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Integral Inequalities for *s*-Convexity via Generalized Fractional Integrals on Fractal Sets

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Abstract: In this study, we establish new integral inequalities of the Hermite–Hadamard type for *s*-convexity via the Katugampola fractional integral. This generalizes the Hadamard fractional integrals and Riemann–Liouville into a single form. We show that the new integral inequalities of Hermite–Hadamard type can be obtained via the Riemann–Liouville fractional integral. Finally, we give some applications to special means.

Keywords: Katugampola fractional integrals; s-convex function; Hermite–Hadamard inequality; fractal space

1. Introduction

Fractional calculus, whose applications can be found in many disciplines including economics, life and physical sciences, as well as engineering, can be considered as one of the modern branches of mathematics [1–4]. Many problems of interests from these fields can be analyzed through fractional integrals, which can also be regarded as an interesting sub-discipline of fractional calculus. Some of the applications of integral calculus can be seen in the following papers [5–10], through which problems in physics, chemistry, and population dynamics were studied. The fractional integrals were extended to include the Hermite–Hadamard inequality, which is classically given as follows.

Consider a convex function, $h : E \subseteq \mathbb{R} \to \mathbb{R}$, $w, z \in E$ if, and only if,

$$h\left(\frac{w+z}{2}\right) \le \frac{1}{z-w} \int_w^z h(x) dx \le \frac{h(w)+h(z)}{2}.$$
(1)

Following this, many important generalizations of Hermite–Hadamard inequality were studied [11–17], some of which were formulated via generalized *s*-convexity, which is defined as follows.

Definition 1. Let $0 < s \le 1$. The function $h : [w, z] \subset \mathbb{R}_+ \to \mathbb{R}^{\alpha}$ is said to be generalized s-convex on fractal sets \mathbb{R}^{α} ($0 < \alpha < 1$) in the second sense if

$$h(tw + (1-t)z) \le (t)^{\alpha s} h(w) + (1-t)^{\alpha s} h(z).$$

This class of function is denoted by GK_s^2 (see Mo and Sui [18]).

Hermite–Hadamard-type inequalities have been extended to include fractional integrals. For example, Chen and Katugampola [19] generalized Equation (1) via generalized fractional integrals. Other important extensions of Equation (1) include the work of Mehran and Anwar [20], who studied the Hermite–Hadamard-type inequalities for *s*-convex functions involving generalized fractional integrals. The definitions of the generalized fractional integrals were given in [21], and we present them as follows.

Definition 2. Suppose $[w, z] \subset \mathbb{R}$ is a finite interval. For order $\alpha > 0$, the two sides of Katugampola fractional integrals for $h \in X_c^p(w, z)$ are defined by

$${}^{\rho}I^{\alpha}_{w}h(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{w}^{x} (x^{\rho}-t^{\rho})^{\alpha-1}t^{\rho-1}h(t)dt,$$

and

$${}^{\rho}I_{z-}^{\alpha}h(x)=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{x}^{z}\left(t^{\rho}-x^{\rho}\right)^{\alpha-1}t^{\rho-1}h(t)dt,$$

where w < x < z, $\rho > 0$, and $X_c^p(w, z)(c \in \mathbb{R}, 1 \le p \le \infty)$ represents the space of complex-valued Lebesgue measurable functions h on [w, z] for $\|h\|_{X_c^p < \infty}$. The norm is given as

$$\|h\|_{X^p_c} = \left(\int_w^z |t^c h(t)|^p \frac{dt}{t}\right)^{1/p} < \infty$$

for $1 \le p < \infty, c \in \mathbb{R}$. For the case $p = \infty$, we get

$$\|h\|_{X_c^\infty} = \operatorname{ess}\sup_{w \leq t \leq z} \left[t^c |h(t)|\right],$$

whereby ess sup is the essential supremum.

Even though Katugampola fractional integrals have been used to generalize many inequalities, such as Grüss [22,23], Hermite–Hadamard [24], and Lyapunov [25], this work generalizes Hermite–Hadamard inequality involving Katugampola on fractal sets.

When improving the results in Mehran and Anwar [20], we used Definition 2 together with the following lemma.

Lemma 1. [19] Suppose that $h : [w^{\rho}, z^{\rho}] \subset \mathbb{R}_+ \to \mathbb{R}$ is a differentiable function on (w^{ρ}, z^{ρ}) , where $0 \le w < z$ for $\alpha > 0$ and $\rho > 0$. If the fractional integrals exist, we get

$$\frac{h\left(w^{\rho}\right)+h\left(z^{\rho}\right)}{2}-\frac{\alpha\rho^{\alpha}\Gamma(\alpha+1)}{2\left(z^{\rho}-w^{\rho}\right)^{\alpha}}\left[{}^{\rho}I_{w^{+}}^{\alpha}h\left(z^{\rho}\right)+{}^{\rho}I_{z^{-}}^{\alpha}h\left(w^{\rho}\right)\right]=\frac{z^{\rho}-w^{\rho}}{2}\int_{0}^{1}\left[\left(1-t^{\rho}\right)^{\alpha}-\left(t^{\rho}\right)^{\alpha}\right]t^{\rho-1}h'\left(t^{\rho}w^{\rho}+\left(1-t^{\rho}\right)z^{\rho}\right)dt.$$

