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Novel Parametric Solutions for the Ideal and Non-Ideal Prouhet Tarry Escott Problem

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Abstract: The present study aims to develop novel parametric solutions for the Prouhet Tarry Escott problem of second degree with sizes 3, 4 and 5. During this investigation, new parametric representations for integers as the sum of three, four and five perfect squares in two distinct ways are identified. Moreover, a new proof for the non-existence of solutions of ideal Prouhet Tarry Escott problem with degree 3 and size 2 is derived. The present work also derives a three parametric solution of ideal Prouhet Tarry Escott problem of degree three and size two. The present study also aimed to discuss the Fibonacci-like pattern in the solutions and finally obtained an upper bound for this new pattern.

Keywords: Diophantine equations; Prouhet Tarry Escott problem; Fibonacci pattern

1. Introduction

The Diophantine problem endeavors a vast area for research in number theory because of its diversity as well as its characteristic property of having immense ways to find the solutions. Thus, Diophantine problems attract number theorists all the time. Some recent studies on Diophantine problems especially on the generalization of Pell equations to higher degrees and the relationship of Diophantine equations with the ring of algebraic integers can be seen in [1–4]. Another remarkable work in the field of Diophantine equations is by Shang [5] in which a necessary condition of solvability of Diophantine equation $W^n + X^n + Y^n = Z^n$ over $M_2(\mathbb{Q})$ has been derived. The Prouhet Tarry Escott problem of size n and degree k ((n,k)-PTE problem) focuses on determining two disjoint sets of integers, say, $\{x_i\}_{i=1}^{n}$ and $\{y_i\}_{i=1}^{n}$ where these two sets satisfy the Diophantine system $\sum_{i=1}^{n} x_i^k = \sum_{i=1}^{n} y_i^k$; $k = 1, 2, ...s$. If $s = n-1$, it is called the ideal solutions and otherwise called non-ideal. It was Bastein who proved the impossibility of $\{x_1, x_2, \cdots, x_n\} =_n \{y_1, y_2, \cdots, y_n\}$ where $x_i's$ do not form a permutation of $y_i's$. He applied a result from the elementary symmetric function which states that the two distinct sets of roots of a polynomial equation of degree n have the same elementary symmetric functions [6]. Later, Tarry proved that the first $2^n(2a+1)$ integers can be split up into two equivalent classes each consists of $2^{n-1}(2a+1)$ integers where the sum of the t^{th} powers in one class will be equal to that of the second class for $t = 1, 2, \cdots, n$. It is to be noted that the system of equations $\sum_i a_i^k = \sum_i b_i^k; i, k = 1, 2, \cdots, n$ is equivalent to the system $\sum_i a_i = \sum_i b_i$, $\sum a_1 a_2 = \sum b_1 b_2, \cdots, \sum a_1 a_2 \cdots a_n = \sum b_1 b_2 \cdots b_n$. Thus, the PTE problem can be reformulated as the problem of detecting two polynomial equations of same degree such that both the equations have the same integral roots and the first n coefficients are equal to each other [6].

Several works exists in evaluating the solutions of the PTE problem [7–19]. Choudhary [20,21] studied PTE problem with the additional condition of equal product of integers and then established the complete ideal solutions for the fourth degree case. Dickson [22] established a method for finding all integral solutions of the $(3,2)$ and $(4,2)$-PTE problem. Later, Gopalan and Srikanth [23] found a general form of parametric soluitons of non-ideal $(4,2)$-PTE problem. The different approaches to the 2nd degree problem and its related problems over finite field can be seen in [24,25]. Bolker et al. [26] first observed the relation between the PTE problem and the Prouhet-Thue-Morse (PTM) sequence. Later, Nguyen [27] has derived the solutions of general PTE problem by using the product generating formula for PTM sequence. Recently, Srikanth and Veena [28] performed a detailed survey on the PTE problem and addressed the difficulties as well as future directions of the problem systematically.

The PTE problem is the most general case of easier waring problem which concerns the integral solutions of the equation $n = x_1^k + x_2^k + \cdots + x_m^k$. Ramanujan [29] provided the integral solutions of the equation $n = ax_1^2 + bx_2^2 + cx_3^2 + dx_4^2$ for the given natural numbers a, b, c, d. Rabin and Shallit [30] constructed a randomized polynomial-time algorithm for finding one representation of the given integer n as $n = x_1^2 + x_2^2 + x_3^2 + x_4^2$. Elia [31] proved that the prime numbers can be proclaimed as the sum of four squares. Recently, Borkovich and Jagy [32] discovered a new design for intgers as the sum of three squares.

In the present study, the authors aim to develop some new parametric forms of solutions of the $(3,2)$, $(4,2)$ and $(5,2)$ PTE problems and also to study the Fibonacci like pattern in the solutions of non-ideal PTE problem. In Section 2, a new proof for the non-existence of solutions of $(3,2)$-PTE problem is presented.

2. On the Ideal PTE Problem

Theorem 1. *The system of Diophantine equations* $\sum_{i=1}^{3} x_i = \sum_{i=1}^{3} y_i$ *and* $\sum_{i=1}^{3} x_i^2 = \sum_{i=1}^{3} y_i^2 = 4^a(8b+7)$ *has no integer solutions.*

Proof. Legendre [6] showed that for any positive integer $m \in \mathbb{N}$, the set of all positive integers can be exemplified as $m = \alpha^2 + \beta^2 + \gamma^2$ for some integers (α, β, γ) if and only if $m \neq 4^a(8b+7)$ where $a \in \mathbb{N}$ and b is any integer. Therefore, the equation $x_1^2 + x_2^2 + x_3^2 = 4^a(8b+7)$ has no integral solutions. □

Lemma 1. *A new parametric form of solutions of the* $(3,2)$*-PTE problem is given by* $x_1 = e_1$, $x_2 = e_2$, $x_3 = k - e_1 - e_2$, $y_1 = e_3$, $y_2 = e_4$, $y_3 = k - e_3 - e_4$; *where* $e_i \in \mathbb{Z}$, *the set of all integers for* $i = 1, \cdots, 4$, *and* $k = \left(\sum_{i=1}^{4} e_i\right) - \frac{e_1e_2 - e_3e_4}{(e_1+e_2)-(e_3+e_4)} \in \mathbb{Z}$.

Proof. Let $\sum_{i=1}^{3} x_i = \sum_{i=1}^{3} y_i = k$. Take $x_1 = e_1$, $x_2 = e_2$ where e_1 and e_2 are any integers. Thus $x_3 = k - e_1 - e_2$. Similarly, if $y_1 = e_3$ and $y_2 = e_4$, then $y_3 = k - e_3 - e_4$. Now, applying these values in second degree equation it provides

$$\begin{aligned} 2e_1^2 + 2e_2^2 + 2e_1e_2 - 2ke_1 - 2ke_2 &= 2e_3^2 + 2e_4^2 + 2e_3e_4 - 2ke_3 - 2ke_4 \\ e_1^2 + e_2^2 + e_1e_2 - e_3^2 - e_4^2 - e_3e_4 &= k\left(\sum_{i=1}^{2} e_i - \sum_{i=3}^{4} e_i\right). \end{aligned}$$

Therefore,

$$\begin{aligned} k &= \frac{e_1^2 + e_2^2 + e_1e_2 - e_3^2 - e_4^2 - e_3e_4}{\left(\sum_{i=1}^{2} e_i\right) - \left(\sum_{i=3}^{4} e_i\right)} \\ &= \frac{(e_1+e_2)^2 - e_1e_2 - [(e_3+e_4)^2 - e_3e_4]}{\left(\sum_{i=1}^{2} e_i\right) - \left(\sum_{i=3}^{4} e_i\right)} \\ &= \frac{(e_1+e_2)^2 - (e_3+e_4)^2}{\left(\sum_{i=1}^{2} e_i\right) - \left(\sum_{i=3}^{4} e_i\right)} - \frac{e_1e_2 - e_3e_4}{\left(\sum_{i=1}^{2} e_i\right) - \left(\sum_{i=3}^{4} e_i\right)}. \end{aligned}$$

