) $\mu(u_a, s_a) > 1/2$ *and* $\mu(s_a, v_a) > 1/2$, *for all* $a \in A$.

Proposition 1 ([3]). *Let X be a foset. The o_F-convergence of sequences in X has the following properties:*

(1) *If* $(s_n)_{n\in\mathbb{N}}\uparrow$, *then* $(s_n)_{n\in\mathbb{N}} \xrightarrow{o_F} x$ *if and only if* $(s_n)_{n\in\mathbb{N}}\uparrow x$.
(2) *If* $(s_n)_{n\in\mathbb{N}}\downarrow$, *then* $(s_n)_{n\in\mathbb{N}} \xrightarrow{o_F} x$ *if and only if* $(s_n)_{n\in\mathbb{N}}\downarrow x$.
(3) *Any o_F-convergent sequence is bounded.*
(4) *If* $\mu(s_n, t_n) > 1/2$, *for all* $n \in \mathbb{N}$, *and* $(s_n)_{n\in\mathbb{N}} \xrightarrow{o_F} x$, $(t_n)_{n\in\mathbb{N}} \xrightarrow{o_F} y$, *then* $\mu(x, y) > 1/2$.
(5) *If* $(s_n)_{n\in\mathbb{N}} \xrightarrow{o_F} x$ *and* $(s_n)_{n\in\mathbb{N}} \xrightarrow{o_F} y$, *then* $x = y$.
(6) *If* $(s_n)_{n\in\mathbb{N}} \xrightarrow{o_F} x$, *then any subsequence of* $(s_n)_{n\in\mathbb{N}}$ *o_F-converges to the same limit.*
(7) *If* $\mu(t_n, s_n) > 1/2$ *and* $\mu(s_n, r_n) > 1/2$, *for all* $n \in \mathbb{N}$ *and* $(t_n)_{n\in\mathbb{N}} \xrightarrow{o_F} x$, $(r_n)_{n\in\mathbb{N}} \xrightarrow{o_F} x$, *then* $(s_n)_{n\in\mathbb{N}} \xrightarrow{o_F} x$.

Definition 7 ([3]). *A (real) linear space X is said to be a fuzzy ordered linear space if X is a foset and the following conditions both hold:*

(1) *If* $x_1, x_2 \in X$ *such that* $\mu(x_1, x_2) > 1/2$, *then* $\mu(x_1, x_2) \leqslant \mu(x_1 + x, x_2 + x)$, *for all* $x \in X$.
(2) *If* $x_1, x_2 \in X$, *such that* $\mu(x_1, x_2) > 1/2$, *then* $\mu(x_1, x_2) \leqslant \mu(rx_1, rx_2)$, *for all* $r \geqslant 0$.

Proposition 2 ([3]). *Let X be a fuzzy ordered linear space. The o_F-convergence of sequences in X has the following properties:*

(1) *If* $(s_n)_{n\in\mathbb{N}} \xrightarrow{o_F} x$ *and* $(t_n)_{n\in\mathbb{N}} \xrightarrow{o_F} y$, *then* $(s_n + t_n)_{n\in\mathbb{N}} \xrightarrow{o_F} x + y$.
(2) *If* $(s_n)_{n\in\mathbb{N}} \xrightarrow{o_F} x$ *and* $r \in \mathbb{R}$, *then* $(rs_n)_{n\in\mathbb{N}} \xrightarrow{o_F} rx$.

3. *o*-Convergence with Respect to the Fuzzy Order Relation

In this section we will introduce and study a notion of convergence in fosets, named o-convergence (the primary version for posets is due, in essence, to McShane [14]). The importance of o-convergence lies in the fact that, in addition to successfully generalizing o_F-convergence, is closer to our understanding of the concept of "convergence", as we will see below.

Definition 8. *Let X be a foset. We say that a net $(s_a)_{a\in A}$ in X is o-converging to a point $x \in X$ and we write $(s_a)_{a\in A} \xrightarrow{o} x$ if there exist a directed to the right subset D of X and a directed to the left subset F of X, such that*

(1) $\vee D = \wedge F = x$ and
(2) for every $d \in D$ and every $f \in F$, $\mu(d, s_a) > 1/2$ and $\mu(s_a, f) > 1/2$, eventually.

Proposition 3. *Let X be a foset, $(s_a)_{a\in A}$ be a net in X and $x \in X$. If $(s_a)_{a\in A} \xrightarrow{o_F} x$, then $(s_a)_{a\in A} \xrightarrow{o} x$.*

Proof. By hypothesis there exists a pair of nets $(u_a)_{a\in A}$ and $(v_a)_{a\in A}$, in X, such that

(a) $(u_a)_{a\in A}\uparrow x$, $(v_a)_{a\in A}\downarrow x$ and
(b) $\mu(u_a, s_a) > 1/2$ and $\mu(s_a, v_a) > 1/2$, for all $a \in A$.

