mathematics

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Venture Capital Contracting with Ambiguity Sharing and Effort Complementarity Effect

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Received: 1 December 2019; Accepted: 16 January 2020; Published: 19 January 2020

Abstract: In this paper, we established a continuous-time agency model in which an ambiguity-averse venture capitalist (VC) employs an ambiguity-neutral entrepreneur (EN) to manage an innovative project. We analyzed the connection between ambiguity sharing and incentives under double moral hazard. Applying a stochastic dynamic programming approach, we solved the VC's maximization problem and obtained the Hamilton–Jacobi–Bellman (HJB) equation under a special form of the value function. We showed that the optimal pay-performance sensitivity was a fixed point of a nonlinear equation. The model ambiguity on the probability measure induced a tradeoff between ambiguity sharing and the incentive compensation that improved the EN's pay-performance sensitivity level. Besides, we simulated the model and showed that when two efforts were complementary, the VC's effort did not monotonically decrease with respect to the pay-performance sensitivity, while the EN's effort did not monotonically increase in the pay-performance sensitivity level. More importantly, we found that as efforts tended to be more complementary, the optimal pay-performance sensitivity tended to approach those that maximized the efforts exerted by the EN and the VC.

Keywords: venture capitalist; ambiguity sharing; effort complementarity; pay-performance sensitivity; relative performance evaluation

1. Introduction

The venture capital market plays an important role in financing and nurturing innovative start-ups. Many highly successful companies, such as Google, Facebook, Amazon, and Alibaba, receive venture capital funding in their early stages of development. According to a report released by Crunchbase in January 2019, more than 56% of private technology companies complete financing through large-scale venture capital in 2018. There is a typical principal-agent relationship between the venture capitalist (VC) and the entrepreneur (EN). Signing an investment contract is an important sign for the two parties to reach a formal agency relationship.

Traditionally, financial contract models rely on the assumption that partners in two-party contracting problems have the same beliefs on the uncertainty output [1–4]. However, the lack of information in start-up enterprises may lead to the ambiguity of individual knowledge about future enterprise performance; that is, there are multiple possible distributions on firm value. The famous Ellsberg Paradox [5] shows that people treat ambiguity and risk from different perspectives. Nowadays, it is well-accepted that economic future outcomes can be subject to 'risk' and 'ambiguity/Knightian' uncertainties. Risk refers to the situation in which the true probability distribution of the uncertain outcome is known, whereas ambiguity refers to the case in which the true probability distribution is unknown.

Since the project of a start-up company is innovative and has few performance records, it is plausible that the VC may not have a single prior belief about the project's success probability and, therefore, face ambiguity. Recently, Kim and Wagman [6] proposed a theoretic start-up financing model in which the VC and EN have ambiguous beliefs about the success probability. Lukas et al. [7] demonstrated that the high uncertainty would result in a larger stake in the venture. Miao and Rivera [8] studied two types of robust contract problems with output uncertainty; they dealt the principal's maximization problem under the worst-case scenario and found ambiguity aversion lowers outside securities value while increases the credit yield spread. Wu et al. [9] extended the Holmstrom and Milgrom [1] model by incorporating model uncertainty to study robust long-term contracting and explore the connection between ambiguity sharing and relative performance evaluation. In addition, they provided a theoretical explanation of paying for luck through writing the compensation contracts on additional signals, such as industry average performance. Liu et al. [10] investigated a principal-agent model in which the information on future firm performance is ambiguous, and the agent is both, ambiguity averse and risk-averse. Assuming the product market outcome is ambiguous, Beauchêne [11] found that ambiguity averse companies tend to invest in more projects, while risk-averse companies invest in fewer projects.

Using a robust optimization approach, this paper studied the optimal venture capital contracting problem in which there are multiple possible probability distributions of future project revenues, and VC is ambiguity-averse. Both the risk-neutral VC and the risk-averse EN commit to the long-term relationship. The VC designs a robust contract to maximize his utility in the worst-case scenario. Our analysis contributed to the contract literature, seeking to explain the well-documented lack of relative performance evaluation in chief executive officer (CEO) compensation. Aggarwal and Samwick [12] assumed that the industry average performance might be affected by the agent's action and showed that strategic interactions among firms could explain the lack of relative performance-based incentives. Wu et al. [9] found that the ambiguity reduces the use of relative performance evaluation and then increases the exposure of the common shocks in the agent's point of view. In contrast, we found that the cash flow evolves with the joint effect of the common shock and the idiosyncratic shock. In the presence of ambiguity uncertainty, the relative performance evaluation would be less considered in order to increase the ambiguity sharing; in other words, the VC's ambiguity on model uncertainty would lead to the lack of relative performance evaluation, which would result in an increase in compensation performance beyond his control.

An appealing feature of our model is considering the EN's effort and the VC's effort simultaneously, both of which play a significant role in the success of a start-up company. An EN holds an innovative project but has a lack of commercial experience and initial capital, which could be provided by a VC. Besides offering finance, the VC could also provide market operations, networking, suggestion, and experience in business management, which would improve the success probability and the revenue of the project [13–15]. Both the EN and the VC are crucial and irreplaceable for the VC-backed new project [16]. As a follower in the principal-agent relationship, the EN has the motive to seek private savings due to the asymmetry of his effort. This generates a moral hazard problem. As a result, the incentive contract designed by the VC is particularly important. Furthermore, this paper analyzed a special case of a double-sided moral hazard problem in which the VC's effort is also unobservable to the EN.

This paper was related to the recent literature on venture capital contracting problem with double-sided moral hazard. Hori and Osano [17] examined a continuous-time agency model with double moral hazard, in which both a risk-neutral EN and a risk-neutral VC provide unobservable value-adding efforts. The cumulative cash flow generated by the risk project is affected by the efforts of VC and EN. Vergara et al. [18] studied an optimal contract design problem and analyzed how the complementarity of efforts between an EN and a VC affects the equity share that the EN is willing to allocate to the VC. Chang and Hu [19] explored the combined impact of double-sided moral hazard and the EN's fairness concerns on venture capital contracting. However, these pieces of literature assume that VC knows the true probability distribution of project cash flow; hence there is no ambiguity uncertainty.

The novelty of this paper focused on the effect of the efforts complementarity and the optimal pay-performance sensitivity that the VC would allocate to the EN. Following [14,19,20], we assumed the VC's investment in the project is endogenous, and the VC plays a leadership role in the game relationship. Our model took a similar angle with [18] on considering the complementarity effect but departed from it in three ways. First, we established the model and solved the optimization problem, where the VC holds bargaining power. Second, we extended the contract analysis to a continuous-time situation, in which cash flow and the compensation awarded to the EN are dynamic. Third, we used a project revenue function proposed by [17] to simulate the model, instead of the constant elasticity of production function in [18].

Another paper closely related to our study is that of Wu et al. [9], who introduced probability measure ambiguity to analyze the continuous-time contract and focus on the research of relative performance evaluation. They found ambiguity induces a tradeoff between ambiguity sharing and incentives. However, many important models in finance and economics do not consider synergy between partners in the project [9,21–23]. In contrast, we analyzed how complementarity of efforts affects the optimal pay-performance sensitivity level that the VC is willing to award to the EN. We simulated the model in the scenario that two efforts are complementary and found that the synergy improves the project's output and increases the pay-performance sensitivity allocated to the EN.

The rest of this paper is structured as follows. Section 2 characterizes the key elements of contracting and deduces the evolution process of the value function. In Section 3, we have solved the contracting problem and provide several applications. Finally, Section 4 concludes the paper.

2. Model Setup and Optimal Contracting

In this section, firstly, we studied the contract designing problem with the cash flow process. Second, we dealt with the belief distortion on model uncertainty by employing the discounted relative entropy. Third, we derived the Hamilton–Jacobi–Bellman (HJB) equation that characterizes the optimal contract.