Now $(e_1+e_2)^2-(e_3+e_4)^2$ is of the form a^2-b^2 which is equal to $(a+b)(a-b)$. Thus $(e_1+e_2)^2-(e_3+e_4)^2=(e_1+e_2+e_3+e_4)(e_1+e_2-e_3-e_4)$. Hence

$$\begin{aligned}
k &= \frac{(e_1+e_2+e_3+e_4)(e_1+e_2-e_3-e_4)}{e_1+e_2-e_3-e_4} - \frac{e_1e_2-e_3e_4}{e_1+e_2-e_3-e_4} \\
&= \frac{(\sum_{i=1}^{4} e_i)(e_1+e_2-(e_3+e_4))}{\left(\sum_{i=1}^{2} e_i\right)-\left(\sum_{i=3}^{4} e_i\right)} - \frac{e_1e_2-e_3e_4}{\left(\sum_{i=1}^{2} e_i\right)-\left(\sum_{i=3}^{4} e_i\right)} \\
&= \left(\sum_{i=1}^{4} e_i\right) - \frac{e_1e_2-e_3e_4}{\left(\sum_{i=1}^{2} e_i\right)-\left(\sum_{i=3}^{4} e_i\right)}.
\end{aligned}$$

This proves the Lemma 1. □

Lemma 2. *All integral solutions of the relation* $\frac{xy-zw}{(x+y)-(z+w)} \in \mathbb{Z}$ *satisfy the following conditions*

1. $y = w + \alpha_1$.
2. $z = w - \beta_1 + (\beta_1+\alpha_1)T$,

where β_1 *is an integer,* $T = \frac{t-w}{\alpha_1}$, α_1 *is any divisor of* $t-w$ *and* t *is an integer.*

Proof. Consider the form $\frac{xy-zw}{(x+y)-(z+w)}$. Let $x = z+\beta_1$ and $y = w+\alpha_1$ where α_1 and β_1 are integers. Then

$$\begin{aligned}
\frac{xy-zw}{(x+y)-(z+w)} &= \frac{(z+\beta_1)(w+\alpha_1)-zw}{z+\beta_1+w+\alpha_1-z-w} \\
&= \frac{z\alpha_1+w\beta_1+a_1\beta_1}{\beta_1+\alpha_1} \\
&= \frac{w\beta_1+w\alpha_1-w\alpha_1+z\beta_1+\beta_1\alpha_1}{\beta_1+\alpha_1} \\
&= \frac{w(\beta_1+\alpha_1)+\alpha_1(z-w)+\beta_1\alpha_1}{\beta_1+\alpha_1} \\
&= w + \frac{(\alpha_1(z-w+\beta_1))}{(\beta_1+\alpha_1)}.
\end{aligned}$$

It is clear that $\frac{xy-zw}{(x+y)-(z+w)} \in \mathbb{Z}$ if and only if $k = \frac{xy-zw}{(x+y)-(z+w)} \in \mathbb{Z}$ if and only if $z = w-\beta_1+(\beta_1+\alpha_1)T$, where $T = \frac{t-w}{\alpha_1}$, α_1 is any divisor of $t-w$ and β_1 and w are any integers. Now, take $z = w-\beta_1+(\beta_1+\alpha_1)T$, where t is any integer, α_1 is any divisor of t and $T = \frac{t-w}{\alpha_1}$. i.e.,

$$\begin{aligned}
\frac{(\alpha_1(z-w+\beta_1))}{(\beta_1+\alpha_1)} &= \frac{w-\beta_1+(\beta_1+\alpha_1)T-w+\beta_1}{\beta_1+\alpha_1}\alpha_1 \\
&= T\alpha_1 \\
&= t-w
\end{aligned}$$

Thus, we obtain $\frac{xy-zw}{(x+y)-(z+w)} = t$. Hence the proof. □

Replacing x by e_1, y by e_2, z by e_3, w by e_4 in Lemma 2 and combining the results of both Lemma 1 and Lemma 2, we obtain Theorem 2.

Theorem 2. *The parametric form $x_1 = e_1$, $x_2 = e_2$, $x_3 = k - e_1 - e_2$, $y_1 = e_3$, $y_2 = e_4$ and $y_3 = k - e_3 - e_4$, where*

1. e_i; $i = 1, \cdots, 4$ *satisfy the relation* $\frac{e_1e_2 - e_3e_4}{(e_1+e_2)-(e_3+e_4)} \in \mathbb{Z}$
2. $k = \left(\sum_{i=1}^{4} e_i\right) - \frac{e_1e_2-e_3e_4}{\left(\sum_{i=1}^{2} e_i\right)-\left(\sum_{i=3}^{4} e_i\right)}$
3. $e_2 = e_4 + \alpha_1$
4. $e_3 = e_4 - \beta_1 + (\beta_1 + \alpha_1)T$
5. $T = \frac{t-e_4}{\alpha_1}$
6. $e_1 = e_3 + \beta_1$
7. α_1 *and* β_1 *are integers such that* $\alpha_1 | (t - e_4)$ *where* t *is any interger*

provides a new parametric form of the integral solutions of the $(3, 2)$*-PTE problem.*

Proof. By Lemma 1, the parametric form of all integral solutions of $(3, 2)$-PTE problem $x_1 = e_1$, $x_2 = e_2$, $x_3 = k - e_1 - e_2$, $y_1 = e_3$, $y_2 = e_4$, $y_3 = k - e_3 - e_4$; where $e_i \in \mathbb{Z}$; $i = 1, \cdots, 4$ and $k = \left(\sum_{i=1}^{4} e_i\right) - \frac{e_1e_2-e_3e_4}{(e_1+e_2)-(e_3+e_4)} \in \mathbb{Z}$. Thus, the proof completes immediately if we replace x, y, z and w by e_1, e_2, e_3 and e_4 respectively in Lemma 2. □

Example 1. *Let $\beta_1 = 7$, $t = 5$ and $e_4 = 3$. Then α_1 be any divisor of 2. Take $\alpha_1 = 1$. Then we have $e_1 = 19$, $e_2 = 4$, $e_3 = 12$ and $e_4 = 33$. So, $x_1 = e_1 = 19$, $x_2 = e_2 = 4$, $x_3 = k - e_1 - e_2 = 10$, $y_1 = e_3 = 12$, $y_2 = e_4 = 3$, and $y_3 = k - e_3 - e_4 = 18$. Thus we obtain solution sets as $\{19, 4, 10\}$ and $\{12, 3, 18\}$.*

Corollary 1. *If we take $e_1 = e_3$ or $e_2 = e_4$ in Theorem 2, then the Diophantine system $\sum_{i=1}^{3} x_i = \sum_{i=1}^{3} y_i$ and $\sum_{i=1}^{3} x_i^2 = \sum_{i=1}^{3} y_i^2$ does not posess any integral solutions.*

Proof. Assume $e_1 = e_3$ in Theorem 2. Then, we get $\frac{e_1e_2-e_3e_4}{(e_1+e_2)-(e_3+e_4)} = 0$ and $\beta_1 = 0$. Thus the solutions becomes $x_1 = t$, $x_2 = e_4 + \alpha_1$, $x_3 = s_4$, $y_1 = t$, $y_2 = e_4$ and $y_3 = e_4 + \alpha_1$ where t, e_4 and α_1 are any integers with $\alpha_1 | (t - e_4)$. i.e., we obtain solution sets as $X = \{t, e_4 + \alpha_1, e_4\}$ and $Y = \{t, e_4, e_4 + \alpha_1\}$ where X and Y are not distinct as one is a permutation of other. □

Theorems 3 and 4 also provide different forms of parametric solutions of $(3, 2)$-PTE problem in such a way that Theorem 3 provides two parametric solutions and Theorem 4 provides three parametric solutions.

