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Mathematical Analysis of a Thermostatted Equation with a Discrete Real Activity Variable

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Abstract: This paper deals with the mathematical analysis of a thermostatted kinetic theory equation. Specifically, the assumption on the domain of the activity variable is relaxed allowing for the discrete activity to attain real values. The existence and uniqueness of the solution of the related Cauchy problem and of the related non-equilibrium stationary state are established, generalizing the existing results.

Keywords: real activity variable; thermostat; nonlinearity; complex systems; Cauchy problem

1. Introduction

The mathematical analysis of a differential equation is usually based on the existence and uniqueness of a positive solution of the related Cauchy problem and on the dependence on the initial data (well-posed problem) [1,2]. The well-posed problem is analyzed under some (usually strongly) assumptions. However, if the differential equation is proposed as a general paradigm for the derivation of a mathematical model for a complex system [3,4], the definition of the assumptions is a delicate issue considering the restrictions that can be required on the system under consideration.

The present paper aims at generalizing the mathematical analysis of the discrete thermostatted kinetic theory framework recently proposed in [5,6] and employed in [7] for the modeling of the pedestrian dynamics into a metro station. The mathematical framework consists of a nonlinear differential equations system derived considering the balancing into the elementary volume of the microscopic states (space, velocity, strategy or activity) of the gain and loss particle-flows. The mathematical framework contains also a dissipative term, called thermostat, for balancing the action of an external force field which acts on the complex system, thus moving the system out-of-equilibrium [8,9]. The thermostat term allows for the existence and thus the modeling of the non-equilibrium stationary (possibly steady) states; see, among others, [10–15].

The mathematical analysis presented in [5] is based on the assumption that the discrete activity variable is greater than 1. In order to model complex systems, such as social and economical systems [16–20], the possibility for the activity variable to also attain negative values should be planned. Accordingly, this paper is devoted to a further generalization of the existence and uniqueness of the solution and of the non-equilibrium stationary solution for a real activity variable.

The present paper is organized as follows: after this introduction, Section 2 is devoted to the main definitions of the differential framework and the related non equilibrium stationary states; Section 3 deals with the new mathematical results.

2. The Mathematical Framework

Let $\mathbb{E}_p > 0$, $I_u = \{u_1, u_2, \ldots, u_n\}$, $u_i \in \mathbb{R}$, $\eta_{h,k} : I_u \times I_u \to \mathbb{R}^+$, $B^i_{hk} : I_u \times I_u \times I_u \to \mathbb{R}^+$, and $F_i : [0, +\infty[\to \mathbb{R}^+$, for $i, h, k \in \{1, 2, \ldots, n\}$.

This paper is devoted to the mathematical analysis of the solutions $f_i : [0, +\infty[\to \mathbb{R}^+$ of the following system of n nonlinear ordinary differential equations (called discrete thermostatted kinetic framework):

$$\frac{df_i}{dt}(t) = J_i[\mathbf{f}](t) + T_i[\mathbf{f}](t), \qquad i \in \{1, 2, \ldots, n\}, \tag{1}$$

where $\mathbf{f}(t) = (f_1(t), f_2(t), \ldots, f_n(t))$ is the vector solution, and $J_i[\mathbf{f}](t) := G_i[\mathbf{f}](t) - L_i[\mathbf{f}](t)$ and $T_i[\mathbf{f}](t)$ are the operators defined as follows:

$$G_i[\mathbf{f}] := \sum_{h=1}^{n} \sum_{k=1}^{n} B^i_{hk}\, \eta_{hk}\, f_h(t) f_k(t);$$

$$L_i[\mathbf{f}] := f_i(t) \sum_{k=1}^{n} \eta_{hk}\, f_k(t);$$

$$T_i[\mathbf{f}](t) := F_i - \left(\frac{\displaystyle\sum_{j=1}^{n} u_j^p \left(J_j[\mathbf{f}] + F_j \right)}{\mathbb{E}_p} \right) f_i(t).$$