2.1. General Model

We studied a continuous-time contracting problem. A risk-neutral VC (she) hires a risk-averse EN (he) to operate an innovative project. Both of their actions (efforts) are crucial to the projected revenue. The project produces cash flow Y_t per unit of time, where Y_t follows the process

$$dY_t = (a_t + \zeta e_t + \alpha a_t e_t)dt + \rho\sigma dB_t + \sqrt{1-\rho^2}\sigma dZ_t. \tag{1}$$

Here, we interpreted a_t and e_t as the EN's and the VC's effort choice at the time t, respectively. As an agent, the EN has the motivation to hide his action (effort), we assumed the EN's effort is unobservable and unverifiable to the VC. ζ is a constant nonnegative scale adjusted parameter, which describes the efficiency of the VC's effort in the project's output. α measures efforts complementarity, with $\alpha \in [0, \hat{\alpha})$ and $\hat{\alpha} < 2$. De Bettignies [20] pointed out that when $\alpha = 0$, both efforts are perfect substituted, but two efforts become more complementary as α increases. $\sigma > 0$ is the volatility of cash flow. The last parameter ρ measures the elasticity (impact) of the common shock. Similar to most project management literature, we adopted the quadratic cost function for both a_t and e_t in the forms of $g(e_t) = \delta_1 e_t^2/2$ and $g(a_t) = \delta_2 a_t^2/2$, δ_1 and δ_2 are positive constants, which measure the effects of being more or less efficient in the delivery of efforts. B_t denotes common shock, while Z_t denotes idiosyncratic shock. In addition, we assumed that (B, Z) is a two-dimensional independent standard Brownian motion under probability measure $\mathbb{P}$.

Suppose, following [1,9,24], that in addition to project performance, compensation contracts can be written on another variable that is correlated with the noise component of project performance.

This variable is assumed to be uninformative about the EN's action. An example would be the performance of other start-ups in the given industry or in the market as a whole under the assumption that the actions of an EN do not affect the performance of other firms in his industry. Similar to [9], we introduced the industry average performance into a contract designing problem, which is not affected by efforts exerted by the EN and the VC. For a given start-up, the industry average performance refers to the average performance of other start-ups engaged in a similar business within a particular industry. Denote M_t as the industry's average performance, which follows a martingale process in the following form

$$dM_t = \sigma dB_t. \tag{2}$$

Both the EN and the VC discount future cash flow at the market interest rate r. Following [1], we assume that the EN has a constant absolute risk aversion (CARA) utility function

$$U(c_t, a_t) = -\frac{1}{\gamma}\exp[-\gamma(c_t - \frac{\delta_2}{2}a_t^2)], \tag{3}$$

where $\gamma > 0$ is the EN's absolute risk aversion coefficient, and c_t is the EN's wage at the time t.

The VC provides a long-term compensation contract $\Pi(c_t, a_t)$ to the EN based on the past rates of project return. $\Pi(c_t, a_t)$ specifies the EN's wage policy $\{c_t\}$ and the recommended effort process $\{a_t\}$. In order to avoid confusion, we used $\{\hat{c}_t, \hat{a}_t\}$ that indicates EN's actual contract policies.

For simplicity, we assumed the EN's initial wealth as $S_0 = 0$. Then, the contract problem faced by the EN is

$$\max_{\{\hat{c}_t,\ \hat{a}_t\}} \mathbb{E}_0^{\hat{a}_t}\left[\int_0^\infty e^{-rt}U(\hat{c}_t, \hat{a}_t)dt\right], \tag{4}$$

subject to

$$dY_t = (\hat{a}_t + \zeta e_t + \alpha\hat{a}_t e_t)dt + \rho\sigma dB_t + \sqrt{1-\rho^2}\sigma dZ_t, \tag{5}$$

$$dS_t = (rS_t + c_t - \hat{c}_t)dt, \qquad S_0 = 0. \tag{6}$$

The first constraint represents the actual cash flows faced by the EN when he exerts actual effort $\{\hat{a}_t\}$. The second constraint states that, the change of the EN's saving dS_t is the interest accrual $rS_t dt$ plus the wage deposit $c_t dt$ and minus the consumption withdrawal $\hat{c}_t dt$. To save, the EN can set his consumption $\hat{c}_t$ strictly below the wage c_t.

The problem faced by the VC is expressed as

$$\max_{\{c_t, a_t, e_t\}} \mathbb{E}_0^{a_t}\left[\int_0^\infty e^{-rt}(dY_t - c_t dt - \frac{\delta_1}{2}e_t^2 dt)\right], \tag{7}$$

subject to

$$dY_t = (a_t + \zeta e_t + \alpha a_t e_t)dt + \rho\sigma dB_t + \sqrt{1-\rho^2}\sigma dZ_t, \tag{8}$$

$$E_0^{a_t}\left[\int_0^\infty e^{-rt}U(c_t, a_t)dt\right] \geq V_0. \tag{9}$$

The value V_0 could be interpreted as the reservation utility that the EN would achieve in the best alternative offer he has. Following Holmström and Milgrom [1], under the CARA assumption framework, the constraint (9) is bind.

2.2. Discounted Relative Entropy

Due to the model ambiguity, we considered belief distortion on the possibility measure. The VC does not trust the probability measure $\mathbb{P}$ and considers alternative models under probability measure $\widetilde{\mathbb{P}}$ to protect him from probability measure ambiguity. Defining two real-valued density generators $\{h_t\}$

and $\{g_t\}$, satisfying Novikov-condition $\exp\{\frac{1}{2}\int_0^t (h_t^2 + g_t^2)dt\} < \infty$ for all $t > 0$, we had Radon-Nikodym derivative with respect to $\mathbb{P}$.

$$\xi_t = \exp\{\int_0^t h_s dB_s - \frac{1}{2}\int_0^t h_s^2 ds + \int_0^t g_s dB_s - \frac{1}{2}\int_0^t g_s^2 ds\}, \tag{10}$$

where $\xi_0 = 1$.

According to the Cameron-Martin-Girsanov Theorem, we knew $\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = \xi_t$ and, two processes $\left\{B_t^h\right\}$ and $\left\{Z_t^g\right\}$ are defined by

$$dB_t^h = dB_t - h_t dt, \tag{11}$$

$$dZ_t^g = dZ_t - g_t dt, \tag{12}$$

where B_t^h and Z_t^g are standard Brownian motions under the probability measure $\widetilde{\mathbb{P}}$.

Under probability measure $\widetilde{\mathbb{P}}$, the cash flow and the industry average performance could be rewritten as

$$dY_t = (a_t + \zeta e_t + \alpha a_t e_t + \rho\sigma h_t + \sqrt{1-\rho^2}\sigma g_t)dt + \rho\sigma dB_t^h + \sqrt{1-\rho^2}\sigma dZ_t^g, \tag{13}$$

$$dM_t = \sigma h_t dt + \sigma dB_t^h. \tag{14}$$

Similar to [25,26], we employed the discounted relative entropy to measure the discrepancy between $\mathbb{P}$ and $\widetilde{\mathbb{P}}$,

$$r\mathbb{E}_0^{\mathbb{P}}\left[\int_0^\infty e^{-rt}\xi_t \ln \xi_t dt\right] = \frac{1}{2}\mathbb{E}_0^{\widetilde{\mathbb{P}}}\left[\int_0^\infty e^{-rt}(h_t^2 + g_t^2)_t dt\right]. \tag{15}$$

To incorporate concerns on the robustness of probability measure ambiguity, in our paper, the VC's optimization problem could be written as

$$\max_{\{a_t,e_t\}}\inf_{\{h_t,g_t\}} \mathbb{E}_0^{\widetilde{\mathbb{P}}}\left[\int_0^\infty e^{-rt}(dY_t - c_t dt - \frac{\delta_1}{2}e_t^2 dt)\right] + \frac{1}{2\theta}\mathbb{E}_0^{\widetilde{\mathbb{P}}}\left[\int_0^\infty e^{-rt}(h_t^2 + g_t^2)dt)\right]. \tag{16}$$

The parameter $\theta > 0$ could be interpreted as the ambiguity aversion coefficient. A large value of θ implies a high degree of probability measure ambiguity or a large degree of concern for robustness. When θ converges to zero, the VC's optimization problem is reduced to be the case without probability measure ambiguity.