Theorem 3. *A two parametric form of integral solutions of the Diophantine system $\sum_{i=1}^{3} x_i = \sum_{i=1}^{3} y_i$ and $\sum_{i=1}^{3} x_i^2 = \sum_{i=1}^{3} y_i^2$ is given by $x_1 = t_1$, $x_2 = t_2$, $x_3 = 2t_2 - t_1 - 3$, $y_1 = t_1 + 1$, $y_2 = 2t_2 - t_1 - 2$ and $y_3 = t_2 - 2$.*

Proof. Let $x_1 = t_1$, $x_2 = t_2$, $x_3 = 2t_2 - t_1 - 3$, $y_1 = t_1 + 1$, $y_2 = 2t_2 - t_1 - 2$ and $y_3 = t_2 - 2$. Then,

$$x_1 + x_2 + x_3 = t_1 + t_2 + 2t_2 - t_1 - 3 = 3t_2 - 3.$$

Similarly,

$$y_1 + y_2 + y_3 = t_1 + 1 + 2t_2 - t_1 - 2 + t_2 - 2 = 3t_2 - 3.$$

Thus we obtain $x_1 + x_2 + x_3 = y_1 + y_2 + y_3$. Now, consider

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 &= t_1^2 + t_2^2 + (2t_2 - t_1 - 3)^2 \\ &= 2_1^2 + 5t_2^2 - 12t_2 + 6t_1 - 4t_1t_2 + 9 \end{aligned}$$

and

$$\begin{aligned} y_1^2 + y_2^2 + y_3^2 &= (t_1+1)^2 + (2t_2 - t_1 - 2)^2 + (t_2 - 2)^2 \\ &= 2_1^2 + 5t_2^2 - 12t_2 + 6t_1 - 4t_1t_2 + 9. \end{aligned}$$

Thus, we obtain $x_1^2 + x_2^2 + x_3^2 = y_1^2 + y_2^2 + y_3^2$. Hence the proof. □

Example 2. *Let* $t_1 = 5$ *and* $t_2 = -3$ *in Theorem 3. Then we have* $x_1 = 5$, $x_2 = -3$, $x_3 = -14$, $y_1 = 6$, $y_2 = -13$ *and* $y_3 = -5$ *where these* x_i'*s and* y_i'*s satisfy* $5 - 3 - 14 = -12 = 6 - 13 - 5$ *and* $5^2 + (-3)^2 + (-14)^2 = 6^2 + (-13)^2 + (-5)^2$.

Theorem 4. *A three parametric form of integral solutions of the Diophantine system* $\sum_{i=1}^3 x_i = \sum_{i=1}^3 y_i$ *and* $\sum_{i=1}^3 x_i^2 = \sum_{i=1}^3 y_i^2$ *is given by* $x_1 = 5t_1$, $x_2 = 6t_3 - t_2 - 4t_1$, $x_3 = 2t_1 - 3t_3 - 2t_2$, $y_1 = 5t_3$, $y_2 = 6t_1 - t_2 - 4t_3$ *and* $y_3 = 2t_3 - 2t_2 - 3t_1$ *provided* $t_1 - t_3 \neq 0$.

Proof. Let $\sum_{i=1}^3 x_i = \sum_{i=1}^3 y_i = p$. Take $x_1 = 5t_1$. Then

$$x_2 + x_3 = p - 5t_1. \tag{1}$$

Let α and β be two constants such that

$$1 \times \alpha - 1 \times \beta = 1. \tag{2}$$

Then $\alpha = 2$ and $\beta = 1$. Multiply (2) by $p - 5t_1$. Then, we obtain

$$2(p - 5t_1) - (p - 5t_1) = p - 5t_1. \tag{3}$$

(1) and (3) gives $[x_2 - 2(p - 5t_1)] + [x_3 + (p - 5t_1)] = 0$. i.e.,

$$[x_2 - 2(p - 5t_1) - 5t_2] + [x_3 + (p - 5t_1) + 5t_2] = 0.$$

Thus we have

$$x_2 = 2(p - 5t_1) + 5t_2$$

and

$$x_3 = -(p - 5t_1) - 5t_2.$$

Similarly take $y_1 + y_2 + y_3 = p$ and $y_1 = 5t_3$. Then as in the previous case we obtain $y_1 = 5t_3$, $y_2 = 2(p - 5t_3) + 5t_4$ and $y_3 = -(p - 5t_3) - 5t_4$. Now, apply these general values in the second degree equation and simplifying we obtain

$$150t_1^2 + 50t_2^2 - 50pt_1 + 30pt_2 - 150t_1t_2 = 150t_3^2 + 50t_4^2 - 50pt_3 + 30pt_4 - 150t_3t_4.$$

Put $t_2 = t_4$. Then we have

$$\begin{aligned} 150(t_1^2 - t_3^2) - 150t_2(t_1 - t_3) &= 50p(t_1 - t_3) \\ 3(t_1 + t_3) - 3t_2 &= p; \text{provided } t_1 - t_3 \neq 0 \\ p &= 3(t_1 + t_3 - t_2). \end{aligned}$$

Hence $x_1 = 5t_1$, $x_2 = 6t_3 - t_2 - 4t_1$, $x_3 = 2t_1 - 3t_3 - 2t_2$, $y_1 = 5t_3$, $y_2 = 6t_1 - t_2 - 4t_3$ and $y_3 = 2t_3 - 2t_2 - 3t_1$. $\square$

Example 3. *Let $t_1 = 5$, $t_3 = 10$ and $t_2 = 0$. Then $t_1 - t_3 \neq 0$. So we obtain $x_1 = 25$, $x_2 = 40$, $x_3 = -20$, $y_1 = 50$, $y_2 = -10$, $y_3 = 5$ where $25 + 40 - 20 = 45 = 50 - 10 + 5$ and $25^2 + 40^2 + (-20)^2 = 2625 = 50^2 + (-10)^2 + 5^2$.*

According to Frolov [10], if $\{x_1, x_2, \cdots, x_n\} =_k \{y_1, y_2, \cdots, y_n\}$, then $\{Mx_1 + K, Mx_2 + K, \cdots, Mx_n + K\} =_k \{My_1 + K, My_2 + K, \cdots, My_n + K\}$. Thus by the repeated application of this, infinite number of solutions can be generated.

Theorem 5. *Let t, β_1, e_4 are any integers and α_1 be any divisor of $t - e_4$. Then, the integer $2e_4^2 + t^2 + \alpha_1^2 + \beta_1^2 + \frac{2\beta_1^2}{\alpha_1^2}(t - e_4)^2 + 2e_4(\alpha_1 - \beta_1) + \frac{2\beta_1}{\alpha_1}(t^2 - e_4^2) + \frac{2\beta_1^2}{\alpha_1}(e_4 - t)$ can be written as the sum of three perfect squares in two disparate ways.*

Proof. As per the assumptions in Theorem 2, choose the integers $t, e_i; i = 1, \cdots, 4, \alpha_1, \beta_1$ and k. Then, we obtain

$$k = 2e_4 + \alpha_1 - \beta_1 + t + \frac{2\beta_1}{\alpha_1}(t - e_4).$$

Let $x_1 = e_1 = e_4 + (\beta_1 + \alpha_1)\frac{(t-e_4)}{\alpha_1}$, $x_2 = e_2 = e_4 + \alpha_1$, $y_1 = e_3 = e_4 - \beta_1 + (\beta_1 + \alpha_1)\frac{(t-e_4)}{\alpha_1}$ and $y_2 = e_4$. Consider the equation $\sum_{i=1}^{3} x_i = \sum_{i=1}^{3} y_i = k$.

