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# Decision-Making Approach under Pythagorean Fuzzy Yager Weighted Operators

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Received: 12 November 2019; Accepted: 24 December 2019; Published: 2 January 2020

**Abstract:** In fuzzy set theory, $t$-norms and $t$-conorms are fundamental binary operators. Yager proposed respective parametric families of both $t$-norms and $t$-conorms. In this paper, we apply these operators for the analysis of Pythagorean fuzzy sets. For this purpose, we introduce six families of aggregation operators named Pythagorean fuzzy Yager weighted averaging aggregation, Pythagorean fuzzy Yager ordered weighted averaging aggregation, Pythagorean fuzzy Yager hybrid weighted averaging aggregation, Pythagorean fuzzy Yager weighted geometric aggregation, Pythagorean fuzzy Yager ordered weighted geometric aggregation and Pythagorean fuzzy Yager hybrid weighted geometric aggregation. These tools inherit the operational advantages of the Yager parametric families. They enable us to study two multi-attribute decision-making problems. Ultimately we can choose the best option by comparison of the aggregate outputs through score values. We show this procedure with two practical fully developed examples.

**Keywords:** Yager operators; aggregation operators; arithmetic; geometric; decision-making

## 1. Introduction

Decision-making is an act of choosing a best choice among different alternatives. There are a lot of decisions that are taken by human begins in daily routine. If there is only one option, then there is no need for decision-making but it helps when there are two or more than two choices. Multicriteria decision-making (MCDM) is an operational research that handles with unique outcome by particularly evaluating the feasible alternatives over inconsistent several criteria in decision-making. It is an improbable supposition that the exact numerical information is proper to model the real world decision-making schemes which are composed of inherent uncertainty in human decisions therefore, Zadeh [1] introduced the idea of fuzzy sets (FSs) to handle the imprecise information. Attanasov [2] studied intuitionistic fuzzy sets (IFSs) by adding non-membership function with constraint $0 \leq \mu + \nu \leq 1$ with indeterminacy part $\varpi = 1 - \mu - \nu$. Yager [3] discussed the model of Pythagorean fuzzy sets (PFSs) to handle imprecise information. The main characteristic of this model is that it relaxes the constraint of IFSs with the condition $0 \leq \mu^2 + \nu^2 \leq 1$. Zhang and Xu [4] established the idea of Pythagorean fuzzy number (PFN). In decision-making problems, Garg [5,6] considered the applications of PFSs.

The main problem arises in MCDM problems is that how we can get a unique decision for alternative when there is a list of attributes for a given information. To handle such type of difficulties, the concept of operators was introduced. The different operators are helpful in getting a unique value by the list of values. The idea of IF weighted averaging, ordered weighted averaging and hybrid averaging operators was studied by Xu [7]. Xu and Yager [8] studied some geometric weighted, geometric ordered weighted and geometric hybrid operators under IFSs environment. The concept of generalized ordered weighted averaging operators under IFSs was discussed by Li [9]. Xu and Xia [10] introduced the concept of induced generalized aggregation operators under IF environments. The dynamic IF multiple-attribute decision-making (MADM) problems were investigated by Wei [11]. The aggregation of infinite chains of IF sets has been recently achieved by Alcantud et al. [12], who used their operators to make decisions in a temporal intuitionistic setting. Wei [13] proposed the induced geometric aggregation operators under IF information and their application in group decision-making. IF Hamacher aggregation operators were examined by Huang [14]. Zhao and Wei [15] studied IF Einstein hybrid aggregation operators and their application in MADM. Liu et al. [16] take advantage of aggregation operators to study centroid transformations of intuitionistic fuzzy values. To handle multiple-attribute group decision-making problem under IF environment, Liu et al. [17] used Dombi Bonferroni mean operator. Yager [18] studied some aggregation operators such as weighted averaging, weighted geometric, ordered weighted averaging and ordered weighted geometric operators under PF environment. The fundamental properties of PF aggregation operators were discussed by Peng and Yaun [19]. Zeng et al. [20] developed a hybrid method for Pythagorean fuzzy MADM. The MADM problems under PF interaction aggregation operators were handled by Wei [21]. Peng and Yang [22] studied the fundamental properties of interval-valued PF aggregation operators. Wei and Lu [23] developed the concept of PF power aggregation operators in MADM. The multiple-attribute group decision-making applications under PF Einstein weighted geometric aggregation operators were discussed by Rahman et al. [24]. Akram et al. [25] proposed the Pythagorean Dombi fuzzy aggregation operators with applications. For further notations and applications, the readers are referred to [26–33]. In this article, we discuss Yager aggregation operators under PF environment. The motivation of this article is described as follows:

1. The main purpose of this article is to establish some aggregation operators under PF data called PF Yager aggregation for assessing the distinct preferences of the choice among the decision-making process.
2. PFs are more flexible to handle uncertainty where IFSs fail.
3. Yager aggregation operators make the decision results more precise and exact when applied to real-life MADM based on PF environment.

The structure of paper is as follows: In Section 2, we will recall some basic definitions. In Section 3, we will study the Pythagorean fuzzy Yager weighted averaging aggregation (PFYWAA), Pythagorean fuzzy Yager ordered weighted averaging aggregation (PFYOWAA), Pythagorean fuzzy Yager hybrid weighted averaging aggregation operators (PFYHWAA) and some results of these operators. In Section 4, we will discuss Pythagorean fuzzy Yager weighted geometric aggregation (PFYWGA), Pythagorean fuzzy Yager ordered weighted geometric aggregation (PFYOWGA) and Pythagorean fuzzy Yager hybrid weighted geometric aggregation (PFYHWGA) operators. In Section 5, we will describe two MADM problems under these operators and choose best option by comparison through score values. In Section 6, we will discuss the comparative analysis of our model with another existing model. In Section 7, we will conclude our results related to our proposed model.

## 2. Preliminaries

In this section, we recall some basic definitions.

**Definition 1** ([3]). *A PFS $p$ on non-empty set $\mathcal{V}$ is defined as*

$$p = \{\langle h, \mu_p(h), \nu_p(h)\rangle\},$$

*where $\mu_p : \mathcal{V} \to [0,1]$ and $\nu_p : \mathcal{V} \to [0,1]$ indicate the membership and non-membership degrees of an element $h \in \mathcal{V}$, respectively. $\varpi_p(h) = \sqrt{1 - (\mu_p(h))^2 - (\nu_p(h))^2}$ is indeterminacy degree of an element $h \in \mathcal{V}$. Zhang and Xu [4] considered $\langle \mu_p(h)), \nu_p(h)\rangle$ as PFN represented by $p = \langle \mu_p, \nu_p\rangle$.*

**Definition 2** ([4])**.** *Consider two PFNs $p_1 = \langle \mu_{p_1}, \nu_{p_1} \rangle$ and $p_2 = \langle \mu_{p_2}, \nu_{p_2} \rangle$. The operational laws on PFNs are*

1. $p_1 \oplus p_2 = \left\langle \sqrt{\mu_{p_1}^2 + \mu_{p_2}^2 - \mu_{p_1}^2 \mu_{p_2}^2}, \nu_{p_1}\nu_{p_2} \right\rangle$,
2. $p_1 \otimes p_2 = \left\langle \mu_{p_1}\mu_{p_2}, \sqrt{\nu_{p_1}^2 + \nu_{p_2}^2 - \nu_{p_1}^2 \nu_{p_2}^2} \right\rangle$,
3. $\lambda p_1 = \left\langle \sqrt{1 - (1 - \mu_{p_1}^2)^{\lambda}}, \nu_{p_1}^{\lambda} \right\rangle$,
4. $p_1^{\lambda} = \left\langle \mu_{p_1}^{\lambda}, \sqrt{1 - (1 - \nu_{p_1}^2)^{\lambda}} \right\rangle$, *where $\lambda$ is a scalar and $\lambda > 0$.*

**Definition 3** ([4])**.** *Consider a PFN $p = \langle \mu_p, \nu_p \rangle$. The score $\mathcal{S}(p)$ and accuracy functions $\mathcal{A}(p)$ of $p$ are*

$$\mathcal{S}(p) = \mu_p^2 - \nu_p^2, \text{where } \mathcal{S}(p) \in [-1, 1],$$

$$\mathcal{A}(p) = \mu_p^2 + \nu_p^2, \text{where } \mathcal{A}(p) \in [0, 1].$$

**Definition 4** ([4])**.** *Consider two PFNs $p_1 = \langle \mu_{p_1}, \nu_{p_1} \rangle$ and $p_2 = \langle \mu_{p_2}, \nu_{p_2} \rangle$. Then*

1. *If $\mathcal{S}(p_1) < \mathcal{S}(p_2)$, then $p_1 < p_2$;*
2. *If $\mathcal{S}(p_1) > \mathcal{S}(p_2)$, then $p_1 > p_2$;*
3. *If $\mathcal{S}(p_1) = \mathcal{S}(p_2)$, then accuracy functions are compared as:*
   (a) *If $\mathcal{A}(p_1) < \mathcal{A}(p_2)$, then $p_1 < p_2$;*
   (b) *If $\mathcal{A}(p_1) > \mathcal{A}(p_2)$, then $p_1 > p_2$;*
   (c) *If $\mathcal{A}(p_1) = \mathcal{A}(p_2)$, then $p_1 \sim p_2$.*

**Definition 5** ([34])**.** *For any two real numbers $h$ and $k$, Yager's t-norms and Yager's t-conorms are given as:*

$$\mathcal{T}(h, k) = 1 - \min(1, \left((1-h)^{\vartheta} + (1-k)^{\vartheta}\right)^{\frac{1}{\vartheta}}), \tag{1}$$

$$\mathcal{T}'(h, k) = \min(1, (h^{\vartheta} + k^{\vartheta})^{\frac{1}{\vartheta}}),\ \vartheta \in (0, \infty). \tag{2}$$

## 3. Pythagorean Fuzzy Yager Hybrid Weighted Arithmetic Aggregation Operators

In this section, we define Yager weighted arithmetic aggregation operators under PF environment.

**Definition 6.** *Let $p_1 = \langle \mu_1, \nu_1 \rangle$ and $p_2 = \langle \mu_2, \nu_2 \rangle$ be two PFNs, $\vartheta > 0$ and $\lambda > 0$. Then, Yager t-norm and t-conorm operations of PFNs are defined by*

1. $p_1 \oplus p_2 = \left\langle \sqrt{\min(1, (\mu_1^{2\vartheta} + \mu_2^{2\vartheta})^{\frac{1}{\vartheta}})}, \sqrt{1 - \min(1, \left((1-\nu_1^2)^{\vartheta} + (1-\nu_2^2)^{\vartheta}\right)^{\frac{1}{\vartheta}})} \right\rangle$,
2. $p_1 \otimes p_2 = \left\langle \sqrt{1 - \min(1, \left((1-\mu_1^2)^{\vartheta} + (1-\mu_2^2)^{\vartheta}\right)^{\frac{1}{\vartheta}})}, \sqrt{\min(1, (\nu_1^{2\vartheta} + \nu_2^{2\vartheta})^{\frac{1}{\vartheta}})} \right\rangle$,
3. $\lambda p_1 = \left\langle \sqrt{\min(1, (\lambda\mu_1^{2\vartheta})^{\frac{1}{\vartheta}})}, \sqrt{1 - \min(1, \left(\lambda(1-\nu_1^2)^{\vartheta}\right)^{\frac{1}{\vartheta}})} \right\rangle$,
4. $p_1^{\lambda} = \left\langle \sqrt{1 - \min(1, \left(\lambda(1-\mu_1^2)^{\vartheta}\right)^{\frac{1}{\vartheta}})}, \sqrt{\min(1, (\lambda\nu_1^{2\vartheta})^{\frac{1}{\vartheta}})} \right\rangle$.

