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w-Distances on Fuzzy Metric Spaces and Fixed Points

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Abstract: We propose a notion of w-distance for fuzzy metric spaces, in the sense of Kramosil and Michalek, which allows us to obtain a characterization of complete fuzzy metric spaces via a suitable fixed point theorem that is proved here. Our main result provides a fuzzy counterpart of a renowned characterization of complete metric spaces due to Suzuki and Takahashi.

Keywords: fuzzy metric space; complete; w-distance; fixed point

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1. Introduction

In their relevant article [1], Kada, Suzuki and Takahashi gave and discussed a concept of w-distance for metric spaces. In fact, they generalized several important theorems including, among others, Caristi's fixed point theorem and Ekeland's Variational Principle, with the help of this concept. Almost simultaneously, Suzuki and Takahashi [2] obtained a characterization of metric completeness through a generalization of the Banach Contraction Principle that uses w-distances. Since then, many authors have used and extended w-distances, mainly in the context of the fixed point theory and from different views (see, e.g., [3–10] and their bibliographies).

We remind that a w-distance on a metric space (S, ρ) is a function $w : S \times S \to R^+$ (the set of non-negative real numbers) such that for every $a, b, c \in S$ the following conditions hold:

(w1) $w(a, b) \leq w(a, c) + w(c, b)$;
(w2) $w(a, \cdot) : S \to R^+$ is a lower semicontinuous function;
(w3) for each $\varepsilon > 0$ there is $\delta > 0$ such that $w(a, b) \leq \delta$ and $w(a, c) \leq \delta$ imply $\rho(b, c) \leq \varepsilon$.

Obviously, every metric ρ on a set S is a w-distance on the metric space (S, ρ) ([1], Example 1). Other interesting instances of w-distances on a metric space can be seen in [1,2].

In accordance with [2] a weakly contraction on a metric space (S, ρ) is a self map f of S for which there are a w-distance w on (S, ρ) and a constant $r \in (0, 1)$ verifying, for any $a, b \in S$, $w(fa, fb) \leq rw(a, b)$.

Then, Suzuki and Takahashi proved their renowned and aforementioned theorem which should be stated in the following way.

Theorem 1 ([2], Theorem 4)**.** *A necessary and sufficient condition for a metric space to be complete is that every weakly contraction on it has a fixed point.*

In this paper, we propose a notion of w-distance for fuzzy metric spaces, in the sense of Kramosil and Michalek [11], which allows us to obtain a fuzzy counterpart of Suzuki and Takahashi theorem (Theorem 1 above). For our approach, (fuzzy) contractions in the sense of Hicks [12] will play a fundamental role. Thus, in Section 2 we remind some meaningfull notions and properties that we shall use throughout this note. In Section 3 we introduce our notion of w-distance in the setting of fuzzy metric spaces and present several pertinent examples. Section 4 is devoted to prove a fixed point theorem, with the help of w-distances, which will provides the "only if" part for the characterization of fuzzy metric completeness that will be obtained in Section 5.

Our study is mainly motivated by the recent articles [13–15] where fuzzy extensions of the famous characterizations of metric completeness due to Kirk [16], Hu [17] and Subrahmanyam [18], respectively, were obtained.

2. Preliminaries

By ω we denote the set of non-negative integer numbers and by $*$ any continuous t-norm (a deep and extensive study of continuous t-norms may be found in [19]). Many results and examples on fuzzy metric spaces and related structures are given in [20].

As we have point out we shall consider fuzzy metric spaces in the sense of Kramosil and Michalek.

Thus, and following the current terminology (see, e.g., [14,21]), by a fuzzy metric space we mean a triple $(S, \mathsf{m}, *)$ where S is a set, m is a function from $S \times S \times R^+$ to $[0,1]$, i.e., a fuzzy set in $S \times S \times R^+$, and $*$ is a continuous t-norm, such that for every $a, b, c \in S$ the next conditions are satisfied:

(fm1) $\mathsf{m}(a, b, 0) = 0$;
(fm2) $\mathsf{m}(a, b, t) = \mathsf{m}(b, a, t)$ for every $t > 0$;
(fm3) $a = b \Longleftrightarrow \mathsf{m}(a, b, t) = 1$ for every $t > 0$;
(fm4) $\mathsf{m}(a, b, s + t) \geq \mathsf{m}(a, c, s) * \mathsf{m}(c, b, t)$ for every $s, t > 0$;
(fm5) $\mathsf{m}(a, b, \cdot) : R^+ \to [0, 1]$ is a left continuous function.

