mathematics

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On Canonical Almost Geodesic Mappings of Type $\pi_2(e)$

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Received: 14 November 2019; Accepted: 24 December 2019; Published: 1 January 2020

Abstract: In the paper, we consider canonical almost geodesic mappings of type $\pi_2(e)$. We have found the conditions that must be satisfied for the mappings to preserve the Riemann tensor. Furthermore, we consider canonical almost geodesic mappings of type $\pi_2(e)$ of spaces with affine connections onto symmetric spaces. The main equations for the mappings are obtained as a closed mixed system of Cauchy-type Partial Differential Equations. We have found the maximum number of essential parameters which the solution of the system depends on.

Keywords: canonical almost geodesic mappings; Cauchy-type PDEs; Riemann tensor; symmetric space

MSC: 53B05; 35R01

1. Introduction

The paper develops some new ideas in the theory of almost geodesic mappings of spaces with the affine connection. This theory can be dated back to the paper [1] of T. Levi-Civita, where he formulated and solved the problem of finding a Riemannian space with common geodesics, note that the problem was solved in a special coordinate system. It is worth noting that this problem is related to the study of equations of dynamics of mechanical systems.

Other problems and ideas in the theory of geodesic mappings were developed by T. Thomas, H. Weyl, P.A. Shirokov, A.S. Solodovnikov, N.S. Sinyukov, A.V. Aminova, J. Mikeš, and others.

Issues, arisen by the exploration, were studied by V.F. Kagan, G. Vrançeanu, Ya.L. Shapiro, and D.V. Vedenyapin et al. The authors discovered special classes of $(n-2)$-flat spaces.

The first person to introduce the notion of quasi-geodesic mappings was Petrov, see [2]. Principally, the holomorphically projective mappings of Kählerian spaces are special quasi-geodesic mappings; these were examined by T. Otsuki and Y. Tashiro, M. Prvanović, and others.

The class of almost geodesic mappings is a natural generalization of the class of geodesic mappings. Sinyukov defined almost geodesic mappings (see [3–6]) and he also specified three kinds of these mappings, in particular, π_1, π_2, π_3.

The theory of almost geodesic mappings was developed by V.S. Sobchuk [7,8], V.S. Shadnyi [9], N.V. Yablonskaya [10], V.E. Berezovski, J. Mikeš et al. [11–25], M.Z. Petrović, Lj.S. Velimirović, N. Vesić, M.S. Stankovič, and M.L. Zlatanović [26–32] et al. The results that follow were presented in the monographs [33,34] and in the review [19,35,36].

In 1962, A.Z. Petrov [2] studied quasi-geodesic mappings, where he showed that it is possible to simulate physical processes and electromagnetic fields. Similar results are presented in the paper of

C.-L. Bejan and O. Kowalski [37]. The mappings $\pi_2(e)$ are similar to those mentioned above. All these spaces are connected with some affinor structure F which can be interpreted as a force field.

In a 2019 paper [38] by A. Kozak and A. Borowiec, the authors studied a new physical interpretation of almost geodesic mappings, that are special transformations which genuinely preserve geodesics on the space-time.

In 1954, N.S. Sinyukov [39] (see [5,33,34]) proved that a (pseudo-) Riemannian space which admits geodesic mapping onto an equiaffine symmetric space is space of constant curvature. This result was generalized by V.E. Fomin [40] for infinity dimension spaces and by I. Hinterleitner and J. Mikeš [41] for geodesic mappings of Weyl space onto symmetric space. Almost geodesic mappings of symmetric mappings were studied by V.S. Sobchuk [7] and V.S. Sobchuk, J. Mikeš, and O. Pokorná [8].

Special almost geodesic mappings π_2 are mappings of type $\pi_2(e)$, which are related to e-structure F ($F^2 = e \cdot Id$, $e = \pm 1, 0$), defined on the manifold, see [5]. The paper is devoted to studying the conditions guaranteeing that the Riemann tensor is invariant with respect to the canonical almost geodesic mappings of type $\pi_2(e)$. Additionally, we study canonical almost geodesic mappings of type $\pi_2(e)$ of spaces with affine connections onto symmetric spaces. The main equations for the mappings are obtained as a closed, mixed system of Cauchy-type Partial Differential Equations in covariant derivatives.

The investigations use local coordinates. We assume that all functions under consideration are sufficiently differentiable.

