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On a New Formula for Fibonacci's Family m-step Numbers and Some Applications

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Abstract: In this work, we obtain a new formula for Fibonacci's family m-step sequences. We use our formula to find the nth term with less time complexity than the matrix multiplication method. Then, we extend our results for all linear homogeneous recurrence m-step relations with constant coefficients by using the last few terms of its corresponding Fibonacci's family m-step sequence. As a computational number theory application, we develop a method to estimate the square roots.

Keywords: Fibonacci's m-step numbers; time complexity; linear homogeneous recurrence relations

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1. Introduction

Currently, in modern science, extensive work has been done in the area of recurrence relations and their applications (see, e.g., [1–5]).

In [6,7], the authors developed a transformation method of Tribonacci sequence and Tetranacci sequence to find the nth term of any Tribonacci-Like sequence and Tetranacci-Like sequence, respectively. In [8], the authors extended the previous transformation method to any Fibonacci-Like m-step sequence.

In [9], the authors used matrix multiplication to find the nth term of Fibonacci's family m-step sequences. The time complexity of their result is of order $m^3 \log n$ times the time of multiplying two n-digit integers.

We generalize the transformation methods in [6,7] for Fibonacci's family m-step sequences. We also use matrix method as in [9] to obtain the closed form of our new formula. However, the time complexity of our formula is of order $m^2 \log n$ times the time of multiplying two n-digit integers. As a computational number theory application, we develop a method to estimate the square root. The paper is organized as follows:

- Section 2 contains the notations and definitions related to this work. Section 2.1 gives a look into Fibonacci sequences and their properties. One of the most important features linked to the evaluation of iterative methods is the order of convergence or the time complexity. Time complexity shows how fast the algorithm converges to the solution. This aspect is discussed briefly in Section 2.2.
- Section 3 provides the main results and is organized as follows. In Section 3.1, we state and prove our main results. In Section 3.2, we provide a method of finding the nth term of any linear homogeneous recurrence relation. In Section 3.3, we illustrate our method by a numerical example.
- Section 4 deals with the computational number theory application; in particular, we give a different method of approximating the square roots.

2. Definitions and Notations

2.1. Fibonacci Primer

The Fibonacci sequence shows a certain numerical pattern. This pattern turns out to have an interest and importance far beyond what its inventor imagined. It can be used to model or describe an amazing variety of phenomena, in mathematics, science, and art (see, e.g., [3,8]).

The well-known Fibonacci sequence of numbers which are defined by the recurrence

$$F_n = F_{n-1} + F_{n-2},\ n \geq 2,$$

with the initial values $F_0 = 0$ and $F_1 = 1$, is an example of a linear homogeneous recurrence sequence.

Definition 1. *The linear homogeneous recurrence m-step sequence $\{H_n\}$, for $m \geq 2$ an arbitrary integer, is defined by the recurrence*

$$H_n = k_1 H_{n-1} + k_2 H_{n-2} + \cdots + k_m H_{n-m}, \quad n \geq m+1, \tag{1}$$

where $k_1, \cdots, k_m$ are constants and $H_1, \cdots, H_m$ are the initial values.

Miles [4] appears to be the first who studied such sequences, with constants $k_i = 1,\ 1 \leq i \leq m$ and initial values $H_1 = 0,\ H_2 = 0, \cdots, H_{m-1} = 0,\ H_m = 1$.

Definition 2. *The Fibonacci m-step sequence, for $m \geq 2$ an arbitrary fixed integer, is defined by the recurrence*

$$U_n = U_{n-1} + U_{n-2} + \cdots + U_{n-m},\ n \geq 2,$$

with the initial values $U_{2-m} = U_{1-m} = \cdots = U_0 = 0$, and $U_1 = 1$.

Definition 3. *The Fibonacci family m-step sequence, for $m \geq 2$ an arbitrary fixed integer, is defined by the recurrence*

$$U_n = k_1 U_{n-1} + k_2 U_{n-2} + \cdots + k_m U_{n-m},\ n \geq 2, \tag{2}$$

with the initial values $U_{2-m} = U_{1-m} = \cdots = U_0 = 0$, and $U_1 = 1$.

Definition 4. *The characteristic equation of Fibonacci m-step sequence is*

$$x^m - x^{m-1} - \cdots - x - 1 = 0. \tag{3}$$

It is well known that such sequence has the following property:

Property 1. $\lim\limits_{n\to\infty} \frac{U_{n+1}}{U_n}$ *is equal to the leading root of Equation (3).*

2.2. The Time Complexity

In [9], the authors used matrix multiplication to find the nth term of Fibonacci's family m-step sequences. The time complexity of their method is $O(m^3 \times \log(n) \times M(n \times n))$, where $M(n \times n)$ denotes the time of multiplying two n-digit integers. We also use matrix notation to obtain the closed form of our formula. However, the time complexity of this new formula is $O(m^2 \times \log(n) \times M(n \times n))$. Notice that, in calculating the time complexity of iterative processes, it is often assumed that the arithmetic operations of addition and multiplication can be computed in constant times. This assumption is invalid if the number of digits depends on the index of the term n as the computation proceeds. Therefore, it is important to distinguish between the process of multiplying two terms and term by a constant. Our actual time complexity is $O(m^3 \times \log(1 \times n) \times M(1 \times n) + m^2 \times \log(n) \times$

$M(n \times n))$, based on the assumption that the terms of the sequence are integer numbers. As $m << n$, our time complexity can be consider as $O(m^2 \times \log(n) \times M(n \times n))$.