This paper is aimed at establishing some new integral inequalities for generalized *s*-convexity via Katugampola fractional integrals on fractal sets linked with Equation (1). We presented some inequalities for the class of mappings whose derivatives in absolute value are the generalized *s*-convexity. In addition, we obtained some new inequalities linked with convexity and generalized *s*-convexity via classical integrals as well as Riemann–Liouville fractional integrals in form of a corollary. As an application, the inequalities for special means are derived.

2. Main Results

Hermite–Hadamard inequality for *s*-convexity via generalized fractional integral can be written with the aid of the following theorem.

Theorem 1. Let $h : [w^{\rho}, z^{\rho}] \subset \mathbb{R}_+ \to \mathbb{R}^{\alpha}$ be a positive function for $0 \le w < z$ and $h \in X_c^p(w^{\rho}, z^{\rho})$ for $\alpha > 0$ and $\rho > 0$. If h is a generalized s-convex function on $[w^{\rho}, z^{\rho}]$, then

$$2^{\alpha(s-1)}h\left(\frac{w^{\rho}+z^{\rho}}{2}\right) \leq \frac{\rho^{\alpha}\Gamma(\alpha+1)}{2(z^{\rho}-w^{\rho})^{\alpha}} \left[{}^{\rho}I^{\alpha}_{w^{+}}h(z^{\rho}) + {}^{\rho}I^{\alpha}_{z^{-}}h(w^{\rho})\right]$$
$$\leq \left[\frac{1}{\rho(1+s)} + \alpha\beta(\alpha,\alpha s+1)\right]\frac{h(w^{\rho}) + h(z^{\rho})}{2}.$$
(2)

Proof. Since *h* is generalized *s*-convex function on $[w^{\rho}, z^{\rho}]$, for $t \in [0, 1]$, we get

$$h(t^{\rho}w^{\rho} + (1 - t^{\rho})z^{\rho}) \le (t^{\rho})^{\alpha s}h(w^{\rho}) + (1 - t^{\rho})^{\alpha s}h(z^{\rho}),$$

and

$$h(t^{\rho}z^{\rho} + (1 - t^{\rho})w^{\rho}) \le (t^{\rho})^{\alpha s} h(z^{\rho}) + (1 - t^{\rho})^{\alpha s} h(w^{\rho})$$

Combining the above inequalities, we have

$$h(t^{\rho}w^{\rho} + (1 - t^{\rho})z^{\rho}) + h(t^{\rho}z^{\rho} + (1 - t^{\rho})w^{\rho}) \le ((t^{\rho})^{\alpha s} + (1 - t^{\rho})^{\alpha s})[h(w^{\rho}) + h(z^{\rho})].$$
(3)

Multiplying both sides of Equation (3) by $t^{\alpha\rho-1}$, for $\alpha > 0$ and integrating it over [0, 1] with respect to t, we obtain

$$\frac{\rho^{\alpha-1}\Gamma(\alpha)}{(z^{\rho}-w^{\rho})^{\alpha}} \left[{}^{\rho}I^{\alpha}_{w^{+}}h\left(z^{\rho}\right) + {}^{\rho}I^{\alpha}_{z^{-}}h\left(w^{\rho}\right) \right] \leq \int_{0}^{1} t^{\alpha\rho-1} \left((t^{\rho})^{\alpha s} + (1-t^{\rho})^{\alpha s} \right) \left[h\left(w^{\rho}\right) + h\left(z^{\rho}\right) \right] dt.$$
(4)

Since

$$\int_0^1 t^{\alpha s \rho + \alpha \rho - 1} dt = \frac{1}{\alpha \rho(s+1)},$$

applying the change of variable $t^{\rho} = a$ gives the following

$$\int_0^1 t^{\alpha \rho - 1} \left(1 - t^{\rho} \right)^{\alpha s} dt = \frac{\beta(\alpha, \alpha s + 1)}{\rho}.$$

Thus, Equation (4) becomes

$$\frac{\rho^{\alpha}\Gamma(\alpha+1)}{2\left(z^{\rho}-w^{\rho}\right)^{\alpha}}\left[{}^{\rho}I_{w^{+}}^{\alpha}h\left(z^{\rho}\right)+{}^{\rho}I_{z^{-}}^{\alpha}h\left(w^{\rho}\right)\right] \leq \left[\frac{1}{\rho(1+s)}+\alpha\beta(\alpha,\alpha s+1)\right]\frac{h\left(w^{\rho}\right)+h\left(z^{\rho}\right)}{2}.$$

In order to prove the first part of Equation (2), since *h* is generalized *s*-convex function on $[w^{\rho}, z^{\rho}]$, the following inequality is obtained:

$$h\left(\frac{x^{\rho}+y^{\rho}}{2}\right) \le \frac{h\left(x^{\rho}\right)+h\left(y^{\rho}\right)}{2^{\alpha s}},\tag{5}$$

for $x^{\rho}, y^{\rho} \in [w^{\rho}, z^{\rho}], \alpha \ge 0$.