Let $\mathbb{E}_p[\mathbf{f}](t)$ be the pth-order moment:

$$\mathbb{E}_p[\mathbf{f}](t) = \sum_{i=1}^{n} u_i^p\, f_i(t), \qquad p \in \mathbb{N},$$

and $\mathcal{R}^p$ the following function space:

$$\mathcal{R}^p := \left\{ \mathbf{f} \in C\left([0, +\infty[; (\mathbb{R}^+)^n \right) : \mathbb{E}_p[\mathbf{f}](t) = \mathbb{E}_p \right\}.$$

Let $\mathbf{f}^0 \in \mathcal{R}^p$, the Cauchy problem related to Equation (1) reads:

$$\begin{cases} \dfrac{d\mathbf{f}(t)}{dt} = \mathbf{J}[\mathbf{f}](t) + \mathbf{T}[\mathbf{f}](t) \quad t \in [0, +\infty[\\[2ex] \mathbf{f}(0) = \mathbf{f}^0, \end{cases} \tag{2}$$

where $\mathbf{J}[\mathbf{f}] = \mathbf{G}[\mathbf{f}] - \mathbf{L}[\mathbf{f}] = (J_1[\mathbf{f}], J_2[\mathbf{f}], \ldots, J_n[\mathbf{f}]) = (G_1[\mathbf{f}] - L_1[\mathbf{f}], G_2[\mathbf{f}] - L_2[\mathbf{f}], \ldots, G_n[\mathbf{f}] - L_n[\mathbf{f}])$ and $\mathbf{T}[\mathbf{f}] = (T_1[\mathbf{f}], T_2[\mathbf{f}], \ldots, T_n[\mathbf{f}])$.

The existence and uniqueness of the solution of the Cauchy problem (2) have been proved in [5], under the main assumption $u_i \geq 1$, for $i \in \{1, 2, \ldots, n\}$. This paper aims at generalizing the result of [5] when $u_i \in \mathbb{R}$.

A non-equilibrium stationary state of the framework (1) is a constant function f_i, for $i \in \{1, 2, \ldots, n\}$, solution of the following problem:

$$J_i[\mathbf{f}] - \left(\frac{\displaystyle\sum_{j=1}^{n} u_j^p (J_j[\mathbf{f}] + F_j)}{\mathbb{E}_p} \right) f_i = 0. \tag{3}$$

The existence and uniqueness of the non-equilibrium stationary state have been shown in [6] under the assumption $u_i \geq 1$, for all $i \in \{1, 2, \dots, n\}$. This result can be relaxed as stated in Theorem 2.

Remark 1. *Nonlinear systems* (1) *are a mathematical framework proposed in [5] for the modeling of a complex system $\mathcal{C}$ homogeneous with respect to the* mechanical variables *(*space *and* velocity*), where u, called* activity*, models the states of the particles. The function f_i, for $i \in \{1, 2, \dots, n\}$, denotes the* distribution function *of the ith functional subsystem.*

3. The Generalized Results

Let $\|x\|_p$ be the ℓ^p-norm on $\mathbb{R}^n$:

$$\|x\|_p := \left(\sum_{i=1}^{n} x_i^p \right)^{\frac{1}{p}},$$

and $\bar{\mathbb{E}}_p$ the following number:

$$\bar{\mathbb{E}}_p := \sup_{t>0} \left(\sum_{i=1}^{n} |u_i|^p f_i(t) \right). \tag{4}$$

It is worth stressing that, if p is even, then $\bar{\mathbb{E}}_p = \mathbb{E}_p$; if p is odd, then $\mathbb{E}_p \leq \bar{\mathbb{E}}_p$.

The main result of the paper follows.

