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Sufficient Criteria for the Absence of Global Solutions for an Inhomogeneous System of Fractional Differential Equations

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Abstract: A nonlinear inhomogeneous system of fractional differential equations is investigated. Namely, sufficient criteria are obtained so that the considered system has no global solutions. Furthermore, an example is provided to show the effect of the inhomogeneous terms on the blow-up of solutions. Our results are extensions of those obtained by Furati and Kirane (2008) in the homogeneous case.

Keywords: absence of global solutions; inhomogeneous system; fractional differential equations

MSC: 26A33; 34A08

1. Introduction

The theory of fractional calculus provides useful mathematical tools for modeling several phenomena from Science and Engineering (see e.g., [1–7]). This fact has motivated researchers to investigate fractional differential equations in various directions including theory (existence and uniqueness of solutions [8–10], comparison principles [11,12], blow-up profile of solutions [13,14]), and numerical methods (see e.g., [15–19]).

In this paper, we study the inhomogeneous system of fractional differential equations

$$\begin{cases} \dfrac{dx}{dt}(t) + \dfrac{dx}{dt^{\alpha}}(t) & = & |y(t)|^q + z_1(t), \\ \dfrac{dy}{dt}(t) + \dfrac{dy}{dt^{\beta}}(t) & = & |x(t)|^p + z_2(t), \\ (x(0), y(0)) & = & (x_0, y_0), \end{cases} \tag{1}$$

where $t > 0$, $p, q > 1$, $z_1, z_2 \in C([0,\infty))$, $z_1, z_2 \geq 0$, $0 < \alpha, \beta < 1$, and $\frac{d}{dt^{\tau}}$, $\tau \in \{\alpha, \beta\}$, is the derivative of order τ in the sense of Caputo. Namely, we obtain sufficient criteria for which global solutions for Equation (1) do not exist, i.e., a finite time blow-up occurs.

The study of the absence of global solutions for differential equations (or fractional differential equations) furnishes important indications on limiting behaviors of many physical systems. In industry, the knowledge of the finite time blow-up can prevent accidents and malfunctions. It can also be useful for the improvement of the performance of machines and the extension of their life-span.

In [20], Furati and Kirane investigated the system

$$\begin{cases} \frac{dx}{dt}(t) + \frac{dx}{dt^{\alpha}}(t) & = & |y(t)|^{q}, \\ \frac{dy}{dt}(t) + \frac{dy}{dt^{\beta}}(t) & = & |x(t)|^{p}, \\ (x(0), y(0)) & = & (x_0, y_0), \end{cases} \tag{2}$$

which is a special case of Equation (1) with $z_i \equiv 0$, $i = 1, 2$. They proved that, if $x_0, y_0 > 0$ and

$$1 - \frac{1}{pq} \leq \max\left\{\alpha + \frac{\beta}{p}, \beta + \frac{\alpha}{q}\right\},$$

then Equation (2) has no global solutions. Next, Kirane and Malik [14] studied the profile of the blowing up solutions of Equation (2).

Let us mention that several results related to the finite time blow-up of solutions of fractional differential equations were obtained in previous contributions (see e.g., [13,21–25] and references therein). However, systems of the type seen in Equation (1) were not studied previously. Here, our aim is to study the effect of the inhomogenous terms $z_i(t)$, $i = 1, 2$, on the blow-up of solutions of Equation (2).

Before presenting the main results, we first recall briefly certain standard notions on fractional calculus that will be used throughout this paper. For more details, see e.g., [9].

Let $\xi > 0$. The fractional integrals of order ξ of a function $\eta \in C([0, \mu])$, $\mu > 0$, are given by

$$\mathcal{I}_0^{\xi}\eta(t) = \frac{1}{\Gamma(\xi)} \int_0^t (t-\tau)^{\xi-1}\eta(\tau)\, d\tau \quad \text{(left-sided fractional integral)}$$

and

$$\mathcal{I}_{\mu}^{\xi}\eta(t) = \frac{1}{\Gamma(\xi)} \int_t^{\mu} (\tau-t)^{\xi-1}\eta(\tau)\, d\tau \quad \text{(right-sided fractional integral),}$$

for all $t \in [0, \mu]$.