**Example 1.** *Let $p_1 = \langle 0.8, 0.5 \rangle$, $p_2 = \langle 0.7, 0.6 \rangle$ be two PFNs, then by Yager operations on PFNs using Definition 6 for $\vartheta = 3,\ \lambda = 4$ are:*

$$
\begin{aligned}
1.\ p_1 \oplus p_2 &= \left\langle \sqrt{\min(1, (0.8^6 + 0.7^6)^{\frac{1}{3}})}, \sqrt{1 - \min(1, \left((1-0.5^2)^3 + (1-0.6^2)^3\right)^{\frac{1}{3}})} \right\rangle \\
&= \langle 0.85, 0.35 \rangle. \\
2.\ p_1 \otimes p_2 &= \left\langle \sqrt{1 - \min(1, \left((1-0.8^2)^3 + (1-0.7^2)^3\right)^{\frac{1}{3}})}, \sqrt{\min(1, (0.5^6 + 0.6^6)^{\frac{1}{3}})} \right\rangle \\
&= \langle 0.66, 0.63 \rangle. \\
3.\ 4p_1 &= \left\langle \sqrt{\min(1, (4(0.8)^6)^{\frac{1}{3}})}, \sqrt{1 - \min(1, \left(4(1-0.5^2)^3\right)^{\frac{1}{3}})} \right\rangle \\
&= \langle 1, 0 \rangle. \\
4.\ p_1^4 &= \left\langle \sqrt{1 - \min(1, \left(4(1-0.8^2)^3\right)^{\frac{1}{3}})}, \sqrt{\min(1, (4(0.5)^6)^{\frac{1}{3}})} \right\rangle \\
&= \langle 0.66, 0.63 \rangle.
\end{aligned}
$$

**Theorem 1.** *Let $p = \langle \mu, \nu \rangle$, $p_1 = \langle \mu_1, \nu_1 \rangle$, $p_2 = \langle \mu_2, \nu_2 \rangle$ be three PFNs, then*

1. $p_1 \oplus p_2 = p_2 \oplus p_1$,
2. $p_1 \otimes p_2 = p_1 \otimes p_2$,
3. $\lambda(p_1 \oplus p_2) = \lambda p_1 \oplus \lambda p_2$,
4. $(\lambda_1 + \lambda_2)p = \lambda_1 p \oplus \lambda_2 p$,
5. $(p_1 \otimes p_2)^\lambda = p_1^\lambda \otimes p_2^\lambda, \lambda > 0$,
6. $p^{\lambda_1} \otimes p^{\lambda_2} = p^{(\lambda_1 + \lambda_2)},\ \lambda_1, \lambda_2 > 0$.

**Proof.** For three PFNs $p, p_1, p_2$ and $\lambda, \lambda_1, \lambda_2 > 0$, by Definition 6, we can get

$$
\begin{aligned}
1.\ p_1 \oplus p_2 &= \left\langle \sqrt{\min(1, (\mu_1^{2\vartheta} + \mu_2^{2\vartheta})^{\frac{1}{\vartheta}})}, \sqrt{1 - \min(1, \left((1-\nu_1^2)^\vartheta + (1-\nu_2^2)^\vartheta\right)^{\frac{1}{\vartheta}})} \right\rangle \\
&= \left\langle \sqrt{\min(1, (\mu_2^{2\vartheta} + \mu_1^{2\vartheta})^{\frac{1}{\vartheta}})}, \sqrt{1 - \min(1, \left((1-\nu_2^2)^\vartheta + (1-\nu_1^2)^\vartheta\right)^{\frac{1}{\vartheta}})} \right\rangle \\
&= p_2 \oplus p_1. \\
2.\ p_1 \otimes p_2 &= \left\langle \sqrt{1 - \min(1, \left((1-\mu_1^2)^\vartheta + (1-\mu_2^2)^\vartheta\right)^{\frac{1}{\vartheta}})}, \sqrt{\min(1, (\nu_1^{2\vartheta} + \nu_2^{2\vartheta})^{\frac{1}{\vartheta}})} \right\rangle \\
&= \left\langle \sqrt{1 - \min(1, \left((1-\mu_2^2)^\vartheta + (1-\mu_1^2)^\vartheta\right)^{\frac{1}{\vartheta}})}, \sqrt{\min(1, (\nu_2^{2\vartheta} + \nu_1^{2\vartheta})^{\frac{1}{\vartheta}})} \right\rangle \\
&= p_2 \otimes p_1. \\
3.\ \lambda(p_1 \oplus p_2) &= \lambda \left\langle \sqrt{\min(1, (\mu_1^{2\vartheta} + \mu_2^{2\vartheta})^{\frac{1}{\vartheta}})}, \sqrt{1 - \min(1, \left((1-\nu_1^2)^\vartheta + (1-\nu_2^2)^\vartheta\right)^{\frac{1}{\vartheta}})} \right\rangle \\
&= \left\langle \sqrt{\min(1, (\lambda\mu_1^{2\vartheta} + \lambda\mu_2^{2\vartheta})^{\frac{1}{\vartheta}})}, \sqrt{1 - \min(1, \left(\lambda(1-\nu_1^2)^\vartheta + \lambda(1-\nu_2^2)^\vartheta\right)^{\frac{1}{\vartheta}})} \right\rangle, \\
\lambda p_1 \oplus \lambda p_2 &= \left\langle \sqrt{\min(1, (\lambda\mu_1^{2\vartheta})^{\frac{1}{\vartheta}})}, \sqrt{1 - \min(1, \left(\lambda(1-\nu_1^2)^\vartheta\right)^{\frac{1}{\vartheta}})} \right\rangle \oplus \\
&\quad \left\langle \sqrt{\min(1, (\lambda\mu_2^{2\vartheta})^{\frac{1}{\vartheta}})}, \sqrt{1 - \min(1, \left(\lambda(1-\nu_2^2)^\vartheta\right)^{\frac{1}{\vartheta}})} \right\rangle \\
&= \left\langle \sqrt{\min(1, (\lambda\mu_1^{2\vartheta} + \lambda\mu_2^{2\vartheta})^{\frac{1}{\vartheta}})}, \sqrt{1 - \min(1, \left(\lambda(1-\nu_1^2)^\vartheta + \lambda(1-\nu_2^2)^\vartheta\right)^{\frac{1}{\vartheta}})} \right\rangle \\
&= \lambda(p_1 \oplus p_2). \\
4.\ \lambda_1 p \oplus \lambda_2 p &= \left\langle \sqrt{\min(1, (\lambda_1\mu^{2\vartheta})^{\frac{1}{\vartheta}})}, \sqrt{1 - \min(1, \left(\lambda_1(1-\nu^2)^\vartheta\right)^{\frac{1}{\vartheta}})} \right\rangle \oplus \\
&\quad \left\langle \sqrt{\min(1, (\lambda_2\mu^{2\vartheta})^{\frac{1}{\vartheta}})}, \sqrt{1 - \min(1, \left(\lambda_2(1-\nu^2)^\vartheta\right)^{\frac{1}{\vartheta}})} \right\rangle \\
&= \left\langle \sqrt{\min(1, ((\lambda_1 + \lambda_2)\mu^{2\vartheta})^{\frac{1}{\vartheta}})}, \sqrt{1 - \min(1, \left((\lambda_1 + \lambda_2)(1-\nu^2)^\vartheta\right)^{\frac{1}{\vartheta}})} \right\rangle \\
&= (\lambda_1 + \lambda_2)p.
\end{aligned}
$$

Similarly, other properties can be verified. Here, we omit their proofs. □

**Definition 7.** *Let* $p_i = \langle \mu_i, \nu_i \rangle (i = 1, 2, \cdots, \mathfrak{s})$ *be a collection of PFNs. The PFYWAA operator is a function* $\mathcal{P}^{\mathfrak{s}} \to \mathcal{P}$ *such that*

$$PFYWAA_{\alpha}(p_1, p_2, \cdots, p_{\mathfrak{s}}) = \bigoplus_{i=1}^{\mathfrak{s}} (\alpha_i p_i),$$

*where* $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_{\mathfrak{s}})^T$ *is the weight vector of* $p_i$ *with* $\alpha_i > 0$ *and* $\sum_{i=1}^{\mathfrak{s}} \alpha_i = 1$.

**Theorem 2.** *Let* $p_i = (\mu_i, \nu_i)(i = 1, 2, \cdots, \mathfrak{s})$ *be a collection of PFNs, then aggregated value of them by the PFYWAA operation is a PFN and*

$$\begin{aligned} PFYWAA_{\alpha}(p_1, p_2, \cdots, p_{\mathfrak{s}}) &= \bigoplus_{i=1}^{\mathfrak{s}} (\alpha_i p_i) \\ &= \left\langle \sqrt{\min(1, (\sum_{i=1}^{\mathfrak{s}} (\alpha_i \mu_i^{2\vartheta}))^{\frac{1}{\vartheta}})}, \sqrt{1 - \min(1, (\sum_{i=1}^{\mathfrak{s}} (\alpha_i (1 - \nu_i^2)^{\vartheta}))^{\frac{1}{\vartheta}})} \right\rangle, \end{aligned} \tag{3}$$

*where* $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_{\mathfrak{s}})^T$ *is the weight vector of* $p_i$ *with* $\alpha_i > 0$ *and* $\sum_{i=1}^{\mathfrak{s}} \alpha_i = 1$.

**Proof.** The mathematical induction is used to prove the Theorem.

(i) when $\mathfrak{s} = 2$,

As

$$\begin{aligned} \alpha_1 p_1 &= \left\langle \sqrt{\min(1, (\alpha_1 \mu_1^{2\vartheta})^{\frac{1}{\vartheta}})}, \sqrt{1 - \min(1, (\alpha_1 (1 - \nu_1^2)^{\vartheta})^{\frac{1}{\vartheta}})} \right\rangle, \\ \alpha_2 p_2 &= \left\langle \sqrt{\min(1, (\alpha_2 \mu_2^{2\vartheta})^{\frac{1}{\vartheta}})}, \sqrt{1 - \min(1, (\alpha_2 (1 - \nu_2^2)^{\vartheta})^{\frac{1}{\vartheta}})} \right\rangle. \end{aligned}$$

Therefore,

$$\begin{aligned} \alpha_1 p_1 \oplus \alpha_2 p_2 &= \left\langle \sqrt{\min(1, (\alpha_1 \mu_1^{2\vartheta})^{\frac{1}{\vartheta}})}, \sqrt{1 - \min(1, (\alpha_1 (1 - \nu_1^2)^{\vartheta})^{\frac{1}{\vartheta}})} \right\rangle \oplus \\ &\quad \left\langle \sqrt{\min(1, (\alpha_2 \mu_2^{2\vartheta})^{\frac{1}{\vartheta}})}, \sqrt{1 - \min(1, (\alpha_2 (1 - \nu_2^2)^{\vartheta})^{\frac{1}{\vartheta}})} \right\rangle \\ &= \left\langle \sqrt{\min(1, (\alpha_1 \mu_1^{2\vartheta} + \alpha_2 \mu_2^{2\vartheta})^{\frac{1}{\vartheta}})}, \sqrt{1 - \min(1, (\alpha_1 (1 - \nu_1^2)^{\vartheta} + \alpha_2 (1 - \nu_2^2))^{\frac{1}{\vartheta}})} \right\rangle \\ &= \left\langle \sqrt{\min(1, (\sum_{i=1}^{2} (\alpha_i \mu_i^{2\vartheta}))^{\frac{1}{\vartheta}})}, \sqrt{1 - \min(1, (\sum_{i=1}^{2} (\alpha_i (1 - \nu_i^2)^{\vartheta}))^{\frac{1}{\vartheta}})} \right\rangle. \end{aligned}$$

Hence, Equation (3) is true for $\mathfrak{s} = 2$.