2. Basic Definitions of Almost Geodesic Mappings of Spaces with Affine Connections

Let us recall the basic definition, formulas, and theorems of the theory presented in [5,6,33–35]. Consider a space A_n with an affine connection ∇ without torsion. The space is referred to with coordinates $x = (x^1, x^2, \dots, x^n)$.

A curve ℓ: $x = x(t)$ in the space A_n is a *geodesic* when its tangent vector $\lambda(t) = dx(t)/dt$ satisfies the equations $\nabla_t \lambda = \rho(t) \cdot \lambda$, where $\rho(t)$ is a certain function of t and ∇_t is a derivative along ℓ. Now, more often used are equations in the form $\nabla_t \lambda = 0$. From our point of view, the parameter t is canonical, for more detail see [34] (pp. 118–121). A curve ℓ in the space A_n is an *almost geodesic* when its tangent vector λ satisfies the equations

$$\nabla_t \nabla_t \lambda = a(t)\lambda + b(t)\nabla_t \lambda,$$

where $a(t)$ and $b(t)$ are certain functions of t.

A diffeomorphism f: $A_n \to \overline{A}_n$ is called a *geodesic mapping* if any geodesic of A_n is mapped under f onto a geodesic in $\overline{A}_n$.

A diffeomorphism f: $A_n \to \overline{A}_n$ is called an *almost geodesic* if any geodesic curve of A_n is mapped under f onto an almost geodesic curve in $\overline{A}_n$.

Suppose that a space A_n with affine connection ∇ admits a mapping f onto space $\overline{A}_n$ with affine connection $\overline{\nabla}$, and the spaces are referred to with the common coordinate system $x = (x^1, x^2, \dots, x^n)$.

The tensor $P = \bar{\nabla} - \nabla$ is called the *deformation tensor* of the connections ∇ and $\bar{\nabla}$ with respect to the mapping f; in common coordinates x, components of P have the following form:

$$P^h_{ij}(x) = \overline{\Gamma}^h_{ij}(x) - \Gamma^h_{ij}(x),$$

where $\Gamma^h_{ij}(x)$ and $\overline{\Gamma}^h_{ij}(x)$ are components of affine connections of the spaces A_n and $\overline{A}_n$, respectively.

According to [5], a necessary and sufficient condition for the mapping f: $A_n \to \overline{A}_n$ to be almost geodesic is that the deformation tensor $P^h_{ij}(x)$ of the mapping f must satisfy the condition

$$A^h_{\alpha\beta\gamma}\lambda^\alpha\lambda^\beta\lambda^\gamma = a \cdot P^h_{\alpha\beta}\lambda^\alpha\lambda^\beta + b \cdot \lambda^h,$$

where λ^h is an arbitrary vector and a and b are certain functions of variables $x^1, x^2, \ldots, x^n$ and $\lambda^1, \lambda^2, \ldots, \lambda^n$. The tensor A^h_{ijk} is defined as

$$A^h_{ijk} = P^h_{ij,k} + P^\alpha_{ij} P^h_{\alpha k}.$$

We denote by comma " , " a covariant derivative with respect to the connection of the space A_n.

Almost geodesic mappings of spaces with affine connections were introduced by N. S. Sinyukov in [5]. He distinguished three kinds of almost geodesic mappings, namely, π_1, π_2, and π_3, characterized by following conditions for the deformation tensor P:

$$\begin{array}{lll} \pi_1: & A^h_{(ijk)} = \delta^h_{(i} a_{jk)} + b_{(i} P^h_{jk)}\,, & \\ \pi_2: & P^h_{ij} = \delta^h_{(i}\psi_{j)} + F^h_{(i}\varphi_{j)}, & F^h_{(i,j)} + F^h_\alpha F^\alpha_{(i}\varphi_{j)} = \delta^h_{(i}\mu_{j)} + F^h_{(i}\rho_{j)}\,, \\ \pi_3: & P^h_{ij} = \delta^h_{(i}\psi_{j)} + \theta^h a_{ij}, & \theta^h_{,i} = \rho \cdot \delta^h_i + \theta^h a_i, \end{array}$$

where δ^h_i is the Kronecker symbol, the round parentheses of indices denote an operation called symmetrization without division, and $F^h_i, \theta^h, a_{ij}, a_i, \psi_i, \varphi_i, \mu_i, \rho_i, \rho$ are tensors.