Then, we have

$$\begin{aligned} x_3 &= k - e_1 - e_2 \\ &= e_4 - \beta_1 + \frac{\beta_1}{\alpha_1}(t - e_4) \end{aligned}$$

and

$$\begin{aligned} y_3 &= k - e_3 - e_4 \\ &= e_4 + \alpha_1 + \frac{\beta_1}{\alpha_1}(t - e_4). \end{aligned}$$

We know that $x_1 = e_1$ where $e_1 = e_4 + (\beta_1 + \alpha_1)\frac{(t-e_4)}{\alpha_1}$. Thus

$$\begin{aligned} x_1 &= e_4 + (\beta_1 + \alpha_1)\frac{(t - e_4)}{\alpha_1} \\ &= e_4 + \frac{t - e_4}{\alpha_1}\beta_1 + \frac{(t - e_4)}{\alpha_1}\alpha_1 \\ &= e_4 + \frac{t - e_4}{\alpha_1}\beta_1 + t - e_4 \\ &= t + \frac{t - e_4}{\alpha_1}\beta_1 \end{aligned}$$

Now, consider

$$\begin{aligned} x_1^2+x_2^2+x_3^2 &= [e_4+\frac{\beta_1}{\alpha_1}(t-e_4)]^2+[e_4+\alpha_1]^2+[e_4-\beta_1+\frac{\beta_1}{\alpha_1}(t-e_4)]^2 \\ &= 2e_4^2+t^2+\alpha_1^2+\beta_1^2+\frac{2\beta_1^2}{\alpha_1^2}(t-e_4)^2+2e_4(\alpha_1-\beta_1)+ \\ &\quad \frac{2\beta_1}{\alpha_1}(t^2-e_4^2)+\frac{2\beta_1^2}{\alpha_1}(e_4-t). \end{aligned}$$

Similarly, we know that $y_1=e_3$ where $e_3=e_4-\beta_1+(\beta_1+\alpha_1)\frac{(t-e_4)}{\alpha_1}$. Thus

$$\begin{aligned} y_1 &= e_4-\beta_1+(\beta_1+\alpha_1)\frac{(t-e_4)}{\alpha_1} \\ &= e_4-\beta_1+\frac{(t-e_4)}{\alpha_1}\beta_1+\frac{(t-e_4)}{\alpha_1}\alpha_1) \\ &= e_4-\beta_1+\frac{(t-e_4)}{\alpha_1}\beta_1+t-e_4 \\ &= t-\beta_1+\frac{\beta_1}{\alpha_1}(t-e_4), \end{aligned}$$

and thus

$$\begin{aligned} y_1^2+y_2^2+y_3^2 &= [t-\beta_1+\frac{\beta_1}{\alpha_1}(t-e_4)]^2+e_4^2+[e_4+\alpha_1+\frac{\beta_1}{\alpha_1}(t-e_4)]^2 \\ &= 2e_4^2+t^2+\alpha_1^2+\beta_1^2+\frac{2\beta_1^2}{\alpha_1^2}(t-e_4)^2+2e_4(\alpha_1-\beta_1)+ \\ &\quad \frac{2\beta_1}{\alpha_1}(t^2-e_4^2)+\frac{2\beta_1^2}{\alpha_1}(e_4-t). \end{aligned}$$

Comparing the values of $x_1^2+x_2^2+x_3^2$ and $y_1^2+y_2^2+y_3^2$ we get,

$$x_1^2+x_2^2+x_3^2=2e_4^2+t^2+\alpha_1^2+\beta_1^2+\frac{2\beta_1^2}{\alpha_1^2}(t-e_4)^2+2e_4(\alpha_1-\beta_1)+\frac{2\beta_1}{\alpha_1}(t^2-e_4^2)+\frac{2\beta_1^2}{\alpha_1}(e_4-t)=y_1^2+y_2^2+y_3^2.$$

Hence the proof. □

3. On the Non-Ideal PTE Problem

Some new parametric forms of solutions of the $(4,2)$-PTE problem and $(5,2)$-PTE problem have been discussed in Section 3.

Lemma 3. *A parametric form of integral solutions of the $(4,2)$-PTE problem is given by $x_1=e_1$, $x_2=e_2$, $x_3=e_3$, $x_4=k-e_1-e_2-e_3$, $y_1=e_4$, $y_2=e_5$, $y_3=e_6$, $y_4=k-e_4-e_5-e_6$ and $k=\left(\sum_{i=1}^6 e_i\right)-\frac{(e_1e_2+e_1e_3+e_2e_3)-(e_4e_5+e_5e_6+e_4e_6)}{(e_1+e_2+e_3)-(e_4+e_5+e_6)}\in\mathbb{Z}$, where $e_i\in\mathbb{Z}, i=1,\cdots,6$.*

Proof. Consider the equation $\sum_{i=1}^4 x_i=\sum_{i=1}^4 y_i$. Let $\sum_{i=1}^4 x_i=k=\sum_{i=1}^4 y_i$. Then we have $x_1+x_2+x_3+x_4=k=y_1+y_2+y_3+y_4$. Let $x_1=e_1$, $x_2=e_2$ and $x_3=e_3$. Then $x_4=k-e_1-e_2-e_3$. Similarly, if $y_1=e_4$, $y_2=e_5$ and $y_3=e_6$, then $y_4=k-e_4-e_5-e_6$. Substituting these values in $\sum_{i=1}^4 x_i^2=\sum_{i=1}^4 y_i^2$,

we obtain $(e_1^2 + e_2^2 + e_3^2 + e_1e_2 + e_1e_3 + e_2e_3) - (e_4^2 + e_5^2 + e_6^2 + e_4e_5 + e_4e_6 + e_5e_6) = k\left[\sum_{i=1}^{3} e_i - \sum_{i=4}^{6} e_i\right]$.
i.e., $(e_1 + e_2 + e_3)^2 - (e_1e_2 + e_1e_3 + e_2e_3) - [(e_4 + e_5 + e_6)^2 - (e_4e_5 + e_4e_6 + e_5e_6)] = k\left[\sum_{i=1}^{3} e_i - \sum_{i=4}^{6} e_i\right]$.

From this, k can be written as

$$k = \frac{(e_1 + e_2 + e_3)^2 - (e_4 + e_5 + e_6)^2}{\left(\sum_{i=1}^{3} e_i\right) - \left(\sum_{i=4}^{6} e_i\right)} - \frac{(e_1e_2 + e_1e_3 + e_2e_3) - (e_4e_5 + e_4e_6 + e_5e_6)}{\left(\sum_{i=1}^{3} e_i\right) - \left(\sum_{i=4}^{6} e_i\right)}$$

i.e.,

$$k = \left(\sum_{i=1}^{6} e_i\right) - \frac{(e_1e_2 + e_1e_3 + e_2e_3) - (e_4e_5 + e_5e_6 + e_4e_6)}{\left(\sum_{i=1}^{3} e_i\right) - \left(\sum_{i=4}^{6} e_i\right)}$$

Hence the proof. □

Lemma 4. *The integral solutions of the relation* $\frac{(a_1b_1+a_1c_1+b_1c_1)-(a_2b_2+b_2c_2+a_2c_2)}{(a_1+b_1+c_1)-(a_2+b_2+c_2)} \in \mathbb{Z}$ *satisfy the following conditions:*

(I) $a_1 = a_2 + \alpha_1$
(II) $b_1 = b_2 + \beta_1$
(III) $c_1 = b_2 + \beta_1 + \alpha_1 T$

where α_1 *is any integer,* $T = \frac{t-b_2}{\beta_1}$ *such that* t, b_2 *are any integers and* $\beta_1 | (t - b_2)$.

Proof. Let $a_1 = a_2 + \alpha_1$, $b_1 = b_2 + \beta_1$ and $c_1 = c_2 - \beta_1$. Then, $(a_1 + b_1 + c_1) - (a_2 + b_2 + c_2) = \alpha_1$ and

$$\begin{aligned}(a_1b_1 + a_1c_1 + b_1c_1) - (a_2b_2 + b_2c_2 + a_2c_2) &= b_2\alpha_1 + c_2\alpha_1 - b_2\beta_1 + c_2\beta_1 - \beta_1^2 \\ &= (c_2 + b_2)\alpha_1 + \beta_1(c_2 - b_2 - \beta_1)\end{aligned}$$

So,

$$\frac{(a_1b_1 + a_1c_1 + b_1c_1) - (a_2b_2 + b_2c_2 + a_2c_2)}{(a_1 + b_1 + c_1) - (a_2 + b_2 + c_2)} = (b_2 + c_2) + \frac{\beta_1}{\alpha_1}(c_2 - b_2 - \beta_1)$$

Thus, $\frac{(a_1b_1+a_1c_1+b_1c_1)-(a_2b_2+b_2c_2+a_2c_2)}{(a_1+b_1+c_1)-(a_2+b_2+c_2)} \in \mathbb{Z}$ if and only if $\frac{\beta_1}{\alpha_1}(c_2 - b_2 - \beta_1) \in \mathbb{Z}$ if and only if $c_2 = b_2 + \beta_1 + \alpha_1 T$; where $T = \frac{t-b_2}{\beta_1}$.
Hence,

$$\frac{(a_1b_1 + a_1c_1 + b_1c_1) - (a_2b_2 + b_2c_2 + a_2c_2)}{(a_1 + b_1 + c_1) - (a_2 + b_2 + c_2)} = c_2 + t$$

□

If the parameters a_1, b_1, c_1, a_2, b_2 and c_2 in Lemma 4 are replaced by e_i ; $i = 1, \cdots, 6$ and then the simultaneous applications of Lemmas 3 and 4, we obtain Theorem 6.