(ii) Let Equation (3) holds for $\mathfrak{s} = k$,

$$\begin{aligned} PFYWAA_{\alpha}(p_1, p_2, \cdots, p_k) &= \bigoplus_{i=1}^{k} (\alpha_i p_i) \\ &= \left\langle \sqrt{\min(1, (\sum_{i=1}^{k} (\alpha_i \mu_i^{2\vartheta}))^{\frac{1}{\vartheta}})}, \sqrt{1 - \min(1, (\sum_{i=1}^{k} (\alpha_i (1 - \nu_i^2)^{\vartheta}))^{\frac{1}{\vartheta}})} \right\rangle. \end{aligned}$$

Now for $\mathfrak{s} = k + 1$.

$$\begin{aligned}
PFYWAA_{\alpha}(p_1, p_2, \cdots, p_{k+1}) &= \left\langle \sqrt{\min(1, (\sum_{\mathrm{i}=1}^{k} (\alpha_{\mathrm{i}}\mu_{\mathrm{i}}^{2\vartheta}))^{\frac{1}{\vartheta}})}, \sqrt{1-\min(1, \sum_{\mathrm{i}=1}^{k} (\alpha_{\mathrm{i}}(1-\nu_{\mathrm{i}}^{2})^{\vartheta})^{\frac{1}{\vartheta}})} \right\rangle \oplus \\
&\quad \left\langle \sqrt{\min(1, (\alpha_{k+1}\mu_{k+1}^{2\vartheta})^{\frac{1}{\vartheta}})}, \sqrt{1-\min(1, (\alpha_{k+1}(1-\nu_{k+1}^{2})^{\vartheta})^{\frac{1}{\vartheta}})} \right\rangle \\
&= \left\langle \sqrt{\min(1, (\sum_{\mathrm{i}=1}^{k+1} (\alpha_{\mathrm{i}}\mu_{\mathrm{i}}^{2\vartheta}))^{\frac{1}{\vartheta}})}, \sqrt{1-\min(1, (\sum_{\mathrm{i}=1}^{k+1} (\alpha_{\mathrm{i}}(1-\nu_{\mathrm{i}}^{2})^{\vartheta}))^{\frac{1}{\vartheta}})} \right\rangle.
\end{aligned}$$

Hence, Equation (3) is true for $\mathfrak{s} = k+1$. Thus, Equation (3) holds for all $\mathfrak{s}$. □

**Example 2.** *Four persons of a home $p_1, p_2, p_3, p_4$ want to evaluate the cooking performance of a chef $\mathfrak{C}$ in making Chinese food. The approximated values of four persons for the chef $\mathfrak{C}$ are given in Pythagorean fuzzy information such as $p_1 = \langle 0.6, 0.5\rangle$, $p_2 = \langle 0.5, 0.5\rangle$, $p_3 = \langle 0.7, 0.6\rangle$ and $p_4 = \langle 1, 0\rangle$ with weight vector $\alpha = (0.2, 0.3, 0.2, 0.3)^T$, where weight vector represents the importance of Chinese food for four persons who are evaluating the performance of a chef and $\vartheta = 2$. By applying Theorem 2, we can aggregate the four PFNs and write a clumped value for the cooking performance of a chef as shown below*

$$\begin{aligned}
PFYWAA_{\alpha}(p_1, p_2, p_3, p_4) &= \bigoplus_{\mathrm{i}=1}^{4} (\alpha_{\mathrm{i}} p_{\mathrm{i}}) \\
&= \left\langle \sqrt{\min\left(1, (\sum_{\mathrm{i}=1}^{4} (\alpha_{\mathrm{i}}\mu_{\mathrm{i}}^{2\vartheta}))^{\frac{1}{\vartheta}}\right)}, \sqrt{1-\min\left(1, (\sum_{\mathrm{i}=1}^{4} (\alpha_{\mathrm{i}}(1-\nu_{\mathrm{i}}^{2})^{\vartheta}))^{\frac{1}{\vartheta}}\right)} \right\rangle \\
&= \Big\langle \sqrt{\min(1, (0.2(0.6)^4 + 0.3(0.5)^4 + 0.2(0.7)^4 + 0.3(1)^4)^{\frac{1}{2}})}, \\
&\quad \sqrt{1-\min(1, (0.2(1-0.5^2)^2 + (0.3(1-0.5^2)^2 + 0.2(1-0.5^2)^2 + 0.3(1-0^2)^2)^{\frac{1}{2}}} \Big\rangle \\
&= \langle 0.80, 0.44\rangle.
\end{aligned}$$

**Theorem 3.** *(Idempotency). If all PFNs are identical, i.e., $p_{\mathrm{i}} = p$ then*

$$PFYWAA(p_1, p_2, \cdots, p_{\mathfrak{s}}) = p.$$

**Proof.** As $p_{\mathrm{i}} = \langle \mu_{\mathrm{i}}, \nu_{\mathrm{i}}\rangle = p(\mathrm{i} = 1, 2, \cdots, \mathfrak{s})$. Then by Equation (3),

$$\begin{aligned}
PFYWAA_{\alpha}(p_1, p_2, \cdots, p_{\mathfrak{s}}) &= \bigoplus_{\mathrm{i}=1}^{\mathfrak{s}} (\alpha_{\mathrm{i}} p_{\mathrm{i}}) \\
&= \left\langle \sqrt{\min\left(1, (\sum_{\mathrm{i}=1}^{\mathfrak{s}} (\alpha_{\mathrm{i}}\mu_{\mathrm{i}}^{2\vartheta}))^{\frac{1}{\vartheta}}\right)}, \sqrt{1-\min\left(1, (\sum_{\mathrm{i}=1}^{\mathfrak{s}} (\alpha_{\mathrm{i}}(1-\nu_{\mathrm{i}}^{2})^{\vartheta}))^{\frac{1}{\vartheta}}\right)} \right\rangle \\
&= \left\langle \sqrt{\min(1, (\mu^{2\vartheta})^{\frac{1}{\vartheta}})}, \sqrt{1-\min(1, \left((1-\nu^{2})^{\vartheta}\right)^{\frac{1}{\vartheta}})} \right\rangle \\
&= \left\langle \sqrt{\min(1, \mu^{2})}, \sqrt{1-\min(1, (1-\nu^{2}))}, \right\rangle \\
&= \left\langle \mu, \nu \right\rangle \\
&= p.
\end{aligned}$$

□

**Theorem 4.** *(Boundedness). Let $p_{\mathrm{i}} = (\mu_{\mathrm{i}}, \nu_{\mathrm{i}})$ be a collection of PFNs. Let $p^{-} = \min(p_1, p_2, \cdots, p_{\mathfrak{s}})$ and $p^{+} = \max(p_1, p_2, \cdots, p_{\mathfrak{s}})$. Then*

$$p^{-} \leq PFYWAA(p_1, p_2, \cdots, p_{\mathfrak{s}}) \leq p^{+}.$$

**Proof.** Suppose that $p^- = \min(p_1, p_2, \cdots, p_{\mathfrak{s}}) = (\mu^-, \nu^-)$ and $p^+ = \max(p_1, p_2, \cdots, p_{\mathfrak{s}}) = (\mu^+, \nu^+)$, where $\mu^- = \min(\mu_{\mathfrak{i}})$, $\nu^- = \max(\nu_{\mathfrak{i}})$, $\mu^+ = \max(\mu_{\mathfrak{i}})$, $\nu^+ = \min(\nu_{\mathfrak{i}})$.

Thus, inequalities for membership value are

$$\begin{aligned}\sqrt{\min\big(1, \big(\sum_{\mathfrak{i}=1}^{\mathfrak{s}} \big(\alpha_{\mathfrak{i}}\mu^{-2\vartheta}\big)\big)^{\frac{1}{\vartheta}}\big)} &\leq \sqrt{\min\big(1, \big(\sum_{\mathfrak{i}=1}^{\mathfrak{s}} \big(\alpha_{\mathfrak{i}}\mu_{\mathfrak{i}}^{2\vartheta}\big)\big)^{\frac{1}{\vartheta}}\big)} \\ &\leq \sqrt{\min\big(1, \big(\sum_{\mathfrak{i}=1}^{\mathfrak{s}} \big(\alpha_{\mathfrak{i}}\mu^{+2\vartheta}\big)\big)^{\frac{1}{\vartheta}}\big)}.\end{aligned}$$

Similarly, for non-membership value

$$\begin{aligned}\sqrt{1-\min\big(1, \big(\sum_{\mathfrak{i}=1}^{\mathfrak{s}} \big(\alpha_{\mathfrak{i}}(1-\nu^{+2})^{\vartheta}\big)\big)^{\frac{1}{\vartheta}}\big)} &\leq \sqrt{1-\min\big(1, \big(\sum_{\mathfrak{i}=1}^{\mathfrak{s}} \big(\alpha_{\mathfrak{i}}(1-\nu_{\mathfrak{i}}^{2})^{\vartheta}\big)\big)^{\frac{1}{\vartheta}}\big)} \\ &\leq \sqrt{1-\min\big(1, \big(\sum_{\mathfrak{i}=1}^{\mathfrak{s}} \big(\alpha_{\mathfrak{i}}(1-\nu^{-2})^{\vartheta}\big)\big)^{\frac{1}{\vartheta}}\big)}.\end{aligned}$$

Therefore, $p^- \leq PFYWAA(p_1, p_2, \cdots, p_{\mathfrak{s}}) \leq p^+$. $\square$

**Theorem 5** (Monotonicity). *Let $p'_{\mathfrak{i}} = \{p'_1, p'_2, \cdots, p'_{\mathfrak{s}}\}$ and $p_{\mathfrak{i}} = \{p_1, p_2, \cdots, p_{\mathfrak{s}}\}$ be two collections of PFNs. If $\mu'_{\mathfrak{i}} \leq \mu_{\mathfrak{i}}$ and $\nu'_{\mathfrak{i}} \geq \nu_{\mathfrak{i}}$, $\forall$ i. Then*

$$PFYWAA(p'_1, p'_2, \cdots, p'_{\mathfrak{s}}) \leq PFYWAA(p_1, p_2, \cdots, p_{\mathfrak{s}}).$$