In this case we say that $(\mathsf{m}, *)$ (or simply m) is a fuzzy metric on S.

It is well known (see ([21], Lemma 4)) that for each $a, b \in S$, $\mathsf{m}(a, b, \cdot)$ is a non-decreasing function on R^+. We shall use this property without explicit reference.

Given a fuzzy metric $(\mathsf{m}, *)$ on a set S put $B(\mathsf{m})(a, r, t) = \{b \in S : \mathsf{m}(a, b, t) > 1 - r\}$ whenever $r \in (0, 1)$ and $t > 0$. Then, the collection $\{B(\mathsf{m})(a, r, t) : a \in S, r \in (0, 1), t > 0\}$ is a base of open sets for a metrizable topology $\tau(\mathsf{m})$ on S, called the topology generated by $(\mathsf{m}, *)$.

Note that a sequence $(a_n)_{n\in\omega}$ converges to an $a \in S$ with respect to $\tau(\mathsf{m})$, if and only if for each $t > 0$, $\mathsf{m}(a, a_n, t) > 1 - t$ eventually.

If a sequence $(a_n)_{n\in\omega}$ converges to an $a \in S$ with respect to $\tau(\mathsf{m})$, we simply write $a_n \to a$ if no confusion arises.

A fuzzy metric space $(S, \mathsf{m}, *)$ is called complete if every Cauchy sequence converges with respect to the topology $\tau(\mathsf{m})$, where a sequence $(a_n)_{n\in\omega}$ in S is a Cauchy sequence assuming that for each $r \in (0, 1)$ and $t > 0$ there is an $n_0 \in \omega$ such that $\mathsf{m}(a_n, a_m, t) > 1 - r$ whenever $n, m \geq n_0$.

At the end of this section we remind the following well-known and paradigmatic example of a fuzzy metric space (see, e.g., ([14], Example 1)).

Example 1. *Given a metric space (S, ρ), denote by m_ρ the fuzzy set in $S \times S \times R^+$ defined as $\mathsf{m}_\rho(a, b, t) = 1$ if $\rho(a, b) < t$, and $\mathsf{m}_\rho(a, b, t) = 0$ if $\rho(a, b) \geq t$. Then $(\mathsf{m}_\rho, *)$ is a fuzzy metric on S for any continuous t-norm $*$, called the induced fuzzy metric. Furthermore, the topology generated by ρ agrees with the topology generated by $(\mathsf{m}_\rho, *)$. We also have that $(S, \mathsf{m}_\rho, *)$ is complete if and only if (S, ρ) is complete.*

3. Fuzzy w-Distances on Fuzzy Metric Spaces

We begin this section by introducing our notion of w-distance for fuzzy metric spaces, after which we will give some remarks and examples to support it.

Definition 1. *A fuzzy w-distance on a fuzzy metric space $(S, \mathsf{m}, *)$ is a fuzzy set w in $S \times S \times R^+$ that satisfies the next conditions for every $a, b, c \in S$:*

(fw1) $\mathsf{w}(a, b, s + t) \geq \mathsf{w}(a, c, s) * \mathsf{w}(c, b, t)$ *for every* $s, t \in R^+$;
(fw2) *if* $a \in S$ *and* $b_n \to b$, *then* $\mathsf{w}(a, b, t + \varepsilon) \geq \limsup_n \mathsf{w}(a, b_n, t)$, *for all* $t > 0$ *and* $\varepsilon \in (0, t)$.

(fw3) *for each* $\varepsilon \in (0,1)$ *there is* $\delta \in (0,1)$ *such that* $\mathsf{w}(a,b,s) \geq 1-\delta$ *and* $\mathsf{w}(a,c,t) \geq 1-\delta$ *imply* $\mathsf{m}(b,c,s+t) \geq 1-\varepsilon$.