Theorem 1. *Let $p \in \mathbb{N}$, $\bar{\mathbb{E}}_p < \infty$ and $\mathbf{f}^0 \in \mathcal{R}^p$. Assume that*

- $u_i \in \mathbb{R} \setminus \{0\}$;
- $\sum_{i=1}^{n} B_{hk}^i = 1$, *for all $h, k \in \{1, 2, \dots, n\}$;*
- *There exists a constant $\eta > 0$ such that $\eta_{hk} \leq \eta$, for all $h, k \in \{1, 2, \dots, n\}$;*
- *There exists a constant $F > 0$ such that $F_i(t) \leq F$, for all $i \in \{1, 2, \dots, n\}$ and $t \geq 0$.*

Then, there exists a unique positive function $\mathbf{f} \in \mathcal{R}^p$ which is solution of the Cauchy problem (2).

Proof. Let $\mathbf{f}, \mathbf{g} \in \mathcal{R}^p$. Since $\sum_{i=1}^{n} B_{hk}^i = 1$, for all $h, k \in \{1, 2, \dots, n\}$, one has:

$$\begin{aligned}
\|\mathbf{G}[\mathbf{f}] - \mathbf{G}[\mathbf{g}]\|_1 &= \sum_{i=1}^{n} |G_i[\mathbf{f}] - G_i[\mathbf{g}]| \\
&= \sum_{i=1}^{n} \left| \sum_{h=1}^{n} \sum_{k=1}^{n} B_{hk}^i \, \eta_{hk} \, f_h(t) \, f_k(t) - \sum_{h=1}^{n} \sum_{k=1}^{n} B_{hk}^i \, \eta_{hk} \, g_h(t) \, g_k(t) \right| \\
&= \sum_{i=1}^{n} \left| \sum_{h=1}^{n} \sum_{k=1}^{n} B_{hk}^i \, \eta_{hk} \left(f_h(t) f_k(t) - g_h(t) g_k(t) \right) \right| \\
&\leq \eta \sum_{h=1}^{n} \sum_{k=1}^{n} |f_h(t) f_k(t) - g_h(t) g_k(t)| \\
&\leq \eta \sum_{h=1}^{n} \sum_{k=1}^{n} |f_h(t) f_k(t) - f_h(t) g_k(t) + f_h(t) g_k(t) - g_h(t) g_k(t)| \\
&\leq \eta \left| \sum_{h=1}^{n} f_h(t) + \sum_{h=1}^{n} g_h(t) \right| \sum_{k=1}^{n} |f_k(t) - g_k(t)| \\
&= \eta \left(\|\mathbf{f}\|_1 + \|\mathbf{g}\|_1 \right) \|\mathbf{f} - \mathbf{g}\|_1 \\
&= \eta \, |\mathbb{E}_0[\mathbf{f}](t) + \mathbb{E}_0[\mathbf{g}](t)| \, \|\mathbf{f} - \mathbf{g}\|_1,
\end{aligned} \tag{5}$$

and

$$\begin{aligned} \|\mathbf{L}[\mathbf{f}]-\mathbf{L}[\mathbf{g}]\|_1 &= \sum_{i=1}^n \left| f_i(t) \sum_{k=1}^n \eta_{ik}\, f_k(t) - g_i(t) \sum_{k=1}^n \eta_{ik}\, g_i(t) \right| \\ &\leq \eta\, |\mathbb{E}_0[\mathbf{f}](t) + \mathbb{E}_0[\mathbf{g}](t)|\, \|\mathbf{f}-\mathbf{g}\|_1. \end{aligned} \tag{6}$$

Since

$$|\mathbb{E}_0[\mathbf{f}]| = \left| \sum_{i=1}^n f_i(t) \right| = \left| \sum_{i=1}^n \frac{u_i^p}{u_i^p}\, f_i(t) \right|, \tag{7}$$

if $L := \max\limits_{0\leq i\leq n} \left\{ \dfrac{1}{|u_i|^p} \right\}$, then, by Equation (7), one has:

$$|\mathbb{E}_0[\mathbf{f}]| \leq L \sum_{i=0}^n |u_i|^p\, f_i(t) \leq L\, \bar{\mathbb{E}}_p. \tag{8}$$