Consider $x^{\rho} = t^{\rho}w^{\rho} + (1 - t^{\rho})z^{\rho}$ and $y^{\rho} = t^{\rho}z^{\rho} + (1 - t^{\rho})w^{\rho}$, where $t \in [0, 1]$. Applying Equation (5), we have

$$2^{\alpha s} h\left(\frac{w^{\rho} + z^{\rho}}{2}\right) \le h\left(t^{\rho} w^{\rho} + (1 - t^{\rho}) z^{\rho}\right) + h\left(t^{\rho} z^{\rho} + (1 - t^{\rho}) w^{\rho}\right).$$
(6)

Multiplying both sides of the Equation (6) by $t^{\alpha \rho - 1}$, for $\alpha > 0$ and integrating over [0, 1] with respect to *t* gives the following:

$$\frac{2^{s}}{\alpha\rho}h\left(\frac{w^{\rho}+z^{\rho}}{2}\right) \leq \int_{0}^{1} t^{\alpha\rho-1}h\left(t^{\rho}w^{\rho}+(1-t^{\rho})z^{\rho}\right)dt + \int_{0}^{1} t^{\alpha\rho-1}h\left(t^{\rho}z^{\rho}+(1-t^{\rho})w^{\rho}\right)dt \\
= \int_{z}^{w}\left(\frac{z^{\rho}-x^{\rho}}{z^{\rho}-w^{\rho}}\right)^{\alpha-1}h\left(x^{\rho}\right)\frac{x^{\rho-1}}{w^{\rho}-z^{\rho}}dx \\
+ \int_{w}^{z}\left(\frac{y^{\rho}-w^{\rho}}{z^{\rho}-w^{\rho}}\right)^{\alpha-1}h\left(y^{\rho}\right)\frac{y^{\rho-1}}{z^{\rho}-w^{\rho}}dy \\
= \frac{\rho^{\alpha-1}\Gamma(\alpha)}{(z^{\rho}-w^{\rho})^{\alpha}}\left[I_{w^{+}}^{\alpha}h\left(z^{\rho}\right)+\rho I_{z^{-}}^{\alpha}h\left(w^{\rho}\right)\right].$$
(7)

Then, it follows that

$$2^{\alpha(s-1)}h\left(\frac{w^{\rho}+z^{\rho}}{2}\right) \leq \frac{\rho^{\alpha}\Gamma(\alpha+1)}{2\left(z^{\rho}-w^{\rho}\right)^{\alpha}}\left[{}^{\rho}I^{\alpha}_{w^{+}}h\left(z^{\rho}\right)+{}^{\rho}I^{\alpha}_{z^{-}}h\left(w^{\rho}\right)\right],$$

where $\beta(w, z)$ is the Beta function. \Box

Remark 1. When substituting $\rho = 1$ and $\alpha = 1$ in Equation (2), we obtained the results reported by Dragomir and Fitzpatrick [11].

Example 1. Consider a function $h : [w^{\rho}, z^{\rho}] \subset \mathbb{R}_+ \to \mathbb{R}^{\alpha}$, such that $h(x) = x^{s\alpha}$ belongs to GK_s^2 , $s \in (0, 1]$ with $h \in X_c^p(w^{\rho}, z^{\rho})$, where $\alpha > 0$ and $\rho > 0$. Suppose w = 0 and z = 1. For $\alpha = 2$, $s = \frac{1}{2}$ and $\rho = 1$, the first, second, and third parts of Equation (2) give 0.25, 0.33 and 0.50, respectively. Thus, the Equation (2) holds. Similarly, when $\alpha = 1$, $s = \frac{1}{2}$ and $\rho = 2$, we get 0.35, 0.50 and 0.80, respectively, which satisfies Theorem 1.

In the next theorem, the new upper bound for the right-hand side of Equation (1) for generalized *s*-convexity is proposed. Thus, the generalized beta function is defined as

$$\beta_{\rho}(w,z) = \int_0^1 \rho(1-x^{\rho})^{b-1} (x^{\rho})^{a-1} x^{\rho-1} dx$$

Note that, as $\rho \to 1$, $\beta_{\rho}(w, z) \to \beta(w, z)$.