Consider the ranges of those nets i.e., the subsets

$$D = \{u_a : a \in A\} \quad \text{and} \quad F = \{v_a : a \in A\}$$

of X. We will show that D is directed to the right (similarly, it can be proved that F is directed to the left). Let $E = \{u_{a_1}, \ldots, u_{a_m}\}$ be a finite subset of D. We will show that $D \cap U(E) \neq \emptyset$. Indeed, since A is an ordinary directed set, there exists $a_0 \in A$ such that $a_0 \geqslant a_i$, for all $i = 1, \ldots, m$. Therefore, $\mu(u_{a_i}, u_{a_0}) > 1/2$, for all $i = 1, \ldots, m$. The last implies that $u_{a_0} \in U(E)$. Evidently, $u_{a_0} \in D$. Furthermore, by hypothesis $\vee D = \wedge F = x$, with respect to the fuzzy order on X. Let now $d \in D$ and $f \in F$ be arbitrary. Then, there exist $a_j, a_k \in A$ such that $d = u_{a_j}$ and $f = v_{a_k}$. Let $a_l \in A$ such that $a_l \geqslant a_j$ and $a_l \geqslant a_k$. Subsequently, $\mu(u_{a_l}, u_a) > 1/2$, for all $a \geqslant a_l$. Because $\mu(u_a, s_a) > 1/2$, for all $a \in A$ and $\mu(u_{a_j}, u_{a_l}) > 1/2$, transitivity yields $\mu(u_{a_j}, s_a) > 1/2$, for all $a \geqslant a_l$. That is $\mu(d, s_a) > 1/2$, for all $a \geqslant a_l$. Analogously, we have $\mu(s_a, f) > 1/2$, for all $a \geqslant a_l$. $\square$

The following examples shows that the converse implication of Proposition 3 does not hold.

Example 1. *Let the set $X = \{a, b, c\}$. Define $\mu : X \times X \to [0, 1]$ by*

$$\mu(x,y) = \begin{cases} 1, & \text{if } x = y \\ 2/3, & \text{if } x = a \text{ and } y = c \\ 3/4, & \text{if } x = b \text{ and } y = c \\ 0, & \text{otherwise.} \end{cases}$$

One can easily check that μ is a fuzzy order relation on X. Let now the sequence $(s_n)_{n\in\mathbb{N}}$, in X, defined by

$$s_n = \begin{cases} a, & \text{if } n = 1 \\ b, & \text{if } n = 2 \\ c, & \text{otherwise.} \end{cases}$$

The subsets $D = \{c\}$ and $F = \{c\}$ of X satisfy all the conditions of the Definition 8 that determines the convergence $(s_n)_{n\in\mathbb{N}} \xrightarrow{o} c$. However, $(s_n)_{n\in\mathbb{N}}$ is not bounded, since it is not bounded below and, therefore, by Proposition 1 (3) $(s_n)_{n\in\mathbb{N}}$ does not o_F-converge.

Example 2. *Let the (real) linear space $X = \mathbb{R}^2$. Define $\mu : X \times X \to [0,1]$ by*

$$\mu(x,y) = \begin{cases} 1, & \text{if } x_1 = y_1 \text{ and } x_2 = y_2 \\ 2/3, & \text{if } x_1 = y_1 \text{ and } x_2 < y_2 \\ 4/5, & \text{if } x_1 < y_1 \text{ and } x_2 \leqslant y_2 \\ 0, & \text{otherwise,} \end{cases}$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. It is straightforward to verify that μ is a fuzzy order relation on X which satisfies the properties of Definition 7. Therefore, X is a fuzzy ordered linear space. Let $\mathbb{Z}_+$ be the set of positive integers ordered as follows

$$1 < 3 < 5 < 7 < \ldots < 2 < 4 < 6 < 8 < \ldots.$$

Clearly, $\mathbb{Z}_+$ is a directed set. Let now $(s_n)_{n\in\mathbb{Z}_+}$ be the net, in X, defined by

$$s_n = \begin{cases} (1/n, 1/n), & \text{if } n \text{ is even} \\ (n,n), & \text{if } n \text{ is odd.} \end{cases}$$

By Definition 8, $(s_n)_{n\in\mathbb{Z}_+} \xrightarrow{o} (0,0)$. However, $(s_n)_{n\in\mathbb{Z}_+}$ does not o_F-converge to $(0,0)$. Indeed, suppose that $(s_n)_{n\in\mathbb{Z}_+} \xrightarrow{o_F} (0,0)$. Subsequently, by Definition 6, there exists a pair of nets $(u_n)_{n\in\mathbb{Z}_+}$ and $(v_n)_{n\in\mathbb{Z}_+}$, in X, such that

(a) $(u_n)_{n\in\mathbb{Z}_+}\uparrow(0,0)$, $(v_n)_{n\in\mathbb{Z}_+}\downarrow(0,0)$ *and*
(b) $\mu(u_n, s_n) > 1/2$ *and* $\mu(s_n, v_n) > 1/2$, *for all* $n \in \mathbb{Z}_+$.