For all $\eta, \lambda \in C([0, \mu])$, one has

$$\int_0^{\mu} \lambda(t)\mathcal{I}_0^{\xi}\eta(t)\, dt = \int_0^{\mu} \eta(t)\mathcal{I}_{\mu}^{\xi}\lambda(t)\, dt. \tag{3}$$

Let

$$\psi(t) = \left(1 - \frac{t}{\mu}\right)^{\kappa}, \quad \text{for all } t \in [0, \mu],$$

where $\kappa \geq 0$. Then

$$\mathcal{I}_{\mu}^{\xi}\psi(t) = \frac{\Gamma(\kappa+1)}{\Gamma(\kappa+\xi+1)}\mu^{-\kappa}(\mu-t)^{\xi+\kappa}, \quad \text{for all } t \in [0, \mu]. \tag{4}$$

Suppose now that $0 < \xi < 1$. The derivative of order ξ of a function $\eta \in C^1([0, \mu])$ in the Caputo sense, is given by

$$\frac{d\eta}{dx^{\xi}}(t) = \mathcal{I}_0^{1-\xi}\left(\frac{d\eta}{dt}\right)(t), \quad \text{for all } t \in [0, \mu]. \tag{5}$$

The pair of functions (x, y), where $x, y \in C^1([0, \infty))$, is a global solution of Equation (1), if it satisfies Equation (1) for all $t > 0$. Let us recall that the system of Equation (1) is investigated under the assumptions:

(A1) $p, q > 1$ and $0 < \alpha, \beta < 1$.
(A2) $z_i \in C([0, \infty))$, $z_i \geq 0$, $i = 1, 2$.

Now, we present our results.

Theorem 1. *Let* $x_0, y_0 \geq 0$. *If*

$$\limsup_{T\to+\infty} T^{\frac{(\alpha-1)pq+\beta q+1}{pq-1}} \int_0^T z_1(s)\,ds = +\infty \quad or \quad \limsup_{T\to+\infty} T^{\frac{(\beta-1)pq+\alpha p+1}{pq-1}} \int_0^T z_2(s)\,ds = +\infty, \tag{6}$$

then Equation (1) has no global solutions.

Consider now the case of a single equation

$$\begin{cases} \dfrac{dx}{dt}(t) + \dfrac{dx}{dt^\alpha}(t) &= |x(t)|^p + z(t), \\ x(0) &= x_0, \end{cases} \tag{7}$$

where $0 < \alpha < 1$, $p > 1$ and $z \in C([0,\infty))$, $z \geq 0$. From Theorem 1, we deduce

Corollary 1. *Let* $x_0 \geq 0$. *If*

$$\limsup_{T\to+\infty} T^{\frac{\alpha p}{p-1}-1} \int_0^T z(s)\,ds = +\infty,$$

then Equation (7) has no global solutions.

The following example shows the effect of the inhomogeneous terms $z_i(t)$, $i = 1, 2$, on the blow-up of solutions of Equation (2)

Example 1. *Consider the system*

$$\begin{cases} \dfrac{dx}{dt}(t) + \dfrac{dx}{dt^\alpha}(t) &= |y(t)|^q + (t+1)^\rho, \\ \dfrac{dy}{dt}(t) + \dfrac{dy}{dt^\beta}(t) &= |x(t)|^p, \\ (x(0), y(0)) &= (x_0, y_0), \end{cases} \tag{8}$$

where $t > 0$, $x_0 \geq 0$,$y_0 \geq 0$, $p > 1$, $q > 1$ *and*

$$\rho > -\min\left\{1, \frac{q(\alpha p + \beta)}{pq - 1}\right\}. \tag{9}$$

System Equation (8) is a special case of Equation (1) with

$$z_1(t) = (t+1)^\rho \quad and \quad z_2(t) = 0,$$

for all $t \geq 0$. *One observes easily that*

$$T^{\frac{(\alpha-1)pq+\beta q+1}{pq-1}} \int_0^T z_1(s)\,ds \sim T^{\frac{(\alpha-1)pq+\beta q+1}{pq-1}+\rho+1}, \quad as\ T \to +\infty.$$

Furthermore, it follows from Equation (9) that

$$\frac{(\alpha-1)pq + \beta q + 1}{pq - 1} + \rho + 1 > 0,$$

which yields

$$\lim_{T\to+\infty} T^{\frac{(\alpha-1)pq+\beta q+1}{pq-1}} \int_0^T z_1(s)\,ds = +\infty.$$

Hence by Theorem 1, one deduces that under condition Equation (9), for all $p, q > 1$, Equation (8) has no global solutions.