The types of almost geodesic mappings π_1, π_2, π_3 can intersect. The problem of completeness of classification had long remained unresolved. Berezovsky and Mikeš [14] proved that, for $n > 5$, other types of almost geodesic mappings except π_1, π_2, and π_3 do not exist.

3. Almost Geodesic Mappings $\pi_2(e)$, $e = \pm 1, 0$

A mapping π_2 satisfies the *mutuality condition* if the inverse mapping is also an almost geodesic of type π_2 and corresponds to the same affinor $F^h_i(x)$.

The mappings π_2 satisfying mutuality condition will be denoted as $\pi_2(e)$, where $e = \pm 1, 0$, see [5], and is characterized by the following equations:

$$P^h_{ij} = \delta^h_{(i}\psi_{j)} + F^h_{(i}\varphi_{j)}, \tag{1}$$

$$F^h_{(i,j)} = F^h_{(i}\mu_{j)} - \delta^h_{(i} F^\alpha_{j)} \mu_\alpha \quad \text{and} \quad F^h_\alpha F^\alpha_i = e\delta^h_i. \tag{2}$$

We remind that F-planar mappings are characterized by Equation (1), these mappings were studied in [33–35,42,43]. These mappings generalize the quasi-geodesic mappings by A.Z. Petrov [2].

As it was proved in [25], in case $e = \pm 1$, the basic equations of the mappings $\pi_2(e)$ can be written as Equation (1), and

$$F^h_{i,j} = F^h_{ij}, \quad F^h_{ij,k} = \overset{6}{\Theta}{}^h_{ijk}, \quad \mu_{i,j} = \mu_{ij}, \quad \mu_{ij,k} = \overset{7}{\Theta}_{ijk}, \tag{3}$$

$$F^h_{(ij)} = F^h_{(i}\mu_{j)} - \delta^h_{(i} F^\alpha_{j)} \mu_\alpha, \qquad F^h_\alpha F^\alpha_i \ = e\delta^h_i, \qquad \mu_{(ij)} = \overset{5}{\Theta}_{ij}, \tag{4}$$

where

$$\begin{aligned} \overset{1}{\Theta}{}^h_{ijk} &\equiv \overset{2}{\Theta}{}^h_{ijk} + \overset{2}{\Theta}{}^h_{kji} - \overset{2}{\Theta}{}^h_{jki} + 2F^h_\alpha R^h_{kji} - F^\alpha_i R^h_{\alpha jk} + F^\alpha_j R^h_{\alpha ik} + F^\alpha_k R^h_{\alpha ij}, \\ \overset{2}{\Theta}{}^h_{ijk} &\equiv \mu_{(i} F^k_{j)k} - \delta^h_{(i} F^\alpha_{j)k} \mu_\alpha, \\ \overset{3}{\Theta}{}^h_{ijk} &\equiv \overset{2}{\Theta}{}^h_{ijk} - \overset{2}{\Theta}{}^h_{kji} + F^\alpha_j R^h_{\alpha ik} - F^h_\alpha j R^\alpha_{jik}, \end{aligned}$$

$$\overset{4}{\Theta}_{jk} \equiv F^{\alpha}_{\beta}\,\overset{1}{\Theta}{}^{\beta}_{\alpha jk} + 2F^{\alpha}_{\beta j}F^{\beta}_{\alpha k},$$
$$\overset{5}{\Theta}_{jk} \equiv \frac{1}{(n-1-F^{\alpha}_{\alpha})^2-1}\big((n-1-F^{\alpha}_{\alpha})\overset{4}{\Theta}_{ij} + \overset{4}{\Theta}_{\alpha\beta}F^{\alpha}_{i}F^{\beta}_{j}\big),$$
$$\overset{6}{\Theta}{}^{h}_{ijk} \equiv \frac{1}{2}\big(F^h_i\mu_{(jk)} + F^h_j\mu_{[jk]} + F^h_k\mu_{[ij]} - \delta^h_i m_{(jk)} - \delta^h_j m_{[ik]} - \delta^h_k m_{[ij]} + \overset{2}{\Theta}{}^{h}_{ikj}\big),$$
$$\overset{7}{\Theta}_{ijk} \equiv \mu_\alpha R^{\alpha}_{kji} + \frac{1}{2}\big(\overset{5}{\Theta}_{ij,k} + \overset{5}{\Theta}_{ik,j} + \overset{5}{\Theta}_{jk,i}\big), \quad m_{ij} \equiv F^{\alpha}_{i}\mu_{\alpha j},$$

F^h_i, F^h_{ij}, μ_i, μ_{ij} are unknown functions, and R^h_{ijk} is the Riemann tensor of the space $\overline{A}_n$. We denote by the brackets $[i,k]$, an operation called antisymmetrization (or alternation) without division with respect to the indices i and k.