By condition (b), $\mu(s_1, v_1) > 1/2$. Let $v_1 = (v_{1,1}, v_{1,2})$ and let $n_0 > 1$ be a positive odd integer such that $v_{1,1} < n_0$ and $v_{1,2} < n_0$. Subsequently, again, by condition (b) $\mu(s_{n_0}, v_{n_0}) > 1/2$, where $s_{n_0} = (n_0, n_0)$. Let $v_{n_0} = (v_{n_0,1}, v_{n_0,2})$. It follows that $v_{n_0,1} \geqslant n_0$ and $v_{n_0,2} \geqslant n_0$, i.e., $\mu(v_1, v_{n_0}) > 1/2$, which contradicts the fact that by condition (a) $(v_n)_{n\in\mathbb{Z}_+}$ is a decreasing net since $v_1 \neq v_{n_0}$. Thus, the net $(s_n)_{n\in\mathbb{Z}_+}$ does not o_F-converge to $(0,0)$. Note that the subnet $(s_{2n})_{n\in\mathbb{Z}_+}$ of $(s_n)_{n\in\mathbb{Z}_+}$ o_F-converges to $(0,0)$. This fact demostrates that, in contrast to our common belief, the existense of additional terms in the "tail" of the net affects its o_F-convergence. Obviously, o-convergence overcomes this pathology.

Remark 1. *From the previous examples, we observe that boundness is a property that is not retained in the case of o-convergence, not even for sequences.*

The o-convergence has, as we will see next, similar properties to o_F-convergence.

Proposition 4. *Let $(s_a)_{a\in A}$ be a net in a foset X. Subsequently,*

(1) $(s_a)_{a\in A}\uparrow x$ *if and only if* $(s_a)_{a\in A}$ *is increasing and* $(s_a)_{a\in A} \xrightarrow{o} x$;
(2) $(s_a)_{a\in A}\downarrow x$ *if and only if* $(s_a)_{a\in A}$ *is decreasing and* $(s_a)_{a\in A} \xrightarrow{o} x$.

Proof. (1) Let $(s_a)_{a\in A}$ be an increasing net and $(s_a)_{a\in A} \xrightarrow{o} x$. We will prove that $x = \vee\{s_a : a \in A\}$ i.e.,

(a) $x \in U(\{s_a : a \in A\})$ and
(b) $y \in U(\{s_a : a \in A\})$ implies $y \in U(x)$.

Fix $b \in A$. By hypothesis there exist a directed to the right subset D of X and a directed to the left subset F of X such that

(c) $\vee D = \wedge F = x$ and
(d) for every $d \in D$ and every $f \in F$, $\mu(d, s_a) > 1/2$ and $\mu(s_a, f) > 1/2$, eventually.

Let $f \in F$ be arbitrary. There exists $a_0 \in A$ such that $\mu(s_a, f) > 1/2$, for every $a \geqslant a_0$. Take any $a_1 \in A$, such that $a_1 \geqslant b$ and $a_1 \geqslant a_0$. It follows that, $\mu(s_b, s_{a_1}) > 1/2$ and $\mu(s_{a_1}, f) > 1/2$. By transitivity we have $\mu(s_b, f) > 1/2$. Since $f \in F$ was arbitrarily chosen, $\mu(s_b, f) > 1/2$, for all $f \in F$. Thus, $s_b \in L(F)$. Since $\wedge F = x$, $s_b \in L(x)$ i.e., $\mu(s_b, x) > 1/2$. The last conclusion does not depend on the choice of $b \in A$, so $\mu(s_b, x) > 1/2$, for all $b \in A$. Thus, $x \in U(\{s_a : a \in A\})$. Let now $y \in U(\{s_a : a \in A\})$. Subsequently, $\mu(s_a, y) > 1/2$, for all $a \in A$. Let $d \in D$ be arbitrary. There exists $a_0' \in A$ such that $\mu(d, s_a) > 1/2$, for all $a \geqslant a_0'$. Thus, by transitivity $\mu(d, y) > 1/2$. Since $d \in D$ was arbitrarily chosen, $\mu(d, y) > 1/2$, for all $d \in D$, which further implies that $y \in U(D)$. On account of $\vee D = x$, $y \in U(x)$. Therefore, conditions (a) and (b) are fulfilled, so $x = \bigvee\{s_a : a \in A\}$. Hence, $(s_a)_{a\in A}\uparrow x$.