**Proof.** Let $PFYWAA(p'_1, p'_2, \cdots, p'_{\mathfrak{s}}) = (\mathcal{G}', \mathcal{K}')$ and $PFYWAA(p_1, p_2, \cdots, p_{\mathfrak{s}}) = (\mathcal{G}, \mathcal{K})$. First, we will show that $\mathcal{G}' \leq \mathcal{G}$. As $\mu'_{\mathfrak{i}} \leq \mu_{\mathfrak{i}}$, $\mu'^2_{\mathfrak{i}} \leq \mu^2_{\mathfrak{i}}$. Moreover,

$$\begin{aligned}\big(\sum_{\mathfrak{i}=1}^{\mathfrak{s}}(\alpha_{\mathfrak{i}}\mu_{\mathfrak{i}}'^{2\vartheta})\big)^{\frac{1}{\vartheta}} &\leq \big(\sum_{\mathfrak{i}=1}^{\mathfrak{s}}(\alpha_{\mathfrak{i}}\mu_{\mathfrak{i}}^{2\vartheta})\big)^{\frac{1}{\vartheta}} \\ \min\big(1, \big(\sum_{\mathfrak{i}=1}^{\mathfrak{s}}(\alpha_{\mathfrak{i}}\mu_{\mathfrak{i}}'^{2\vartheta})\big)^{\frac{1}{\vartheta}}\big) &\leq \min\big(1, \big(\sum_{\mathfrak{i}=1}^{\mathfrak{s}}(\alpha_{\mathfrak{i}}\mu_{\mathfrak{i}}^{2\vartheta})\big)^{\frac{1}{\vartheta}}\big) \\ \sqrt{\min\big(1, \big(\sum_{\mathfrak{i}=1}^{\mathfrak{s}}(\alpha_{\mathfrak{i}}\mu_{\mathfrak{i}}'^{2\vartheta})\big)^{\frac{1}{\vartheta}}\big)} &\leq \sqrt{\min\big(1, \big(\sum_{\mathfrak{i}=1}^{\mathfrak{s}}(\alpha_{\mathfrak{i}}\mu_{\mathfrak{i}}^{2\vartheta})\big)^{\frac{1}{\vartheta}}\big)}.\end{aligned}$$

Hence, $\mathcal{G}' \leq \mathcal{G}$. Similarly, we can prove that $\mathcal{K}' \geq \mathcal{K}$. Thus, we conclude that $(\mathcal{G}', \mathcal{K}') \leq (\mathcal{G}, \mathcal{K})$, i.e., $PFYWAA(p'_1, p'_2, \cdots, p'_{\mathfrak{s}}) \leq PFYWAA(p_1, p_2, \cdots, p_{\mathfrak{s}})$. $\square$

**Theorem 6** (Reducibility). *Let $p_{\mathfrak{i}} = (\mu_{\mathfrak{i}}, \nu_{\mathfrak{i}})(\mathfrak{i} = 1, 2, \cdots, \mathfrak{s})$ be a collection of PFNs with the corresponding weight vector $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_{\mathfrak{s}})^T = (\frac{1}{\mathfrak{s}}, \frac{1}{\mathfrak{s}}, \cdots, \frac{1}{\mathfrak{s}})^T$. Then, PFYWAA operator is*

$$PFYWAA_{\alpha}(p_1, p_2, \cdots, p_{\mathfrak{s}}) = \Big\langle \sqrt{\min\big(1, \frac{1}{\mathfrak{s}}\big(\sum_{\mathfrak{i}=1}^{\mathfrak{s}}(\mu_{\mathfrak{i}}^{2\vartheta})\big)^{\frac{1}{\vartheta}}\big)}, \sqrt{1-\min\big(1, \frac{1}{\mathfrak{s}}\big(\sum_{\mathfrak{i}=1}^{\mathfrak{s}} \big((1-\nu_{\mathfrak{i}}^{2})^{\vartheta}\big)\big)^{\frac{1}{\vartheta}}\big)} \Big\rangle.$$

**Theorem 7** (Commutativity). *Consider the collection $p_{\mathfrak{i}} = (\mu_{\mathfrak{i}}, \nu_{\mathfrak{i}})(\mathfrak{i} = 1, 2, \cdots, \mathfrak{s})$ of PFNs. If $p'_{\mathfrak{i}}$ is the permutation of $p_{\mathfrak{i}}$, then*

$$PFYWAA_{\alpha}(p_1, p_2, \cdots, p_{\mathfrak{s}}) = PFYWAA_{\alpha}(p'_1, p'_2, \cdots, p'_{\mathfrak{s}}).$$

We now define the Pythagorean fuzzy Yager ordered weighted arithmetic aggregation operators.

**Definition 8.** *Let* $p_{\mathfrak{i}} = \langle \mu_{\mathfrak{i}}, \nu_{\mathfrak{i}} \rangle (\mathfrak{i} = 1, 2, \cdots, \mathfrak{s})$ *be a collection of PFNs with the corresponding weight vector* $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_{\mathfrak{s}})^T$ *with* $\alpha_{\mathfrak{i}} > 0$ *and* $\sum\limits_{\mathfrak{i}=1}^{\mathfrak{s}} \alpha_{\mathfrak{i}} = 1$. *The PFYOWAA operator is a function* $\mathcal{P}^{\mathfrak{s}} \to \mathcal{P}$ *such that*

$$PFYOWAA_{\alpha}(p_1, p_2, \cdots, p_{\mathfrak{s}}) = \bigoplus_{\mathfrak{i}=1}^{\mathfrak{s}} (\alpha_{\mathfrak{i}} p_{\varrho(\mathfrak{i})}),$$

*where* $(\varrho(1), \varrho(2), \cdots, \varrho(\mathfrak{s}))$ *is the permutation of* $(\mathfrak{i} = 1, 2, \cdots, \mathfrak{s})$ *such that* $p_{\varrho(\mathfrak{i}-1)} \geq p_{\varrho(\mathfrak{i})}, \forall\, \mathfrak{i} = 1, 2, \cdots, \mathfrak{s}$.

**Theorem 8.** *Let* $p_{\mathfrak{i}} = (\mu_{\mathfrak{i}}, \nu_{\mathfrak{i}})(\mathfrak{i} = 1, 2, \cdots, \mathfrak{s})$ *be a collection of PFNs with the corresponding weight vector* $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_{\mathfrak{s}})^T$ *with* $\alpha_{\mathfrak{i}} > 0$ *and* $\sum\limits_{\mathfrak{i}=1}^{\mathfrak{s}} \alpha_{\mathfrak{i}} = 1$, *then clumped value of them by PFYOWAA operation is a PFN and*

$$\begin{aligned} PFYOWAA_{\alpha}(p_1, p_2, \cdots, p_{\mathfrak{s}}) &= \bigoplus_{\mathfrak{i}=1}^{\mathfrak{s}} (\alpha_{\mathfrak{i}} p_{\varrho(\mathfrak{i})}) \\ &= \left\langle \sqrt{\min\left(1, \left(\sum_{\mathfrak{i}=1}^{\mathfrak{s}} (\alpha_{\mathfrak{i}} \mu^{2\vartheta}_{\varrho(\mathfrak{i})})\right)^{\frac{1}{\vartheta}}\right)}, \sqrt{1 - \min\left(1, \left(\sum_{\mathfrak{i}=1}^{\mathfrak{s}} (\alpha_{\mathfrak{i}} (1 - \nu^2_{\varrho(\mathfrak{i})})^{\vartheta})\right)^{\frac{1}{\vartheta}}\right)} \right\rangle, \end{aligned} \tag{4}$$

*where* $(\varrho(1), \varrho(2), \cdots, \varrho(\mathfrak{s}))$ *is the permutation of* $(\mathfrak{i} = 1, 2, \cdots, \mathfrak{s})$ *such that* $p_{\varrho(\mathfrak{i}-1)} \geq p_{\varrho(\mathfrak{i})}, \forall\, \mathfrak{i} = 1, 2, \cdots, \mathfrak{s}$.

**Example 3.** *Four members of an interview committee assign different Pythagorean fuzzy values* $p_1 = \langle 0.8, 0.3 \rangle$, $p_2 = \langle 0.5, 0.6 \rangle$, $p_3 = \langle 0.7, 0.5 \rangle$ *and* $p_4 = \langle 0.8, 0.5 \rangle$ *with weight vector* $\alpha = (0.1, 0.3, 0.3, 0.3)^T$ *to check the ability of a candidate for the post of a job in the field of pure math. Here, weight vector represents the importance of field of pure math for the committee's members. To find the clumped ability of the candidate, we used PFYOWAA operator. As*

$$\mathcal{S}(p_1) = 0.8^2 - 0.3^2 = 0.55,$$
$$\mathcal{S}(p_2) = 0.5^2 - 0.6^2 = -0.11,$$
$$\mathcal{S}(p_3) = 0.7^2 - 0.5^2 = 0.24,$$
$$\mathcal{S}(p_4) = 0.8^2 - 0.5^2 = 0.39.$$

*Since* $\mathcal{S}(p_1) > \mathcal{S}(p_3) > \mathcal{S}(p_4) > \mathcal{S}(p_2)$, *therefore*

$$p_{\varrho(1)} = p_1 = \langle 0.8, 0.3 \rangle,$$
$$p_{\varrho(2)} = p_4 = \langle 0.8, 0.5 \rangle,$$
$$p_{\varrho(3)} = p_3 = \langle 0.7, 0.5 \rangle,$$
$$p_{\varrho(4)} = p_2 = \langle 0.5, 0.6 \rangle,$$

*Thus, by applying the PFYOWAA operator, we get*

$$\begin{aligned} PFYOWAA_{\alpha}(p_1, p_2, p_3, p_4) &= \bigoplus_{\mathfrak{i}=1}^{4} (\alpha_{\mathfrak{i}} p_{\varrho(\mathfrak{i})}) \\ &= \left\langle \sqrt{\min\left(1, \left(\sum_{\mathfrak{i}=1}^{4} (\alpha_{\mathfrak{i}} \mu^{2\vartheta}_{\varrho(\mathfrak{i})})\right)^{\frac{1}{\vartheta}}\right)}, \sqrt{1 - \min\left(1, \left(\sum_{\mathfrak{i}=1}^{4} (\alpha_{\mathfrak{i}} (1 - \nu^2_{\varrho(\mathfrak{i})})^{\vartheta})\right)^{\frac{1}{\vartheta}}\right)} \right\rangle \\ &= \left\langle \sqrt{\min(1, (0.1(0.8)^4 + 0.3(0.8)^4 + 0.3(0.7)^4 + 0.3(0.5)^4)^{\frac{1}{2}}}, \right. \\ &\quad \left. \sqrt{1 - \min(1, (0.1(1 - 0.3^2)^2 + 0.3(1 - 0.5^2)^2 + 0.3(1 - 0.5^2)^2 + 0.3(1 - 0.6^2)^2)^{\frac{1}{2}}} \right\rangle \\ &= \langle 0.71, 0.51 \rangle. \end{aligned}$$

We state the following properties without their proofs.