Obviously, right-hand sides of Equation (3) depend on unknown functions F^h_i, F^h_{ij}, μ_i, μ_{ij} and on the components Γ^h_{ij} of the space A_n. Then, Equations (3) and (4) form a closed, mixed system of PDEs of Cauchy-type with respect to functions F^h_i, F^h_{ij}, μ_i, μ_{ij}. The general solution of the system, Equations (3) and (4), depends on no more than $\frac{1}{2}\,n(n+1)^2$ essential parameters. In addition, the mapping $\pi_2(e)$ depends on unknown functions ψ_i, φ_j (see Equation (1)).

4. Canonical Almost Geodesic Mappings $\pi_2(e)$ $(e=\pm 1)$ Preserving the Riemann Tensor

An almost geodesic mapping π_2 for which $\psi_i = 0$ is called *canonical*. It is known that any almost geodesic mapping π_2 can be written as the composition of a canonical almost geodesic mapping and a geodesic mapping. The latter may be referred to as a trivial almost geodesic mapping.

Hence, a canonical almost geodesic mapping $\pi_2(e)$ $(e=\pm 1)$ is determined by the equation

$$P^h_{ij} = F^h_i\varphi_j + F^h_j\varphi_i, \tag{5}$$

and also by Equations (3) and (4).

We proved [11] that Riemann tensor is preserved by the diffeomorphism if and only if the tensor A^h_{ijk} satisfies the conditions

$$A^h_{ijk} = A^h_{ikj}. \tag{6}$$

If the deformation tensor P^h_{ij} is expressed by Equation (5), then for $\pi_2(e)$ $(e=\pm 1)$ taking account of (2), (3), and (4) we get

$$A^h_{ijk} = \varphi_{i,k}F^h_j + \varphi_{j,k}F^h_i + \varphi_i\big(F^h_{jk} + \varphi_\alpha F^\alpha_j F^h_k + e\delta^h_j\varphi_k\big) + \varphi_j\big(F^h_{ik} + \varphi_\alpha F^\alpha_i F^h_k + e\delta^h_i\varphi_k\big).$$

Now, we require that A^h_{ijk} satisfies (6). Hence,

$$\varphi_{i,k}F^h_j - \varphi_{i,j}F^h_k + \varphi_{j,k}F^h_i - \varphi_{k,j}F^h_i = B^h_{ijk}, \tag{7}$$

where

$$\begin{aligned} B^h_{ijk} \;=\;& \varphi_k\big(F^h_{ij} + \varphi_\alpha F^\alpha_i F^h_j + e\delta^h_i\varphi_j\big) - \varphi_j\big(F^h_{ik} + \varphi_\alpha F^\alpha_i F^h_k + e\delta^h_i\varphi_k\big) \\ &+\varphi_i\big(F^h_{kj} + \varphi_\alpha F^\alpha_k F^h_j + e\delta^h_k\varphi_j - F^h_{jk} - \varphi_\alpha F^\alpha_j F^h_k - e\delta^h_j\varphi_k\big). \end{aligned}$$

Let us multiply (7) by F^j_h and contract for indices h and j. Hence, we have

$$n\varphi_{i,k} - \varphi_{k,i} = eB^\alpha_{i\beta k}F^\beta_\alpha. \tag{8}$$

Symmetrizing (8) in i and k, we obtain

$$\varphi_{i,k} + \varphi_{k,i} = \frac{e}{n-1}F^\beta_\alpha\big(B^\alpha_{i\beta k} + B^\alpha_{k\beta i}\big). \tag{9}$$

Equations (8) and (9) can be written as

$$\varphi_{i,k} = \frac{e}{n+1} F^{\beta}_{\alpha} \big(B^{\alpha}_{i\beta k} + \frac{1}{n-1} \big(B^{\alpha}_{i\beta k} + B^{\alpha}_{k\beta i} \big) \big). \tag{10}$$

Hence, we get the theorem.