Theorem 6. *The parametric form* $x_1 = e_1$, $x_2 = e_2$, $x_3 = e_3$, $x_4 = k - e_1 - e_2 - e_3$, $y_1 = e_4$, $y_2 = e_5$. $y_3 = e_6$ *and* $y_4 = k - e_4 - e_5 - e_6$ *where*

- e_i; $i = 1, \cdots, 6$ *satisfy* $\frac{(e_1e_2+e_1e_3+e_2e_3)-(e_4e_5+e_5e_6+e_4e_6)}{(e_1+e_2+e_3)-(e_4+e_5+e_6)} \in \mathbb{Z}$
- $k = \left(\sum_{i=1}^{6} e_i\right) - \frac{(e_1e_2+e_1e_3+e_2e_3)-(e_4e_5+e_5e_6+e_4e_6)}{(e_1+e_2+e_3)-(e_4+e_5+e_6)}$
- $e_1 = e_4 + \alpha_1$
- $e_2 = e_5 + \beta_1$

- $e_3 = e_5 + \alpha_1 T$
- $e_6 = e_5 + \beta_1 + \alpha_1 T$
- $T = \frac{t-e_5}{\beta_1}$
- *t, α_1, e_4, e_5 and β_1 are any integers such that β_1 is any divisor of $(t - e_5)$*

provide integral solutions of the non-ideal $(4,2)$-PTE problem.

Proof. By Lemma 3, the parametric form of the integral solutions $(4,2)$-PTE problem is $x_1 = e_1$, $x_2 = e_2$, $x_3 = e_3$, $x_4 = k - e_1 - e_2 - e_3$, $y_1 = e_4$, $y_2 = e_5$, $y_3 = e_6$ and $y_4 = k - e_4 - e_5 - e_6$ where $e_i \in \mathbb{Z}; i = 1, 2, \cdots, 6$ with $k = \left(\sum_{i=1}^{6} e_i\right) - \frac{(e_1e_2+e_1e_3+e_2e_3)-(e_4e_5+e_5e_6+e_4e_6)}{(e_1+e_2+e_3)-(e_4+e_5+e_6)} \in \mathbb{Z}$. If we replace a_1 by e_1, b_1 by e_2, c_1 by e_3, a_2 by e_4, b_2 by e_5, c_2 by e_6 in Lemma 4 we will have the theorem. □

Example 4. *Let $t = 5$, $\alpha_1 = -2$, $e_4 = 3$ and $e_5 = 2$. Then $t - e_5 = 3$. Take $\beta_1 = 1$. Then $e_6 = -3$, $e_1 = 1$, $e_2 = 3$, $e_3 = -4$ and $k = 0$. So $x_1 = 1$, $x_2 = 3$, $x_3 = -4$, $x_4 = 0$, $y_1 = 3$, $y_2 = 2$, $y_3 = -3$, $y_4 = -2$.*

Theorem 7 also provides another parametric solution of $(4,2)$-PTE problem.

Theorem 7. *A four parametric form of integral solutions of $(4,2)$-PTE problem is given by $x_1 = 5t_1$, $x_2 = t_2$, $x_3 = 6t_3 - 4t_1 - t_5$, $x_4 = 2t_1 - 3t_3 - 2t_5$, $y_1 = 5t_3$, $y_2 = t_2$, $y_3 = 6t_1 - 4t_3 - t_5$ and $y_4 = 2t_3 - 3t_1 - 2t_5$; provided $t_1 - t_3 \neq 0$.*

Proof. Consider the equation
$$\sum_{i=1}^{4} x_i = \sum_{i=1}^{4} y_i = k.$$

Let $x_1 = 5t_1$, $x_2 = t_2$. Then we obtain
$$x_3 + x_4 = k - 5t_1 - t_2. \tag{4}$$

Let α and β are two integers such that $1 \times \alpha - 1 \times \beta = 1$. Then $\alpha = 2$ and $\beta = 1$. i.e.,
$$1 \times 2 - 1 \times 1 = 1.$$

Multiplying this with $k - 5t_1 - t_2$, we obtain
$$2(k - 5t_1 - t_2) - (k - 5t_1 - t_2) = k - 5t_1 - t_2. \tag{5}$$

Then, (4) and (5) gives
$$[x_3 - 2(k - 5t_1 - t_2)] + [x_4 + (k - 5t_1 - t_2)] = 0.$$

This implies
$$[x_3 - 2(k - 5t_1 - t_2) - 5t_5] + [x_4 + (k - 5t_1 - t_2) + 5t_5] = 0.$$

Thus, $x_3 = 2(k - 5t_1 - t_2) + 5t_5$ and $x_4 = -(k - 5t_1 - t_2) - 5t_5$. Similarly, we obtain $y_3 = 2(k - 5t_3 - t_4) + 5t_6$ and $y_4 = -(k - 5t_3 - t_4) - 5t_6$. Applying the values of x_i's and y_i's in the second degree equation, we obtain

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 + x_4^2 &= 25t_1^2 + t_2^2 + [2(k - 5t_1 - t_2) + 5t_5]^2 + [(k - 5t_1 - t_2) + 5t_5]^2, \\ y_1^2 + y_2^2 + y_3^2 + y_4^2 &= 25t_3^2 + t_4^2 + [2(k - 5t_3 - t_4) + 5t_6]^2 + [(k - 5t_3 - t_4) + 5t_6]^2. \end{aligned}$$

After elementary simplification, we obtain $150t_1^2+6t_2^2+50t_5^2-50kt_1-10kt_2+50t_1t_2+30kt_5-150t_1t_5-30t_2t_5=150t_3^2+6t_4^2+50t_6^2-50kt_3-10kt_4+50t_3t_4+30kt_6-150t_3t_6-30t_4t_6$. Now take $t_5=t_6$ and $t_2=t_4$ and after simplifying further we obtain

$$\begin{aligned}
150t_1^2-50kt_1+50t_1t_2-150t_1t_5 &= 150t_3^2-50kt_3+50t_2t_3-150t_3t_5\\
150(t_1^2-t_3^2)+50t_2(t_1-t_3)-150t_5(t_1-t_3) &= 50k(t_1-t_3)\\
3(t_1+t_3)+t_2-3t_5 &= k;\ \text{provided}\ t_1-t_3\neq 0\\
k &= 3t_1+3t_3+t_2-3t_5.
\end{aligned}$$

So, by substituting the value of k, we obtain $x_3=6t_3-4t_1-t_5$, $x_4=2t_1-3t_3-2t_5$, $y_3=6t_1-4t_3-t_5$ and $y_4=2t_3-3t_1-2t_5$. This proves the theorem. □

Example 5. *Let $t_1=1$, $t_2=-3$, $t_3=2$ and $t_5=-1$. Then $t_1-t_3\neq 0$. $k=9$. Thus $x_3=9$, $x_4=-2$, $y_3=-1$ and $y_4=3$.*

Theorem 8. *Any integer of the form $45a^2+b^2+45c^2+5d^2-60ac$, where a, b, c, $d\in\mathbb{Z}$ with $a-c\neq 0$ can be disclosed as the sum of four perfect squares in two distinct ways.*