2.3. Contracting Problem

Definition 1. *The contract $\Pi(c_t, a_t)$ offered to the EN is incentive-compatible if the optimal solution of the EN's problem is (c_t, a_t).*

That is, the EN would choose the recommended contract $\Pi(c_t, a_t)$, as this choice would bring him the highest revenue, that is, $(\hat{c}_t, \hat{a}_t) \in \Pi\ (c_t, a_t)$. Thus, $\int_0^t (c_s - \hat{c}_s)ds = 0$. In other words, when the contract is incentive-compatible, there would be no savings.

Given contract $\Pi(c_t, a_t)$, the EN's continuation value at time t is defined as

$$V_t = \mathbb{E}_t^{\widetilde{\mathbb{P}}}\left[\int_t^\infty e^{-r(s-t)}U(c_t, a_t)dt\right]. \tag{17}$$

Under Equation (17), Lemma 1 summarizes the relationship between the EN's expected time utility and his continuation value. This result is documented in [21].

Lemma 1. *(He [21]). When the EN has a CARA utility function, then.*

$$U(c_t, a_t) = rV_t. \tag{18}$$

Lemma 1 indicates that the EN's utility function can be regarded as the time discount form with respect to continuation value. This property is peculiar to the exponential utility and would be convenient to deal with the process V_t.

By Martingale Representation Theorem [27], there would be two measurable processes ϕ_t and $\widetilde{\phi}_t$ which are relevant to continuation value V_t such that

$$\begin{aligned} dV_t &= rVdt - U(c_t, a_t)dt + \phi_t(V_t)(\rho\sigma dB_t + \sqrt{1-\rho^2}\sigma dZ_t) + \widetilde{\phi}_t(V_t)\sigma dB_t \\ &= \phi_t(V_t)(\rho\sigma dB_t + \sqrt{1-\rho^2}\sigma dZ_t) + \widetilde{\phi}_t(V_t)\sigma dB_t \end{aligned} \tag{19}$$

where the second equation is obtained with the help of Lemma 1. In order to derive the specific optimal contract rather than providing the HJB equation of the VC's value function, we assumed simple forms of ϕ_t and $\widetilde{\phi}_t$ as $\phi_t(V_t) = -\beta_t V_t$ and $\widetilde{\phi}_t(V_t) = -\widetilde{\beta}_t V_t$, respectively.

We interpreted the two real variables β_t and $\widetilde{\beta}_t$ as pay-performance sensitivity level and relative performance sensitivity level, which are paid to the EN. While, relative performance evaluation occurs if the optimal contract lists a negative value $\widetilde{\beta}_t$ to the industry average performance.

According to Equations (8) and (19), it could be rewritten as

$$dV_t = (-\beta_t V_t)[dY_t - (a_t + e_t + \alpha a_t e_t)dt] + (-\widetilde{\beta}_t V_t)\sigma dB_t. \tag{20}$$

Now, we turned to deal with the EN's incentive compatibility. It is not difficult to find that the EN's effort $\hat{a}_t$ not only affects his instantaneous $U(c_t, \hat{a}_t)$ but also his continuation value, thus, the EN's optimal effort a_t satisfies

$$a_t = \underset{\hat{a}_t}{\operatorname{argmax}}\{U(c_t, a_t) + (-\beta_t V_t)(a_t + \alpha a_t e_t)\}. \tag{21}$$

Taking the EN's first-order condition of his effort decision, we have

$$U(c_t, a_t)(-\gamma)(-\delta_2 a_t) + (-\beta_t V_t)(1 + \alpha e_t) = 0. \tag{22}$$

Together with Lemma 1, the above equation implies that

$$a_t = \beta_t(1 + \alpha e_t)/(\gamma r \delta_2). \tag{23}$$

Proposition 1. *The contract $\Pi(c_t, a_t)$ is incentive-compatible with respect to the EN if and only if his effort satisfies.*

$$a_t = \beta_t(1 + \alpha e_t)/(\gamma r \delta_2).$$

It is worth noting that, due to the complementarity between the EN and the VC, the EN's effort depends not only on the pay-performance sensitivity level β_t but also VC's effort. In this section, we assume that the EN can observe and verify the VC's effort e_t. Thus, this is a single moral hazard problem.

By Proposition 1, the VC's objective function (16) can be converted into

$$\max_{\{e_t, \beta_t, \widetilde{\beta}_t\}} \inf_{\{h_t, g_t\}} \mathbb{E}_0^{\widetilde{\mathbb{P}}}\Big[\int_0^\infty e^{-rt}\Big\{\zeta e_t + \beta_t(1+\alpha e_t)^2/(\gamma r \delta_2) + \rho\sigma h_t + \sqrt{1-\rho^2}\sigma g_t + \tfrac{1}{2\theta}(h_t^2 + g_t^2) - \tfrac{\delta_1}{2}e_t^2 - c_t\Big\}d\,t\Big]. \tag{24}$$

Based on Equation (3) and Lemma 1, we have $c_t = \frac{\delta_2}{2}a_t^2 - \frac{1}{\gamma}\ln(-\gamma r V_t)$.

Formally, we defined the EN's continuation value at time t as

$$J(V_t) = \mathbb{E}_t^{\widetilde{\mathbb{P}}}\Big[\int_t^{\infty} e^{-r(s-t)}\Big\{\zeta e_s + \beta_s(1+\alpha e_s)^2/(\gamma r\delta_2) + \rho\sigma h_s + \sqrt{1-\rho^2}\sigma g_s + \tfrac{1}{2\theta}(h_s^2+g_s^2) - \tfrac{\delta_1}{2}e_s^2 - c_s\Big\}ds\Big], \tag{25}$$

here V_t evolves as

$$\begin{aligned} dV_t &= (-\beta_t V_t)[dY_t - (a_t + \zeta e_t + \alpha a_t e_t)dt] + (-\widetilde{\beta}_t V_t)\sigma dB_t \\ &= (-V_t)\Big\{\Big[\beta_t(\rho\sigma h_t + \sqrt{1-\rho^2}\sigma g_t) + \widetilde{\beta}_t\sigma h_t\Big]dt + (\beta_t\rho + \widetilde{\beta}_t)\sigma dB_t^h + \beta_t\sqrt{1-\rho^2}\sigma dZ_t^g\Big\}. \end{aligned} \tag{26}$$

Using Equations (25) and (26), under the measure $\widetilde{\mathbb{P}}$, the HJB equation for the VC's problem (24) is

$$\begin{aligned} rJ(V) = \max_{\{e_t,\beta_t,\widetilde{\beta}_t\}}\inf_{\{h_t,g_t\}} &\zeta e_t + \beta_t(1+\alpha e_t)^2/(\gamma r\delta_2) + \rho\sigma h_t + \sqrt{1-\rho^2}\sigma g_t + \tfrac{1}{2\theta}(h_t^2+g_t^2) - \tfrac{\delta_1}{2}e_t^2 - c_t \\ &+ J'(V)(-V_t)[\beta_t(\rho\sigma h_t + \sqrt{1-\rho^2}\sigma g_t) + \widetilde{\beta}_t\sigma h_t] + \tfrac{1}{2}J''(V)(-V_t)^2\varphi(\beta_t,\widetilde{\beta}_t)\sigma, \end{aligned} \tag{27}$$

with

$$c_t = \frac{\delta_2}{2}a_t^2 - \frac{1}{\gamma}\ln(-\gamma r V_t),\ \varphi(\beta_t,\widetilde{\beta}_t) = (\beta_t\rho + \widetilde{\beta}_t)^2 + (1-\rho^2)(\beta_t)^2.$$

3. Model Solution and Applications

3.1. Model Solution

To make further research on the optimal contract terms in addition to the HJB equation that characters the optimal contracting problem, we studied a special form of the value function $J(V)$.