**Theorem 9** (Idempotency). *If all PFNs are identical, i.e., $p_{\mathfrak{i}} = p$ then*

$$PFYOWAA(p_1, p_2, \cdots, p_{\mathfrak{s}}) = p.$$

**Theorem 10** (Boundedness). *Let $p_{\mathfrak{i}} = (\mu_{\mathfrak{i}}, \nu_{\mathfrak{i}})$ be a collection of PFNs. Let $p^- = \min(p_1, p_2, \cdots, p_{\mathfrak{s}})$ and $p^+ = \max(p_1, p_2, \cdots, p_{\mathfrak{s}})$. Then*

$$p^- \leq PFYOWAA(p_1, p_2, \cdots, p_{\mathfrak{s}}) \leq p^+.$$

**Theorem 11** (Monotonicity). *Let $p'_{\mathfrak{i}} = \{p'_1, p'_2, \cdots, p'_{\mathfrak{s}}\}$ and $p_{\mathfrak{i}} = \{p_1, p_2, \cdots, p_{\mathfrak{s}}\}$ be two collections of PFNs. If $\mu'_{\mathfrak{i}} \leq \mu_{\mathfrak{i}}$ and $\nu'_{\mathfrak{i}} \geq \nu_{\mathfrak{i}}$, $\forall$ $\mathfrak{i}$. Then*

$$PFYOWAA(p'_1, p'_2, \cdots, p'_{\mathfrak{s}}) \leq PFYOWAA(p_1, p_2, \cdots, p_{\mathfrak{s}}).$$

**Theorem 12** (Reducibility). *Let $p_{\mathfrak{i}} = (\mu_{\mathfrak{i}}, \nu_{\mathfrak{i}})(\mathfrak{i} = 1, 2, \cdots, \mathfrak{s})$ be a collection of PFNs with the corresponding weight vector $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_{\mathfrak{s}})^T = (\frac{1}{\mathfrak{s}}, \frac{1}{\mathfrak{s}}, \cdots, \frac{1}{\mathfrak{s}})^T$. Then, PFYOWAA operator is*

$$PFYOWAA_{\alpha}(p_1, p_2, \cdots, p_{\mathfrak{s}}) \quad = \quad \left\langle \sqrt{\min\big(1, \tfrac{1}{\mathfrak{s}}\big(\sum_{\mathfrak{i}=1}^{\mathfrak{s}} (\mu^{2\vartheta}_{\varrho(\mathfrak{i})})\big)^{\frac{1}{\vartheta}}\big)}, \sqrt{1 - \min\big(1, \tfrac{1}{\mathfrak{s}}\big(\sum_{\mathfrak{i}=1}^{\mathfrak{s}} \big((1 - \nu^2_{\varrho(\mathfrak{i})})^{\vartheta}\big)\big)^{\frac{1}{\vartheta}}\big)} \right\rangle.$$

**Theorem 13** (Commutativity). *Consider the collection $p_{\mathfrak{i}} = (\mu_{\mathfrak{i}}, \nu_{\mathfrak{i}})(\mathfrak{i} = 1, 2, \cdots, \mathfrak{s})$ of PFNs. If $p'_{\mathfrak{i}}$ is the permutation of $p_{\mathfrak{i}}$, then*

$$PFYOWAA_{\alpha}(p_1, p_2, \cdots, p_{\mathfrak{s}}) = PFYOWAA_{\alpha}(p'_1, p'_2, \cdots, p'_{\mathfrak{s}}).$$

We now define the Pythagorean fuzzy Yager hybrid weighted averaging operators.

**Definition 9.** *A PFYHWAA is a function $P^{\mathfrak{s}} \rightarrow P$, with correlated weight vector $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_{\mathfrak{s}})^T$ with $\alpha_{\mathfrak{i}} > 0$ and $\sum_{\mathfrak{i}=1}^{\mathfrak{s}} \alpha_{\mathfrak{i}} = 1$ such that*

$$\begin{aligned} PFYHWAA_{\alpha}(p_1, p_2, \cdots, p_{\mathfrak{s}}) \quad &= \quad \bigoplus_{\mathfrak{i}=1}^{\mathfrak{s}} (\alpha_{\mathfrak{i}} \dot{p}_{\varrho(\mathfrak{i})}) \\ &= \quad \left\langle \sqrt{\min\big(1, \big(\sum_{\mathfrak{i}=1}^{\mathfrak{s}} (\alpha_{\mathfrak{i}} \dot{\mu}^{2\vartheta}_{\varrho(\mathfrak{i})})\big)^{\frac{1}{\vartheta}}\big)}, \sqrt{1 - \min\big(1, \big(\sum_{\mathfrak{i}=1}^{\mathfrak{s}} \big(\alpha_{\mathfrak{i}}(1 - \dot{\nu}^2_{\varrho(\mathfrak{i})})^{\vartheta}\big)\big)^{\frac{1}{\vartheta}}\big)} \right\rangle, \end{aligned} \tag{5}$$

*where $\dot{p}_{\varrho(\mathfrak{i})}$ is the $\mathfrak{i}$th biggest weighted Pythagorean fuzzy values $\dot{p}_{\mathfrak{i}}(\dot{p}_{\mathfrak{i}} = \mathfrak{s}\alpha_{\mathfrak{i}} p_{\mathfrak{i}}, \mathfrak{i} = 1, 2, \cdots, \mathfrak{s})$ and $\mathfrak{s}$ is the balancing coefficient.*

**Remark 1.** *For $\alpha = (\frac{1}{\mathfrak{s}}, \frac{1}{\mathfrak{s}}, \cdots, \frac{1}{\mathfrak{s}})^T$, PFYWAA and PFYOWAA operators are considered to be a particular Example of PFYHWAA operator. Thus, PFYHWAA operator is a generalization of both operators.*

## 4. Pythagorean Fuzzy Yager Hybrid Weighted Geometric Aggregation Operators

In this section, we define Yager weighted geometric aggregation operators under PF environment.

**Definition 10.** *Let $p_{\mathfrak{i}} = \langle \mu_{\mathfrak{i}}, \nu_{\mathfrak{i}} \rangle (\mathfrak{i} = 1, 2, \cdots, \mathfrak{s})$ be several PFNs. The PFYWGA operator is a function $\mathcal{P}^{\mathfrak{s}} \rightarrow \mathcal{P}$ such that*

$$PFYWGA_{\alpha}(p_1, p_2, \cdots, p_{\mathfrak{s}}) = \bigotimes_{\mathfrak{i}=1}^{\mathfrak{s}} {p_{\mathfrak{i}}}^{\alpha_{\mathfrak{i}}},$$

*where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_{\mathfrak{s}})^T$ is the weight vector of $p_{\mathfrak{i}}$ with $\alpha_{\mathfrak{i}} > 0$ and $\sum_{\mathfrak{i}=1}^{\mathfrak{s}} \alpha_{\mathfrak{i}} = 1$.*

**Theorem 14.** *Let $p_i = \langle \mu_i, \nu_i \rangle (i = 1, 2, \cdots, \mathfrak{s})$ be several PFNs, then aggregated value of them by PFYWGA operation is a PFN and*

$$\begin{aligned} PFYWGA_{\alpha}(p_1, p_2, \cdots, p_{\mathfrak{s}}) &= \bigotimes_{i=1}^{\mathfrak{s}} p_i^{\alpha_i} \\ &= \left\langle \sqrt{1 - \min\left(1, \left(\sum_{i=1}^{\mathfrak{s}} (\alpha_i(1-\mu_i^2)^{\vartheta})\right)^{\frac{1}{\vartheta}}\right)}, \sqrt{\min\left(1, \left(\sum_{i=1}^{\mathfrak{s}} (\alpha_i \nu_i^{2\vartheta})\right)^{\frac{1}{\vartheta}}\right)} \right\rangle, \end{aligned} \tag{6}$$

*where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_{\mathfrak{s}})^T$ is the weight vector of $p_i$ with $\alpha_i > 0$ and $\sum_{i=1}^{\mathfrak{s}} \alpha_i = 1$.*

**Proof.** By using similar arguments as used in Theorem 2, we can prove this result. □

**Example 4.** *Consider Example 2 and by using Theorem 14, the clumped value for the cooking performance of a chef is*

$$\begin{aligned} PFYWGA_{\alpha}(p_1, p_2, p_3, p_4) &= \bigotimes_{i=1}^{4} (p_i)^{\alpha_i} \\ &= \left\langle \sqrt{1 - \min(1, \sum_{i=1}^{4} (\alpha_i(1-\mu_i^2)^{\vartheta})^{\frac{1}{\vartheta}})}, \sqrt{\min(1, \sum_{i=1}^{4} (\alpha_i \nu_i^{2\vartheta})^{\frac{1}{\vartheta}})} \right\rangle \\ &= \left\langle \sqrt{1 - \min(1, (0.2(1-0.6^2)^2 + 0.3(1-0.5^2)^2 + 0.2(1-0.7^2)^2 + 0.3(1-1^2)^2)^{\frac{1}{2}}}, \right. \\ &\quad \left. \sqrt{\min(1, (0.2(0.5)^4 + 0.3(0.5)^4 + 0.2(0.6)^4 + 0.3(0)^4)^{\frac{1}{2}})} \right\rangle \\ &= \langle 0.67, 0.49 \rangle. \end{aligned}$$

**Theorem 15** (Idempotency). *If all PFNs are identical, i.e., $p_i = p$ then*

$$PFYWGA(p_1, p_2, \cdots, p_{\mathfrak{s}}) = p.$$

**Proof.** By using similar arguments as used in Theorem 3, we can prove this result. □

**Theorem 16** (Boundedness). *Let $p_i = (\mu_i, \nu_i)$ be several PFNs. Let $p^- = \min(p_1, p_2, \cdots, p_{\mathfrak{s}})$ and $p^+ = \max(p_1, p_2, \cdots, p_{\mathfrak{s}})$. Then*

$$p^- \leq PFYWGA(p_1, p_2, \cdots, p_{\mathfrak{s}}) \leq p^+.$$

**Proof.** By using similar arguments as used in Theorem 4, we can prove this result. □

**Theorem 17** (Monotonicity). *Consider two collections $p'_i = \{p'_1, p'_2, \cdots, p'_{\mathfrak{s}}\}$ and $p_i = \{p_1, p_2, \cdots, p_{\mathfrak{s}}\}$ of PFNs. If $\mu'_i \leq \mu_i$ and $\nu'_i \geq \nu_i$, $\forall$ i. Then*

$$PFYWGA(p'_1, p'_2, \cdots, p'_{\mathfrak{s}}) \leq PFYWGA(p_1, p_2, \cdots, p_{\mathfrak{s}}).$$

**Proof.** By using similar arguments as used in Theorem 5, we can prove this result. □

**Theorem 18** (Reducibility). *Let $p_i = (\mu_i, \nu_i)(i = 1, 2, \cdots, \mathfrak{s})$ be a collection of PFNs with the corresponding weight vector $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_{\mathfrak{s}})^T = (\frac{1}{\mathfrak{s}}, \frac{1}{\mathfrak{s}}, \cdots, \frac{1}{\mathfrak{s}})^T$. Then, PFYWGA operator is*

$$PFYWGA_{\alpha}(p_1, p_2, \cdots, p_{\mathfrak{s}}) = \left\langle \sqrt{1 - \min\left(1, \frac{1}{\mathfrak{s}}\left(\sum_{i=1}^{\mathfrak{s}} ((1-\mu_i^2)^{\vartheta})\right)^{\frac{1}{\vartheta}}\right)}, \sqrt{\min\left(1, \frac{1}{\mathfrak{s}}\left(\sum_{i=1}^{\mathfrak{s}} (\nu_i^{2\vartheta})\right)^{\frac{1}{\vartheta}}\right)} \right\rangle.$$

**Theorem 19** (Commutativity). *Consider the collection $p_{\mathfrak{i}}=(\mu_{\mathfrak{i}},\nu_{\mathfrak{i}})(\mathfrak{i}=1,2,\cdots,\mathfrak{s})$ of PFNs. If $p'_{\mathfrak{i}}$ is the permutation of $p_{\mathfrak{i}}$, then*

$$PFYWGA_{\alpha}(p_1,p_2,\cdots,p_{\mathfrak{s}})=PFYWGA_{\alpha}(p'_1,p'_2,\cdots,p'_{\mathfrak{s}}).$$

We now define Pythagorean fuzzy Yager ordered weighted geometric aggregation operators.