Conversely, let $(s_a)_{a\in A}\uparrow x$. Afterwards, $(s_a)_{a\in A}$ is increasing and therefore, by Proposition 1 (1) $(s_a)_{a\in A} \xrightarrow{o_F} x$. Hence, by Proposition 3 $(s_a)_{a\in A} \xrightarrow{o} x$.
The proof of (2) is similar to the proof of (1). □

Proposition 5. *Let $(s_a)_{a\in A}$ and $(t_a)_{a\in A}$ be nets in a foset X and $x, y \in X$. If $\mu(s_a, t_a) > 1/2$, for all $a \in A$ and $(s_a)_{a\in A} \xrightarrow{o} x$, $(t_a)_{a\in A} \xrightarrow{o} y$, then $\mu(x, y) > 1/2$.*

Proof. By hypothesis there exist directed to the right subsets D_s, D_t of X and directed to the left subsets F_s, F_t of X such that

(a) $\vee D_s = \wedge F_s = x$ and $\vee D_t = \wedge F_t = y$;
(b) for every $d_s \in D_s$ and every $f_s \in F_s$, $\mu(d_s, s_a) > 1/2$ and $\mu(s_a, f_s) > 1/2$, eventually;
(c) for every $d_t \in D_t$ and every $f_t \in F_t$, $\mu(d_t, t_a) > 1/2$ and $\mu(t_a, f_t) > 1/2$, eventually.

Fix $d_s \in D_s$ and let $f_t \in F_t$ be arbitrary. There exists $a_0 \in A$ such that, for all $a \geqslant a_0$, $\mu(d_s, s_a) > 1/2$, $\mu(s_a, t_a) > 1/2$ and $\mu(t_a, f_t) > 1/2$. Transitivity yields that $\mu(d_s, f_t) > 1/2$. Since $f_t \in F_t$ was arbitrarily chosen, $\mu(d_s, f_t) > 1/2$, for all $f_t \in F_t$. Thus, $d_s \in L(F_t)$. Furthermore, $\wedge F_t = y$ yields $d_s \in L(y)$ i.e., $\mu(d_s, y) > 1/2$. The last conclusion does not depend on the choice of $d_s \in D_s$, so $\mu(d_s, y) > 1/2$, for all $d_s \in D_s$. Thus, $y \in U(D_s)$. Because $\vee D_s = x$, $y \in U(x)$ i.e., $\mu(x, y) > 1/2$. □

Corollary 1. *Let $(s_a)_{a\in A}$ be a net in a foset X and $x, y \in X$. If $(s_a)_{a\in A} \xrightarrow{o} x$ and $(s_a)_{a\in A} \xrightarrow{o} y$, then $x = y$.*

Proof. Applying Proposition 5 by considering $s_a = t_a$, for all $a \in A$, we get $\mu(x, y) > 1/2$ and $\mu(y, x) > 1/2$. Thus, $\mu(x, y) + \mu(y, x) > 1$. Antisymmetry property yields $x = y$. □

Proposition 6. *Let X be a foset. If $(s_a)_{a\in A} \xrightarrow{o} x$ and $(t_\lambda)_{\lambda\in\Lambda}$ is any subnet of $(s_a)_{a\in A}$, then $(t_\lambda)_{\lambda\in\Lambda} \xrightarrow{o} x$.*

Proof. By hypothesis, there exists a directed to the right subset D of X and a directed to the left subset F of X, such that

(a) $\vee D = \wedge F = x$;
(b) for every $d \in D$ and every $f \in F$, $\mu(d, s_a) > 1/2$ and $\mu(s_a, f) > 1/2$, eventually.

Let now $(t_\lambda)_{\lambda\in\Lambda}$ be a subnet of $(s_a)_{a\in A}$. There exists a function $\varphi : \Lambda \to A$ with the following properties:

(c) $t = s \circ \varphi$, or equivalently, $t_\lambda = s_{\varphi(\lambda)}$, for all $\lambda \in \Lambda$.
(d) For every $a \in A$ there exists $\lambda_0 \in \Lambda$ such that $\varphi(\lambda) \geqslant a$, for all $\lambda \geqslant \lambda_0$.

Let $d \in D$ and $f \in F$ be arbitrary. By condition (b) there exists $a_0 \in A$, such that $\mu(d, s_a) > 1/2$ and $\mu(s_a, f) > 1/2$, for all $a \geqslant a_0$. By condition (d), there exists $\lambda_0' \in \Lambda$ such that $\varphi(\lambda) \geqslant a_0$, for all $\lambda \geqslant \lambda_0'$. Thus, $\mu(d, s_{\varphi(\lambda)}) > 1/2$ and $\mu(s_{\varphi(\lambda)}, f) > 1/2$, for all $\lambda \geqslant \lambda_0'$. Because, $t_\lambda = s_{\varphi(\lambda)}$, $\mu(d, t_\lambda) > 1/2$ and $\mu(t_\lambda, f) > 1/2$, for all $\lambda \geqslant \lambda_0'$. Therefore, the directed to the right subset D of X and the directed to the left subset F of X satisfy all the conditions of the Definition 8 that determines the convergence $(t_\lambda)_{\lambda\in\Lambda} \xrightarrow{o} x$. □