By Equations (5), (6) and (8), one has:

$$\begin{aligned} \|\mathbf{J}[\mathbf{f}]-\mathbf{J}[\mathbf{g}]\|_1 &\leq 2\,\eta\, |\mathbb{E}_0[\mathbf{f}] + \mathbb{E}_0[\mathbf{g}]|\, \|\mathbf{f}-\mathbf{g}\|_1 \\ &\leq 4\,\eta\, L\, \bar{\mathbb{E}}_p \|\mathbf{f}-\mathbf{g}\|_1. \end{aligned} \tag{9}$$

Moreover:

$$\begin{aligned} &\|\mathbf{T}[\mathbf{f}]-\mathbf{T}[\mathbf{g}]\|_1 = \sum_{i=1}^n |T_i[\mathbf{f}] - T_i[\mathbf{g}]| \\ &= \sum_{i=1}^n \left| \left(\frac{\sum_{j=1}^n u_j^p\, (J_j[\mathbf{f}] + F_j)}{\mathbb{E}_p} \right) f_i(t) - \left(\frac{\sum_{j=1}^n u_j^p\, (J_j[\mathbf{g}] + F_j)}{\mathbb{E}_p} \right) g_i(t) \right| \\ &\leq \sum_{i=1}^n \left(\frac{\sum_{j=1}^n u_j^p\, F_j}{\mathbb{E}_p} \right) |f_i(t) - g_i(t)| + \sum_{i=1}^n \left| \left(\frac{\sum_{j=1}^n u_j^p\, J_j[\mathbf{f}]}{\mathbb{E}_p} \right) f_i(t) - \left(\frac{\sum_{j=1}^n u_j^p\, J_j[\mathbf{g}]}{\mathbb{E}_p} \right) g_i(t) \right| \\ &= \left(\frac{\sum_{j=1}^n u_j^p\, F_j}{\mathbb{E}_p} \right) \|\mathbf{f}(t) - \mathbf{g}(t)\|_1 + \sum_{i=1}^n \left| \left(\frac{\sum_{j=1}^n u_j^p\, J_j[\mathbf{f}]}{\mathbb{E}_p} \right) f_i(t) - \left(\frac{\sum_{j=1}^n u_j^p\, J_j[\mathbf{g}]}{\mathbb{E}_p} \right) g_i(t) \right|. \end{aligned} \tag{10}$$

Bearing the expressions of the operator $\mathbf{J}[\mathbf{f}]$ in mind, one has:

$$
\begin{aligned}
&\sum_{i=1}^{n}\left|\left(\frac{\sum_{j=1}^{n} u_j^p\, J_j[\mathbf{f}]}{\mathbb{E}_p}\right) f_i(t) - \left(\frac{\sum_{j=1}^{n} u_j^p\, J_j[\mathbf{g}]}{\mathbb{E}_p}\right) g_i(t)\right| \\
&= \sum_{i=1}^{n}\left|\frac{1}{\mathbb{E}_p}\sum_{l=1}^{n} u_l^p\, J_l[\mathbf{f}]\, f_i(t) - \frac{1}{\mathbb{E}_p}\sum_{l=1}^{n} u_l^p\, J_l[\mathbf{g}]\, g_i(t)\right| \\
&= \sum_{i=1}^{n}\left|\frac{1}{\mathbb{E}_p}\sum_{l=1}^{n} u_l^p \sum_{h=1}^{n}\sum_{k=1}^{n} B_{hk}^l\, \eta_{hk}\, f_h(t) f_k(t) f_i(t)\right. \\
&- \frac{1}{\mathbb{E}_p}\sum_{l=1}^{n} u_l^p\, f_l(t)\sum_{k=1}^{n}\eta_{lk}\, f_k(t) f_i(t) \\
&- \frac{1}{\mathbb{E}_p}\sum_{l=1}^{n} u_l^p \sum_{h=1}^{n}\sum_{k=1}^{n} B_{hk}^l\, \eta_{hk}\, g_h(t) g_k(t) g_i(t) \\
&\left. + \frac{1}{\mathbb{E}_p}\sum_{l=1}^{n} u_l^p\, g_l(t)\sum_{k=1}^{n}\eta_{lk}\, g_k(t) g_i(t)\right| \\
&\le \sum_{i=1}^{n}\left|\frac{1}{\mathbb{E}_p}\sum_{l=1}^{n} u_l^p \sum_{h=1}^{n}\sum_{k=1}^{n} B_{hk}^l\, \eta_{hk}\, (f_h(t) f_k(t) f_i(t) - g_h(t) g_k(t) g_i(t))\right| \\
&+ \eta\sum_{i=1}^{n}\sum_{k=1}^{n} |f_k(t) f_i(t) - g_k(t) g_i(t)| \\
&\le \frac{\eta\,\sum_{j=1}^{n} u_j^p}{\mathbb{E}_p}\sum_{i=1}^{n}\sum_{h=1}^{n}\sum_{k=1}^{n} |f_h(t) f_k(t) f_i(t) - g_h(t) g_k(t) g_i(t)| \\
&+ \eta\sum_{i=1}^{n}\sum_{k=1}^{n} |f_h(t) f_i(t) - g_k(t) g_i(t)|\,.
\end{aligned}
\tag{11}
$$

Since $\mathbf{f}$ and $\mathbf{g}$ belong to the space $\mathcal{R}^p$, by Equation (8), one has:

$$
\begin{aligned}
&\sum_{i=1}^{n}\sum_{h=1}^{n}\sum_{k=1}^{n} |f_h(t) f_k(t) f_i(t) - g_h(t) g_k(t) g_i(t)| \\
&= \sum_{i=1}^{n}\sum_{h=1}^{n}\sum_{k=1}^{n}\Big| f_h(t) f_k(t) f_i(t) - f_h(t) f_k(t) g_i(t) + f_h(t) f_k(t) g_i(t) \\
&- g_i(t) g_h(t) f_k(t) + g_i(t) g_h(t) f_k(t) - g_h(t) g_k(t) g_i(t)\Big| \\
&= \sum_{i=1}^{n}\sum_{h=1}^{n}\sum_{k=1}^{n} |f_h(t) f_k(t)(f_i(t) - g_i(t)) + g_i(t) f_k(t)(f_h(t) - g_h(t)) + g_i(t) g_h(t)(f_k(t) - g_k(t))| \\
&\le \sum_{h=1}^{n} f_h(t)\sum_{k=1}^{n} f_k(t)\sum_{i=1}^{n} |f_i(t) - g_i(t)| + \sum_{i=1}^{n} g_i(t)\sum_{k=1}^{n} f_k(t)\sum_{h=1}^{n} |f_h(t) - g_h(t)| \\
&+ \sum_{i=1}^{n} g_i(t)\sum_{h=1}^{n} g_h(t)\sum_{k=1}^{n} |f_k(t) - g_k(t)| \\
&\le \|\mathbf{f}-\mathbf{g}\|_1\left(\mathbb{E}_0^2[\mathbf{f}] + \mathbb{E}_0[\mathbf{f}]\mathbb{E}_0[\mathbf{g}] + \mathbb{E}_0^2[\mathbf{g}]\right) \\
&\le \|\mathbf{f}-\mathbf{g}\|_1\left(\mathbb{E}_0[\mathbf{f}] + \mathbb{E}_0[\mathbf{g}]\right)^2 \\
&\le \|\mathbf{f}-\mathbf{g}\|_1 4\, L^2\, \left(\bar{\mathbb{E}}_p\right)^2.
\end{aligned}
\tag{12}
$$