Proof. Let $x_1=5a$, $x_2=b$, $x_3=6c-4a-d$, $x_4=2a-3c-2d$, $y_1=5c$, $y_2=b$, $y_3=6a-4c-d$ and $y_4=2c-3a-2d$; provided $a-c\neq 0$. Then,

$$\begin{aligned}
x_1^2+x_2^2+x_3^2+x_4^2 &= 25a^2+b^2+(6c-4a-d)^2+(2a-3c-2d)^2\\
&= 45a^2+b^2+45c^2+5d^2-60ac.
\end{aligned}$$

Similarly,

$$\begin{aligned}
y_1^2+y_2^2+y_3^2+y_4^2 &= 25c^2+b^2+(6a-4c-d)^2+(2c-3a-2d)^2\\
&= 45a^2+45c^2+b^2+5d^2-60ac.
\end{aligned}$$

Comparing these two we get

$$x_1^2+x_2^2+x_3^2+x_4^2=y_1^2+y_2^2+y_3^2+y_4^2$$

Hence the proof. □

Theorem 9 is a particular case of the result by Nguyen [11] which illustrates a particular solution of the non-ideal PTE problem $\sum_{i=1}^4 x_i^r=\sum_{i=1}^4 y_i^r$; $r=1,2$ by using PTM sequence.

Theorem 9. *The first eight non-negative integers can be partitioned into two sets of equal size such that the elements in each set satisfy the non-ideal $(4,2)$-PTE problem.*

Proof. Consider the Prouhet-Thue-Morse sequence defined by

$$\nu_2(n)=\left(\sum_{j=0}^{d} n_j\right) \mod 2,$$

where $n = n_0 2^0 + n_1 2^1 + \cdots + n_d 2^d$ is the base-2 expansion of the integer n. Now, consider the first eight non-negative integers, say $\{0, 1, 2, 3, 4, 5, 6, 7\}$. We define two disjoint sets S_0 and S_1 in such a way that

$$n \in S_{\nu_2(n)}.$$

Then, we obtain $S_0 = \{0, 3, 5, 6\}$ and $S_1 = \{1, 2, 4, 7\}$ such that

$$\begin{aligned} 0 + 3 + 5 + 6 &= 1 + 2 + 4 + 7, \\ 0^2 + 3^2 + 5^2 + 6^2 &= 1^2 + 2^2 + 4^2 + 7^2. \end{aligned}$$

Hence the proof. □

Remark 1. *In Theorem 9, if we consider the generalized Prouhet-Thue-Morse sequence under modulo 3, then we can partition the first 27 non-negative integers into three distinct sets of equal size satisfying the relations* $\sum_{i=1}^{9} x_i^r = \sum_{i=1}^{9} y_i^r$; *r=1,2.*

Remark 2. *Consider the assignment* $n \in S_{\nu_2(n)}$ *as in Theorem 9. Now, define* $s_k(m) = \sum_{n \in S_k} n^m$; *for* $m = 1, 2$ *and* $k = 0, 1$. *Let* $A = (a_0, a_1)$ *be a vector consisting of two arbitrary complex values such that* $a_0 + a_1 = 0$. *Define* $F_3(x; A)$ *to be a polynomial of degree 7 whose coefficients belong to A and repeat according to* $\nu_2(n)$. *i.e.,*

$$F_3(x; A) = \sum_{n=0}^{7} a_{\nu_2(n)} x^n.$$

So, we obtain $F_3(x; A) = a_0 + a_1 x + a_1 x^2 + a_0 x^3 + a_1 x^4 + a_0 x^5 + a_0 x^6 + a_1 x^7$. *Put* $x = e^\theta$ *and define* $G_3(\theta) := F_3(e^\theta; A)$. *Then,*

$$G_3(\theta) = a_0 + a_1 e^\theta + a_1 e^{2\theta} + a_0 e^{3\theta} + a_1 e^{4\theta} + a_0 e^{5\theta} + a_0 e^{6\theta} + a_1 e^{7\theta}.$$

Let $G_3^{(m)}(\theta)$ *denotes the* m^{th} *derivative of* $G_3(\theta)$ *for* $m = 1, 2$. *i.e.,*

$$G_3^{(1)}(\theta) = a_1 e^\theta + 2a_1 e^{2\theta} + 3a_0 e^{3\theta} + 4a_1 e^{4\theta} + 5a_0 e^{5\theta} + 6a_0 e^{6\theta} + 7a_1 e^{7\theta}$$

and

$$G_3^{(2)}(\theta) = a_1 e^\theta + 4a_1 e^{2\theta} + 9a_0 e^{3\theta} + 16a_1 e^{4\theta} + 25a_0 e^{5\theta} + 36a_0 e^{6\theta} + 49a_1 e^{7\theta}.$$

At $\theta = 0$, *we obtain*

$$\begin{aligned} G_3^{(1)}(0) &= 14a_0 + 14a_1 \\ &= 14(a_0 + a_1) \\ &= 0; \textit{ since } a_0 + a_1 = 0, \end{aligned}$$

and

$$\begin{aligned} G_3^{(2)}(0) &= 70a_0 + 70a_1 \\ &= 70(a_0 + a_1) \\ &= 0; \textit{ since } a_0 + a_1 = 0. \end{aligned}$$

However, both $G_3^{(1)}(0)$ *and* $G_3^{(2)}(0)$ *can also be written as* $G_3^{(1)}(0) = a_0 s_0(1) + a_1 s_1(1)$ *and* $G_3^{(2)}(0) = a_0 s_0(2) + a_1 s_1(2)$. *Since* $G_3^{(1)}(0) = 0$ *and* $G_3^{(2)}(0)$, *we get* $a_0 s_0(1) + a_1 s_1(1) = 0$ *and* $a_0 s_0(2) + a_1 s_1(2) = 0$. *From the choice of* a_0 *and* a_1, *we get* $s_0(1) = s_1(1)$ *and* $s_0(2) = s_1(2)$ *where* $S_0 = \{0, 3, 5, 6\}$ *and* $S_1 = \{1, 2, 4, 7\}$.

Theorem 10. *The parametric form of all integral solutions of the non-ideal PTE problem*

$$\sum_{i=1}^{5} x_i = \sum_{i=1}^{5} y_i \tag{6}$$

$$\sum_{i=1}^{5} x_i^2 = \sum_{i=1}^{5} y_i^2 \tag{7}$$

is given by $x_1 = 5t_1$, $x_2 = t_2$, $x_3 = t_3$, $x_4 = 6t_4 - 4t_1 - t_7$, $x_5 = 2t_1 - 3t_4 - 2t_7$, $y_1 = 5t_4$, $y_2 = t_2$, $y_3 = t_3$, $y_4 = 6t_1 - 4t_4 - t_7$ *and* $y_5 = 2t_4 - 3t_1 - 2t_7$.