2. Proof of the Main Result

The proof is based on the nonlinear capacity method (see e.g., [26]). More precisely, we first suppose that (x, y) is a global solution of Equation (1). Next, we multiply both equations in Equation (1) by an adequate test function that depends of a parameter $T \gg 1$, and we integrate by parts over the interval $(0, T)$. Using standard integral inequalities, the condition in Equation (6) and passing to the limit as $T \to \infty$, a contradiction follows.

The detailed proof of Theorem 1 is given below.

Proof. We follow the steps mentioned previously.
Step 1 (Multiplication of both equations in Equation (1) by an adequate test function that depends of a parameter $T \gg 1$):

Suppose (x, y) is solution of (1) which is global. For $\kappa, T \gg 1$, let

$$\nu(t) = T^{-\kappa}(T-t)^{\kappa}, \quad \text{for all } 0 \leq t \leq T. \tag{10}$$

After multiplication of the first equation in Equation (1) by $\nu(t)$ and integration over $(0, T)$, we obtain

$$\int_0^T \frac{dx}{dt}\nu \, dt + \int_0^T \frac{dx}{dt^\alpha}\nu \, dt = \int_0^T |y|^q \nu \, dt + \int_0^T z_1 \nu \, dt. \tag{11}$$

Integrating by parts, we have

$$\int_0^T \frac{dx}{dt}\nu \, dt = x(T)\nu(T) - x(0)\nu(0) - \int_0^T x\frac{d\nu}{dt} \, dt.$$

Since $\nu(T) = 0$ and $\nu(0) = 1$, we get

$$\int_0^T \frac{dx}{dt}\nu \, dt = -x(0) - \int_0^T x\frac{d\nu}{dt} \, dt. \tag{12}$$

Using Equations (5) and (3), we have

$$\int_0^T \frac{dx}{dt^\alpha}\nu \, dt = \int_0^T \mathcal{I}_0^{1-\alpha}\left(\frac{dx}{dt}\right)\nu \, dt = \int_0^T \frac{dx}{dt}\mathcal{I}_T^{1-\alpha}\nu \, dt.$$

Again, we integrate by parts, we obtain

$$\int_0^T \frac{dx}{dt^\alpha}\nu \, dt = x(T)\left(\mathcal{I}_T^{1-\alpha}\nu\right)(T) - x(0)\left(\mathcal{I}_T^{1-\alpha}\nu\right)(0) - \int_0^T x\frac{d\mathcal{I}_T^{1-\alpha}\nu}{dt} \, dt.$$

On the other hand, by Equation (4), one has $\left(\mathcal{I}_T^{1-\alpha}\nu\right)(T) = 0$. Therefore, it holds that

$$\int_0^T \frac{dx}{dt^\alpha}\nu \, dt = -x(0)\left(\mathcal{I}_T^{1-\alpha}\nu\right)(0) - \int_0^T x\frac{d\mathcal{I}_T^{1-\alpha}\nu}{dt} \, dt. \tag{13}$$

Further, using Equation (11)–(13), we deduce that

$$x(0)\left(1 + \left(\mathcal{I}_T^{1-\alpha}\nu\right)(0)\right) + \int_0^T |y|^q \nu \, dt + \int_0^T z_1 \nu \, dt \leq \int_0^T |x|\left|\frac{d\nu}{dt}\right| dt + \int_0^T |x|\left|\frac{d\mathcal{I}_T^{1-\alpha}\nu}{dt}\right| dt.$$