Theorem 2. Let $\alpha > 0$ and $\rho > 0$. Let $h : [w^{\rho}, z^{\rho}] \subset \mathbb{R}_+ \to \mathbb{R}^{\alpha}$ be a differentiable function on (w^{ρ}, z^{ρ}) , and $h' \in L^1[w, z]$ with $0 \le w < z$. If $|h'|^q$ is generalized s-convex on $[w^{\rho}, z^{\rho}]$ for $q \ge 1$, we obtain

$$\frac{h(w^{\rho})+h(z^{\rho})}{2} - \frac{\alpha\rho^{\alpha}\Gamma(\alpha+1)}{2(z^{\rho}-w^{\rho})^{\alpha}} \left[{}^{\rho}I_{w^{+}}^{\alpha}h\left(z^{\rho}\right) + {}^{\rho}I_{z^{-}}^{\alpha}h\left(w^{\rho}\right) \right] \bigg| \qquad \leq \frac{z^{\rho}-w^{\rho}}{2} \left(\frac{1}{(\alpha+1)\rho}\right)^{\frac{q-1}{q}} \\ \times \left[\frac{\beta_{\rho}(\alpha s+1,\alpha+1)}{\rho} + \frac{1}{(\alpha\rho(s+1)+1)}\right]^{\frac{1}{q}} \\ \times \left(|h'(w^{\rho})|^{q} + |h'(z^{\rho})|^{q}\right)^{\frac{1}{q}}.$$

Proof. In view of Lemma 1, we have

$$\left|\frac{h(w^{\rho}) + h(z^{\rho})}{2} - \frac{\alpha \rho^{\alpha} \Gamma(\alpha+1)}{2(z^{\rho} - w^{\rho})^{\alpha}} \left[{}^{\rho} I^{\alpha}_{w^{+}} h(z^{\rho}) + {}^{\rho} I^{\alpha}_{z^{-}} h(w^{\rho})\right] \right| = \left|\frac{z^{\rho} - w^{\rho}}{2} \int_{0}^{1} \left[(1 - t^{\rho})^{\alpha} - (t^{\rho})^{\alpha}\right] t^{\rho - 1} \times h'(t^{\rho} w^{\rho} + (1 - t^{\rho}) z^{\rho}) dt \right|.$$
(8)

For the first case, when q = 1, and |h'| is generalized *s*-convex on $[w^{\rho}, z^{\rho}]$, we have

$$h'(t^{\rho}w^{\rho} + (1 - t^{\rho})z^{\rho}) \le (t^{\rho})^{\alpha s} h'(w^{\rho}) + (1 - t^{\rho})^{\alpha s} h'(z^{\rho}).$$

Therefore,

$$\left| \int_{0}^{1} \left[(1-t^{\rho})^{\alpha} - (t^{\rho})^{\alpha} \right] t^{\rho-1} h' \left(t^{\rho} w^{\rho} + (1-t^{\rho}) z^{\rho} \right) dt \right| \leq \int_{0}^{1} \left[(1-t^{\rho})^{\alpha} + (t^{\rho})^{\alpha} \right] t^{\rho-1} \left[(t^{\rho})^{\alpha s} \left| h' (w^{\rho} \right| \right) + (1-t^{\rho})^{\alpha s} \left| h' (z^{\rho} \right| \right] dt = \left| h' (w^{\rho}) \right| \int_{0}^{1} \left[(t^{\rho-1} (t^{\rho})^{\alpha s}) ((1-t^{\rho})^{\alpha} + (t^{\rho})^{\alpha}) \right] dt + \left| h' (z^{\rho}) \right| \int_{0}^{1} \left[(t^{\rho-1} (1-t^{\rho})^{\alpha s}) ((1-t^{\rho})^{\alpha} + (t^{\rho})^{\alpha}) \right] dt = S_{1} + S_{2}.$$

$$(9)$$

Calculating S_1 and S_2 , we get

$$S_{1} = |h'(w^{\rho}|) \left[\int_{0}^{1} (1-t^{\rho})^{\alpha} t^{\rho-1}(t^{\rho})^{\alpha s} dt + \int_{0}^{1} (t^{\rho})^{\alpha(s+1)} t^{\rho-1} dt \right]$$

= $|h'(w^{\rho})| \left[\frac{\beta_{\rho}(\alpha s+1,\alpha+1)}{\rho} + \frac{1}{\rho(\alpha s+\alpha+1)} \right],$ (10)

and

$$S_{2} = |h'(z^{\rho}|) \left[\int_{0}^{1} (1-t^{\rho})^{\alpha(s+1)} t^{\rho-1} dt + \int_{0}^{1} (t^{\rho})^{\alpha} t^{\rho-1} (1-t^{\rho})^{\alpha s} dt \right]$$

= $|h'(z^{\rho})| \left[\frac{1}{\rho(\alpha s + \alpha + 1)} + \frac{\beta_{\rho}(\alpha + 1, \alpha s + 1)}{\rho} \right].$ (11)