Proposition 7. *Let $(s_a)_{a\in A}$, $(t_a)_{a\in A}$ and $(r_a)_{a\in A}$ be nets in a foset X and $x \in X$. If $\mu(s_a, r_a) > 1/2$ and $\mu(r_a, t_a) > 1/2$, for all $a \in A$ and $(s_a)_{a\in A} \xrightarrow{o} x$, $(t_a)_{a\in A} \xrightarrow{o} x$, then $(r_a)_{a\in A} \xrightarrow{o} x$.*

Proof. By hypothesis, there exist directed to the right subsets D_s, D_t of X and directed to the left subsets F_s, F_t of X such that

(a) $\vee D_s = \vee D_t = \wedge F_s = \wedge F_t = x$;
(b) for every $d_s \in D_s$ and every $f_s \in F_s$, $\mu(d_s, s_a) > 1/2$ and $\mu(s_a, f_s) > 1/2$, eventually; and,
(c) for every $d_t \in D_t$ and every $f_t \in F_t$, $\mu(d_t, t_a) > 1/2$ and $\mu(t_a, f_t) > 1/2$, eventually.

Let $d_s \in D_s$ and $f_t \in F_t$ be arbitrary. There exists $a_0 \in A$ such that, for all $a \geqslant a_0$ we have $\mu(d_s, s_a) > 1/2$, $\mu(s_a, r_a) > 1/2$, $\mu(r_a, t_a) > 1/2$ and $\mu(t_a, f_t) > 1/2$. Transitivity yields that $\mu(d_s, r_a) > 1/2$ and $\mu(r_a, f_t) > 1/2$, for all $a \geqslant a_0$. Therefore, the directed to the right subset D_s of X and the directed to the left subset F_t of X satisfy all the conditions of the Definition 8 that determines the convergence $(r_a)_{a\in A} \xrightarrow{o} x$. $\square$

Proposition 8. *Let X be a fuzzy ordered linear space, $(s_a)_{a\in A}$, $(t_a)_{a\in A}$ be nets in X, $x, y \in X$ and $r \in \mathbb{R}$. Subsequently, the following implications hold.*

(1) If $(s_a)_{a\in A} \xrightarrow{o} x$ and $(t_a)_{a\in A} \xrightarrow{o} y$, then $(s_a + t_a)_{a\in A} \xrightarrow{o} x + y$.
(2) If $(s_a)_{a\in A} \xrightarrow{o} x$, then $(rs_a)_{a\in A} \xrightarrow{o} rx$.

Proof. (1) By hypothesis there exist directed to the right subsets D_s, D_t of X and directed to the left subsets F_s, F_t of X such that

(a) $\vee D_s = \wedge F_s = x$ and $\vee D_t = \wedge F_t = y$;
(b) for every $d_s \in D_s$ and every $f_s \in F_s$, $\mu(d_s, s_a) > 1/2$ and $\mu(s_a, f_s) > 1/2$, eventually; and,
(c) for every $d_t \in D_t$ and every $f_t \in F_t$, $\mu(d_t, t_a) > 1/2$ and $\mu(t_a, f_t) > 1/2$, eventually.

We consider the following subsets of X:

$$D = \{d_s + d_t : d_s \in D_s \text{ and } d_t \in D_t\} \quad \text{and} \quad F = \{f_s + f_t : f_s \in F_s \text{ and } f_t \in F_t\}.$$

We will prove that D is directed to the right (similarly, it can be proved that F is directed to the left). Let $E = \{d_{s_1} + d_{t_1}, \ldots, d_{s_k} + d_{t_k}\}$ be a finite subset of D. We will show that $D \cap U(E) \neq \emptyset$. Let the finite subsets $E_s = \{d_{s_1}, \ldots, d_{s_k}\}$ and $E_t = \{d_{t_1}, \ldots, d_{t_k}\}$ of D_s and D_t, respectively. Since D_s and D_t are directed to the right subsets of X, we have that $D_s \cap U(E_s) \neq \emptyset$ and $D_t \cap U(E_t) \neq \emptyset$. The last yields that there exist $d_s \in D_s$ and $d_t \in D_t$ such that $d_s \in U(E_s)$ and $d_t \in U(E_t)$ i.e., for all $i = 1, \ldots, k$,

$$\mu(d_{s_i}, d_s) > 1/2 \quad \text{and} \quad \mu(d_{t_i}, d_t) > 1/2.$$

Therefore, by [3] (Remark 4.4), for all $i = 1, \ldots, k$,

$$\mu(d_{s_i} + d_{t_i}, d_s + d_t) > 1/2.$$

Thus, $d_s + d_t \in U(E)$ and so $D \cap U(E) \neq \emptyset$.