Thanks to the CARA preference, it plays a key role in solving the optimal contract. Following [9], we conjecture that

$$J(V_t) = A - \frac{-1}{\gamma r}\ln(-\gamma r V_t),$$

where A is the constant part of $J(V_t)$. Moreover, it can be easily verified as

$$J'(V_t) = \frac{1}{\gamma r V_t} \text{ and } J''(V_t) = -\frac{1}{\gamma r V_t^2}.$$

Plugging them back into the HJB Equation (27) yields

$$\begin{aligned} r[A - \tfrac{-1}{\gamma r}\ln(-\gamma r V_t)] = \max_{\{e_t,\beta_t,\widetilde{\beta}_t\}}\inf_{\{h_t,g_t\}} &\zeta e_t + \beta_t(1+\alpha e_t)^2/(\gamma r\delta_2) + \rho\sigma h_t + \sqrt{1-\rho^2}\sigma g_t + \tfrac{1}{2\theta}(h_t^2+g_t^2) - \tfrac{\delta_1}{2}e_t^2 - c_t \\ &+ \tfrac{1}{\gamma r V_t}(-V_t)[\beta_t(\rho\sigma h_t + \sqrt{1-\rho^2}\sigma g_t) + \widetilde{\beta}_t\sigma h_t] - \tfrac{1}{2}\tfrac{1}{\gamma r V_t^2}(-V_t)^2\varphi(\beta_t,\widetilde{\beta}_t)\sigma^2. \end{aligned} \tag{28}$$

As $c_t = \frac{\delta_2}{2}a_t^2 - \frac{1}{\gamma}\ln(-\gamma r V_t)$, the equation above yields

$$\begin{aligned} rA = \max_{\{e_t,\beta_t,\widetilde{\beta}_t\}}\inf_{\{h_t,g_t\}} &\zeta e_t + \beta_t(1+\alpha e_t)^2/(\gamma r\delta_2) + \rho\sigma h_t + \sqrt{1-\rho^2}\sigma g_t + \tfrac{1}{2\theta}(h_t^2+g_t^2) - \tfrac{\delta_1}{2}e_t^2 - \tfrac{1}{2}(\beta_t)^2(1+\alpha e_t)^2/(\gamma r)^2\delta_2 \\ &- \tfrac{1}{\gamma r}\Big[\beta_t(\rho\sigma h_t + \sqrt{1-\rho^2}\sigma g_t) + \widetilde{\beta}_t\sigma h_t\Big] - \tfrac{1}{2}\tfrac{1}{\gamma r}\varphi(\beta_t,\widetilde{\beta}_t)\sigma^2. \end{aligned} \tag{29}$$

Let

$$\begin{aligned} H_1 &= \zeta e_t + \beta_t(1+\alpha e_t)^2/(\gamma r\delta_2) + \rho\sigma h_t + \sqrt{1-\rho^2}\sigma g_t + \tfrac{1}{2\theta}(h_t^2+g_t^2) - \tfrac{1}{2}(\beta_t)^2(1+\alpha e_t)^2/(\gamma\, r)^2\delta_2 - \tfrac{1}{2}\tfrac{1}{\gamma\, r}\varphi(\beta_t,\widetilde{\beta}_t)\sigma^2 \\ &\quad - \tfrac{\delta_1}{2}e_t^2 - \tfrac{1}{\gamma r}\Big[\beta_t(\rho\sigma h_t + \sqrt{1-\rho^2}\sigma g_t) + \widetilde{\beta}_t\sigma h_t\Big], \end{aligned}$$

then, applying the Envelope Theorem, we obtained the First-Order-Condition (FOC) of H_1 with respect to e_t as

$$\zeta + 2\alpha\beta_t(1+\alpha e_t)/(\gamma r\delta_2) - \delta_1 e_t - \alpha(\beta_t)^2(1+\alpha e_t)/(\gamma r)^2\delta_2 = 0. \tag{30}$$

Evaluating the equation above, we had

$$e_t = \frac{\zeta\delta_2(\gamma r)^2 + \alpha\beta_t(2\gamma r - \beta_t)}{\delta_1\delta_2(\gamma r)^2 - \alpha^2\beta_t(2\gamma r - \beta_t)}. \tag{31}$$

Based on (23) and (30), we could obtain

$$a_t = \frac{(\delta_1 + \alpha\zeta)\gamma r\beta_t}{(\gamma r)^2\delta_1\delta_2 - \alpha^2\beta_t(2\gamma r - \beta_t)}. \tag{32}$$

Taking the FOCs of H_1 for $(h_t, g_t, \widetilde{\beta_t}, \beta_t)$ and re-ordering, we had

$$h_t = -\theta\sigma(\rho - \frac{1}{\gamma r}\beta_t\rho - \frac{1}{\gamma r}\widetilde{\beta_t}), \tag{33}$$

$$g_t = -\theta(1 - \frac{1}{\gamma r}\beta_t)\sqrt{1-\rho^2}\sigma, \tag{34}$$

$$-\sigma^2(\beta_t\rho + \widetilde{\beta_t}) - \theta\sigma^2(\frac{1}{\gamma r}\beta_t\rho + \frac{1}{\gamma r}\widetilde{\beta_t} - \rho) = 0, \tag{35}$$

$$(1 - \frac{1}{\gamma r}\beta_t)(1+\alpha e)^2/\delta_2 - \sigma^2(\widetilde{\beta_t}\rho + \beta_t) - \theta\sigma^2(\frac{1}{\gamma r}\widetilde{\beta_t}\rho + \frac{1}{\gamma r}\beta_t - 1) = 0. \tag{36}$$

By Equations (35) and (36), we could derive the optimal pay-performance sensitivity and the optimal relative performance sensitivity stated in the following proposition, which characterizes the robust contract items.

Proposition 2. *Suppose that the effort exerted by the EN is unobservable while he can observe and verify the VC's effort. With model uncertainty, the optimal pay-performance sensitivity β_t^* given to the EN is non-linear and at a fixed point takes the form of $\beta_t^* = l(\beta_t^*)$*, where:

$$l(\beta_t^*) = \gamma r \cdot \frac{\delta_2(\delta_1+\zeta\alpha)^2(\gamma r)^4 + \theta\sigma^2(1-\rho^2)[(\gamma r)^2\delta_1\delta_2 - \alpha^2\beta_t^*(2\gamma r - \beta_t^*)]^2}{\delta_2(\delta_1+\zeta\alpha)^2(\gamma r)^4 + [\gamma r\sigma^2(1-\rho^2) + \theta\sigma^2(1-\rho^2)][(\gamma r)^2\delta_1\delta_2 - \alpha^2\beta_t^*(2\gamma r - \beta_t^*)]^2}. \tag{37}$$

The optimal relative performance sensitivity $\widetilde{\beta_t^*}$ is given by

$$\widetilde{\beta_t^*} = \gamma r\frac{\rho\theta}{\gamma r + \theta} - \rho\beta_t^*. \tag{38}$$

It follows from the above proposition that when the project's cash flow is deterministic ($\sigma = 0$) or when the project's performance is perfectly correlated with industry average performance ($\rho^2 = 1$), the EN would obtain the largest pay-performance sensitivity. Meanwhile, the probability measure ambiguity has no impact on the EN's pay-performance sensitivity.

3.2. Applications

In applications, we assumed the project's cash flow is subject to an industry-level shock.

3.2.1. Comparative Statics on Ambiguity

To better understand the impact of the probability measure ambiguity on the optimal contract, we made a robust analysis of ambiguity.