**Definition 11.** *Let $p_{\mathfrak{i}}=\langle\mu_{\mathfrak{i}},\nu_{\mathfrak{i}}\rangle(\mathfrak{i}=1,2,\cdots,\mathfrak{s})$ be a collection of PFNs with the corresponding weight vector $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_{\mathfrak{s}})^T$ with $\alpha_{\mathfrak{i}}>0$ and $\sum_{\mathfrak{i}=1}^{\mathfrak{s}}\alpha_{\mathfrak{i}}=1$. The PFYOWGA operator is a function $\mathcal{P}^{\mathfrak{s}}\to\mathcal{P}$ such that*

$$PFYOWGA_{\alpha}(p_1,p_2,\cdots,p_{\mathfrak{s}})=\bigotimes_{\mathfrak{i}=1}^{\mathfrak{s}}(p_{\varrho(\mathfrak{i})})^{\alpha_{\mathfrak{i}}},$$

*where $(\varrho(1),\varrho(2),\cdots,\varrho(\mathfrak{s}))$ is the permutation of $(\mathfrak{i}=1,2,\cdots,\mathfrak{s})$ such that $p_{\varrho(\mathfrak{i}-1)}\geq p_{\varrho(\mathfrak{i})},\forall\,\mathfrak{i}=1,2,\cdots,\mathfrak{s}$.*

**Theorem 20.** *Let $p_{\mathfrak{i}}=\langle\mu_{\mathfrak{i}},\nu_{\mathfrak{i}}\rangle(\mathfrak{i}=1,2,\cdots,\mathfrak{s})$ be a collection of PFNs with the corresponding weight vector $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_{\mathfrak{s}})^T$ with $\alpha_{\mathfrak{i}}>0$ and $\sum_{\mathfrak{i}=1}^{\mathfrak{s}}\alpha_{\mathfrak{i}}=1$ then clumped value of them by PFYOWGA operation is a PFN and*

$$\begin{aligned}PFYOWGA_{\alpha}(p_1,p_2,\cdots,p_{\mathfrak{s}}) &= \bigotimes_{\mathfrak{i}=1}^{\mathfrak{s}}(p_{\varrho(\mathfrak{i})})^{\alpha_{\mathfrak{i}}}\\ &= \left\langle\sqrt{1-\min\left(1,\left(\sum_{\mathfrak{i}=1}^{\mathfrak{s}}\left(\alpha_{\mathfrak{i}}(1-\mu^2_{\varrho(\mathfrak{i})})^{\vartheta}\right)\right)^{\frac{1}{\vartheta}}\right)},\sqrt{\min\left(1,\left(\sum_{\mathfrak{i}=1}^{\mathfrak{s}}(\alpha_{\mathfrak{i}}\nu^{2\vartheta}_{\varrho(\mathfrak{i})})\right)^{\frac{1}{\vartheta}}\right)}\right\rangle,\end{aligned}\tag{7}$$

*where $(\varrho(1),\varrho(2),\cdots,\varrho(\mathfrak{s}))$ is the permutation of $(\mathfrak{i}=1,2,\cdots,\mathfrak{s})$ such that $p_{\varrho(\mathfrak{i}-1)}\geq p_{\varrho(\mathfrak{i})},\forall\,\mathfrak{i}=1,2,\cdots,\mathfrak{s}$.*

**Proof.** By using similar arguments as used in Theorem 2, we can prove this result. □

**Example 5.** *Consider Example 3 and by using Theorem 20, the clumped value for the ability of a candidate is*

$$\begin{aligned}PFYOWGA_{\alpha}(p_1,p_2,p_3,p_4) &= \bigotimes_{\mathfrak{i}=1}^{4}(p_{\varrho(\mathfrak{i})})^{\alpha_{\mathfrak{i}}}\\ &= \left\langle\sqrt{1-\min\left(1,\left(\sum_{\mathfrak{i}=1}^{4}\left(\alpha_{\mathfrak{i}}(1-\mu^2_{\varrho(\mathfrak{i})})^{\vartheta}\right)\right)^{\frac{1}{\vartheta}}\right)},\sqrt{\min\left(1,\left(\sum_{\mathfrak{i}=1}^{4}(\alpha_{\mathfrak{i}}\nu^{2\vartheta}_{\varrho(\mathfrak{i})})\right)^{\frac{1}{\vartheta}}\right)}\right\rangle\\ &= \Big\langle\sqrt{1-\min(1,(0.1(1-0.8^2)^2+0.3(1-0.8^2)^2+0.3(1-0.7^2)^2+0.3(1-0.5^2)^2)^{\frac{1}{2}})},\\ &\qquad\sqrt{\min(1,(0.1(0.3)^4+0.3(0.5)^4+0.3(0.5)^4+0.3(0.6)^4)^{\frac{1}{2}})}\Big\rangle\\ &= \langle 0.67,0.53\rangle.\end{aligned}$$

We state the following properties without their proofs.

**Theorem 21** (Idempotency). *If all PFNs are identical, i.e., $p_{\mathfrak{i}}=p$ then*

$$PFYOWGA(p_1,p_2,\cdots,p_{\mathfrak{s}})=p.$$

**Theorem 22** (Boundedness). *Let $p_{\mathfrak{i}}=(\mu_{\mathfrak{i}},\nu_{\mathfrak{i}})$ be a collection of PFNs. Let $p^-=\min(p_1,p_2,\cdots,p_{\mathfrak{s}})$ and $p^+=\max(p_1,p_2,\cdots,p_{\mathfrak{s}})$. Then*

$$p^-\leq PFYOWGA(p_1,p_2,\cdots,p_{\mathfrak{s}})\leq p^+.$$

**Theorem 23** (Monotonicity). *Let $p'_{\mathfrak{i}} = \{p'_1, p'_2, \cdots, p'_{\mathfrak{s}}\}$ and $p_{\mathfrak{i}} = \{p_1, p_2, \cdots, p_{\mathfrak{s}}\}$ be two collections of PFNs. If $\mu'_{\mathfrak{i}} \leq \mu_{\mathfrak{i}}$ and $\nu'_{\mathfrak{i}} \geq \nu_{\mathfrak{i}}$, $\forall$ $\mathfrak{i}$. Then*

$$PFYOWGA(p'_1, p'_2, \cdots, p'_{\mathfrak{s}}) \leq PFYOWGA(p_1, p_2, \cdots, p_{\mathfrak{s}}).$$

**Theorem 24** (Reducibility). *Let $p_{\mathfrak{i}} = (\mu_{\mathfrak{i}}, \nu_{\mathfrak{i}})(\mathfrak{i} = 1, 2, \cdots, \mathfrak{s})$ be a collection of PFNs with the corresponding weight vector $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_{\mathfrak{s}})^T = (\frac{1}{\mathfrak{s}}, \frac{1}{\mathfrak{s}}, \cdots, \frac{1}{\mathfrak{s}})^T$. Then, PFYOWGA operator is*

$$PFYOWGA_{\alpha}(p_1, p_2, \cdots, p_{\mathfrak{s}}) = \left\langle \sqrt{\min(1, \tfrac{1}{\mathfrak{s}}(\sum_{\mathfrak{i}=1}^{\mathfrak{s}} (\mu^{2\vartheta}_{\varrho(\mathfrak{i})}))^{\frac{1}{\vartheta}})}, \sqrt{1 - \min\left(1, \tfrac{1}{\mathfrak{s}}(\sum_{\mathfrak{i}=1}^{\mathfrak{s}} ((1 - \nu^2_{\varrho(\mathfrak{i})})^{\vartheta}))^{\frac{1}{\vartheta}}\right)} \right\rangle.$$

**Theorem 25** (Commutativity). *Consider the collection $p_{\mathfrak{i}} = (\mu_{\mathfrak{i}}, \nu_{\mathfrak{i}})(\mathfrak{i} = 1, 2, \cdots, \mathfrak{s})$ of PFNs. If $p'_{\mathfrak{i}}$ is the permutation of $p_{\mathfrak{i}}$, then*

$$PFYOWGA_{\alpha}(p_1, p_2, \cdots, p_{\mathfrak{s}}) = PFYOWGA_{\alpha}(p'_1, p'_2, \cdots, p'_{\mathfrak{s}}).$$

Now we define Pythagorean fuzzy Yager hybrid weighted geometric aggregation operators.

**Definition 12.** *A Pythagorean fuzzy Yager hybrid weighted geometric aggregation operator is a mapping $PFYHWGA : P^{\mathfrak{s}} \to P$, with correlated weight vector $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_{\mathfrak{s}})^T$ with $\alpha_{\mathfrak{i}} > 0$ and $\sum_{\mathfrak{i}=1}^{\mathfrak{s}} \alpha_{\mathfrak{i}} = 1$ such that*

$$\begin{aligned} PFYHWGA_{\alpha}(p_1, p_2, \cdots, p_{\mathfrak{s}}) &= \bigotimes_{\mathfrak{i}=1}^{\mathfrak{s}} (\dot{p}_{\varrho(\mathfrak{i})})^{\alpha_{\mathfrak{i}}} \\ &= \left\langle \sqrt{1 - \min\left(1, (\sum_{\mathfrak{i}=1}^{\mathfrak{s}} (\alpha_{\mathfrak{i}}(1 - \dot{\mu}^2_{\varrho(\mathfrak{i})})^{\vartheta}))^{\frac{1}{\vartheta}}\right)}, \sqrt{\min\left(1, (\sum_{\mathfrak{i}=1}^{\mathfrak{s}} (\alpha_{\mathfrak{i}} \dot{\nu}^{2\vartheta}_{\varrho(\mathfrak{i})}))^{\frac{1}{\vartheta}}\right)} \right\rangle, \end{aligned} \tag{8}$$

*where $\dot{p}_{\varrho(\mathfrak{i})}$ is the $\mathfrak{i}$th biggest weighted Pythagorean fuzzy values $\dot{p}_{\mathfrak{i}}(\dot{p}_{\mathfrak{i}} = p_{\mathfrak{i}}^{\mathfrak{s}\alpha_{\mathfrak{i}}}, \mathfrak{i} = 1, 2, \cdots, \mathfrak{s})$ and $\mathfrak{s}$ is the balancing coefficient.*

## 5. Applications in Multi-Attribute Decision-Making Problems

In this section, we discuss two multi-attribute decision-making problems under PF environment using Yager aggregation operators.

### *5.1. Selection of Top Rank Country in Health Care System*

Basically, health care is the recovery and maintenance of human body through prevention, diagnosis and treatment of diseases and other physical or mental deteriorations. For best health care, health system should be well-ordered and maintained. There are a lot of things which are counted in health care system such as psychology, nursing, medicine, physical therapy and many other things. Every human body needs a good health to spend a happy life. Due to lack of good health care system, people face different difficulties. A good health system is influenced by economic and social factors and a best health care system demands solid funding mechanism, a perfectly competent workforce, well-maintained facilities, etc. The countries which are financially stable with a good accessible health care system. To determine level of health care quality in every country, there are different factors such as care process (preventive care measures, coordinated care, and engagement and patient preferences), access (unavoidability and timeliness), administrative performance, equity, and health care outcomes (population health and disease-specific health outcomes).

The purpose of this application is to rank the top country among different countries in health care system according to "World Health Organization" (WHO) by applying PFYWAA and PFYWGA operators. Let $\mathcal{C} = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3\}$ be a set of alternatives (countries), where

$\mathcal{C}_1 = \{$represents United Kingdom$\}$,
$\mathcal{C}_2 = \{$represents Netherlands$\}$,
$\mathcal{C}_3 = \{$represents Australia$\}$.