Remark 1. *By similarity with the concept of w-distance for metric spaces, we could attempt to reformulate condition (fw2) above as follows: "for each* $a \in S$ *and* $t \in R^+$ *the function* $\mathsf{w}(a,\cdot,t) : S \to R^+$ *is lower semicontinuous". However, Example 4 below provides a useful instance (see the proof of Theorem 4) of a fuzzy w-distance such that* $\mathsf{w}(a,\cdot,t)$ *is not lower semicontinuous for an* $a \in S$*. Notice also that condition (fw2) is neither strange nor artificial; in fact, Grabiec proved in ([21], Lemma 6) that it is satisfied by every fuzzy metric.*

Remark 2. *An antecedent of Definition 1 is in the article [22], where the authors introduced an discussed the concept of r-distance. Hence, and following ([22], Definition 2.1), by an r-distance on a fuzzy metric space* $(S,\mathsf{m},*)$ *we mean a fuzzy set r in* $S \times S \times R^+$ *satisfying conditions (fw1) and (fw3) above and the following one instead of (fw2): "for each* $a \in S$ *and* $t \in R^+$ *the function* $\mathsf{r}(a,\cdot,t) : S \to R^+$ *is continuous". Example 4 (see also Remark 1) shows that condition (fw2) is more appropriate in our framework. We stress that actually the authors defined the concept of r-distance in the slight more general context of Menger probabilistic metric spaces.*

Example 2. *Let* $(S,\mathsf{m},*)$ *be a fuzzy metric space and let T be a closed subset of S. Define a fuzzy set w in* $S \times S \times R^+$ *by* $w(a,b,t) = \mathsf{m}(a,b,t)$ *if* $a,b \in T$ *and* $t \in R^+$*, and* $w(a,b,t) = 0$ *otherwise. We show that w verifies the conditions of Definition 1, so it is a fuzzy w-distance on* $(S,\mathsf{m},*)$.

(fw1) *We first assume that* $a,b,c \in T$*. Hence*

$$\begin{aligned} \mathsf{w}(a,b,s+t) &= \mathsf{m}(a,b,s+t) \geq \mathsf{m}(a,c,s) * \mathsf{m}(c,b,t) \\ &= \mathsf{w}(a,c,s) * \mathsf{w}(a,b,t). \end{aligned}$$

Otherwise, we have $w(a,c,s) = 0$ *or* $w(c,b,t) = 0$*, so* $\mathsf{w}(a,c,s) * \mathsf{w}(a,b,t) = 0$.

(fw2) *Let* $a \in S$, $b_n \to b$, $t > 0$ *and* $\varepsilon \in (0,t)$.

If $a \in S \backslash T$ *or the sequence* $(b_n)_{n\in\omega}$ *is eventually in* $S\backslash T$*, we get* $\mathsf{w}(a,b_n,t) = 0$ *eventually, so* $\limsup_n \mathsf{w}(a,b_n,t) = 0$.

Otherwise, $a \in T$ *and the sequence* $(b_n)_{n\in\omega}$ *has a subsequence* $(b_{n(k)})_{k\in\omega}$ *such that* $b_{n(k)} \in T$ *for all* $k \in \omega$*. Since T is closed we deduce that* $b \in T$*, and hence*
$\mathsf{w}(a,b,t+\varepsilon) = \mathsf{m}(a,b,t+\varepsilon) \geq \limsup_n \mathsf{m}(a,b_n,t) \geq \limsup_n \mathsf{w}(a,b_n,t)$.

(fw3) *Given* $\varepsilon \in (0,1)$*, by the continuity of* $*$*, there is* $\delta \in (0,1)$ *for which* $(1-\delta)*(1-\delta) > 1-\varepsilon$*. If* $\mathsf{w}(a,b,s) \geq 1-\delta$ *and* $w(a,c,t) \geq 1-\delta$*, we get* $a,b,c \in T$, $\mathsf{m}(a,b,s) = \mathsf{w}(a,b,s)$ *and* $\mathsf{m}(a,c,t) = \mathsf{w}(a,c,t)$*. Therefore*
$\mathsf{m}(b,c,s+t) \geq \mathsf{m}(b,a,s) * \mathsf{m}(a,c,t) \geq (1-\delta)*(1-\delta) > 1-\varepsilon$.

Example 3. *Let* $(S,\mathsf{m},*)$ *be a fuzzy metric space. Then* m *is a fuzzy w-distance on* $(S,\mathsf{m},*)$*. Indeed, this fact is an immediate consequence of Example 2 when* $T = S$.

Example 4. *Denote by* $\rho_{|\cdot|}$ *the restriction to* R^+ *of the Euclidean metric. Let* $(\mathsf{m}_{\rho_{|\cdot|}},*)$ *be the induced fuzzy metric as constructed in Example 1, and let* $T := [1,\infty)$*. Since T is closed we can apply the construction done in Example 2 and thus the fuzzy set w in* $R^+ \times R^+ \times R^+$ *given by* $w(a,b,t) = \mathsf{m}_{\rho_{|\cdot|}}(a,b,t)$ *if* $a,b \in T$ *and* $t \in R^+$*, and* $w(a,b,t) = 0$ *otherwise, is a fuzzy w-distance on* $(R^+,\mathsf{m}_{\rho_{|\cdot|}},*)$.