Proposition 3. *For any $\alpha \in [0,2)$, we had$d\beta_t^*/d\theta > 0$.*

Proof. Define
$$h=(\gamma r)^2\delta_1\delta_2-\alpha^2\beta_t^*(2\gamma r-\beta_t^*),$$
then
$$h_\theta=-2\alpha^2(\beta_t^*)_\theta(\gamma r-\beta_t^*).$$

As
$$\beta_t^*=\gamma r\cdot\frac{\delta_2(\delta_1+\zeta\alpha)^2(\gamma r)^4+\theta\sigma^2(1-\rho^2)h^2}{\delta_2(\delta_1+\zeta\alpha)^2(\gamma r)^4+[\gamma r\sigma^2(1-\rho^2)+\theta\sigma^2(1-\rho^2)]h^2},$$
we had
$$\gamma r\Big\{\delta_2(\delta_1+\zeta\alpha)^2(\gamma r)^4+\theta\sigma^2(1-\rho^2)h^2\Big\}=\beta_t^*\Big\{\delta_2(\delta_1+\zeta\alpha)^2(\gamma r)^4+[\gamma r\sigma^2(1-\rho^2)+\theta\sigma^2(1-\rho^2)]h^2\Big\}.$$

Taking the derivative of both sides of the equation above with respect to θ and arranging, it follows that
$$(\beta_t^*)_\theta=\frac{\theta\sigma^2(1-\rho^2)h^2(\gamma r-\beta_t^*)}{\delta_2(\delta_1+\zeta\alpha)^2(\gamma r)^4+[\gamma r\sigma^2(1-\rho^2)+\theta\sigma^2(1-\rho^2)]h^2+4h\alpha^2\theta\sigma^2(1-\rho^2)(\gamma r-\beta_t^*)^2}>0.$$

□

Graphically, Figure 1a,b illustrate the probability measure ambiguity effect for the pay-performance sensitivity and the relative performance sensitivity using the parameters in Table 1. Figure 1a shows the changes between the pay-performance sensitivity and ambiguity aversion, we could find that the pay-performance sensitivity increases in θ, coinciding with Proposition 3. As shown in Figure 1b, it depicts the relative performance sensitivity dynamics at a different level of ambiguity aversion θ. When $\theta>0$, V_t evolves related to the industry average performance. In other words, the adoption of the relative performance evaluation improves the EN's compensation.

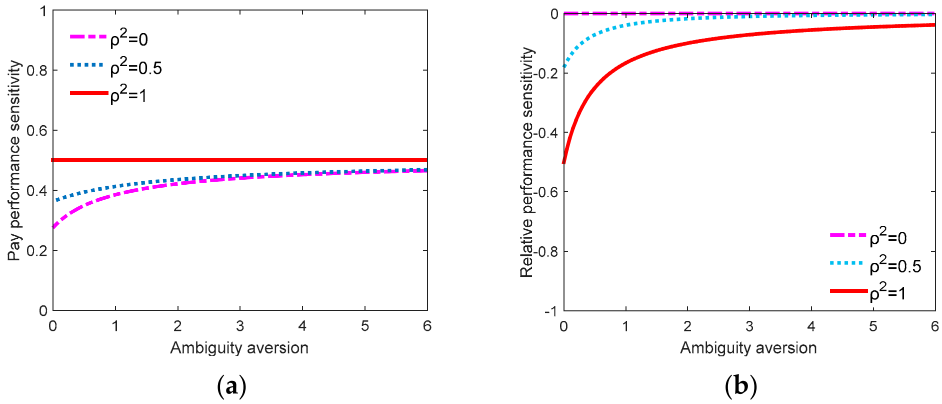

Figure 1. Changes of β_t^* in ambiguity aversion: (**a**) θ for $\rho^2=0,\ 0.5$ and 1. (Note: $\alpha=1$); (**b**) θ for $\rho^2=0,\ 0.5$ and 1. (Note: $\alpha=1$).

Table 1. Simulation parameters.

Parameter	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6
δ_1	2	2	2	2	2	2
δ_2	2	2	2	2	2	2
ζ	0.5	0.5	0.5	0.5	0.5	0.5
γ	1	1	1	1	1	1
r	0.5	0.5	0.5	0.5	0.5	0.5
α	0	0.5	1	1.5	1.8	1.9

3.2.2. Lack of Relative Performance Evaluation

According to the standard principal-agent theory, relative performance evaluation should be employed to improve the efficiency of compensation contracts [10]. However, in the empirical literature, many studies document a lack of relative performance evaluation in compensation contracts [9,12]. Our model provided an explanation for this executive compensation puzzle. As shown in Proposition 2, the adoption of the relative performance evaluation allowed the VC to weaken the effect of the common shock on the project's cash flow.

When there is no model ambiguity on probability measure, that is, $\theta = 0$, we had $\widetilde{\beta_t^*} = -\rho\beta_t^*$, then V_t follows

$$dV_t = -\beta_t^* \sqrt{1-\rho^2}\sigma V_t dZ_t.$$

Intuitively, V_t varies without the common shock B_t, that is, the common shock has no effect on V_t. In the field of investment, an investor would take a short position in order to hedge the market risk. This finding is consistent with a number of studies, where they consider idiosyncratic shock only.

However, with probability measure ambiguity, $\theta > 0$, according to Proposition 2, we had $\left|\frac{\widetilde{\beta_t^*}}{\beta_t^*}\right| < \rho$, which means that the use of the relative performance evaluation is reduced due to probability measure ambiguity. Thus, based on (38), we can rewrite V_t as

$$dV_t = -\beta_t^* \sqrt{1-\rho^2}\sigma V_t dZ_t - \frac{\gamma r \theta}{\gamma r + \theta}\rho\sigma V_t dB_t.$$

It is worth noting that the EN's continuation value V_t is related to the common shock B_t. Moreover, we can find that V_t increases in response to common shocks to performance beyond the EN's control. This result may be related to the empirical evidence of "reward for luck" by Bertrand and Mullainathan [13].

3.2.3. Effort Dynamic and Complementarity Effect

The efforts complementarity plays an important role in allocating compensation equity between the VC and the EN. The revenue function with the complementarity parameter is extensively used in the production field of microeconomics research. As shown in (31), (32), and (37), α is indeed related to the effort dynamics and the optimal contract, thus, we discussed the effect that the degree of complementarity has on the efforts and the pay-performance sensitivity.

Figure 2 depicts the effort dynamics of each partner at a different pay-performance sensitivity level using parameters in Table 1. The aim of this simulation is to observe the complementarity effect that the parameter α has on the dynamics of the two efforts exerted by the VC and the EN. In case 1, the two efforts are perfectly substituted; it can be shown that when the EN obtains the highest pay-performance sensitivity level, $\beta_t = 1$, he would exert maximum effort. On the contrary, the VC would deploy a fixed effort, which we could interpret as capital assets put in this project.

In case 2–case 6, different complementarity levels are assumed. Note that as the degree of effort complementarity increases, for instance, case 6: $\alpha = 1.9$, both the entrepreneur and the VC deploy low efforts if $\beta_t = 0$ or $\beta_t = 1$. A large degree of complementarity means that in order for the project to be successful, both partners must put their efforts at work at the same time, as only one partner making effort would be worthless.

Furthermore, it could be seen that two efforts become concave on the pay-performance sensitivity degree β_t. That is, there would be a pay-performance sensitivity β_t^{EN} assigned to the EN that maximizes his effort. The same holds for the VC. This finding is established in Proposition 4. In addition, the distance between β_t^{EN} and β_t^{VC} falls as the effort complementarity rises, meaning the levels that maximize efforts tend to be equal.