Theorem 1. *In order for space A_n, with affine connection preserving the Riemann tensor, to admit an almost geodesic mappings of type $\pi_2(e)$ $(e = \pm 1)$ onto space $\overline{A}_n$ with affine connection, it is necessary and sufficient that the mixed system of differential equations of Cauchy-type in covariant derivatives (3) and (10) has a solution with respect to unknown functions F^h_i, F^h_{ij}, μ_i, μ_{ij}, φ_i which must satisfy the algebraic conditions (4).*

The general solution of the system (3), (4), and (10) depends on no more than $\frac{1}{2}\, n(n+1)^2 + n$ essential parameters.

5. Canonical Almost Geodesic Mappings $\pi_2(e)$ of Spaces with Affine Connection onto Symmetric Spaces

A space A_n with affine connection is called (locally) *symmetric* if its Riemann tensor is absolutely parallel. Symmetric spaces were introduced by É. Cartan in 1932 [44]. These spaces are also described in many monographs, i.e., S. Helgason [45]. Let us note that in the 1920's, P.A. Shirokov studied spaces where the Riemannian tensor is absolutely parallel, see reference paper [46]. Thus, the symmetric spaces $\overline{A}_n$ are characterized by

$$\overline{R}^h_{ijk|m} = 0, \tag{11}$$

where $\overline{R}^h_{ijk}$ is the Riemann tensor of the space $\overline{A}_n$. By the symbol " $|$ " we denote covariant derivative with respect to the connection of the space $\overline{A}_n$.

Let us consider the canonical almost geodesic mappings of type $\pi_2(e)$ $(e = \pm 1)$ of spaces A_n with affine connection onto symmetric spaces $\overline{A}_n$, which are determined by Equations (5), (3), and (4). Suppose that the spaces are referred to the common coordinate system $x^1, x^2, \ldots, x^n$.

Since

$$\overline{R}^h_{ijk|m} = \frac{\partial \overline{R}^h_{ijk}}{\partial x^m} + \overline{\Gamma}^h_{m\alpha} \overline{R}^\alpha_{ijk} - \overline{\Gamma}^\alpha_{mi} \overline{R}^h_{\alpha jk} - \overline{\Gamma}^\alpha_{mj} \overline{R}^h_{i\alpha k} - \overline{\Gamma}^\alpha_{mk} \overline{R}^h_{ij\alpha},$$

then taking account of (2) we can obtain

$$\overline{R}^h_{ijk|m} = \overline{R}^h_{ijk,m} + P^h_{m\alpha} \overline{R}^\alpha_{ijk} - P^\alpha_{mi} \overline{R}^h_{\alpha jk} - P^\alpha_{mj} \overline{R}^h_{i\alpha k} - P^\alpha_{mk} \overline{R}^h_{ij\alpha}. \tag{12}$$

In what follows, we understand that the space $\overline{A}_n$ is symmetric. Taking account of (5) and (11), we have from (12) that

$$\overline{R}^h_{ijk,m} = \varphi_{(i} F^\alpha_{m)} \overline{R}^h_{\alpha jk} + \varphi_{(j} F^\alpha_{m)} \overline{R}^h_{i\alpha k} + \varphi_{(k} F^\alpha_{m)} \overline{R}^h_{ij\alpha} - \varphi_{(m} F^h_{\alpha)} \overline{R}^\alpha_{ijk}. \tag{13}$$

It is known [5] that the Riemann tensors of the spaces A_n and $\overline{A}_n$ are related to each other by the equations

$$\overline{R}^h_{ijk} = R^h_{ijk} + P^k_{ik,j} - P^h_{ij,k} + P^\alpha_{ik} P^h_{\alpha j} - P^\alpha_{ij} P^h_{\alpha k}. \tag{14}$$