We show that, for $a = 1$ *and any* $t > 0$*, the function* $\mathsf{w}(a,\cdot,t) : R^+ \to R^+$ *is not lower semicontinuous. Obviously* $n/(n+1) \to 1$ *with respect to the topology generated by* $(\mathsf{m}_{\rho_{|\cdot|}},*)$*. Nevertheless, for any* $t > 0$, $\mathsf{w}(a,1,t) = \mathsf{m}_{\rho_{|\cdot|}}(1,1,t) = 1$ *and* $\mathsf{w}(a,n/(n+1),t) = 0$ *because* $n/(n+1) \in R^+\backslash T$ *for all* $n \in \omega$*. Conistenly, for any* $t > 0$*, the function* $\mathsf{w}(a,\cdot,t)$ *is not lower semincontinuous, and thus not continuous.*

The following example provides an efficient method to generate fuzzy w-distances from w-distances on metric spaces.

Example 5. *Let w be a w-distance on a metric space (S,ρ) and let $(S,\mathsf{m}_\rho,*)$ be the fuzzy metric space associated to (S,ρ) as constructed in Example 1. Define a fuzzy set* w *in $S \times S \times R^+$ by*

$\mathsf{w}(a,b,t) = 1$ if $w(a,b) < t$ and $\rho(a,b) < t$, and
$\mathsf{w}(a,b,t) = 0$ otherwise.

We show that w *is a fuzzy w-distance on the fuzzy metric space $(S,\mathsf{m}_\rho,*)$. Indeed,*

(fw1) *If $\mathsf{w}(a,c,s) = 0$ or $\mathsf{w}(c,b,t) = 0$, then $\mathsf{w}(a,c,s) * \mathsf{w}(c,b,t) = 0 \leq \mathsf{w}(a,b,s+t)$.*

Otherwise we have $\mathsf{w}(a,c,s) = \mathsf{w}(c,b,t) = 1$, so $w(a,c) < s$, $\rho(a,c) < s$, $w(c,b) < t$ and $\rho(c,b) < t$. Hence $w(a,b) < s+t$, by condition (w1), and also $\rho(a,b) < s+t$. Therefore $\mathsf{w}(a,b,s+t) = 1 = \mathsf{w}(a,c,s) * \mathsf{w}(c,b,t)$.

(fw2) *Let $a \in S$, $b_n \to b$, $t > 0$ and $\varepsilon \in (0,t)$.*

If $\mathsf{w}(a,b,t+\varepsilon) = 1$, the conclusion is obvious.
If $\mathsf{w}(a,b,t+\varepsilon) = 0$, we get $w(a,b) \geq t+\varepsilon$ or $\rho(a,b) \geq t+\varepsilon$.

In the first case, by condition (w2), there is $n_0 \in \omega$ such that $w(a,b) < \varepsilon + w(a,b_n)$ for all $n \geq n_0$. Hence $w(a,b_n) > t$ for all $n \geq n_0$, so $\limsup_n \mathsf{w}(a,b_n,t) = 0$.
In the second one, the same argument, changing w with ρ, also shows that $\limsup_n \mathsf{w}(a,b_n,t) = 0$.

(fw3) *Given $\varepsilon \in (0,1)$ there is $\delta > 0$ for which condition (w3) is fulfilled. Put $\delta' = \min\{\varepsilon,\delta\}$. Then $\delta' \in (0,1)$ and hence from $\mathsf{w}(a,b,s) \geq 1-\delta'$ and $\mathsf{w}(a,c,t) \geq 1-\delta'$ it follows $\mathsf{w}(a,b,s) = \mathsf{w}(a,c,t) = 1$, i.e., $w(a,b) < s$, $\rho(a,b) < s$, $w(a,c) < t$ and $\rho(a,c) < t$. We get $\rho(b,c) < s+t$, so $\mathsf{m}_\rho(b,c,s+t) = 1 > 1-\varepsilon$.*