By Equation (8), one has:

$$\sum_{i=1}^{n}\sum_{k=1}^{n}|f_h(t)f_i(t)-g_h(t)g_i(t)| \le (\mathbb{E}_0[\mathbf{f}]+\mathbb{E}_0[\mathbf{g}])\,\|\mathbf{f}-\mathbf{g}\|_1 \\ \le 2\,L\,\bar{\mathbb{E}}_p\|\mathbf{f}-\mathbf{g}\|_1. \tag{13}$$

By Equations (10)–(13), one has:

$$\|\mathbf{T}[\mathbf{f}]-\mathbf{T}[\mathbf{g}]\|_1 \\ \le \left[\left(\frac{\sum_{j=1}^{n} u_j^p\,F_j}{\mathbb{E}_p}\right)+4\,\eta\sum_{j=1}^{n}u_j^p\,L^2\,\frac{(\bar{\mathbb{E}}_p)^2}{\mathbb{E}_p}+2\,\eta\,L\,\bar{\mathbb{E}}_p\right]\|\mathbf{f}-\mathbf{g}\|_1. \tag{14}$$

According to Equation (9) and Equation (14), the operators $\mathbf{J}[\mathbf{f}]$ and $\mathbf{T}[\mathbf{f}]$ are locally Lipschitz in $\mathbf{f}$, uniformly in t. Then, there exists a unique local solution of the Cauchy problem (2), and the solution $\mathbf{f}$ belongs to the space $\mathcal{R}^p$ (see Theorem 4.1 of [5]). The global existence of the solution is gained because $\mathbf{f}$ is globally bounded, for all $t>0$, i.e., by Equation (7), one has:

$$\left|\sum_{i=1}^{n} f_i(t)\right| \le L\,\bar{\mathbb{E}}_p < +\infty, \quad \forall t>0.$$

Then, the proof is gained . □

Remark 2. *If $u_i=0$, Theorem 1 holds true if f_i is a bounded function, i.e., $\exists K>0$ such that*

$$|f_i(t)|\le K, \quad \forall t>0.$$

Indeed, if l is such that $u_l=0$, the estimates inequalities (8), (9), and (14) rewrite:

$$|\mathbb{E}_0[\mathbf{f}]| \le L\sum_{i=1|u_i\neq 0}^{n}|u_i|^p\,f_i(t)+K\le L\,\bar{\mathbb{E}}_p+K,$$

$$\|\mathbf{J}[\mathbf{f}]-\mathbf{J}[\mathbf{g}]\|_1\le 4\,M(L\bar{\mathbb{E}}_p+K)^2\|\mathbf{f}-\mathbf{g}\|_1,$$

$$\|\mathbf{T}[\mathbf{f}]-\mathbf{T}[\mathbf{g}]\|_1 \le \\ \le\left[\left(\frac{\sum_{j=1}^{n}u_j^p\,F_j}{\mathbb{E}_p}\right)+\frac{4\,M\sum_{j=1}^{n}u_j^p}{\mathbb{E}_p}(L\,\bar{\mathbb{E}}_p+K)^2+2\,M(L\,\bar{\mathbb{E}}_p+K)\right]\|\mathbf{f}-\mathbf{g}\|_1.$$

Theorem 2. *Let $p\in\mathbb{N}$. Assume that*

- $u_i\in\mathbb{R}\setminus\{0\}$;
- $\sum_{i=1}^{n}B_{hk}^i=1$, *for all* $h,k\in\{1,2,\ldots,n\}$;
- *There exists a constant $\eta>0$ such that $\eta_{hk}\le\eta$, for all $h,k\in\{1,2,\ldots,n\}$;*
- *There exists a constant $F>0$ such that $F_i\le F$, for all $i\in\{1,2,\ldots,n\}$;*
- *The following bound holds true:*

$$F>\eta\left[\frac{2\,\mathbb{E}_p^2}{\sum_{j=1}^{n}u_j^p}\,L+4\,L^2\,\mathbb{E}_p^2\right], \tag{15}$$

where $L := \max_{u_i \neq 0} \left\{ \frac{1}{|u_i|^p} \right\}$.
Then, there exists a unique positive nonequilirbium stationary solution $\mathbf{f} = (f_1, f_2, \dots, f_n) \in \mathcal{R}^p$ *of the* (3).