Since the deformation tensor of the mapping $P^h_{ij}(x)$ is represented by Equation (5), it follows from (14) that

$$\varphi_{i,j} F^h_k + \varphi_{k,j} F^h_i - \varphi_{i,k} F^h_j - \varphi_{j,k} F^h_i = D^h_{ijk}, \tag{15}$$

where

$$D^h_{ijk} = \overline{R}^h_{ijk} - R^h_{ijk} - \varphi_i \big(F^h_{kj} + \varphi_\alpha F^\alpha_k F^h_j + e\delta^h_k \varphi_j - F^h_{jk} - \varphi_\alpha F^\alpha_j F^h_k - e\delta^h_j \varphi_k \big)$$
$$+ \varphi_k \big(F^h_{ij} + \varphi_\alpha F^\alpha_i F^h_j \big) - \varphi_j \big(F^h_{ik} + \varphi_\alpha F^\alpha_i F^h_k \big).$$

Let us multiply (15) by F_h^k and contract for h and k. Hence, we have

$$n\varphi_{i,j} - \varphi_{j,i} = eD^{\alpha}_{i\beta j}F^{\beta}_{\alpha}. \tag{16}$$

Symmetrizing (16) in i and j, we obtain

$$\varphi_{i,j} + \varphi_{j,i} = \frac{e}{n-1}F^{\beta}_{\alpha}\left(D^{\alpha}_{i\beta j} + D^{\alpha}_{j\beta i}\right). \tag{17}$$

Equations (16) and (17) can be written as

$$\varphi_{i,j} = \frac{e}{n+1}F^{\beta}_{\alpha}\left(D^{\alpha}_{i\beta j} + \frac{1}{n-1}\left(D^{\alpha}_{i\beta j} + D^{\alpha}_{j\beta i}\right)\right). \tag{18}$$

Obviously, Equations (3), (13), and (18) form a closed, mixed system of PDEs of Cauchy-type with respect to functions F_i^h, F_{ij}^h, μ_i, μ_{ij}, $\overline{R}^h_{ijk}$, φ_i, and the functions F_i^h, F_{ij}^h, μ_i, μ_{ij} must satisfy the algebraic conditions (4). The algebraic conditions for the functions $\overline{R}^h_{ijk}$ are the Bianci identity

$$\overline{R}^h_{ijk} + \overline{R}^h_{ikj} = 0, \quad \text{and} \quad \overline{R}^h_{ijk} + \overline{R}^h_{jki} + \overline{R}^h_{kij} = 0. \tag{19}$$

Hence, we have proved the theorem.

Theorem 2. *In order for space A_n with affine connection to admit an almost geodesic mappings of type $\pi_2(e)$ $(e = \pm 1)$ onto symmetric space $\overline{A}_n$, it is necessary and sufficient that the mixed system of differential equations of Cauchy-type in covariant derivatives* (3), (13), *and* (18) *has a solution with respect to unknown functions F_i^h, F_{ij}^h, μ_i, μ_{ij}, $\overline{R}^h_{ijk}$, φ_i which must satisfy the algebraic conditions* (4) *and* (19).

It is obvious that the general solution of the mixed system of Cauchy-type depends on no more than

$$\frac{1}{3}\,n^2(n^2-1) + \frac{1}{2}\,n(n+1)^2 + n$$

essential parameters.

Author Contributions: All authors contributed equally and significantly in writing this article. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the grant IGA PrF 2019015 at Palacky University in Olomouc.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Levi-Civita, T. Sulle trasformazioni dello equazioni dinamiche. *Annali di Matematica Pura ed Applicata* **1896**, *24*, 252–300. [CrossRef]
2. Petrov, A.Z. Modeling of physical fields. *Gravitation Gen. Relat.* **1968**, *4*, 7–21.
3. Sinyukov, N.S. Almost geodesic mappings of affinely connected and Riemannian spaces. *Sov. Math.* **1963**, *4*, 1086–1088.
4. Sinyukov, N.S. Almost-geodesic mappings of affinely-connected spaces and e-structures. *Math. Notes* **1970**, *7*, 272–278. [CrossRef]
5. Sinyukov, N.S. *Geodesic Mappings of Riemannian Spaces*; Nauka: Moscow, Russia, 1979.
6. Sinyukov, N.S. Almost-geodesic mappings of affinely connected and Riemann spaces. *J. Sov. Math.* **1984**, *25*, 1235–1249. [CrossRef]
7. Sobchuk, V.S. Almost geodesic mappings of Riemannian spaces onto symmetric Riemannian spaces. *Mat. Zametki* **1975**, *17*, 757–763.
8. Sobchuk, V.S.; Mikeš, J.; Pokorná, O. On almost geodesic mappings π_2 between semisymmetric Riemannian spaces. *Novi Sad J. Math.* **1999**, *9*, 309–312.