Example 6. *Let $S = [1,\infty)$ and let m be the fuzzy set in $S \times S \times R^+$ given by $m(a,b,0) = 0$, and $m(a,b,t) = \min\{a,b\}/\max\{a,b\}$ for all $a,b \in S$ and $t > 0$. It is well known that $(S,m,*)$ is a fuzzy metric space where $*$ is the usual product on [0, 1] (see, e.g., ([20], Example 10.1.3)). Furthermore, the topology $\tau(m)$ agrees with the usual topology on S.*

We show that the fuzzy set w *in $S \times S \times R^+$ given by $\mathsf{w}(a,b,0) = 0$, and $\mathsf{w}(a,b,t) = 1/b$ for all $a,b \in S$, $t > 0$, is a fuzzy w-distance on $(S,m,*)$. Indeed,*

(fw1) *We get, for $s,t > 0$,*

$$\mathsf{w}(a,b,s+t) = 1/b \geq 1/cb = \mathsf{w}(a,c,s)\mathsf{w}(c,b,t) = \mathsf{w}(a,c,s) * \mathsf{w}(a,b,t).$$

(fw2) *Let $a \in S$, $b_n \to b$, $t > 0$ and $\varepsilon \in (0,t)$. Notice that $b_n \to b$ for the usual topology on S and thus $1/b_n \to 1/b$ for the usual topology on S. Therefore*

$$\mathsf{w}(a,b,t+\varepsilon) = 1/b = \lim_n 1/b_n = \lim_n \mathsf{w}(a,b_n,t) = \limsup_n \mathsf{w}(a,b_n,t).$$

(fw3) *Given $\varepsilon \in (0,1)$, suppose $\mathsf{w}(a,b,s) \geq 1-\varepsilon$ and $\mathsf{w}(a,c,t) \geq 1-\varepsilon$. Then $1/b \geq 1-\varepsilon$ and $1/c \geq 1-\varepsilon$. Consequently $m(b,c,t+s) = \min\{b,c\}/\max\{b,c\} \geq 1/\max\{b,c\} \geq 1-\varepsilon$.*

4. A Fixed Point Theorem in Terms of Fuzzy w-Distances

In his article [12], Hicks introduced a relevant notion of contraction under the name of C-contraction. Hicks' notion was called Hicks contraction by several authors (see, e.g., [14]).

A self map f of a fuzzy metric space $(S,\mathsf{m},*)$ is a Hicks contraction provided that there is $r \in (0,1)$ such that, for each $a,b \in S$ and $t > 0$,

$$\mathsf{m}(a,b,t) > 1-t \Longrightarrow \mathsf{m}(fa,fb,rt) > 1-rt.$$

We generalize this notion to our setting in a natural fashion as follows.

Definition 2. *A self map f of a fuzzy metric space $(S, \mathsf{m}, *)$ is a w-Hicks contraction provided that there are a fuzzy w-distance* w *on $(S, \mathsf{m}, *)$ and an $r \in (0,1)$ such that for each $a, b \in S$ and $t > 0$,*

$$\mathsf{w}(a,b,t) > 1 - t \Longrightarrow \mathsf{w}(fa, fb, rt) > 1 - rt.$$

Next we prove a fixed point result in terms of w-Hicks contractions.

Theorem 2. *Every w-Hicks contraction on a complete fuzzy metric space has a fixed point.*

Proof. We point out that the starting idea for the construction of a Cauchy sequence of iterates, constructed below, comes from [23].

Let $(S, \mathsf{m}, *)$ be a complete fuzzy metric space and let f be a w-Hicks contraction on S. Then, there exist a fuzzy w-distance w on S and a constant $r \in (0,1)$ such that for any $a, b \in S$ and $t > 0$,

$$\mathsf{w}(a,b,t) > 1 - t \Longrightarrow \mathsf{w}(fa, fb, rt) > 1 - rt. \tag{1}$$

Fix $t_0 > 1$. Let $a, b \in S$. Since $\mathsf{w}(a, b, t_0) > 1 - t_0$, we get $\mathsf{w}(fa, fb, rt_0) > 1 - rt_0$. Applying again condition (1), we deduce that $\mathsf{w}(f^2a, f^2b, c^2 f_0) > 1 - r^2 t_0$, and following this process we obtain

$$\mathsf{w}(f^n a, f^n b, r^n t_0) > 1 - r^n t_0, \tag{2}$$

for all $n \in \omega$.

Now fix $a_0 \in S$ and put $a_n := f^n a_0$ for all $n \in \omega$.