Since $x(0) \geq 0$ and $\left(\mathcal{I}_T^{1-\alpha}\nu\right)(0) \geq 0$ by Equation (4), it holds that

$$\int_0^T |y|^q \nu \, dt + \int_0^T z_1 \nu \, dt \leq \int_0^T |x| \left|\frac{d\nu}{dt}\right| dt + \int_0^T |x| \left|\frac{d\mathcal{I}_T^{1-\alpha}\nu}{dt}\right| dt. \tag{14}$$

Similarly, after multiplication of the second equation in Equation (1) by $\nu(t)$ and integration over $(0,T)$, using that $y(0) \geq 0$ and $\left(\mathcal{I}_T^{1-\beta}\nu\right)(0) \geq 0$, we obtain

$$\int_0^T |x|^p \nu \, dt + \int_0^T z_2 \nu \, dt \leq \int_0^T |y| \left|\frac{d\nu}{dt}\right| dt + \int_0^T |y| \left|\frac{d\mathcal{I}_T^{1-\beta}\nu}{dt}\right| dt. \tag{15}$$

Next, we set

$$\mathbb{I} = \int_0^T |x|^p \nu \, dt \quad \text{and} \quad \mathbb{J} = \int_0^T |y|^q \nu \, dt.$$

We use Hölder's inequality to get

$$\int_0^T |x| \left|\frac{d\nu}{dt}\right| dt = \int_0^T |x| \nu^{\frac{1}{p}} \nu^{\frac{-1}{p}} \left|\frac{d\nu}{dt}\right| dt \leq \mathbb{I}^{\frac{1}{p}} \left(\int_0^T \nu^{\frac{-1}{p-1}} \left|\frac{d\nu}{dt}\right|^{\frac{p}{p-1}} dt\right)^{\frac{p-1}{p}} \tag{16}$$

and

$$\int_0^T |x| \left|\frac{d\mathcal{I}_T^{1-\alpha}\nu}{dt}\right| dt \leq \mathbb{I}^{\frac{1}{p}} \left(\int_0^T \nu^{\frac{-1}{p-1}} \left|\frac{d\mathcal{I}_T^{1-\alpha}\nu}{dt}\right|^{\frac{p}{p-1}} dt\right)^{\frac{p-1}{p}}. \tag{17}$$

Similarly, one has

$$\int_0^T |y| \left|\frac{d\nu}{dt}\right| dt \leq \mathbb{J}^{\frac{1}{q}} \left(\int_0^T \nu^{\frac{-1}{q-1}} \left|\frac{d\nu}{dt}\right|^{\frac{q}{q-1}} dt\right)^{\frac{q-1}{q}} \tag{18}$$

and

$$\int_0^T |y| \left|\frac{d\mathcal{I}_T^{1-\beta}\nu}{dt}\right| dt \leq \mathbb{J}^{\frac{1}{q}} \left(\int_0^T \nu^{\frac{-1}{q-1}} \left|\frac{d\mathcal{I}_T^{1-\beta}\nu}{dt}\right|^{\frac{q}{q-1}} dt\right)^{\frac{q-1}{q}}. \tag{19}$$

Next, setting

$$\mathbb{A}_1 = \int_0^T \nu^{\frac{-1}{p-1}} \left|\frac{d\nu}{dt}\right|^{\frac{p}{p-1}} dt \quad \text{and} \quad \mathbb{A}_2 = \int_0^T \nu^{\frac{-1}{p-1}} \left|\frac{d\mathcal{I}_T^{1-\alpha}\nu}{dt}\right|^{\frac{p}{p-1}} dt,$$

using Equations (14), (16), and (17), one deduces that

$$\mathbb{J} + \int_0^T z_1 \nu \, dt \leq \mathbb{I}^{\frac{1}{p}} \left(\mathbb{A}_1^{\frac{p-1}{p}} + \mathbb{A}_2^{\frac{p-1}{p}}\right). \tag{20}$$