Proof. The non-equilibrium stationary problem (3) can be rewritten, for $i \in \{1, 2, \dots, n\}$, as the following fixed point problem (see [6]):

$$f_i = S_i[\mathbf{f}] := \frac{\eta}{F} \left(\frac{\mathbb{E}_p}{\|\mathbf{U}^p\|_1} - f_i \right) \left(\sum_{h=1}^{n} \sum_{k=1}^{n} B^i_{hk} f_h f_k \right) + \frac{\mathbb{E}_p}{\sum_{j=1}^{n} u_j^p}. \tag{16}$$

By straightforward calculations, one has:

$$\begin{aligned} \|\mathbf{S}[\mathbf{f}] - \mathbf{S}[\mathbf{g}]\|_1 \leq \frac{\eta}{F} & \left(\frac{\mathbb{E}_p}{\sum_{j=1}^{n} u_j^p} \sum_{h=1}^{n} \sum_{k=1}^{n} |f_h f_k - g_h g_k| \right) \\ & + \frac{\eta}{F} \sum_{i=1}^{n} \left| (f_i - g_i) \sum_{h=1}^{n} \sum_{k=1}^{n} B^i_{hk} (f_h f_k - g_h g_k) \right|. \end{aligned} \tag{17}$$

Furthermore, by the same arguments of the Theorem 1, one has:

$$\begin{aligned} \sum_{h=1}^{n} \sum_{k=1}^{n} |f_h f_k - g_h g_k| & \leq \|\mathbf{f} - \mathbf{g}\|_1 \left(\mathbb{E}_0[\mathbf{f}] + \mathbb{E}_0[\mathbf{g}] \right) \\ & \leq 2 \|\mathbf{f} - \mathbf{g}\|_1 L \, \bar{\mathbb{E}}_p. \end{aligned} \tag{18}$$

Moreover,

$$\begin{aligned} & \sum_{i=1}^{n} \left| (f_i - g_i) \sum_{h=1}^{n} \sum_{k=1}^{n} B^i_{hk} (f_h f_k - g_h g_k) \right| \\ & \leq \|\mathbf{f} - \mathbf{g}\|_1 \left(\mathbb{E}_0^2[\mathbf{f}] + \mathbb{E}_0^2[\mathbf{g}] \right) \\ & \leq 4 \|\mathbf{f} - \mathbf{g}\|_1 L^2 \left(\bar{\mathbb{E}}_p \right)^2. \end{aligned} \tag{19}$$

Finally, by Equations (18) and (19), (17) rewrites:

$$\|\mathbf{S}[\mathbf{f}] - \mathbf{S}[\mathbf{g}]\|_1 \leq \frac{\eta}{F} \left[\frac{\mathbb{E}_p}{\sum_{j=1}^{n} u_j^p} 2 \, L \, \bar{\mathbb{E}}_p + 4 \, L^2 \left(\bar{\mathbb{E}}_p \right)^2 \right] \|\mathbf{f} - \mathbf{g}\|_1, \tag{20}$$

and, by assumption Equation (15), there exists a unique fixed point of the problem (16) (see [21]). Then, there exists a unique non-equilibrium stationary state for problem (3). □

Remark 3. *If* $u_i = 0$, *the Theorem 2 holds true if the* ℓ_1*-norm of* $\mathbf{f}$ *is bounded, i.e.,*

$$\|\mathbf{f}\|_1 \leq K.$$

Indeed, condition (15) *rewrites:*

$$F > \eta \left[\frac{\mathbb{E}_p}{\sum_{j=1}^{n} u_j^p} 2(L \, \bar{\mathbb{E}}_p + K) + 4(L \, \bar{\mathbb{E}}_p + K)^2 \right].$$

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