We will prove that $\vee D = x + y$. (Similarly, it can be proved that $\wedge F = x + y$.) Indeed, by [3] (Proposition 4.8),

$$\vee D = \bigvee_{\substack{d_s \in D_s \\ d_t \in D_t}} (d_s + d_t) = \bigvee_{d_s \in D_s} d_s + \bigvee_{d_t \in D_t} d_t = \vee D_s + \vee D_t = x + y.$$

Let now $d_s + d_t \in D$ and $f_s + f_t \in F$ be arbitrary. By condition (b) there exists $a_1 \in A$ such that, for all $a \geqslant a_1$,

$$\mu(d_s, s_a) > 1/2 \quad \text{and} \quad \mu(s_a, f_s) > 1/2.$$

By condition (c), there exists $a_2 \in A$ such that, for all $a \geqslant a_2$,

$$\mu(d_t, t_a) > 1/2 \quad \text{and} \quad \mu(t_a, f_t) > 1/2.$$

Let $a_0 \in A$, such that $a_0 \geqslant a_1$ and $a_0 \geqslant a_2$. By [3] (Remark 4.4), for all $a \geqslant a_0$,

$$\mu(d_s + d_t, s_a + t_a) > 1/2 \quad \text{and} \quad \mu(s_a + t_a, f_s + f_t) > 1/2.$$

Hence, the directed to the right subset D of X and the directed to the left subset F of X satisfy all of the conditions of the Definition 8 determines the convergence $(s_a + t_a)_{a\in A} \xrightarrow{o} x + y$.

(2) By hypothesis, there exist a directed to the right subset D of X and a directed to the left subset F of X, such that

(d) $\vee D = \wedge F = x$ and
(e) for every $d \in D$ and every $f \in F$, $\mu(d, s_a) > 1/2$ and $\mu(s_a, f) > 1/2$, eventually.

We consider the following subsets of X:

$$rD = \{rd : d \in D\} \quad \text{and} \quad rF = \{rf : f \in F\}.$$

Let $r > 0$. We will prove that rD is directed to the right (similarly, it can be proved that rF is directed to the left). Let $rE = \{rd_1, \dots, rd_k\}$ be a finite subset of rD, where $E = \{d_1, \dots, d_k\}$ is a finite subset of D. We will show that $rD \cap U(rE) \neq \emptyset$. Because D is directed to the right subset of X, we have that $D \cap U(E) \neq \emptyset$. The last yields that there exists $d \in D$ such that $d \in U(E)$ i.e.,

$$\mu(d_i, d) > 1/2, \text{ for all } i = 1, \dots, k.$$

Therefore, by Definition 7 (2),

$$\mu(rd_i, rd) > 1/2, \text{ for all } i = 1, \dots, k.$$

Thus, $rd \in U(rE)$ and so $rD \cap U(rE) \neq \emptyset$.

We will prove that $\vee rD = rx$ (similarly, it can be proved that $\wedge rF = rx$). Indeed, by [3] [Proposition 4.10],

$$\vee rD = \bigvee_{d\in D} (rd) = r\left(\bigvee_{d\in D} d\right) = r(\vee D) = rx.$$

Let now $rd \in rD$ and $rf \in rF$ be arbitrary. By condition (e) there exists $a_1 \in A$, such that, for all $a \geqslant a_1$,

$$\mu(d, s_a) > 1/2 \quad \text{and} \quad \mu(s_a, f) > 1/2.$$

Thus, for all $a \geqslant a_1$,

$$\mu(rd, rs_a) > 1/2 \quad \text{and} \quad \mu(rs_a, rf) > 1/2.$$

Hence, the directed to the right subset rD of X and the directed to the left subset rF of X satisfy all of the conditions of the Definition 8 that determine the convergence $(rs_a)_{a\in A} \xrightarrow{o} rx$.

Let $r < 0$. We will prove that rD is directed to the left (similarly, it can be proved that rF is directed to the right). Let $rE = \{rd_1, \dots, rd_k\}$ be a finite subset of rD, where $E = \{d_1, \dots, d_k\}$ is a finite subset of D. We will show that $rD \cap L(rE) \neq \emptyset$. Because D is directed to the right subset of X, we have that $D \cap U(E) \neq \emptyset$. The last yields that there exists $d \in D$ such that $d \in U(E)$ i.e.,

$$\mu(d_i, d) > 1/2, \text{ for all } i = 1, \dots, k.$$

Therefore, by [3] (Proposition 4.5 (4)),

$$\mu(rd, rd_{i},) > 1/2, \text{ for all } i = 1, \ldots, k.$$

Thus, $rd \in L(rE)$ and so $rD \cap L(rE) \neq \emptyset$.

We will prove that $\wedge rD = rx$ (similarly, it can be proved that $\vee rF = rx$). Indeed, by [3] (Corollary 4.11),

$$\wedge rD = \bigwedge_{d \in D} (rd) = r\left(\bigvee_{d \in D} d\right) = r(\vee D) = rx.$$

Let now $rd \in rD$ and $rf \in rF$ be arbitrary. By condition (e), there exists $a_1 \in A$, such that, for all $a \geqslant a_1$,

$$\mu(d, s_a) > 1/2 \quad \text{and} \quad \mu(s_a, f) > 1/2.$$

Thus, by [3] (Proposition 4.5 (4)), for all $a \geqslant a_1$,

$$\mu(rf, rs_a) > 1/2 \quad \text{and} \quad \mu(rs_a, rd) > 1/2.$$

Hence, the directed to the left subset rD of X and the directed to the right subset rF of X satisfy all of the conditions of the Definition 8 that determines the convergence $(rs_a)_{a \in A} \xrightarrow{o} rx$.