Let $\mathcal{Y} = \{\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3\}$ be a set of three attributes, where

$\mathcal{Y}_1 = \{$represents Care process$\}$,
$\mathcal{Y}_2 = \{$represents Access$\}$,
$\mathcal{Y}_3 = \{$represents Equity$\}$,

and $\alpha = (0.3, 0.4, 0.3)^T$ be the weight vector of the attributes given by the decision-maker such that $\sum_{i=1}^{3} \alpha_i = 1$. The Pythagorean fuzzy values $(\mu_{ts}, \nu_{ts})$ of the alternatives according to different attributes are given in Table 1, where $\mu_{ts}$ is the positive membership degree for which alternative satisfies the given attribute, given by the decision-maker and $\nu_{ts}$ is the membership degree for which alternative does not satisfy the given attribute, where $\mu_{ts}, \nu_{ts} \in [0, 1]$ and $0 \leq \mu_{ts}^2 + \nu_{ts}^2 \leq 1$.

**Table 1.** Pythagorean fuzzy numbers.

| Countries | Care Process | Access | Equity |
|---|---|---|---|
| UK | (0.8, 0.3) | (0.7, 0.3) | (0.7, 0.3) |
| Netherlands | (0.7, 0.5) | (0.9, 0.1) | (0.7, 0.4) |
| Australia | (0.8, 0.5) | (0.7, 0.5) | (0.5, 0.6) |

Applying the PFYWAA operator given in Equation 3, to find the aggregate value $\xi_i$ of each country corresponding to given attributes and take $\vartheta = 2$.

$$\xi_1 = (0.96, 0),$$

$$\xi_2 = (1, 0),$$

$$\xi_3 = (0.89, 0).$$

Now we find the score values of $\xi_i$ which are given as,

$$\mathcal{S}(\xi_1) = (0.96)^2 - (0)^2 = 0.92,$$

$$\mathcal{S}(\xi_2) = (1)^2 - (0)^2 = 1,$$

$$\mathcal{S}(\xi_3) = (0.89)^2 - (0)^2 = 0.79.$$

As $\mathcal{S}(\xi_2) > \mathcal{S}(\xi_1) > \mathcal{S}(\xi_3)$. Hence, $\mathcal{C}_2 > \mathcal{C}_1 > \mathcal{C}_3$. Thus, Netherlands is on the top rank country in health care system among other two countries.

Now, applying the PFYWGA operator given in Equation 6 to find the aggregate value $\xi_i$ of each country corresponding to given attributes

$$\xi_1 = (0.78, 0.82),$$

$$\xi_2 = (0.57, 0.49),$$

$$\xi_3 = (0.26, 0.71).$$

Now we find the score values of $\xi_i$ which are given as,

$$\mathcal{S}(\xi_1) = (0.78)^2 - (0.82)^2 = -0.06,$$

$$\mathcal{S}(\xi_2) = (0.57)^2 - (0.49)^2 = 0.08,$$

$$\mathcal{S}(\xi_3) = (0.26)^2 - (0.71)^2 = -0.45.$$

As $\mathcal{S}(\xi_2) > \mathcal{S}(\xi_1) > \mathcal{S}(\xi_3)$. Hence, $\mathcal{C}_2 > \mathcal{C}_1 > \mathcal{C}_3$. Thus, Netherlands is the top rank country in health care system among other two countries.

We conclude from Table 2 that by applying both operators, Netherlands is on the top of rank. Therefore, we can use any operator between these two operators.

**Table 2.** Comparison analysis of PFYWAA and PFYWGA operators.

| Operators | $\mathcal{S}(\xi_1)$ | $\mathcal{S}(\xi_2)$ | $\mathcal{S}(\xi_3)$ | Ranking Order |
|---|---|---|---|---|
| PFYWAA | 0.92 | 1 | 0.79 | $\mathcal{C}_2 > \mathcal{C}_1 > \mathcal{C}_3$ |
| PFYWGA | −0.06 | 0.08 | −0.45 | $\mathcal{C}_2 > \mathcal{C}_1 > \mathcal{C}_3$ |

The whole method which we have adopted in this application is given in Figure 1.

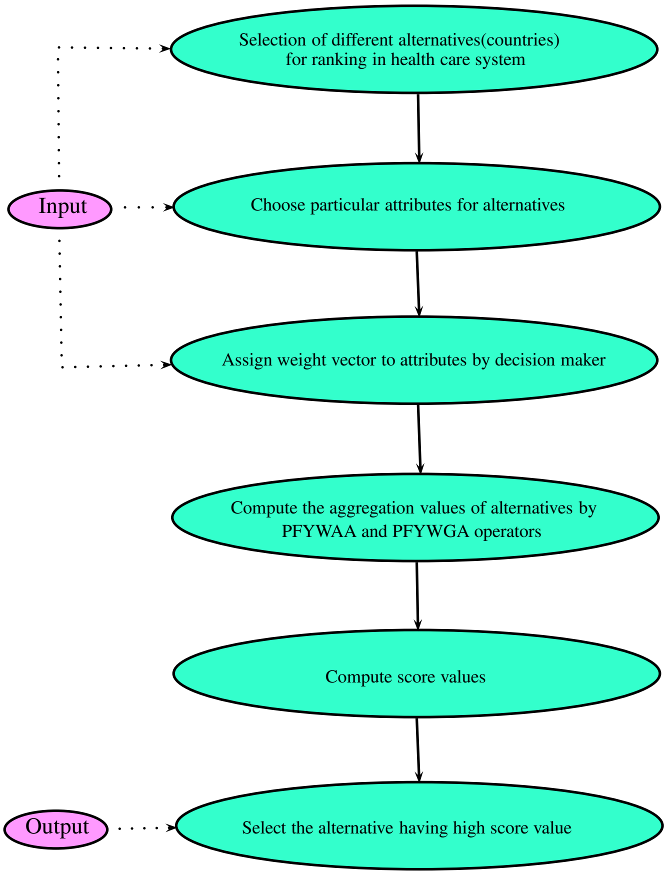


**Figure 1.** Flow chart to select high rank country in health care system.

### *5.2. Selection of Top Rank University in Education Sector*

Ranking is the mean of selecting the high position of anything or person in any field. Basically, ranking is a relationship between a set of different sources, in which it can be judged which is superior than others. All other fields, raking in education sector is too much important. Rankings are the significantly help maintain and build institutional position and reputation. The high rank of institute promotes the chances of falling into a prospective student shortlisting process. There are a lot of ways to help students in identifying the best colleges and universities for them. By visiting different institutes in particular settings will help to evaluate what feels right. Rankings can help the students to sort out which institutes are considered the most selective and important. National and international partnerships and collaborations are also affected by the university rankings. For the university with high rank, it will increase the chances of willingness of others to partner with them or support their membership in helping the academic or professional issues. There is an increasing impact of university rankings on the higher education associations and their environment, influencing, the decisions of the future students in their elect of schools, the government policy of financing higher education institutions along with the way of managing the universities. It is exacting for the ranking organizations to give the objective picture of the position of different universities in relation to one another.

The objective of this application is to select the top rank university among different universities by applying PFYWAA and PFYWGA operators. Let $\mathcal{U} = \{\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3\}$ be a set of alternatives (universities), where

$\mathcal{U}_1 = \{$represents University of Punjab Lahore$\}$,
$\mathcal{U}_2 = \{$represents Quaid-e-Azam University Islamabad$\}$,
$\mathcal{U}_3 = \{$represents University of Education Lahore$\}$.

Let $\mathcal{Y} = \{\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3\}$ be a set of three attributes for the selection of universities, where

$\mathcal{Y}_1 = \{$represents teaching quality$\}$,
$\mathcal{Y}_2 = \{$represents quality assurance$\}$,
$\mathcal{Y}_3 = \{$represents research in natural sciences fields $\}$,

and $\alpha = (0.3, 0.2, 0.5)^T$ be the weight vector of the attributes given by the decision-maker such that $\sum_{i=1}^{3} \alpha_i = 1$. The PF values $(\mu_{ts}, \nu_{ts})$ of the alternatives according to different attributes are given in Table 3, where $\mu_{ts}$ is the positive membership degree for which alternative satisfies the given attribute and $\nu_{ts}$ is the membership degree for which alternative does not satisfy the given attribute, where $\mu_{ts}, \nu_{ts} \in [0,1]$ and $0 \leq \mu_{ts}^2 + \nu_{ts}^2 \leq 1$.

**Table 3.** Pythagorean fuzzy numbers.

| Universities | Teaching Quality | Quality Assurance | Research in Natural Sciences Fields |
|---|---|---|---|
| University of Punjab Lahore | (0.8, 0.3) | (0.7, 0.5) | (1, 0) |
| Quaid-e-Azam University Islamabad | (0.7, 0.3) | (0.8, 0.3) | (0.7, 0.4) |
| University of Education Lahore | (0.5, 0.6) | (0.5, 0.5) | (0.6, 0.5) |

Applying the PFYWAA operator given in Equation 3, to find the aggregate value $\xi_i$ of each university corresponding to given attributes and take $\vartheta = 2$.

$$\xi_1 = (1, 0),$$

$$\xi_2 = (0.95, 0),$$

$$\xi_3 = (0.71, 0).$$

The score values of $\xi_i$ are,

$$\mathcal{S}(\xi_1) = (1)^2 - (0)^2 = 1,$$

$$\mathcal{S}(\xi_2) = (0.95)^2 - (0)^2 = 0.90,$$

$$\mathcal{S}(\xi_3) = (0.71)^2 - (0)^2 = 0.50.$$

As $\mathcal{S}(\xi_1) > \mathcal{S}(\xi_2) > \mathcal{S}(\xi_3)$. Hence, $\mathcal{U}_1 > \mathcal{U}_2 > \mathcal{U}_3$. Thus, University of Punjab Lahore is the top rank university in education sector among other two universities.
Now, applying the PFYWGA operator given in Equation 6 to find the aggregate value $\xi_i$ of each country corresponding to given attributes

$$\xi_1 = (0.75, 0.40),$$

$$\xi_2 = (0.45, 0.45),$$

$$\xi_3 = (0, 0.70).$$

Now we find the score values of $\xi_i$ which are given as,

$$\mathcal{S}(\xi_1) = (0.75)^2 - (0.40)^2 = 0.40,$$

$$\mathcal{S}(\xi_2) = (0.45)^2 - (0.45)^2 = 0,$$

$$\mathcal{S}(\xi_3) = (0)^2 - (0.70)^2 = -0.49.$$

As $\mathcal{S}(\xi_1) > \mathcal{S}(\xi_2) > \mathcal{S}(\xi_3)$. Hence, $\mathcal{U}_1 > \mathcal{U}_2 > \mathcal{U}_3$. Thus, University of Punjab Lahore is the top rank university in education sector among other two universities.

We conclude from Table 4 that by applying both operators, University of Punjab Lahore is on the top of rank. Therefore, we can use any operator between these two operators.

**Table 4.** Comparison analysis of PFYWAA and PFYWGA operators.

| **Operators** | $\mathcal{S}(\xi_1)$ | $\mathcal{S}(\xi_2)$ | $\mathcal{S}(\xi_3)$ | **Ranking Order** |
|---|---|---|---|---|
| PFYWAA | 1 | 0.90 | 0.50 | $\mathcal{U}_1 > \mathcal{U}_2 > \mathcal{U}_3$ |
| PFYWGA | 0.40 | 0 | $-0.49$ | $\mathcal{U}_1 > \mathcal{U}_2 > \mathcal{U}_3$ |

The whole method which we have adopted in this application is given in Figure 2.