Similarly, setting

$$\mathbb{B}_1 = \int_0^T \nu^{\frac{-1}{q-1}} \left|\frac{d\nu}{dt}\right|^{\frac{q}{q-1}} dt \quad \text{and} \quad \mathbb{B}_2 = \int_0^T \nu^{\frac{-1}{q-1}} \left|\frac{d\mathcal{I}_T^{1-\beta}\nu}{dt}\right|^{\frac{q}{q-1}} dt,$$

using Equations (15), (18), and (19), one deduces that

$$\mathbb{I}+\int_0^T z_2\nu\,dt \le \mathbb{J}^{\frac{1}{q}}\left(\mathbb{B}_1^{\frac{q-1}{q}}+\mathbb{B}_2^{\frac{q-1}{q}}\right). \tag{21}$$

Consider now the case

$$\limsup_{T\to+\infty} T^{\frac{(\alpha-1)pq+\beta q+1}{pq-1}}\int_0^T z_1(s)\,ds=+\infty. \tag{22}$$

Since

$$\int_0^T z_2\nu\,dt\ge 0,$$

it follows from Equation (21) that

$$\mathbb{I}\le \mathbb{J}^{\frac{1}{q}}\left(\mathbb{B}_1^{\frac{q-1}{q}}+\mathbb{B}_2^{\frac{q-1}{q}}\right).$$

The above inequality with Equation (20) yields

$$\mathbb{J}+\int_0^T z_1\nu\,dt \le \mathbb{J}^{\frac{1}{pq}}\left(\mathbb{B}_1^{\frac{q-1}{q}}+\mathbb{B}_2^{\frac{q-1}{q}}\right)^{\frac{1}{p}}\left(\mathbb{A}_1^{\frac{p-1}{p}}+\mathbb{A}_2^{\frac{p-1}{p}}\right). \tag{23}$$

Step 2 (Estimates and conclusion):

Further, we shall estimate the term in right-hand side of the above inequality. Using the inequality

$$(a_1+a_2)^\gamma\le 2^\gamma(a_1^\gamma+a_2^\gamma),\quad 0<\gamma<1,\ a_1,a_2>0,$$

one obtains

$$\left(\mathbb{B}_1^{\frac{q-1}{q}}+\mathbb{B}_2^{\frac{q-1}{q}}\right)^{\frac{1}{p}}\left(\mathbb{A}_1^{\frac{p-1}{p}}+\mathbb{A}_2^{\frac{p-1}{p}}\right)\le 2^{\frac{1}{p}}\left(\mathbb{A}_1^{\frac{p-1}{p}}\mathbb{B}_1^{\frac{q-1}{pq}}+\mathbb{A}_2^{\frac{p-1}{p}}\mathbb{B}_1^{\frac{q-1}{pq}}+\mathbb{A}_1^{\frac{p-1}{p}}\mathbb{B}_2^{\frac{q-1}{pq}}+\mathbb{A}_2^{\frac{p-1}{p}}\mathbb{B}_2^{\frac{q-1}{pq}}\right). \tag{24}$$

On the other hand, using Equation (10), elementary calculations yield

$$\mathbb{A}_1=\frac{1}{\kappa-\frac{1}{p-1}}T^{\frac{-1}{p-1}}\quad\text{and}\quad \mathbb{B}_1=\frac{1}{\kappa-\frac{1}{q-1}}T^{\frac{-1}{q-1}}. \tag{25}$$

Similarly, using Equations (4) and (10), we obtain

$$\mathbb{A}_2=\left[\frac{\Gamma(\kappa+1)}{\Gamma(\kappa+1-\alpha)}\right]^{\frac{p}{p-1}}\frac{T^{1-\frac{\alpha p}{p-1}}}{\kappa-\frac{\alpha p}{p-1}+1}\quad\text{and}\quad \mathbb{B}_2=\left[\frac{\Gamma(\kappa+1)}{\Gamma(\kappa+1-\beta)}\right]^{\frac{q}{q-1}}\frac{T^{1-\frac{\beta q}{q-1}}}{\kappa-\frac{\beta q}{q-1}+1}. \tag{26}$$