Thus, if we use Equations (10) and (11) in (9), we obtain

$$\left| \int_{0}^{1} \left[(1-t^{\rho})^{\alpha} - (t^{\rho})^{\alpha} \right] t^{\rho-1} h' \left(t^{\rho} w^{\rho} + (1-t^{\rho}) z^{\rho} \right) dt \right| \leq |h'(w^{\rho})| \left[\frac{\beta_{\rho}(\alpha s+1,\alpha+1)}{\rho} + \frac{1}{\rho(\alpha s+\alpha+1)} \right] + |h'(z^{\rho})| \left[\frac{1}{\rho(\alpha s+\alpha+1)} + \frac{\beta_{\rho}(\alpha+1,\alpha s+1)}{\rho} \right].$$

$$(12)$$

Obtaining Equations (8) and (12) completes the proof for this case. Consider the second case, q > 1. Using Equation (8) and the power mean inequality, we obtain

$$\begin{aligned} \left| \int_{0}^{1} \left[(1-t^{\rho})^{\alpha} - (t^{\rho})^{\alpha} \right] t^{\rho-1} h' \left(t^{\rho} w^{\rho} + (1-t^{\rho}) z^{\rho} \right) dt \right| &\leq \left(\int_{0}^{1} |(1-t^{\rho})^{\alpha} - (t^{\rho})^{\alpha}| t^{\rho-1} dt \right)^{1-\frac{1}{q}} \\ &\times \left(\int_{0}^{1} |(1-t^{\rho})^{\alpha} - (t^{\rho})^{\alpha}| t^{\rho-1} |h'(t^{\rho} w^{\rho} + (1-t^{\rho}) z^{\rho})|^{q} dt \right)^{\frac{1}{q}} \\ &\leq \int_{0}^{1} \left(\left[(1-t^{\rho})^{\alpha} + (t^{\rho})^{\alpha} \right] t^{\rho-1} dt \right)^{1-\frac{1}{q}} \\ &\times \left(\int_{0}^{1} [(1-t^{\rho})^{\alpha} + (t^{\rho})^{\alpha}] t^{\rho-1} [(t^{\rho})^{\alpha s}| h'(w^{\rho})|^{q} \\ &+ (1-t^{\rho})^{\alpha s}| h'(z^{\rho})|^{q} \right)^{\frac{1}{q}} \end{aligned}$$

$$(13)$$

$$&= \left(\frac{1}{\rho(\alpha s+1,\alpha s+1)} \right) \frac{q^{-\frac{1}{q}}}{p} \\ &\times \left(\left(\frac{\beta_{\rho}(\alpha s+1,\alpha s+1)}{\rho} + \frac{\beta_{\rho}(\alpha s+1,\alpha s+1)}{\rho} \right) |h'(w^{\rho})|^{q} \\ &+ \frac{1}{\rho(\alpha s+\alpha s+1)} + \frac{\beta_{\rho}(\alpha s+1,\alpha s+1)}{\rho} |h'(z^{\rho})|^{q} \right)^{\frac{1}{q}}.$$

The Equations (8) and (13) complete the proof. \Box

Corollary 1. *Using the similar assumptions given in Theorem 2.*

1. If $\rho = 1$, we get

$$\begin{split} \left| \frac{h(w) + h(z)}{2} - \frac{\alpha \Gamma(\alpha + 1)}{2(z - w)^{\alpha}} [I_{w^+}^{\alpha} h(z) + I_{z^-}^{\alpha} h(w)] \right| &\leq \frac{z - w}{2} \left(\frac{1}{\alpha + 1}\right)^{\frac{q - 1}{q}} \\ &\times \left[\beta(\alpha s + 1, \alpha + 1) + \frac{1}{1 + \alpha(s + 1)} \right]^{\frac{1}{q}} \\ &\times (|h'(w)| + |h'(z)|). \end{split}$$

2. If $\rho = 1$ and s = 1, then

$$\begin{split} \left| \frac{h(w) + h(z)}{2} - \frac{\alpha \Gamma(\alpha + 1)}{2(z - w)^{\alpha}} [I_{w^+}^{\alpha} h(z) + I_{z^-}^{\alpha} h(w)] \right| &\leq \frac{z - w}{2} \left(\frac{1}{1 + \alpha}\right)^{\frac{q - 1}{q}} \\ & \times \left(\beta(\alpha + 1, \alpha + 1) + \frac{1}{1 + 2\alpha}\right)^{\frac{1}{q}} \\ & \times (|h'(w)|^q + |h'(z)|^q). \end{split}$$

3. If $\rho = 1$, s = 1 and $\alpha = 1$, we obtain

$$\left|\frac{h(w)+h(z)}{2} - \frac{1}{z-w}\int_w^z h(x)dx\right| \le \frac{z-w}{2}\left(\frac{1}{2}\right)^{\frac{q-1}{q}} \left(\frac{h'(w)|^q+h'(z)|^q}{2}\right)^{\frac{1}{q}}.$$