9. Shadnyi, V.S. Almost geodesic maps of Riemannian spaces onto spaces of constant curvature. *Math. Notes* **1979**, *25*, 151–153. [CrossRef]
10. Yablonskaya, N.V. Special groups of almost geodesic transformations of spaces with affine connection. *Sov. Math.* **1986**, *30*, 105–108.
11. Berezovski, V.; Bácsó, S.; Mikeš, J. Almost geodesic mappings of affinely connected spaces that preserve the riemannian curvature. *Ann. Math. Inf.* **2015**, *45*, 3–10.
12. Berezovskii, V.E.; Guseva, N.I.; Mikeš, J. On special first-type almost geodesic mappings of affine connection spaces preserving a certain tensor. *Math. Notes* **2015**, *98*, 515–518. [CrossRef]
13. Berezovski, V.E.; Jukl, M.; Juklová, L. Almost geodesic mappings of the first type onto symmetric spaces. In Proceedings of the 16th Conference on Applied Mathematics (APLIMAT 2017), Bratislava, Slovakia, 31 January–2 February 2017.
14. Berezovski, V.E.; Mikeš, J. On the classification of almost geodesic mappings of affine-connected spaces. In Proceedings of the Differential Geometry and Applications Conference, Dubrovnik, Yugoslavia, 26 June–3 July 1988; pp. 41–48.
15. Berezovski, V.E.; Mikeš, J. On a classification of almost geodesic mappings of affine connection spaces. *Acta Univ. Palacki. Olomuc. Math.* **1996**, *35*, 21–24.
16. Berezovski, V.E.; Mikeš, J. On almost geodesic mappings of the type π_1 of Riemannian spaces preserving a system n-orthogonal hypersurfaces. *Rend. Circ. Mat. Palermo* **1999**, *II*, 103–108.
17. Berezovski, V.E.; Mikeš, J. Almost geodesic mappings of type π_1 onto generalized Ricci-symmetric manifolds. *Uch. zap. Kazan. Univ. Ser. Fiz.-Math.* **2009**, *151*, 9–14.
18. Berezovski, V.E.; Mikeš, J. On canonical almost geodesic mappings of the first type of affinely connected spaces. *Russ. Math.* **2014**, *58*, 1–5. [CrossRef]
19. Berezovski, V.E.; Mikeš, J. Almost geodesic mappings of spaces with affine connection. *J. Math. Sci.* **2015**, *207*, 389–409. [CrossRef]
20. Berezovski, V.E.; Mikeš, J.; Vanžurová, A. Almost geodesic mappings onto generalized Ricci-Symmetric manifolds. *Acta Math. Acad. Paedag. Nyiregyhaziensis* **2010**, *26*, 221–230.
21. Berezovski, V.E.; Mikeš, J.; Vanžurová, A. Fundamental PDE's of the canonical almost geodesic mappings of type π_1. *Bull. Malays. Math. Sci. Soc.* **2014**, *2*, 647–659.
22. Berezovski, V.E.; Cherevko, Y.; Rýparová, L. Conformal and geodesic mappings onto some special spaces. *Mathematics* **2019**, *7*, 664. [CrossRef]
23. Mikeš, J.; Pokorná, O.; Starko, G.A.; Vavříková, H. On almost geodesic mappings $\pi_2(e)$, $e = \pm 1$. In Proceedings of the APLIMAT 2005 Conference, Bratislava, Slovakia, 1–4 February 2005; pp. 315–321.
24. Škodová, M.; Mikeš, J.; Pokorná, O. On holomorphically projective mappings from equiaffine symmetric and recurrent spaces onto Kählerian spaces. *Rend. Circ. Mat. Palermo. Ser. II* **2005**, *75*, 309–316.
25. Vavříková, H.; Mikeš, J.; Pokorná, O.; Starko, G. On fundamental equations of almost geodesic mappings $\pi_2(e)$. *Russ. Math.* **2007**, *1*, 8–12. [CrossRef]
26. Petrović, M.Z.; Stanković, M.S. Special almost geodesic mappings of the first type of non-symmetric affine connection spaces. *Bull. Malays. Math. Sci. Soc.* **2017**, *40*, 1353–1362. [CrossRef]
27. Petrović, M.Z. Canonical almost geodesic mappings of type ${}_\theta\pi_2(0, F)$, $\theta \in \{1, 2\}$ between generalized parabolic Kähler manifolds. *Miskolc Math. Notes* **2018**, *19*, 469–482. [CrossRef]
28. Petrović, M.Z. Special almost geodesic mappings of the second type between generalized Riemannian spaces. *Bull. Malays. Math. Sci. Soc.* **2019**, *42*, 707–727. [CrossRef]
29. Stanković, M.S. On canonic almost geodesic mappings of the second type of affine spaces. *Filomat* **1999**, *13*, 105–144.
30. Stanković, M.S.; Zlatanović, M.L.; Vesić, N.O. Basic equations of G-almost geodesic mappings of the second type, which have the property of reciprocity. *Czech. Math. J.* **2015**, *65*, 787–799. [CrossRef]
31. Vesić, N.O.; Stanković, M.S. Invariants of special second-type almost geodesic mappings of generalized Riemannian space. *Mediterr. J. Math.* **2018**, *15*, 60. [CrossRef]
32. Vesić, N.O.; Velimirović, L.S.; Stanković, M.S. Some invariants of equitorsion third type almost geodesic mappings. *Mediterr. J. Math.* **2016**, *13*, 4581–4590. [CrossRef]
33. Mikeš, J.; Stepanova, E.; Vanžurová, A.; Bácsó, S.; Berezovski, V.E.; Chepurna, O.; Chodorová, M.; Chudá, H.; Gavrilchenko, M.L.; Haddad, M.; et al. *Differential Geometry of Special Mappings*; Palacky Univ. Press: Olomouc, Czech Republic, 2015.