The case $r = 0$ is trivial. □

4. Coincidence of the Two Notions of Convergence

In this section, we will show that in the special context of complete F-lattices the notion of o-convergence can be restated in terms of the notions of limit inferior and limit superior, with respect to the fuzzy order relation, which will be introduced in the sequel. Apart from the fact that o-convergence is characterized by another form that may be sometimes more useful and convenient, in this way it can also be shown that it reduces to o_F-convergence.

Definition 9. *Let $(s_a)_{a \in A}$ be a net in a complete F-lattice X. Subsequently, we may define the related nets $(u_a)_{a \in A}$ and $(v_a)_{a \in A}$ such that*

$$u_a = \bigwedge_{b \geqslant a} s_b \quad \text{and} \quad v_a = \bigvee_{b \geqslant a} s_b.$$

The limit inferior and the limit superior (or lower limit and upper limit) of the net $(s_a)_{a \in A}$, denoted by $\liminf s_a$ and $\limsup s_a$, respectively, are defined by

$$\liminf s_a = \bigvee_{a \in A} u_a = \bigvee_{a \in A} \bigwedge_{b \geqslant a} s_b$$

and

$$\limsup s_a = \bigwedge_{a \in A} v_a = \bigwedge_{a \in A} \bigvee_{b \geqslant a} s_b.$$

Proposition 9. *Let $(s_a)_{a \in A}$ be a net in a complete F-lattice X. If u_a and v_a are the nets mentioned in Definition 9, then*

(1) $\mu(u_a, s_a) > 1/2$ and $\mu(s_a, v_a) > 1/2$, for all $a \in A$;
(2) $(u_a)_{a \in A}$ is increasing and $(v_a)_{a \in A}$ is decreasing; and,
(3) $\mu(\liminf s_a, \limsup s_a) > 1/2$.

Proof. (1) Let $a \in A$ be arbitrary. Because $u_a = \bigwedge_{b \geqslant a} s_b$, $u_a \in L(\{s_b : b \geqslant a\})$. Therefore, $\mu(u_a, s_a) > 1/2$. Similarly, we can prove that $\mu(s_a, v_a) > 1/2$, for all $a \in A$.

(2) Let $a_1, a_2 \in A$ be arbitrary and suppose that $a_1 \leqslant a_2$. Since $u_{a_1} = \bigwedge\limits_{b \geqslant a_1} s_b$, $u_{a_1} \in L(\{s_b : b \geqslant a_1\})$. Hence, $\mu(u_{a_1}, s_b) > 1/2$, for all $b \geqslant a_1$. Therefore, $\mu(u_{a_1}, s_b) > 1/2$, for all $b \geqslant a_2$, which further implies $u_{a_1} \in L(\{s_b : b \geqslant a_2\})$. Because $u_{a_2} = \bigwedge\limits_{b \geqslant a_2} s_b$, $u_{a_1} \in L(u_{a_2})$ i.e., $\mu(u_{a_1}, u_{a_2}) > 1/2$. Similarly, we can prove that $\mu(v_{a_2}, v_{a_1}) > 1/2$.

(3) By (2) $(u_a)_{a \in A}$ is increasing. By Definition 9, $\liminf s_a = \bigvee\limits_{a \in A} u_a$, thus $(u_a)_{a \in A} \uparrow \liminf s_a$. By Proposition 4 (1), $(u_a)_{a \in A} \xrightarrow{o} \liminf s_a$. Similarly, we can prove that $(v_a)_{a \in A} \xrightarrow{o} \limsup s_a$. Taking into account (1), transitivity yields $\mu(u_a, v_a) > 1/2$, for all $a \in A$. Therefore, Proposition 5 applies and so, $\mu(\liminf s_a, \limsup s_a) > 1/2$. $\square$

Lemma 1. *Let $(u_a)_{a \in A}$ be a net in a foset X and $a_0 \in A$. Then $(u_a)_{a \in A} \uparrow x$ (resp. $(u_a)_{a \in A} \downarrow x$) implies $\bigvee\limits_{a \geqslant a_0} u_a = x$ (resp. $\bigwedge\limits_{a \geqslant a_0} u_a = x$).*