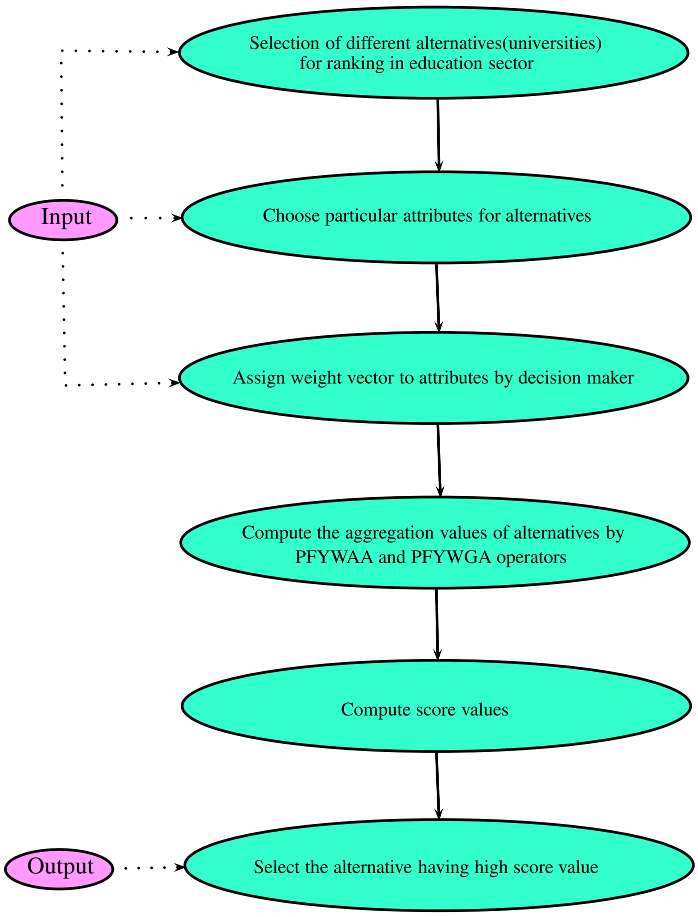


**Figure 2.** Flow chart to select the high rank university in education sector

## 6. Comparison Analysis

This section provides the comparison analysis of the proposed Yager aggregation operators under PF numbers with other existing operators. We compared the results of Pythagorean fuzzy Yager aggregation operators with Pythagorean fuzzy Dombi aggregation operators [29]. The results obtained from both operators for Section 5.1 are summarized in Table 5 and Figure 3. Similarly, the results obtained from both operators for Section 5.2 are summarized in Table 6 and Figure 4.

It is clear that results of Pythagorean fuzzy Dombi weighted geometric aggregation (PFDWGA) operator are similar to our proposed PFYWAA and PFYWGA operators but results of PFDWAA operator are different from results of our methods. It is clear that PFDWAA and PFDWGA operators are representing different results but results are same from both PFYWAA and PFYWGA operators. Thus, our proposed methods are more general and more flexible than other existing methods to deal with Pythagorean fuzzy MADM problems.

**Table 5.** Comparison analysis for Section 5.1 with Pythagorean fuzzy Dombi weighted averaging aggregation (PFDWAA) and Pythagorean fuzzy Dombi weighted geometric aggregation (PFDWGA) operators.

| Methods | $\mathcal{S}(\xi_1)$ | $\mathcal{S}(\xi_2)$ | $\mathcal{S}(\xi_3)$ | Ranking Order |
|---|---|---|---|---|
| PFYWAA | 0.92 | 1 | 0.79 | $\mathcal{C}_2 > \mathcal{C}_1 > \mathcal{C}_3$ |
| PFYWGA | $-0.06$ | 0.08 | $-0.45$ | $\mathcal{C}_2 > \mathcal{C}_1 > \mathcal{C}_3$ |
| PFDWAA | 0.57 | 0.25 | 0.77 | $\mathcal{C}_3 > \mathcal{C}_1 > \mathcal{C}_2$ |
| PFDWGA | $-0.57$ | $-0.53$ | $-0.77$ | $\mathcal{C}_2 > \mathcal{C}_1 > \mathcal{C}_3$ |

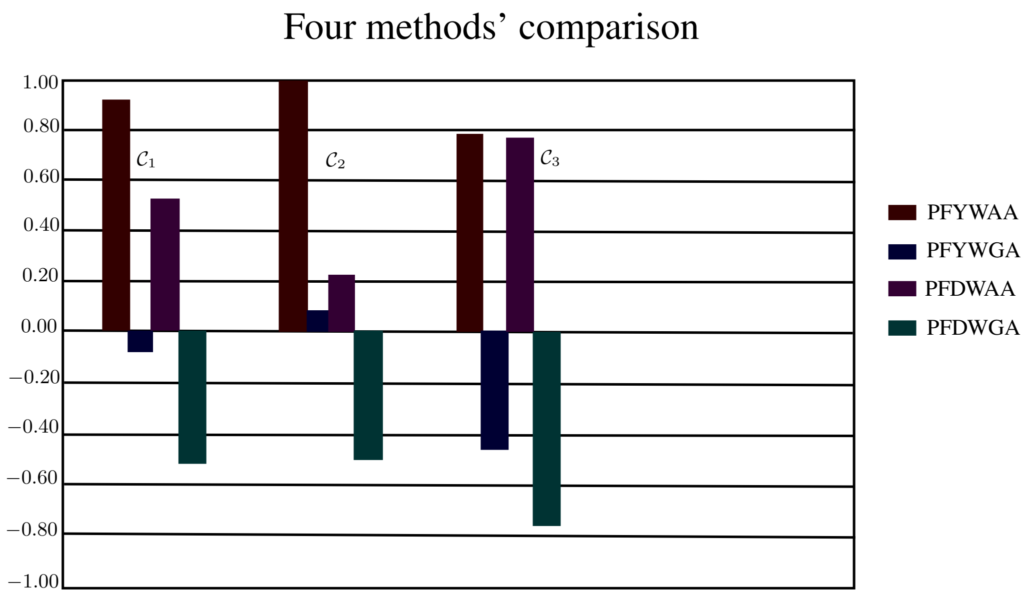


**Figure 3.** Four methods' comparison.

**Table 6.** Comparison analysis for Section 5.2 with Pythagorean fuzzy Dombi weighted averaging aggregation (PFDWAA) and Pythagorean fuzzy Dombi weighted geometric aggregation (PFDWGA) operators.

| Methods | $\mathcal{S}(\xi_1)$ | $\mathcal{S}(\xi_2)$ | $\mathcal{S}(\xi_3)$ | Ranking Order |
|---|---|---|---|---|
| PFYWAA | 1 | 0.90 | 0.50 | $\mathcal{U}_1 > \mathcal{U}_2 > \mathcal{U}_3$ |
| PFYWGA | 0.40 | 0 | $-0.49$ | $\mathcal{U}_1 > \mathcal{U}_2 > \mathcal{U}_3$ |
| PFDWAA | 0.40 | 0.58 | 0.84 | $\mathcal{U}_3 > \mathcal{U}_2 > \mathcal{U}_1$ |
| PFDWGA | $-0.39$ | $-0.58$ | $-0.84$ | $\mathcal{U}_1 > \mathcal{U}_2 > \mathcal{U}_3$ |

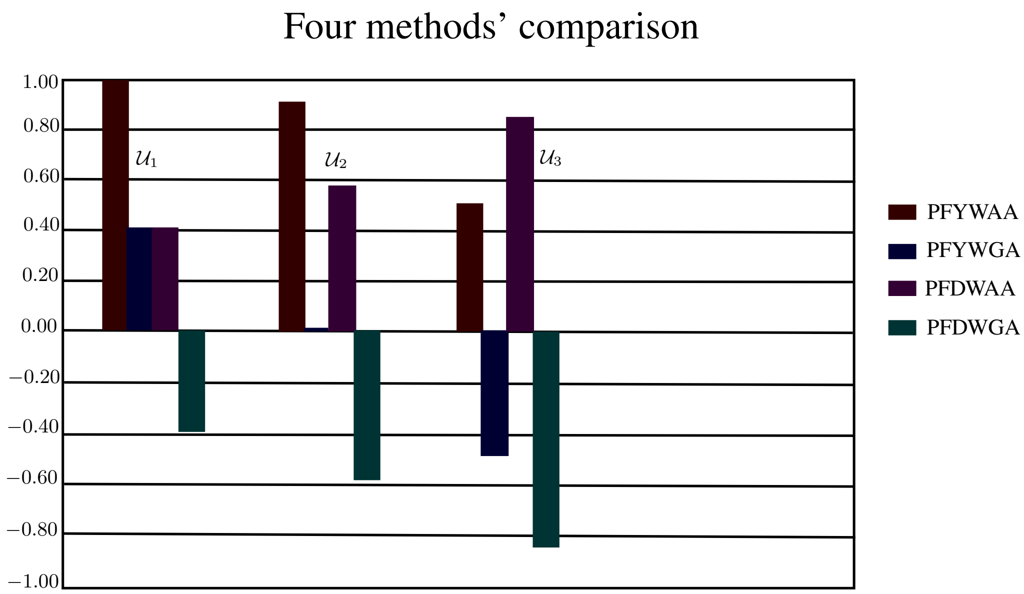


**Figure 4.** Four methods' comparison.

## 7. Conclusions and Future Directions

Aggregation operators are useful to associate a unique representative value for each alternative, when there are various attributes that apply to any given case. PFSs, a remarkable extension of IFSs, permit modeling of situations with higher more generality than IFSs, because they still apply in cases where the membership $\mu$ and non-membership $\nu$ values sum up to more than 1 but they satisfy $\mu^2 + \nu^2 \leq 1$. In this paper, we have benefited from the performance of the Yager $t$-norms and conorms to propose six aggregation operators that associate PFSs with finite families of PFSs. They are the PFYWAA, PFYOWAA, PFYHWAA, PFYWGA, PFYOWGA and PFYHWGA operators. We have also discussed two MADM problems, i.e., selection of a top-ranked country in health care systems and top-ranked university according to different attributes. In these problems we have applied both the PFYWAA and PFYWGA operators to summarize the information corresponding to each alternative. Then we have derived the appropriate results by the recourse to score functions. These operators allowed us to assess the value of each alternative in a comparable fashion. We have observed that the results are the same whether we use the PFYWAA or the PFYWGA operator. We are extending our study to (i) $m$-polar fuzzy Yager hybrid weighted operators; (ii) Pythagorean fuzzy soft Yager hybrid weighted operators, (iii) $m$-polar fuzzy soft Yager hybrid weighted operators, (iv) Rough $m$-polar fuzzy soft Yager hybrid weighted operators, (v) Rough neutrosophic Yager hybrid operators, and (vi) $q$-rung picture fuzzy Yager hybrid operators.

**Author Contributions:** Investigation, G.S., M.A. and A.N.A.-K.; writing—original draft, G.S. and M.A.; writing—review and editing, A.N.A.-K. All authors have read and agreed to the published version of the manuscript.

**Funding:** This project was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, under grant No. (D-107-130-1441).

**Acknowledgments:** The authors gratefully acknowledge DSR technical and financial support.

**Conflicts of Interest:** The authors declare no conflict of interest.

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