Proof. Consider

$$\sum_{i=1}^{5} x_i = \sum_{i=1}^{5} y_i = k.$$

Let $x_1 = 5t_1$, $x_2 = t_2$, $x_3 = t_3$, $y_1 = 5t_4$, $y_2 = t_5$ and $y_6 = t_6$.Then we get

$$x_4 + x_5 = k - 5t_1 - t_2 - t_3 \tag{8}$$

and

$$y_4 + y_5 = k - 5t_4 - t_5 - t_6.$$

Let α and β be two integers such that $1 \times \alpha - 1 \times \beta = 1$. Then $\alpha = 2$ and $\beta = 1$. So $1 \times 2 - 1 \times 1 = 1$. Multiplying by $k - 5t_1 - t_2 - t_3$, we get

$$2(k - 5t_1 - t_2 - t_3) - 1(k - 5t_1 - t_2 - t_3) = k - 5t_1 - t_2 - t_3. \tag{9}$$

(8) and (9) we get

$$[x_4 - 2(k - 5t_1 - t_2 - t_3)] + [x_5 + (k - 5t_1 - t_2 - t_3)] = 0. \tag{10}$$

Adding and subtracting $5t_7$ we obtain

$$[x_4 - 2(k - 5t_1 - t_2 - t_3) - 5t_7] + [x_5 + (k - 5t_1 - t_2 - t_3) + 5t_7] = 0. \tag{11}$$

Thus, $x_4 = 2(k - 5t_1 - t_2 - t_3) + 5t_7$ and $x_5 = -(k - 5t_1 - t_2 - t_3) - 5t_7$. Similarly, we get $y_4 = 2(k - 5t_4 - t_5 - t_6) + 5t_8$ and $y_5 = -(k - 5t_4 - t_5 - t_6) - 5t_8$. Now, substitute the values of x_i's and y_i's in (7), we get $x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 25t_1^2 + t_2^2 + t_3^2 + [2(k - 5t_1 - t_2 - t_3) + 5t_7]^2 + [(k - 5t_1 - t_2 - t_3) + 5t_7]^2$ and $y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2 = 25t_4^2 + t_5^2 + t_6^2 + [2(k - 5t_4 - t_5 - t_6) + 5t_8]^2 + [(k - 5t_4 - t_5 - t_6) + t_8]^2$. Take $t_7 = t_8$, $t_2 = t_5$ and $t_3 = t_6$. After performing elementary calculations we obtain

$$\begin{aligned} 3(t_1^2 - t_4^2) + t_2(t_1 - t_4) + t_3(t_1 - t_4) - 3t_7(t_1 - t_4) &= k(t_1 - t_4) \\ k &= 3(t_1 + t_4 - t_7) + t_2 + t_3 \end{aligned}$$

provided $t_1 - t_4 \neq 0$. Thus, $x_4 = -4t_1 + 6t_4 - t_7$, $x_5 = 2t_1 - 3t_4 - 2t_7$, $y_4 = 6t_1 - 4t_4 - t_7$ and $y_5 = -3t_1 + 2t_4 - 2t_7$. Hence the proof. □

Example 6. *Let $t_1 = 1$, $t_4 = 2$, $t_7 = -1$, $t_2 = 1$ and $t_3 = -2$. Then $k = 11$. Thus, $x_4 = 9$, $x_5 = -2$, $y_4 = -1$ and $y_5 = 3$. Hence $\{5, 1, -2, 9, -2\} =_2 \{10, 1, -2, -1, 3\}$.*

Theorem 11. *If a_1, a_2, a_3, a_4 and a_5 are any integers with $a_1 - a_4 \neq 0$, then the integer of the form $45a_1^2 + a_2^2 + a_3^2 + 45a_4^2 + 5a_5^2 - 60a_1a_4$ can be represented as the sum of five perfect squares in two distinct ways.*

Proof. Let $x_1 = 5a_1$, $x_2 = a_2$, $x_3 = a_3$, $x_4 = 6a_4 - 4a_1 - a_5$, $x_5 = 2a_1 - 3a_4 - 2a_5$, $y_1 = 5a_4$, $y_2 = a_2$, $y_3 = a_3$, $y_4 = 6a_1 - 4a_4 - a_5$ and $y_5 = 2a_4 - 3a_1 - 2a_5$.Then,

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 &= 25a_1^2 + a_2^2 + a_3^2 + (6a_4 - 4a_1 - a_5)^2 + (2a_1 - 3a_4 - 2a_5)^2 \\ &= 45a_1^2 + a_2^2 + a_3^2 + 45a_4^2 + 5a_5^2 - 60a_1a_4. \end{aligned}$$

Similarly,

$$\begin{aligned} y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2 &= 25a_4^2 + a_2^2 + a_3^2 + (6a_1 - 4a_4 - a_7)^2 + (2a_4 - 3a_1 - 2a_5)^2 \\ &= 45a_4^2 + a_2^2 + a_3^2 + 45a_1^2 + 5a_5^2 - 60a_1a_4. \end{aligned}$$

Comparing these two we get

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2.$$

Hence the proof. □

4. On Fibonacci Like Pattern in PTE Problem

Because of the uncertainty in nature, the Fibonacci numbers are always become fishy to the mathematicians. One can expound the Fibonacci sequence of numbers using the looping as

$$\begin{gathered} F_1 = 1, F_2 = 1 \text{ and} \\ F_n = F_{n-1} + F_{n-2}; \text{ for } n > 2. \end{gathered}$$

Some remarkable studies on Fibonacci numbers and their applications can be seen in [33–35]. In Section 4, the Fibonacci like pattern appearing in the solutions of the $(4, 2)$-PTE problem is analyzed by impossing an additonal condition $x_i = x_{i-1} + x_{i-2}, y_i = y_{i-1} + y_{i-2}$ for $i \geq 3$ to the problem.

Note 1. *Consider the Diophantine equation*

$$\sum_{i=1}^{3} x_i = \sum_{i=1}^{3} y_i, \tag{12}$$

where $x_3 = x_1 + x_2$ and $y_3 = y_1 + y_2$. If we replace x_3 and y_3 in (12) by $x_1 + x_2$ and $y_1 + y_2$, we obtain

$$\sum_{i=1}^{2} x_i = \sum_{i=1}^{2} y_i,$$

which is the $(2, 1)$-PTE problem. Thus, any solution of $\sum_{i=1}^{2} x_i = \sum_{i=1}^{2} y_i$ will provide solution to (12).

Note 2. *It does not guarantee that the solutions of ideal PTE problem satisfy the Fibonacci like pattern. This is because, in the $(3, 2)$-PTE problem, if $x_3 = x_1 + x_2$ and $y_3 = y_1 + y_2$, then after some algebraic operations the system will reduce to $(2, 2)$-PTE problem which has no solutions.*

Theorem 12. *A two parameter family of infinitely many integral solutions of the system of equations* $\sum_{i=1}^{4} x_i = \sum_{i=1}^{4} y_i$ *and* $\sum_{i=1}^{4} x_i^2 = \sum_{i=1}^{4} y_i^2$ *where* $x_i = x_{i-1} + x_{i-2}, y_i = y_{i-1} + y_{i-2}$ *for* $i \geq 3$*, is given by* $x_1 = 8t_1$, $y_1 = 8t_2$, $x_2 = 5t_2 - t_1$, $y_2 = 5t_1 - t_2$, $x_3 = 5t_2 + 7t_1$, $y_3 = 5t_1 + 7t_2$, $x_4 = 10t_2 + 6t_1$ *and* $y_4 = 10t_1 + 6t_2$, *provided* $t_2 - t_1 \neq 0$*.*

Proof. Consider the following equations

$$\sum_{i=1}^{4} x_i = \sum_{i=1}^{4} y_i, \tag{13}$$

$$\sum_{i=1}^{4} x_i^2 = \sum_{i=1}^{4} y_i^2. \tag{14}$$

Put $x_3 = x_1 + x_2$, $x_4 = x_2 + x_3 = x_1 + 2x_2$, $y_3 = y_1 + y_2$ and $y_4 = y_2 + y_3 = y_1 + 2y_2$. Then, we have

$$\begin{aligned} x_1 + x_2 + (x_1 + x_2) + (x_1 + 2x_2) &= y_1 + y_2 + (y_1 + y_2) + (y_1 + 2y_2), \\ 3x_1 + 4x_2 &= 3y_1 + 4y_2. \end{aligned}$$

Similarly

$$\begin{aligned} x_1^2 + x_2^2 + (x_1 + x_2)^2 + (x_1 + 2x_2)^2 &= y_1^2 + y_2^2 + (y_1 + y_2)^2 + (y_1 + 2y_2)^2 \\ 3x_1^2 + 6x_2^2 + 6x_1x_2 &= 3y_1^2 + 6y_2^2 + 6y_1y_2 \\ x_1^2 + 2x_2^2 + 2x_1x_2 &= y_1^2 + 2y_2^2 + 2y_1y_2 \\ (x_1 + x_2)^2 + x_2^2 &= (y_1 + y_2)^2 + y_2^2. \end{aligned}$$