Theorem 3. With the similar assumptions stated in Theorem 2, we get the following inequality:

$$\left| \frac{h(w^{\rho}) + h(z^{\rho})}{2} - \frac{\alpha \rho^{\alpha} \Gamma(\alpha + 1)}{2(z^{\rho} - w^{\rho})^{\alpha}} \left[{}^{\rho} I_{w^{+}}^{\alpha} h(z^{\rho}) + {}^{\rho} I_{z^{-}}^{\alpha} h(w^{\rho}) \right] \right| \leq \left(\frac{1}{\rho} \right)^{1 - \frac{1}{q}} \frac{z^{\rho} - w^{\rho}}{2} \\
\times \left[\frac{\beta_{\rho}(\alpha s + 1, \alpha + 1)}{\rho} + \frac{1}{(\alpha (s + 1) + 1)\rho} \right]^{\frac{1}{q}} \qquad (14)$$

$$\times \left(|h'(w^{\rho})|^{q} + |h'(z^{\rho})|^{q} \right)^{\frac{1}{q}}.$$

Proof. Using the fact $|h'|^q$, a generalized *s*-convex on $[w^{\rho}, z^{\rho}]$ with $q \ge 1$, we get

$$h'(t^{\rho}w^{\rho} + (1 - t^{\rho})z^{\rho}) \le (t^{\rho})^{\alpha s} h'(w^{\rho}) + (1 - t^{\rho})^{\alpha s} h'(z^{\rho}).$$

Applying Equation (8) together with the power mean inequality, we get

$$\begin{split} \left| \int_{0}^{1} \left[(1-t^{\rho})^{\alpha} - (t^{\rho})^{\alpha} \right] t^{\rho-1} h' \left(t^{\rho} w^{\rho} + (1-t^{\rho}) z^{\rho} \right) dt \right| &\leq \left(\int_{0}^{1} t^{\rho-1} dt \right)^{1-\frac{1}{q}} \\ &\times \left(\int_{0}^{1} \left| (1-t^{\rho})^{\alpha} - (t^{\rho})^{\alpha} \right| t^{\rho-1} \left| h'(t^{\rho} w^{\rho} + (1-t^{\rho}) z^{\rho}) \right|^{q} dt \right)^{\frac{1}{q}} \\ &\leq \left(\frac{1}{\rho} \right)^{1-\frac{1}{q}} \left(\int_{0}^{1} \left[(1-t^{\rho})^{\alpha} + (t^{\rho})^{\alpha} \right] t^{\rho-1} \left[(t^{\rho})^{\alpha s} \left| h'(w^{\rho}) \right|^{q} \\ &+ (1-t^{\rho})^{\alpha s} \left| h'(z^{\rho}) \right|^{q} dt \right)^{\frac{1}{q}} \\ &\leq \left(\frac{1}{\rho} \right)^{1-\frac{1}{q}} \left(\left| h'(w^{\rho}) \right|^{q} \int_{0}^{1} \left[(1-t^{\rho})^{\alpha} (t^{\rho})^{\alpha s} t^{\rho-1} + (t^{\rho})^{\alpha} (t^{\rho})^{\alpha s} t^{\rho-1} \right] dt \\ &+ \left| h'(z^{\rho}) \right|^{q} \int_{0}^{1} \left[(1-t^{\rho})^{\alpha} t^{\rho-1} (1-t^{\rho})^{\alpha s} + (t^{\rho})^{\alpha} (1-t^{\rho})^{\alpha s} t^{\rho-1} \right] dt \right)^{\frac{1}{q}} \\ &= \left(\frac{1}{\rho} \right)^{1-\frac{1}{q}} \\ &\times \left(\left| h'(w^{\rho}) \right|^{q} \left[\frac{\beta \rho (\alpha s+1,\alpha s+1)}{\rho} + \frac{1}{\rho (\alpha s+\alpha s+1)} \right] \right)^{\frac{1}{q}}. \end{split}$$

Remark 2. Choosing $\rho = 1$ in Theorem 3, we get the following

$$\begin{aligned} \left| \frac{h(w) + h(z)}{2} - \frac{\alpha \Gamma(\alpha + 1)}{2(z - w)^{\alpha}} [I_{w^+}^{\alpha} h(z) + I_{z^-}^{\alpha} h(w)] \right| &\leq \frac{z - w}{2} \\ & \times \left[\beta(\alpha s + 1, \alpha + 1) + \frac{1}{\alpha(s + 1) + 1} \right]^{\frac{1}{q}} \\ & \times (|h'(w)| + |h'(z)|). \end{aligned}$$

Remark 3. When choosing $\rho = 1$ and $s = \frac{1}{2}$ in Theorem 3, we get

$$\left|\frac{h(w) + h(z)}{2} - \frac{\alpha \Gamma(\alpha + 1)}{2(z - w)^{\alpha}} [I_{w^+}^{\alpha} h(z) + I_{z^-}^{\alpha} h(w)]\right| \leq \frac{z - w}{2} \left(\beta \left(\frac{\alpha}{2} + 1, \alpha + 1\right) + \frac{1}{\frac{3}{2}\alpha + 1}\right)^{\frac{1}{q}} \times (|h'(w)|^q + |h'(z)|^q).$$

Corollary 2. *Choosing* $\rho = 1$, s = 1 *and* $\alpha = 1$ *in* Theorem 3, we obtain

$$\left|\frac{h(w)+h(z)}{2} - \frac{1}{z-w}\int_w^z h(x)dx\right| \le \frac{z-w}{2}\left(\frac{h'(w)|^q+h'(z)|^q}{2}\right)^{\frac{1}{q}}.$$

The other type is given by the next theorem.