Proposition 4. *If the two efforts e_t and a_t are complementary,* *(**a**) There is a pay-performance sensitivity level β_t^{VC} allocated to the VC that maximizes his effort in the form of $\beta_t^{\mathrm{VC}} = \min\{\gamma r, 1\}$.*

*(**b**) There is a pay-performance sensitivity level β_t^{EN} allocated to the EN that maximizes his effort in the form of $\beta_t^{\mathrm{EN}} = \min\left\{(\gamma r)\sqrt{\delta_1\delta_2}/\alpha, 1\right\}$.*

Proof. Proof of (**a**):

According to (31), we had

$$\begin{aligned}\frac{de_t}{d\beta_t} &= \frac{2\alpha(\gamma r-\beta_t)\left[\delta_1\delta_2(\gamma r)^2-\alpha^2\beta_t(2\gamma r-\beta_t)\right]-\left[\zeta\delta_2(\gamma r)^2+\alpha\beta_t(2\gamma r-\beta_t)\right](-2\alpha^2)(\gamma r-\beta_t)}{\left[\delta_1\delta_2(\gamma r)^2-\alpha^2\beta_t(2\gamma r-\beta_t)\right]^2}\\ &= \frac{2\alpha\delta_2(\gamma r)^2(\delta_1+\zeta\alpha)(\gamma r-\beta_t)}{\left[\delta_1\delta_2(\gamma r)^2-\alpha^2\beta_t(2\gamma r-\beta_t)\right]^2}.\end{aligned}$$

As $\alpha \geq 0$, $\delta_1 > 0$, $\delta_2 > 0$ and $\gamma r > 0$, then, if $\gamma r > 1$, $\forall\beta_t \in [0,1]$, we had $de_t/d\beta_t \geq 0$, $\beta_t = 1$ is the maximum point of e_t; if $0 < \gamma\, r < 1$, $\beta_t \in [0,\gamma r]$, $de_t/d\beta_t \geq 0$, while $\beta_t \in (\gamma r, 1]$, $de_t/d\beta_t \leq 0$, then e_t reaches a maximum at $\beta_t = \gamma r$. On the whole, $\beta_t^{\mathrm{VC}} = \min\{\gamma r, 1\}$ assigned to the EN maximizes the VC's effort.

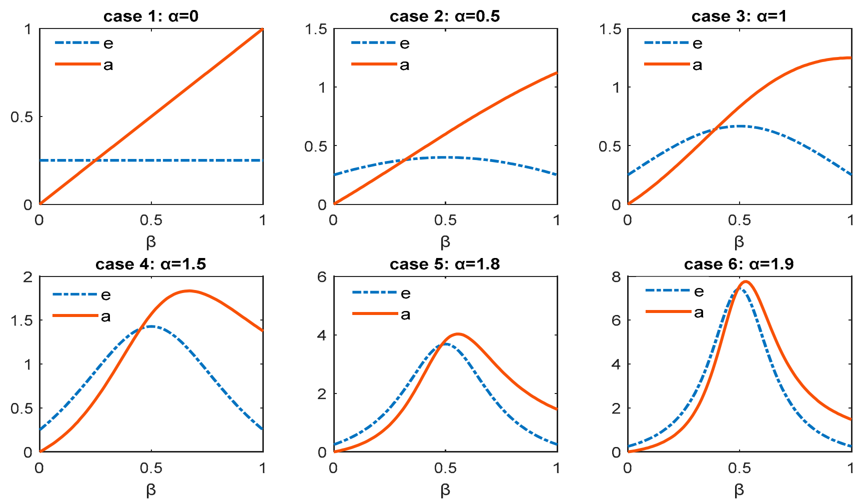

Figure 2. Efforts dynamics for different complementarity levels α.

Proof of (**b**): Taking the first derivative of (23) with respect to β_t, it yields

$$\begin{aligned}\frac{da_t}{d\beta_t} &= \frac{1}{\gamma r\delta_2}\left(1+\alpha e_t+\alpha\beta_t\frac{de_t}{d\beta_t}\right)\\ &= \frac{1}{\gamma r\delta_2}\,\frac{\left[\delta_1\delta_2-\alpha^2\beta_t^{\lambda}(2-\beta_t^{\lambda})\right](\delta_1\delta_2+\alpha\zeta\delta_2)+2\alpha^2\beta_t^{\lambda}(\delta_1\delta_2+\alpha\zeta\delta_2)\left(1-\beta_t^{\lambda}\right)}{\left[\delta_1\delta_2(\gamma r)^2-\alpha^2\beta_t(2\gamma r-\beta_t)\right]^2}\\ &= \frac{(\gamma r)(\delta_1+\zeta\alpha)\left[\delta_1\delta_2(\gamma r)^2-\alpha^2(\beta_t)^2\right]}{\left[\delta_1\delta_2(\gamma r)^2-\alpha^2\beta_t(2\gamma r-\beta_t)\right]^2}.\end{aligned}$$

The FOC of a_t with respect to β_t is a quadratic equation

$$\frac{(\gamma r)(\delta_1+\zeta\alpha)\left[\delta_1\delta_2(\gamma r)^2-\alpha^2(\beta_t)^2\right]}{\left[\delta_1\delta_2(\gamma r)^2-\alpha^2\beta_t(2\gamma r-\beta_t)\right]^2} = 0.$$

As $\delta_1 + \zeta\alpha > 0$ and $\gamma r > 0$, then $(\beta_t)_1 = (\gamma r)\sqrt{\delta_1\delta_2}/\alpha$ and $(\beta_t)_2 = -(\gamma r)\sqrt{\delta_1\delta_2}/\alpha$ are two solutions to the quadratic equation above. Since $\beta_t \in [0,1]$, thus $\beta_t^{\mathrm{EN}} = \min\left\{(\gamma r)\sqrt{\delta_1\delta_2}/\alpha,\ 1\right\}$ is the maximum point of the EN's effort. □

Notice that β_t^{EN} and β_t^{VC} do not solve the VC's optimal problem. The optimal pay-performance sensitivity level β_t^* allocated to the EN is expressed in Proposition 2 at a fixed point of $\beta_t^* = l(\beta_t^*)$. The result of Proposition 2 is further illustrated in Figure 3. The graphs depict the result of simulation using parameters in Table 1. The equilibrium point that solves the VC's maximization problem occurs when functions β_t^* and $l(\beta_t^*)$ intersect the 45° line. As shown in Figure 3, the intersection point β_t^* stays on the left of the imaginary line $\beta_t = \beta_t^{VC}$, this implies that $\beta_t^* < \beta_t^{VC}$. It is crucial to point out that this phenomenon is somehow distinct from the traditional static contracting models, in which $\beta_t^* > \beta_t^{VC}$ [20]. The driving force behind this result is that according to the project performance, dynamic contracts could be adjusted at any period, whereas the static contract terms are designed at the beginning.

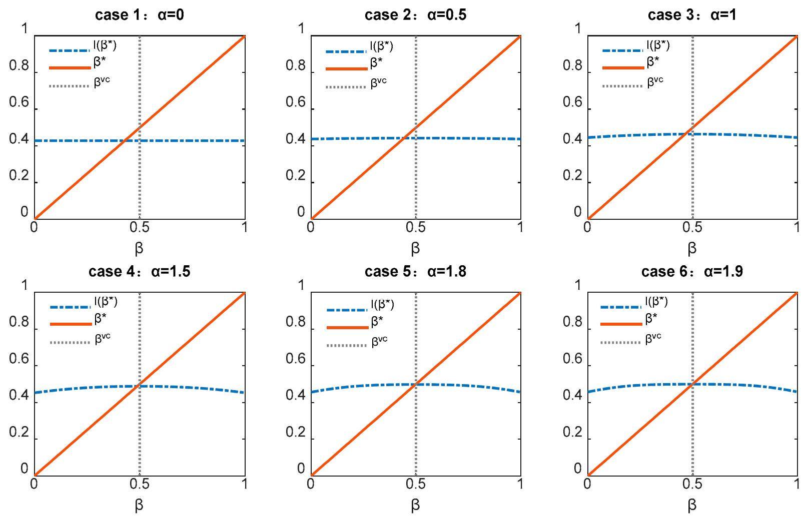

Figure 3. Optimal pay-performance sensitivity level for a different degree of complementarity α.

Intuitively, an increase in the complementarity of efforts causes β_t^* to approach β_t^{VC}. In other words, the VC would put more effort, when the degree of effort complementarity rises.

Proposition 5. In the context that the two efforts are complementary.

*(**a**)* $\beta_t^* < \beta_t^{VC}$. *In particular, if* $\sqrt{\delta_1\delta_2} > \alpha$, *then* $\beta_t^* < \beta_t^{VC} < \beta_t^{EN}$.

*(**b**)* $d\beta_t^*/d\alpha > 0$.