Hence, using Equations (24)–(26), one deduces easily that

$$\left(\mathbb{B}_1^{\frac{q-1}{q}}+\mathbb{B}_2^{\frac{q-1}{q}}\right)^{\frac{1}{p}}\left(\mathbb{A}_1^{\frac{p-1}{p}}+\mathbb{A}_2^{\frac{p-1}{p}}\right)\le CT^{1-\alpha-\frac{\beta}{p}-\frac{1}{pq}},$$

where $C>0$ is a constant. The above inequality with Equation (23) yield

$$\mathbb{J}+\int_0^T z_1\nu\,dt\le C\mathbb{J}^{\frac{1}{pq}}T^{1-\alpha-\frac{\beta}{p}-\frac{1}{pq}}. \tag{27}$$

Next, using Young's inequality

$$ab \leq \frac{1}{pq} a^{pq} + \frac{pq-1}{pq} b^{\frac{pq}{pq-1}}, \quad a, b > 0,$$

we get

$$C\mathbb{J}^{\frac{1}{pq}} T^{1-\alpha-\frac{\beta}{p}-\frac{1}{pq}} \leq \frac{1}{pq}\mathbb{J} + \left(\frac{pq-1}{pq}\right) C^{\frac{pq}{pq-1}} T^{\frac{pq}{pq-1}\left(1-\alpha-\frac{\beta}{p}-\frac{1}{pq}\right)},$$

together with Equation (27) implies

$$\int_0^T z_1 \nu \, dt \leq \widetilde{C} T^{\frac{pq}{pq-1}\left(1-\alpha-\frac{\beta}{p}-\frac{1}{pq}\right)}, \tag{28}$$

where $\widetilde{C} = \left(\frac{pq-1}{pq}\right) C^{\frac{pq}{pq-1}}$. On the other hand, by Equation (10), one has

$$\int_0^T z_1 \nu \, dt = T^{-\kappa} \int_0^T z_1(t)(T-t)^{\kappa} \, dt \geq \frac{1}{2^{\kappa}} \int_0^{\frac{T}{2}} z_1(t) \, dt.$$

The above inequality with (28) yields

$$\frac{1}{2^{\kappa}} T^{\frac{(\alpha-1)pq+\beta q+1}{pq-1}} \int_0^{\frac{T}{2}} z_1(t) \, dt \leq \widetilde{C},$$

which contradicts Equation (22).

Next, consider the case

$$\limsup_{T \to +\infty} T^{\frac{(\beta-1)pq+\alpha p+1}{pq-1}} \int_0^T z_2(s) \, ds = +\infty. \tag{29}$$

Using similar argument as above, one obtains

$$\frac{1}{2^{\kappa}} T^{\frac{(\beta-1)pq+\alpha p+1}{pq-1}} \int_0^{\frac{T}{2}} z_2(t) \, dt \leq \widetilde{C}',$$

for some constant $\widetilde{C}' > 0$, which contradicts (29). □

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References

1. Arias, S.I.R.; Munoz, D.R.; Moreno, J.S.; Cardoso, S.; Ferreira, R.; Freitas, P.J. Fractional modeling of the AC large-signal frequency response in magnetoresistive current sensors. *Sensors* **2013**, *13*, 17516–17533. [CrossRef] [PubMed]
2. Barkai, E.; Metzler, R.; Klafter, J. From continuous time random walks to the fractional Fokker-Planck equation. *Phys. Rev. E* **2000**, *61*, 132–138. [CrossRef] [PubMed]
3. Chaves, A. Fractional diffusion equation to describe Levy flights. *Phys. Lett. A* **1998**, *239*, 13–16. [CrossRef]
4. Drapaca, C.S.; Sivaloganathan, S. A fractional model of continuum mechanics. *J. Elast* **2012**, *107*, 105–123. [CrossRef]
5. Gorenflo, R.; Mainardi, F.; Scalas, E.; Raberto, M. Fractional calculus and continuous-time finance III: The diffusion limit. In *Mathematical Finance (Konstanz, 2000)*; Birkhäuser: Basel, Switzerland, 2001; pp. 171–180.