Theorem 4. Let $\alpha > 0$ and $\rho > 0$. Let $h : [w^{\rho}, z^{\rho}] \subset \mathbb{R}_+ \to \mathbb{R}^{\alpha}$ be a differentiable function on (w^{ρ}, z^{ρ}) , where $h' \in L^1[w, z]$ with $0 \le w < z$. For q > 1, if $|h'|^q$ is generalized s-convex on $[w^{\rho}, z^{\rho}]$, we get

$$\begin{split} \left| \frac{h\left(w^{\rho}\right) + h\left(z^{\rho}\right)}{2} - \frac{\alpha \rho^{\alpha} \Gamma(\alpha + 1)}{2\left(z^{\rho} - w^{\rho}\right)^{\alpha}} \left[^{\rho} I_{w^{+}}^{\alpha} h\left(z^{\rho}\right) + {}^{\rho} I_{z^{-}}^{\alpha} h\left(w^{\rho}\right) \right] \right| &\leq \frac{z^{\rho} - w^{\rho}}{2} \left(\frac{1}{p(\rho - 1) + 1}\right)^{\frac{1}{p}} \\ &\times \left[\frac{\beta_{\rho}(\alpha s + 1, \alpha + 1)}{\rho} + \frac{1}{\rho(\alpha s + \alpha + 1)} \right]^{\frac{1}{q}} \\ &\times \left(|h'(w^{\rho})|^{q} + |h'(z^{\rho})|^{q} \right)^{\frac{1}{q}}, \end{split}$$

with $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Using the Hölder's inequality, we obtain the following:

$$\left| \int_0^1 \left[(1-t^{\rho})^{\alpha} - (t^{\rho})^{\alpha} \right] t^{\rho-1} h' \left(t^{\rho} w^{\rho} + (1-t^{\rho}) z^{\rho} \right) dt \right| \leq \left(\int_0^1 (t^{\rho-1})^p dt \right)^{\frac{1}{p}} \times \left(\int_0^1 \left[(1-t^{\rho})^{\alpha} + (t^{\rho})^{\alpha} \right] t^{\rho-1} |h'(t^{\rho} w^{\rho} + (1-t^{\rho}) z^{\rho})|^q dt \right)^{\frac{1}{q}}.$$

The fact |h'| is generalized *s*-convex, and it can be used to obtain the following:

$$\begin{split} \left| \int_{0}^{1} \left[(1-t^{\rho})^{\alpha} - (t^{\rho})^{\alpha} \right] t^{\rho-1} h' \left(t^{\rho} w^{\rho} + (1-t^{\rho}) z^{\rho} \right) dt \right| &\leq \left(\frac{1}{p(\rho-1)+1} \right)^{\frac{1}{p}} \\ &\times \left(\int_{0}^{1} \left[(1-t^{\rho})^{\alpha} + (t^{\rho})^{\alpha} \right] t^{\rho-1} \left[(t^{\rho})^{\alpha s} |h'(w^{\rho})|^{q} \right] \\ &+ (1-t^{\rho})^{\alpha s} |h'(z^{\rho})|^{q} dt \right)^{\frac{1}{q}} \\ &\leq \left(\frac{1}{p(\rho-1)+1} \right)^{\frac{1}{p}} \\ &\times \left(|h'(w^{\rho})|^{q} \int_{0}^{1} [t^{\rho-1}(t^{\rho})^{\alpha s} (1-t^{\rho})^{\alpha} + t^{\rho-1}(t^{\rho})^{\alpha} (t^{\rho})^{\alpha s} \right] dt \\ &+ |h'(z^{\rho})|^{q} \int_{0}^{1} [t^{\rho-1}(1-t^{\rho})^{\alpha} (1-t^{\rho})^{\alpha s} \\ &+ t^{\rho-1}(t^{\rho})^{\alpha} (1-t^{\rho})^{\alpha s} \\ &+ t^{\rho-1}(t^{\rho})^{\alpha} (1-t^{\rho})^{\alpha s} \right] dt \\ &= \left(\frac{1}{p(\rho-1)+1} \right)^{\frac{1}{q}} \\ &= \left(\frac{1}{p(\rho-1)+1} \right)^{\frac{1}{q}} \\ &\times \left(|h'(w^{\rho})|^{q} \left[\frac{\beta_{\rho}(\alpha s+1,\alpha +1)}{\rho} + \frac{1}{(\alpha (s+1)+1)\rho} \right] \\ &+ |h'(z^{\rho})|^{q} \left[\frac{1}{\rho (\alpha (s+1)+1)} + \frac{\beta_{\rho}(\alpha +1,\alpha s+1)}{\rho} \right] \right)^{\frac{1}{q}}. \end{split}$$