Proof. Proof of (**a**): $\forall \beta_t \in [0,1]$, we had $\beta_t = l(\beta_t) < \gamma r = \beta_t^{VC}$. If $\sqrt{\delta_1\delta_2} > \alpha$, then $\beta_t^{VC} < \beta_t^{EN}$ follows. As a result, $\beta_t^* < \beta_t^{VC} < \beta_t^{EN}$.

Proof of (**b**):

$$\begin{aligned} l(\beta_t, \alpha) &= \gamma r \cdot \frac{\delta_2(\delta_1+\zeta\alpha)^2(\gamma r)^4 + \theta\sigma^2(1-\rho^2)[(\gamma r)^2\delta_1\delta_2 - \alpha^2\beta_t(2\gamma r - \beta_t)]^2}{\delta_2(\delta_1+\zeta\alpha)^2(\gamma r)^4 + [\gamma r\sigma^2(1-\rho^2)+\theta\sigma^2(1-\rho^2)][(\gamma r)^2\delta_1\delta_2 - \alpha^2\beta_t(2\gamma r - \beta_t)]^2} \\ &= \gamma r \cdot \frac{\delta_2[\frac{(\delta_1+\zeta\alpha)(\gamma r)^2}{(\gamma r)^2\delta_1\delta_2 - \alpha^2\beta_t(2\gamma r-\beta_t)}]^2 + \theta\sigma^2(1-\rho^2)}{\delta_2[\frac{(\delta_1+\zeta\alpha)(\gamma r)^2}{(\gamma r)^2\delta_1\delta_2 - \alpha^2\beta_t(2\gamma r-\beta_t)}]^2 + [\gamma r\sigma^2(1-\rho^2)+\theta\sigma^2(1-\rho^2)]}. \end{aligned}$$

Define

$$\Phi = \frac{(\delta_1 + \zeta\alpha)(\gamma r)^2}{(\gamma r)^2\delta_1\delta_2 - \alpha^2\beta_t(2\gamma r - \beta_t)},$$

we had $\Phi > 0$ as $\beta_t > 0$, then, it could be easily shown that $l(\beta_t, \alpha)$ increases in Φ. Note that $d\Phi/d\alpha > 0$, $d^2\Phi/d\beta_t^2 < 0$, therefore, $l(\beta_t, \alpha)$ increases with respect to α and is concave of β_t.

Now, together with the fact $l(\beta_t, \alpha)\big|_{\beta_t=0} > \beta_t\big|_{\beta_t=0}$, our result follows. □

3.2.4. Effort Dynamic and Complementarity Effect

Given the results obtained in Proposition 2, we rewrote (20) as

$$dV_t = (-V_t)\left[\gamma r \frac{\rho\sigma\theta}{\gamma r + \theta} dB_t + \beta_t^* \sqrt{1-\rho^2}\sigma dZ_t\right]. \tag{39}$$

Denote $\Delta\beta_t^* = (\beta_t^*)|_{\alpha>0} - (\beta_t^*)|_{\alpha=0}$, then, dV_t follows

$$dV_t = (-V_t)\left[(\beta_t^*)|_{\alpha=0} \sqrt{1-\rho^2}\sigma dZ_t + \gamma r \frac{\rho\sigma\theta}{\gamma r + \theta} dB_t + \Delta\beta_t^* \sqrt{1-\rho^2}\sigma dZ_t\right]. \tag{40}$$

As shown in Proposition 5, β_t^* increases in α, then $\Delta\beta_t^* > 0$, intuitively, we had

$$dV_t > (\beta_t^*)|_{\alpha=0} (-V_t) \sqrt{1-\rho^2}\sigma dZ_t,$$

where the right hand of the inequality above is the evolution process of the EN's value function in the traditional principal-agent model with no reward for observable luck. However, in the presence of probability measure uncertainty and complementarity, as shown in (40), there are two elements, which positively increase the EN's continuation value in response to favorable shocks on performance beyond the EN's control. One is the second term in the square brackets of the right hand of (40) related to ambiguity; it presents the impact of common shocks. Another is the third term in the right hand of (40), which expresses the effect of the efforts complementarity.

3.2.5. Double Moral Hazard

Now, we dealt with the context that both efforts are not observable and verifiable by the other partner. For simplicity, we assumed that the effort exerted by the EN is binary, that is $a_t \in \{0, 1\}$. The EN chooses to shirk, then $a_t = 0$, while to work, $a_t = 1$. Based on (3) and (19), the continuation value of the EN evolves as

$$\begin{aligned} dV_t &= \phi_t(V_t)(\rho\sigma dB_t + \sqrt{1-\rho^2}\sigma dZ_t) + \widetilde{\phi}_t(V_t)\sigma dB_t \\ &= \phi_t(V_t)[dY_t - (a_t + \zeta e_t + \alpha a_t e_t)dt] + \widetilde{\phi}_t(V_t)\sigma dB_t \end{aligned} \tag{41}$$

When the contract is incentive compatible, that is, the EN implements $a_t(V_t) = 1$ if and only if

$$\phi_t(V_t)(1 + \zeta e_t + \alpha e_t) - \frac{\delta_2}{2} \geq \phi_t(V_t)\zeta e_t.$$

That is,

$$\phi_t(V_t) \geq \frac{1}{2} \cdot \frac{\delta_2}{1 + \alpha e_t}. \tag{42}$$

Without ambiguity ($\theta = 0$), the incentive compatible for the VC's effort is determined by the following maximization problem

$$(e_t, \phi_t(V_t)) = \underset{\phi_t(V_t) \geq \frac{1}{2} \cdot \frac{\delta_2}{1+\alpha e_t}}{\text{argmax}} \; 1 + (\zeta + \alpha)e_t - \frac{\delta_1}{2}e_t^2 + \frac{1}{2}J''(V_t)\left[\left(\phi_t(V_t)\rho + \widetilde{\phi}_t(V_t)\right)^2 + \phi_t^2(V_t)(1-\rho^2)\right]\sigma^2. \tag{43}$$

As $J''(V_t) < 0$, the VC would optimally choose $\phi_t(V_t) = \frac{1}{2} \cdot \frac{\delta_2}{1+\alpha e_t} \triangleq \phi_0$. Note that the VC could choose his effort at each point of V_t without considering the EN's incentive compatibility constraint after providing the contract choices for the EN. Then, the optimal effort the VC made is $e_t = (\zeta + \alpha)/\delta_1$.

Noteworthily, if the VC's effort could be observed by the EN, we would have

$$e_t = \frac{1}{\delta_1}\left\{\zeta + \alpha + J''(V_t)\left(\rho(\phi_0\rho + \widetilde{\phi}_t(V_t))\frac{\partial\phi_0}{\partial e_t} + \frac{\partial\phi_0}{\partial e_t}(1-\rho^2)\right)\sigma^2\right\}.$$

As $J''(V_t) < 0$, $\partial\phi_0/\partial e_t < 0$, then, we had

$$e_t = \frac{1}{\delta_1}\left[\zeta + \alpha + J''(V_t)\left[\rho(\phi_0\rho + \widetilde{\phi}_t(V_t))\frac{\partial\phi_0}{\partial e_t} + \frac{\partial\phi_0}{\partial e_t}(1-\rho^2)\right]\sigma^2\right] > \frac{\zeta+\alpha}{\delta_1}, \tag{44}$$

which implies that the VC would exert lower effort in the context of double moral hazard.

However, in the presence of ambiguity ($\theta > 0$), the VC's maximization problem is

$$\begin{aligned}(e_t, \phi_t(V_t)) = \underset{\phi_t(V_t)\geq\phi_0}{\operatorname{argmax}}\ & 1 + (\zeta+\alpha)e_t - \tfrac{\delta_1}{2}e_t^2 + J'(V_t)\left[\phi_t(V_t)(\rho\sigma h_t + \sqrt{1-\rho^2}\sigma g_t) + \widetilde{\phi}_t(V_t)\sigma h_t\right] \\ & + \tfrac{1}{2}J''(V_t)\left[(\phi_t(V_t)\rho + \widetilde{\phi}_t(V_t))^2 + \phi_t^2(V_t)(1-\rho^2)\right]\sigma^2,\end{aligned} \tag{45}$$

where $J'(V_t) > 0$, $J''(V_t) < 0$.