Proof. Let $a_0 \in A$. Because $x \in U(\{u_a : a \in A\})$, $\mu(u_a, x) > 1/2$, for all $a \in A$ and thus $\mu(u_a, x) > 1/2$, for all $a \geqslant a_0$. Hence, $x \in U(\{u_a : a \geqslant a_0\})$. Let $y \in U(\{u_a : a \geqslant a_0\})$. Fix any $a \in A$. There exists $a_1 \in A$ such that $a_1 \geqslant a_0$ and $a_1 \geqslant a$. Then, $\mu(u_a, u_{a_1}) > 1/2$ and $\mu(u_{a_1}, y) > 1/2$. By transitivity, $\mu(u_a, y) > 1/2$. However, $a \in A$ was arbitrarily chosen, thus $\mu(u_a, y) > 1/2$, for all $a \in A$ i.e., $y \in U(\{u_a : a \in A\})$. Because, $\bigvee\limits_{a \in A} u_a = x$, $y \in U(x)$. Therefore, $\bigvee\limits_{a \geqslant a_0} u_a = x$. The proof of the other implication is analogous. $\square$

Theorem 3. *Let $(s_a)_{a \in A}$ be a net in a complete F-lattice X and $x \in X$. Subsequently, the following conditions are equivalent:*

(1) $(s_a)_{a \in A} \xrightarrow{o} x$.
(2) $\liminf s_a = \limsup s_a = x$.
(3) $(s_a)_{a \in A} \xrightarrow{o_F} x$.

Proof. (1) $\Rightarrow$ (2) Let $(u_a)_{a \in A}$ and $(v_a)_{a \in A}$ be the nets mentioned in Definition 9 with $u = \bigvee\limits_{a \in A} u_a$ and $v = \bigwedge\limits_{a \in A} v_a$. Subsequently, By Proposition 9 (2)

$$(u_a)_{a \in A} \uparrow u \quad \text{and} \quad (v_a)_{a \in A} \downarrow v.$$

It will suffice to prove that $u = v = x$. We will prove that $u = x$ (analogously, it can be proved that $v = x$). By hypothesis there exist a directed to the right subset D of X and a directed to the left subset F of X such that

(a) $\vee D = \wedge F = x$ and
(b) for every $d \in D$ and every $f \in F$, $\mu(d, s_a) > 1/2$ and $\mu(s_a, f) > 1/2$, eventually.

Let $d \in D$ be arbitrary. By condition (b), there exists $a_1 \in A$, such that $\mu(d, s_a) > 1/2$, for all $a \geqslant a_1$ i.e., $d \in L(\{s_a : a \geqslant a_1\})$. Taking into account that $\bigwedge\limits_{a \geqslant a_1} s_a = u_{a_1}$, $d \in L(u_{a_1})$. Thus, $\mu(d, u_{a_1}) > 1/2$. Because $u = \bigvee\limits_{a \in A} u_a$, $\mu(u_a, u) > 1/2$, for all $a \in A$ and so $\mu(u_{a_1}, u) > 1/2$. Consequently, transitivity yields, $\mu(d, u) > 1/2$. The last conclusion does not depend on the choice of $d \in D$, so $\mu(d, u) > 1/2$, for all $d \in D$. Thus, $u \in U(D)$, which further implies $u \in U(x)$. Therefore, $\mu(x, u) > 1/2$.

Now, let $f \in F$ be arbitrary. By condition (b), there exists $a_2 \in A$, such that, $\mu(s_a, f) > 1/2$, for all $a \geqslant a_2$. Because $\mu(u_a, s_a) > 1/2$, for all $a \in A$, transitivity yields $\mu(u_a, f) > 1/2$, for all $a \geqslant a_2$. Thus, $f \in U(\{u_a : a \geqslant a_2\})$. By Lemma 1, $u = \bigvee\limits_{a \geqslant a_2} u_a$ which in turn implies that $f \in U(u)$. Hence, $\mu(u, f) > 1/2$. The last conclusion does not depend on the choice of $f \in F$, so $\mu(u, f) > 1/2$, for all $f \in F$. Thus, $u \in L(F)$, which further implies $u \in L(x)$. Therefore, $\mu(u, x) > 1/2$.

Consequently, $\mu(x,u) + \mu(u,x) > 1$ and, thus, by antisymmetry, $u = x$.

(2) $\Rightarrow$ (3) Let $(u_a)_{a\in A}$ and $(v_a)_{a\in A}$ be the nets that are mentioned in Definition 9. By hypothesis,

$$\bigvee_{a\in A} u_a = \bigwedge_{a\in A} v_a = x.$$

Therefore, $(u_a)_{a\in A}$ and $(v_a)_{a\in A}$ satisfy all of the conditions in the Definition 6, which determines the convergence $(s_a)_{a\in A} \xrightarrow{o_F} x$.

(3) $\Rightarrow$ (1) See Proposition 3. □

5. Conclusions

In the present paper, o-convergence is inserted as a generalization of o_F-convergence. Many of the properties of o-convergence are proved to be much alike the properties of o_F-convergence with the advantage that the notion of o-convergence is, in our opinion, closer to our initiation to the concept of "convergence". In addition, the coincidence of the two notions is established in the area of complete F-lattices. Future research options may investigate the notion of o-convergence and its applications in the context of fuzzy Riesz spaces (see [18]) and, in a more theoretical perspective, exploring the topological nature of the two notions of convergence and their correlation, while taking into account the relevant induced topologies, in different types of fosets.

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