Corollary 3. From Theorems 2–4, for q > 1, we obtain the following inequality:

$$\left|\frac{h(w^{\rho}) + h(z^{\rho})}{2} - \frac{\alpha \rho^{\alpha} \Gamma(\alpha + 1)}{2(z^{\rho} - w^{\rho})^{\alpha}} \left[{}^{\rho} I_{w^{+}}^{\alpha} h(z^{\rho}) + {}^{\rho} I_{z^{-}}^{\alpha} h(w^{\rho})\right]\right| \leq \min(M_{1}, M_{2}, M_{3}) \frac{(z^{\rho} - w^{\rho})}{2},$$

where

$$M_{1} = \left(\frac{1}{\rho(\alpha+1)}\right)^{\frac{q-1}{q}} \left[\frac{\beta_{\rho}(\alpha s+1,\alpha+1)}{\rho} + \frac{1}{((s+1)\alpha+1)\rho}\right]^{\frac{1}{q}} (|h'(w^{\rho})|^{q} + |h'(z^{\rho})|^{q})^{\frac{1}{q}},$$

$$M_{2} = \left(\frac{1}{\rho}\right)^{\frac{q-1}{q}} \left[\frac{\beta_{\rho}(\alpha s+1,\alpha+1)}{\rho} + \frac{1}{\rho(\alpha(s+1)+1)}\right]^{\frac{1}{q}} (|h'(w^{\rho})|^{q} + |h'(z^{\rho})|^{q})^{\frac{1}{q}},$$

and

$$M_{3} = \left(\frac{1}{1+(\rho-1)p}\right)^{\frac{1}{p}} \left[\frac{\beta_{\rho}(\alpha s+1,\alpha+1)}{\rho} + \frac{1}{(\alpha(s+1)+1)\rho}\right]^{\frac{1}{q}} (|h'(w^{\rho})|^{q} + |h'(z^{\rho})|^{q})^{\frac{1}{q}}$$

3. Applications to Special Means

The applications to special means for positive real numbers w and z can be studied through the results obtained.

1. The arithmetic mean:

$$A = A(w, z) = \frac{w+z}{2}$$

- 2. The logarithmic mean: $L(w,z) = \frac{z-w}{\log z - \log w}.$
- $L(w, z) = \frac{z w}{\log z \log w}.$ 3. The generalized logarithmic mean: $L_i(w, z) = \left[\frac{z^{i+1} - w^{i+1}}{(z - w)(i+1)}\right]^{\frac{1}{i}}; i \in \mathbb{Z} \setminus \{-1, 0\}.$

Applying the results in Section 2, together with the applications of means, gives the following propositions.

Proposition 1. Let $i \in \mathbb{Z}$, $|i| \ge 2$ and $w, z \in \mathbb{R}$ where 0 < w < z. For $q \ge 1$, we obtain the following:

$$\left|A(w^{i},z^{i})-L_{i}^{i}(w,z)\right| \leq \frac{(z-w)|i|}{2^{\frac{q-1}{q}+1}}A^{\frac{1}{q}}(|w|^{q(i-1)},|z|^{q(i-1)}).$$

Proof. This follows from Corollary 1 (iii) when applied on $h(w) = w^i$. \Box

Proposition 2. Let $i \in \mathbb{Z}$, $|i| \ge 2$ and $w, z \in \mathbb{R}$, where 0 < x < y. For $q \ge 1$, we obtain the following:

$$\left|A(w^{i},z^{i})-L_{i}^{i}(w,z)\right| \leq \frac{(z-w)|i|}{2}A^{\frac{1}{q}}(|w|^{q(i-1)},|z|^{q(i-1)}).$$

Proof. This follows from Corollary 2 when applied on $h(w) = w^i$. \Box

Proposition 3. Let $w, z \in \mathbb{R}$, where 0 < w < z. For $q \ge 1$, we obtain

$$\left|A(w^{-1}, z^{-1}) - L(w, z)\right| \le \frac{(z - w)}{2^{\frac{q - 1}{q}} + 1} A^{\frac{1}{q}}(|w|^{-2q}, |z|^{-2q}).$$

Proof. This follows from Corollary 1 (iii) when applied on $h(w) = \frac{1}{w}$.

Proposition 4. Let $w, z \in \mathbb{R}$, where 0 < w < z. For $q \ge 1$, we obtain

$$\left|A(w^{-1}, z^{-1}) - L(w, z)\right| \le \frac{(z-w)}{2} A^{\frac{1}{q}}(|w|^{-2q}, |z|^{-2q}).$$

Proof. This follows from Corollary 2 when applied for $h(w) = \frac{1}{w}$. \Box

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