Let $\varepsilon \in (0,1)$. We are going to prove the following two assertions (A) and (B):

(A) *There exists $k(\varepsilon) \in \omega$ such that $\mathsf{m}(a_n, a_m, \varepsilon) > 1 - \varepsilon$ for all $n, m \in \omega$ with $k(\varepsilon) < n < m$.*

Thus, since ε is arbitrary, the sequence $(a_n)_{n \in \omega}$ will be a Cauchy sequence in $(S, \mathsf{m}, *)$.

(B) $\mathsf{m}(a, fa, \varepsilon) > 1 - \varepsilon$, *where $a \in S$ is the limit of the sequence $(a_n)_{n \in \omega}$.*

Thus, since ε is arbitrary, a will be a fixed point of f.

Let $\delta \in (0,1)$ for which condition (fw3) in Definition 1 holds with respect to $\varepsilon/2$. Choose $k = k(\varepsilon)$ such that $r^k t_0 < \min\{\varepsilon/2, \delta\}$. By inequality (2) we have

$$\mathsf{w}(a_k, a_n, r^k t_0) > 1 - r^k t_0 \quad \text{and} \quad \mathsf{w}(a_k, a_m, r^k t_0) > 1 - r^k t_0,$$

whenever $k < n < m$.

Since $1 - r^k t_0 > 1 - \delta$, it follows from condition (fw3), taking $t = s = r^k t_0$, that

$$\mathsf{m}(a_n, a_m, 2r^k t_0) \geq 1 - \frac{\varepsilon}{2}.$$

Therefore

$$\mathsf{m}(a_n, a_m, \varepsilon) > 1 - \varepsilon,$$

whenever $k < n < m$.

We have shown assertion (A). Consequently $(a_n)_{n \in \omega}$ is a Cauchy sequence in $(S, \mathsf{m}, *)$.

Since $(S, \mathsf{m}, *)$ is complete there exists $a \in X$ such that $a_n \to a$

Next we prove assertion (B).

Since $r^k t_0 < \min\{\varepsilon/2, \delta\}$ we can choose $\eta \in (0, r^{k+1} t_0)$ such that $r^k t_0 + \eta < \min\{\varepsilon/2, \delta\}$. Then, by condition (fw2), in Definition 1,

$$\mathsf{w}(a_k, a, r^k t_0 + \eta) \geq \limsup_n \mathsf{w}(a_k, a_n, r^k t_0),$$

and

$$\mathsf{w}(a_{k+1}, a, r^{k+1} t_0 + \eta) \geq \limsup_n \mathsf{w}(a_{k+1}, a_n, r^{k+1} t_0).$$

Consequently, we can find $i > k$ and $j > k+1$ such that

$$\eta + \mathsf{w}(a_k, a, r^k t_0 + \eta) > \mathsf{w}(a_k, a_i, r^k t_0),$$

and

$$\eta + \mathsf{w}(a_{k+1}, a, r^{k+1} t_0 + \eta) > \mathsf{w}(a_{k+1}, a_j, r^{k+1} t_0).$$

Hence

$$\mathsf{w}(a_k, a, r^k t_0 + \eta) > \mathsf{w}(a_k, a_i, r^k t_0) - \eta > 1 - (r^k t_0 + \eta),$$

which implies by (1) that

$$\mathsf{w}(a_{k+1}, fa, r(r^k t_0 + \eta)) > 1 - r(r^k t_0 + \eta). \tag{3}$$

Moreover

$$\mathsf{w}(a_{k+1}, a, r^{k+1} t_0 + \eta) > \mathsf{w}(a_{k+1}, a_j, r^{k+1} t_0) - \eta > 1 - (r^{k+1} t_0 + \eta). \tag{4}$$

Since $1 - r(r^k t_0 + \eta) > 1 - (r^{k+1} t_0 + \eta) > 1 - \delta$, we deduce from inequalities (3) and (4) and condition (fw3) that

$$\mathsf{m}(a, fa, s+t) \geq 1 - \frac{\varepsilon}{2},$$

where $s = r^{k+1} t_0 + \eta$ and $t = r(r^k t_0 + \eta)$. Since $\varepsilon > 2(r^k t_0 + \eta)$ we get $\varepsilon > s + t$, so

$$\mathsf{m}(a, fa, \varepsilon) > 1 - \varepsilon.$$

We have proved that a is a fixed point of f. □

As an immediate consequence of Example 3 and Theorem 2 we get the next enhancement of Hicks' fixed point theorem (see [24,25]).