6. Sapora, A.; Cornetti, P.; Carpinteri, A. Wave propagation in nonlocal elastic continua modelled by a fractional calculus approach. *Commun. Nonlinear Sci. Numer. Simul.* **2013**, *18*, 63–74. [CrossRef]
7. Tarasov, V.E. Review of some promising fractional physical models. *Int. J. Mod. Phys. B* **2013**, *27*, 1330005. [CrossRef]
8. Lakshmikantham, V.; Vatsala, A.S. Basic theory of fractional differential equations. *Nonlinear Anal.* **2008**, *69*, 2677–2682. [CrossRef]
9. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Fractional Integrals and Derivatives: Theory and Applications*; Gordon and Breach Science Publishers: Yverdon-les-Bains, Switzerland, 1993.
10. Shuqin, Z. Monotone iterative method for initial value problem involving Riemann-Liouville fractional derivatives. *Nonlinear Anal.* **2009**, *71*, 2087–2093.
11. Al-Refai, M.; Luchko, Y. Maximum principles for the fractional diffusion equations with the Riemann-Liouville fractional derivative and their applications. *Fract. Calc. Appl. Anal.* **2014**, *17*, 483–498. [CrossRef]
12. Luchko, Y. Maximum principle for the generalized time-fractional diffusion equation. *J. Math. Anal. Appl.* **2009**, *351*, 218–223. [CrossRef]
13. Kirane, M.; Kadem, A.; Debbouche, A. Blowing-up solutions to two-times fractional differential equations. *Math. Nachr.* **2013**, *286*, 1797–1804. [CrossRef]
14. Kirane, M.; Malik, S.A. The profile of blowing-up solutions to a nonlinear system of fractional differential equations. *Nonlinear Anal.* **2010**, *73*, 3723–3736. [CrossRef]
15. Donatelli, M.; Mazza, M.; Serra-Capizzano, S. Spectral analysis and structure preserving preconditioners for fractional diffusion equation. *J. Comput. Phys.* **2016**, *307*, 262–279. [CrossRef]
16. Bu, W.; Liu, X.; Tang, Y.; Yang, J. Finite element multigrid method for multi-term time fractional advection diffusion equations. *Int. J. Mod. Sim. Sci. Comp.* **2015**, *6*, 1540001. [CrossRef]
17. Lin, X.-L.; Ng, M.K.; Sun, H.-W. A multigrid method for linear systems arising from time-dependent two-dimensional space-fractional diffusion equations. *J. Comput. Phys.* **2017**, *336*, 69–86. [CrossRef]
18. Meerschaert, M.M.; Tadjeran, C. Finite difference approximations for two-sided space-fractional partial differential equations. *Appl. Numer. Math.* **2006**, *56*, 80–90. [CrossRef]
19. Parvizi, M.; Eslahchi, M.R.; Dehghan, M. Numerical solution of fractional advection-diffusion equation with a nonlinear source term. *Numer. Algorithms* **2015**, *68*, 601–629. [CrossRef]
20. Furati, K.M.; Kirane, M. Necessary conditions for the existence of global solutions to systems of fractional differential equations. *Fract. Calc. Appl. Anal.* **2008**, *11*, 281–298.
21. Bai, Z.; Chen, Y.; Lian, H.; Sun, S. On the existence of blow up solutions for a class of fractional differential equations. *Fract. Calc. Appl. Anal.* **2014**, *17*, 1175–1187. [CrossRef]
22. Henderson, J.; Luca, R. Nonexistence of positive solutions for a system of coupled fractional boundary value problems. *Bound. Value Probl.* **2015**, *2015*, 138. [CrossRef]
23. Kassim, M.D.; Furati K.M.; Tatar, N.E. Non-existence for fractionally damped fractional differential problems. *Acta Math. Sci.* **2017**, *37*, 119–130. [CrossRef]
24. Mennouni, A.; Youkana, A. Finite time blow-up of solutions for a nonlinear system of fractional differential equations. *Electron. J. Differ. Equ.* **2017**, *152*, 1–15.
25. Samet, B. Nonexistence of global solutions for a class of sequential fractional differential inequalities. *Eur. Phys. J. Spec. Top.* **2018**, *226*, 3513–3524. [CrossRef]
26. Mitidieri, E.; Pohozaev, S.I. Nonexistence of weak solutions for some degenerate and singular hyperbolic problems on $\mathbb{R}^N$. *Proc. Steklov Inst. Math.* **2001**, *232*, 240–259.