Intuitively, the solution of problem (45) is related to the two density generators $h_t \in \mathbb{R}$ and $g_t \in \mathbb{R}$, which exist due to the ambiguity on probability measure. In particular, the optimal choice of (45) is similar with (43) when $\rho h_t + \sqrt{1-\rho^2}g_t \leq 0$, that is, the existence of ambiguity would not affect the optimal pay-performance sensitivity awarded to the EN. However, when $\rho h_t + \sqrt{1-\rho^2}g_t > 0$, we had

$$\phi_t(V_t) = \max\left\{\phi_0,\ -\frac{J'(V_t)}{J''(V_t)\sigma}(\rho h + \sqrt{1-\rho^2}g) - \rho\widetilde{\phi}_t(V_t)\right\},$$

which implies $\phi_t(V_t) \geq \phi_0$. That is, the existence of ambiguity would bring the EN a large pay-performance sensitivity level.

4. Conclusions

This paper explored a continuous-time model with probability measure ambiguity in which both the VC and the EN make complementary efforts into a project. We analyzed the connection between ambiguity sharing and incentives and investigated efforts of complementarity effect.

Similar to Wu et al. [9], we assumed that the EN is ambiguity-neutral. Due to the lack of information, the VC considers alternative models to protect him from probability measure ambiguity. Our analysis showed that the probability measure ambiguity induces a trade-off between ambiguity sharing and incentives. Moreover, the probability measure ambiguity enlarges the pay-performance sensitivity level awarded to the EN.

The effort of the VC in the dynamic contract research is seldom taken into account. In this study, we considered the efforts complementarity using continuous-time principal-agent theory and analyzed the effect of efforts complementarity on the optimal pay-performance sensitivity level. We found that as efforts tend to be more complimentary, the optimal pay-performance sensitivity tends to approach those that maximize the efforts exerted by the two partners. Our model is much more complicated but is closer to reality.

There are two possible directions, deserving future research. First, in the present study, we assumed that only the VC faces ambiguity about project value. It is interesting to assume that EN also faces ambiguity about project value and investigate the effects of asymmetric ambiguity between the EN and the VC. Second, it could be seen from the existing literature that the volatility of the project cash flow importantly affects both risk-sharing and incentives in contracting [8,26]. Therefore, another possible work is to examine the effects of ambiguity uncertainties about the mean and volatility of the projected revenue on optimal contracting.

Author Contributions: Conceptualization, Z.H.; methodology, J.C.; Formal analysis and validation, J.C. and Z.H.; Writing—original draft, J.C.; writing—review and editing, Z.H. and H.Y.; supervision and investigation, H.Y.; funding acquisition, Z.H. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the National Natural Science Foundations of China (no. 71361003) and Natural Science Foundations of Guizhou Province (no. [2018]3002).

Acknowledgments: The authors thank the anonymous reviewers for their careful reading and constructive comments.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

1. Holmstrom, B.; Milgrom, P. Aggregation and linearity in the provision of intertemporal incentives. *Econometrica* **1987**, *55*, 303–328. [CrossRef]
2. Cadenillas, A.; Cvitanic, J.; Zapatero, F. Optimal risk-sharing with effort and project choice. *J. Econ. Theory* **2007**, *133*, 403–440. [CrossRef]
3. DeMarzo, P.M.; Sannikov, Y. Optimal security design and dynamic capital structure in a continuous time agency model. *J. Financ.* **2006**, *61*, 681–724. [CrossRef]
4. Wang, C. Renegotiation-proof dynamic contracts with private information. *Rev. Econ. Dyn.* **2000**, *3*, 396–422. [CrossRef]
5. Ellsberg, D. Risk, ambiguity and the Savage axioms. *Q. J. Econ.* **1961**, *75*, 643–669. [CrossRef]
6. Kim, J.H.; Wagman, L. Early-stage entrepreneurial financing: A signaling perspective. *J. Bank. Financ.* **2016**, *67*, 12–22. [CrossRef]
7. Lukas, E.; Mölls, S.; Welling, A. Venture capital, staged financing and optimal funding policies under uncertainty. *Eur. J. Oper. Res.* **2016**, *250*, 305–313. [CrossRef]
8. Miao, J.; Rivera, A. Robust contracts in continuous time. *Econometrica* **2016**, *84*, 1405–1440. [CrossRef]
9. Wu, Y.; Yang, J.; Zou, Z. Ambiguity sharing and the lack of relative performance evaluation. *Econ. Theory* **2018**, *66*, 141–157. [CrossRef]
10. Liu, Q.; Lu, L.; Sun, B. Incentive contracting under ambiguity aversion. *Econ. Theory* **2018**, *66*, 929–950. [CrossRef]
11. Beauchêne, D. Is ambiguity aversion bad for innovation? *J. Econ. Theory* **2019**, *183*, 1154–1176. [CrossRef]
12. Aggarwal, R.K.; Samwick, A.A. Executive compensation, strategic competition, and relative performance evaluation: Theory and evidence. *J. Financ.* **1999**, *54*, 1999–2043. [CrossRef]
13. Bertrand, M.; Mullainathan, S. Are CEOs rewarded for luck? The ones without principals are. *Q. J. Econ.* **2001**, *116*, 901–930. [CrossRef]
14. Casamatta, C. Financing and advising: Optimal financial contracts with VCs. *J. Financ.* **2003**, *58*, 2059–2085. [CrossRef]
15. Fairchild, R. An EN's choice of VC or angel-financing: A behavioral game theoretic approach. *J. Bus. Ventur.* **2011**, *26*, 359–374. [CrossRef]
16. Hochberg, Y.V.; Ljungqvist, A.; Lu, Y. Whom you know matters: Venture capital networks and investment performance. *J. Financ.* **2007**, *62*, 251–301. [CrossRef]
17. Hori, K.; Osano, H. Managerial incentives and the role of advisors in the continuous-time agency model. *Rev. Financ. Stud.* **2013**, *26*, 2620–2647. [CrossRef]
18. Vergara, M.; Bonilla, C.A.; Sepulveda, J.P. The complementarity effect: Effort and sharing in the EN and venture capital contract. *Eur. J. Oper. Res.* **2016**, *254*, 1017–1025. [CrossRef]
19. Chang, J.; Hu, Z. Venture capital contracting with double-sided moral hazard and fairness concerns. *Math. Probl. Eng.* **2018**, *2018*. [CrossRef]
20. De Bettignies, J.E. Financing the entrepreneurial venture. *Manag. Sci.* **2008**, *54*, 151–166. [CrossRef]
21. He, Z.; Wei, B.; Yu, J.; Gao, F. Optimal long-term contracting with learning. *Rev. Financ. Stud.* **2017**, *30*, 2006–2065. [CrossRef]
22. Cvitanic, J.; Wan, X.H.; Yang, H.L. Dynamics of contract design with screening. *Manag. Sci.* **2013**, *59*, 1229–1244. [CrossRef]
23. He, Z.G. A model of dynamic compensation and capital structure. *J. Financ. Econ.* **2011**, *100*, 351–366. [CrossRef]
24. Aggarwal, R.K.; Andrew, A. The other side of the trade-off: The impact of risk on executive compensation. *J. Political Econ.* **1999**, *107*, 65–105. [CrossRef]

25. Anderson, E.W.; Hansen, L.P.; Sargent, T.J. A quartet of semi-groups for model specification, robustness, prices of risk, and model detection. *J. Eur. Econ. Assoc.* **2003**, *1*, 68–123. [CrossRef]
26. Mastrolia, T.; Possamai, D. Moral hazard under ambiguity. *J. Optim. Theory Appl.* **2018**, *179*, 452–500. [CrossRef]
27. Sannikov, Y. A continuous-time version of the principal-agent problem. *Rev. Econ. Stud.* **2008**, *75*, 957–984. [CrossRef]