Corollary 1. *Every Hicks contraction on a complete fuzzy metric space has a unique fixed point.*

Next we present and example where we Theorem 2 works but not Corollary 1. The next result will be of great help for our purposes.

Lemma 1 ([1,2]). *Let (S, ρ) be a metric space and T be a bounded and closed subset of S with $|T| \geq 2$. If K is a positive constant such that $K \geq \operatorname{diameter}(T)$, then, the function $w : S \times S \to R^+$ defined by $w(a,b) = \rho(a,b)$ if $a, b \in T$ and $w(a,b) = K$ otherwise, is a w-distance on (S, ρ).*

Example 7. *Let $(R^+, \mathsf{m}_{\rho_{|\cdot|}}, *)$ be the fuzzy metric space constructed in Example 4. In fact, it is complete because the metric space $(R^+, \rho_{|\cdot|})$ it is.*

Define $f : R^+ \to R^+$ as $fa = 1/2$ if $a \in [0, 1/2]$, and $fa = 0$ if $a > 1/2$.

Since f is not continuous (at $a = 1/2$), it is not a Hicks contraction on R^+.

Now put $T = [0, 1/2]$ and $K = 2$. By Lemma 1, the function $w : R^+ \times R^+ \to R^+$ defined by $w(a,b) = \rho_{|\cdot|}(a,b)$ if $a, b \in T$ and $w(a,b) = 2$ otherwise, is a w-distance on $(R^+, \rho_{|\cdot|})$.

We show that f is a w-Hicks contraction on $(R^+, \mathsf{m}_{\rho_{|\cdot|}}, *)$ (with constant $r = 1/2$), for the fuzzy w-distance w *on $(R^+, \mathsf{m}_{\rho_{|\cdot|}}, *)$ given by (compare Example 5)*

$\mathsf{w}(a,b,t) = 1$ *if* $w(a,b) < t$ *and* $\rho_{|\cdot|}(a,b) < t$, *and*
$\mathsf{w}(a,b,t) = 0$ *otherwise.*

We differentiate the three next cases.

Case 1. $a, b \in T$. Then, for any $t > 0$ we get

$$\mathsf{w}(fa, fb, rt) = \mathsf{w}(1/2, 1/2, t/2) = 1 > 1 - rt.$$

Case 2. $a, b \in R^+ \backslash T$. Then, for any $t > 0$ we get

$$\mathsf{w}(fa, fb, rt) = \mathsf{w}(0, 0, t/2) = 1 > 1 - rt.$$

Case 3. $a \in T$ and $b \in R^+ \backslash T$. (we do not need to consider the case $a \in R^+ \backslash T$. and $b \in T$ because w *is symmetric). Let $t > 0$ such that $\mathsf{w}(a, b, t) > 1 - t$.*

If $\mathsf{w}(a, b, t) = 1$ it follows $w(a, b) < t$, i.e., $2 < t$, so $1 - t/2 < 0$. Therefore

$$\mathsf{w}(fa, fb, rt) > 1 - rt.$$

If $\mathsf{w}(a, b, t) = 0$ it follows $t > 1$. Suppose that $\mathsf{w}(fa, fb, rt) = 0$. Then $w(1/2, 0) \geq rt$ or $\rho_{|\cdot|}(a, b) \geq rt$. Since $w(1/2, 0) = \rho_{|\cdot|}(1/2, 0) = 1/2$ we deduce that $1 \geq t$, a contradiction. Hence

$$\mathsf{w}(fa, fb, rt) > 1 - rt.$$

Therefore, Theorem 2 works for this example.

5. A Necessary and Sufficient Condition for Completeness of Fuzzy Metric Spaces

In this section we prove our fuzzy counterpart of Theorem 1. In fact, we show that the fixed point theorem obtained in Theorem 2 jointly with the following characterization of fuzzy metric completeness obtained in [14] yield such a counterpart.

Theorem 3. *A necessary and sufficient condition for a fuzzy metric space to be complete is that every Hicks contraction on any of its closed subsets has a fixed point.*

Theorem 4. *A necessary and sufficient condition for a fuzzy metric space to be complete is that every w-Hicks contraction on it has a fixed point.*

Proof. Necessity: It follows from Theorem 2.

Sufficiency: Let T be a closed subset of a fuzzy metric space $(S, \mathsf{m}, *)$ and f be a Hicks contraction on T (with constant $r \in (0, 1)$).