34. Mikeš, J.; Bácsó, S.; Berezovski, V.E.; Chepurna, O.; Chodorová, M.; Chudá, H.; Formella, S.; Gavrilchenko, M.L.; Haddad, M.; Hinterleitner, I.; et al. *Differential Geometry of Special Mappings*; Palacky Univ. Press: Olomouc, Czech Republic, 2019.
35. Mikeš, J. Holomorphically projective mappings and their generalizations. *J. Math. Sci.* **1998**, *89*, 1334–1353. [CrossRef]
36. Berezovskii, V.E.; Mikeš, J.; Chudá, H.; Chepurna, O.Y. On canonical almost geodesic mappings which preserve the Weyl projective tensor. *Russ. Math.* **2017**, *61*, 1–5. [CrossRef]
37. Bejan, C.-L.; Kowalski, O. On generalization of geodesic and magnetic curves. *Note Mat.* **2017**, *37*, 49–57.
38. Kozak, A.; Borowiec, A. Palatini frames in scalar-tensor theories of gravity. *Eur. Phys. J.* **2019**, 79, 335. [CrossRef]
39. Sinyukov, N.S. On geodesic mappings of Riemannian manifolds onto symmetric spaces. *Dokl. Akad. Nauk SSSR* **1954**, *98*, 21–23.
40. Fomin, V.E. On geodesic mappings of infinite-dimmensional Riemannian spaces onto symmetric spaces of an affine connection. *Tr. Geom. Semin. Kazan* **1979**, *11*, 93–99.
41. Hinterleitner, I.; Mikeš, J. Geodesic mappings onto Weyl manifolds. *J. Appl. Math.* **2009**, *2*, 125–133.
42. Mikeš, J. Special F-planar mappings of affinely connected spaces onto Riemannian spaces. *Vestn. Mosk. Univ.* **1994**, *3*, 18–24. *Mosc. Univ. Math. Bull.* **1994**, *49*, 15–21.
43. Mikeš, J.; Sinyukov, N.S. On quasiplanar mappings of spaces of affine connection. *Iz. VUZ. Matematika* **1983**, *27*, 55–61. *Sov. Math.* **1983**, *27*, 63–70.
44. Cartan, É. Les espaces riemanniens symétriques. *Verhandlungen Kongress Zürich* **1932**, *1*, 152–161.
45. Helgason, S. *Differential Geometry, Lie Groups, and Symmetric Spaces*; AMS: Providence, RI, USA, 1978.
46. Shirokov, A.P. P.A. Shirokov's work on the geometry of symmetric spaces. *J. Math. Sci.* **1998**, *89*, 1253–1260. [CrossRef]