Thus (13) and (14) becomes

$$\begin{aligned} 3x_1 + 4x_2 &= 3y_1 + 4y_2 \\ (x_1 + x_2)^2 + x_2^2 &= (y_1 + y_2)^2 + y_2^2. \end{aligned}$$

Let

$$3x_1 + 4x_2 = 3y_1 + 4y_2 = p.$$

Take $x_1 = 8t_1$ and $y_1 = 8t_2$. Then, we have $x_2 = \frac{p-24t_1}{4}$ and $y_2 = \frac{p-24t_2}{4}$. Substituting these values, we get

$$\begin{aligned} \left[8t_1 + \frac{p - 24t_1}{4}\right]^2 + \left[\frac{p - 24t_1}{4}\right]^2 &= \left[8t_2 + \frac{p - 24t_2}{4}\right]^2 + \left[\frac{p - 24t_2}{4}\right]^2 \\ (8t_1 + p)^2 + (p - 24t_1)^2 &= (8t_2 + p)^2 + (p - 24t_2)^2 \\ p[32t_2 - 32t_1] &= 64(t_2^2 - t_1^2) + 24^2(t_2^2 - t_1^2) \\ p &= 20(t_2 + t_1);\ t_2 - t_1 \neq 0. \end{aligned}$$

So,

$$\begin{aligned} y_2 &= \frac{p-24t_2}{4} \\ &= \frac{20t_2+20t_1-24t_2}{4} \\ &= 5t_1-t_2 \end{aligned}$$

and

$$\begin{aligned} x_2 &= \frac{p-24t_1}{4} \\ &= \frac{20t_2+20t_1-24t_1}{4} \\ &= 5t_2-t_1. \end{aligned}$$

Thus the solutions of the problem is given by $x_1 = 8t_1$, $y_1 = 8t_2$, $x_2 = 5t_2 - t_1$, $y_2 = 5t_1 - t_2$, $x_3 = 5t_2 + 7t_1$, $y_3 = 5t_1 + 7t_2$, $x_4 = 10t_2 + 6t_1$ and $y_4 = 10t_1 + 6t_2$ provided $t_2 - t_1 \neq 0$. □

Example 7. *Let $t_1 = 2$ and $t_2 = 3$. Then $t_2 - t_1 = 1 \neq 0$. $x_1 = 16$, $x_2 = 13$, $x_3 = 29$, $x_4 = 42$, $y_1 = 24$, $y_2 = 7$, $y_3 = 31$ and $y_4 = 38$.*

Corollary 2. *The primes 2 and 3 divides $\prod_{i=1}^{4}(x_i - y_i)$.*

Proof. Let $C = \prod_{i=1}^{4}(x_i - y_i)$. As per the assumptions in Theorem 12, we have $x_1 = 8t_1$, $y_1 = 8t_2$, $x_2 = 5t_2 - t_1$, $y_2 = 5t_1 - t_2$, $x_3 = 5t_2 + 7t_1$, $y_3 = 5t_1 + 7t_2$, $x_4 = 10t_2 + 6t_1$ and $y_4 = 10t_1 + 6t_2$. Then

$$\begin{aligned} C &= \prod_{i=1}^{4}(x_i - y_i) \\ &= (8t_1 - 8t_2) \times (6t_2 - 6t_1) \times (2t_1 - 2t_2) \times (4t_2 - 4t_1) \\ &= 8 \times 6 \times 2 \times 4 \times (t_1 - t_2)^4. \end{aligned}$$

So, the primes 2 and 3 divides C. The other prime divisors of C will obtain accordingly as the number $(t_1 - t_2)$ since t_1 and t_2 can take any integers such that $t_1 - t_2 \neq 0$. □

If $N(k)$ denote the least positive integer such that the Diophantine system $\sum_{i=1}^{s} x_i^k = \sum_{i=1}^{s} y_i^k$; $k = 1, 2, \cdots, n$; and $x_i = x_{i-1} + x_{i-2}$; $y_i = y_{i-1} + y_{i-2}$ for $i \geq 3$, possess nontrivial integer solutions, then from Note 1 and Theorem 12 we obtain $N(1) = 3$ and $N(2) = 4$. Thus, we arrive at the Theorem 13.

Theorem 13. $N(k) \leq [\frac{1}{2}k(k+1)] + 2$.

Proof. Let $n > s^k s!$. Define

$$A = \{(x_1, x_2, \cdots, x_s) : 1 \leq x_i \leq n;\ i = 1, 2, \cdots, s \text{ and } x_s = x_{s-1} + x_{s-2};\ s \geq 3\}.$$

Then there are $(n-1)!$ elements in A. Let (a_i), $(b_i) \in A$. Then we can take (a_i), (b_i) as $(a_i) = (x_1, x_2, \cdots, x_s)$ and $(b_i) = (y_1, y_2, \cdots, y_s)$ for some integers x_i and y_i; $i = 1, 2, ...s$. We define an equivalence relation on A by $(a_i) \sim (b_i)$ if and only if $(a_i) := (x_1, x_2, \cdots, x_s)$ is a permutation of $(b_i) := (y_1, y_2, \cdots, y_s)$. For example, if $(a_i) = (1, 2, 3)$ is an element in A, then all its six permutations like $(2, 3, 1)$, $(3, 1, 2)$, $\cdots$ are equivalent to (a_i). The reason for defining the equivalence relation like this is that generally in PTE problems if $(1,2,3)$ satisfies the left hand side equality then the right hand side solution should not be the permutation of $(1,2,3)$. Since $(x_1, x_2, \cdots, x_s)$ has atmost $s!$ distinct permutations, there are $\frac{(n-1)!}{s!}$ distinct classes in $A/\sim$. Define

$$(S_j(a_i)) = x_1^j + x_2^j + \cdots + x_s^j$$

for $j = 1, 2, \cdots, k$. Note that $s \leq (S_j(a_i)) \leq sn^j$. So there are atmost $\prod_{j=1} k(sn^j - s + 1) < \prod_{j=1} ksn^j = s^k n^{\frac{k(k+1)}{2}}$ distinct sets $((S_1(a_i)), (S_2(a_i), \cdots, (S_k(a_i))$. Choose $s = [\frac{1}{2}k(k+1)] + 2$. Then, we have

$$s^k n^{\frac{k(k+1)}{2}} = s^k s^{n-2} < \frac{n}{s!} n^{s-2} = \frac{n^{s-1}}{s!} < \frac{(n-1)!}{s!}$$

since $n > s^k s!$. So the number of possible $((S_1(a_i)), (S_2(a_i), \cdots, (S_k(a_i))$ is less than the number of distinct (a_i). Thus, the two distinct sets $\{x_1, x_2, \cdots, x_s\}$ and $\{y_1, y_2, \cdots, y_s\}$ form a solution of degree k. $\square$

5. Conclusions

In the present study, a new parametric solution of the non-ideal PTE problem $\sum_{i=1}^{s} x_i^r = \sum_{i=1}^{s} y_i^r; r = 1, 2$ with $s = 3, 4$ and 5 has been developed. The main significance of the present solution is that the method adopted is very simple and the parametric solutions are new compared to other works in this area. It is also noteworthy that a new proof of the non-existence of solutions of $(3,2)$-PTE problem has been derived. In the present work, a three parametric solution of $(3,2)$-PTE problem has been obtained which is new in the study of the solutions of PTE problem. Moreover, a new parametric form of positive integers that can be expressed as the sum of three, four and five perfect squares in two distinct ways has been determined. Another significance of this study is that a parametric form of solutions of the PTE problem in which solutions satisfying Fibonacci like pattern has been formulated and then obtained a bound for the size of this particular PTE problem. Further, it is observed that the arithmetic function derived from the PTM sequence is non-multiplicative. The present study has been done for order two with sizes three, four! and five. It can be further extended for higher degrees and higher sizes. One of the applications of the $(4,2)$-PTE problem is in the combinatorics where the PTE partitions are used to pour the same volume of coffee from a container into a finite number of cups so that each gets almost the same amount of caffeine, as discussed in [26]. In general, the solutions of the PTE problem play a major role in fields like combinatorics, the easier waring problems, and in finding the rational points on elliptic curves.

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