Fix $u \in T$ and construct a self map g of S as follows:

$ga = fa$ for all $a \in T$, and
$ga = fu$ for all $a \in S \backslash T$.

Following the construction given in Example 2, let w be the fuzzy w-distance on $(S, \mathsf{m}, *)$ given by

$\mathsf{w}(a, b, t) = \mathsf{m}(a, b, t)$ if $a, b \in T$ and $t > 0$, and
$\mathsf{w}(a, b, t) = 0$, otherwise.

We are going to show that g is a w-Hicks contraction on $(S, \mathsf{m}, *)$ for this fuzzy w-distance (with constant r).

Indeed, let $a, b \in S$ and $t > 0$ such that $\mathsf{w}(a, b, t) > 1 - t$.

We distinguish three cases:

Case 1. $a, b \in T$. Then $\mathsf{m}(a, b, t) = \mathsf{w}(a, b, t) > 1 - t$, so $\mathsf{m}(fa, fb, rt) > 1 - rt$, and thus

$$\mathsf{w}(ga, gb, rt) = \mathsf{m}(fa, fb, rt) > 1 - rt.$$

Case 2. $a, b \in S \backslash T$. Then, we get

$$\mathrm{w}(ga, gb, rt) = \mathrm{w}(fu, fu, rt) = \mathrm{m}(fu, fu, rt) = 1 > 1 - rt.$$

Case 3. $a \in T$ and $b \in S \backslash T$ (we do not need to consider the case $a \in S \backslash T$ and $b \in T$ because w is symmetric). Then $\mathrm{w}(a, b, t) = 0$, so $t > 1$, and thus $\mathrm{m}(a, u, t) > 1 - t$. Since f is a Hicks contraction on T we deduce that $\mathrm{m}(fa, fu, rt) > 1 - rt$. Consequently

$$\mathrm{w}(ga, gb, rt) = \mathrm{w}(fa, fu, rt) = \mathrm{m}(fu, fu, rt) > 1 - rt.$$

We conclude that g is a w-Hicks contraction on $(S, \mathrm{m}, *)$, so, by assumption it has a fixed point $a_0 \in S$. By the definition of g, $a_0 \in T$, and consequently a_0 is a fixed point of f. We have proved that every self map of T has a fixed point. Hence, by Theorem 3, $(S, \mathrm{m}, *)$ is complete. $\square$

We finish the paper by studying a case that illustrates the preceding theorem.

Example 8. *Let $(S, \mathrm{m}, *)$ be the fuzzy metric space where $S = [0, 1)$, $*$ is any continuous t-norm and m is the fuzzy metric on S given, for any $a, b \in S$, by $\mathrm{m}(a, b, 0) = 0$, $\mathrm{m}(a, a, t) = 1$ if $t > 0$, and $\mathrm{m}(a, b, t) = \min\{a, b\}$ if $a \neq b$, $t > 0$.*

Since $(n/(n+1))_{n \in \omega}$ is a non convergent Cauchy sequence, it follows that $(S, \mathrm{m}, *)$ is not complete, so, by Theorem 4, there exist w-Hicks contractions on $(S, \mathrm{m}, *)$ without fixed points. We proceed to explicitly construct one of such contractions.

Let f be the self map of S given by $fa = (a+1)/2$ for all $a \in S$. Now let w be the w-distance on $(S, \mathrm{m}, *)$ given by $\mathrm{w}(a, b, 0) = 0$ and $\mathrm{w}(a, b, t) = b$ for all $a, b \in S$, $t > 0$. Suppose $\mathrm{w}(a, b, t) > 1 - t$, with $t > 0$. Then $b > 1 - t$, so $(b+1)/2 > 1 - t/2$, and hence $\mathrm{w}(fa, fb, t/2) > 1 - t/2$. We conclude that f is a w-Hicks contraction without fixed point. Observe that f is also a w-Hicks contraction with respect to the fuzzy metric m.

6. Conclusions

We have proposed and discussed a notion of w-distance for fuzzy metric spaces in the sense of Kramosil and Michalek. We have presented several examples to accredit its suitability, and proved a fixed point theorem which has allowed us to obtain a characterization of fuzzy metric completeness that provides a fuzzy counterpart of a celebrated characterization of metric completeness due to Suzuki and Takahashi. For further work related to the extension and possible applications of the obtained results, the articles [26–28], suggested by one of the reviewers, could be of interest.

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