3. Main Results

Let $H_n = \sum_{i=1}^{m} k_i H_{n-i}$, where $n \geq m+1$, be a linear homogenous recurrence m-step relation with constant coefficients k_i, $1 \leq i \leq m$, and initial values $H_1, H_2, \cdots, H_m$. Let $U_n = \sum_{i=1}^{m} k_i U_{n-i}$, where $m \geq 2$, be a linear recurrence m-step relation with $U_i = 0$, for $2-m \leq i \leq 0$ and $U_1 = 1$, i.e., $\{U_n\}$ is a Fibonacci family m-step sequence. We define the matrices $A_{m\times m}, K_{m\times m}, {}^tH_{1\times m}, {}^tU_{1\times m}$ as follows:

$$A_{m\times m} := \begin{pmatrix} k_1 & k_2 & k_3 & \cdots & k_m \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 1 & \cdots & 0 \end{pmatrix}, \quad K_{m\times m} := \begin{pmatrix} k_1 & k_2 & \cdots & k_{m-1} & k_m \\ k_2 & k_3 & \cdots & k_m & 0 \\ k_3 & k_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ k_m & 0 & \cdots & 0 & 0 \end{pmatrix}$$

$${}^tH_{1\times m} := \begin{bmatrix} H_t & H_{t-1} & \cdots & H_{t-m+1} \end{bmatrix}, \quad {}^tU_{1\times m} := \begin{bmatrix} U_t & U_{t-1} & \cdots & U_{t-m+1} \end{bmatrix}.$$

The following technical proposition are used to prove our main result.

Proposition 1. *For positive integers n, m, t with $n \geq m+1$, we have*

$$H_n = \sum_{i=1}^{m} \left(H_{n-i+1-t} \sum_{j=i}^{m} k_j U_{t+i-j} \right)$$

Alternately, $H_n = {}^{(n-t)}H_{1\times m} \times K_{m\times m} \times {}^tU^T_{1\times m}$, where ${}^tU^T$ denotes the transpose of the matrix ${}^tU$.

Proof. Let $H_n = \sum_{i=1}^{m} L_{i,t} H_{n-i+2-t}$, where $L_{i,t}$ is the coefficient of $H_{n-i+2-t}$. The following table shows that

$$U_t = L_{1,t}, \text{ for } 2-m \leq t \leq 1. \tag{4}$$

t	$\sum_{i=1}^{m} L_{i,t} H_{n-i+2-t}$	$L_{1,t}$
1	$\sum_{i=1}^{m} L_{i,1} H_{n-i+1} = (1)H_n + \sum_{i=1}^{m-1} (0)H_{n-i}$	$L_{1,1} = 1$
0	$\sum_{i=1}^{m} L_{i,0} H_{n-i+2} = (0)H_{n+1} + (1)H_n + \sum_{i=1}^{m-2} (0)H_{n-i}$	$L_{1,0} = 0$
-1	$\sum_{i=1}^{m} L_{i,-1} H_{n-i+3} = \sum_{i=1}^{2} (0)H_{n+3-i} + (1)H_n + \sum_{i=1}^{m-3} (0)H_{n-i}$	$L_{1,-1} = 0$
$\vdots$	$\vdots$	$\vdots$
$3-m$	$\sum_{i=1}^{m} L_{i,3-m} H_{n-i+m-1} = \sum_{i=1}^{m-2} (0)H_{n+m-1-i} + (1)H_n + (0)H_{n-1}$	$L_{1,3-m} = 0$
$2-m$	$\sum_{i=1}^{m} L_{i,2-m} H_{n-i+m} = \sum_{i=1}^{m-1} (0)H_{n+m-i} + (1)H_n$	$L_{1,2-m} = 0$

Now,

$$\begin{aligned}
H_n &= \sum_{i=1}^{m} L_{i,t} H_{n-i+2-t} \\
&= L_{1,t} H_{n+1-t} + \sum_{i=2}^{m} L_{i,t} H_{n-i+2-t} \\
&= L_{1,t} \sum_{i=1}^{m} k_i H_{n+1-t-i} + \sum_{i=2}^{m} L_{i,t} H_{n-i+2-t} \\
&= \sum_{i=1}^{m} k_i L_{1,t} H_{n+1-t-i} + \sum_{i=1}^{m-1} L_{i+1,t} H_{n-i+1-t} \\
&= \sum_{i=1}^{m-1} (k_i L_{1,t} + L_{i+1,t}) H_{n-i+1-t} + k_m L_{1,t} H_{n-m+1-t} \\
&= \sum_{i=1}^{m-1} (k_i L_{1,t} + L_{i+1,t}) H_{n-i+2-(t+1)} + k_m L_{1,t} H_{n-m+2-(t+1)} \qquad (*)
\end{aligned}$$

Since $H_n = \sum_{i=1}^{m} L_{i,t+1} H_{n-i+2-(t+1)} = \sum_{i=1}^{m-1} L_{i,t+1} H_{n-i+2-(t+1)} + L_{m,t+1} H_{n-m+2-(t+1)}$, then from (*), we have $L_{m,t+1} = k_m L_{1,t}$ and

$$L_{i,t+1} = k_i L_{1,t} + L_{i+1,t} \quad \text{for} \quad 1 \leqslant i \leqslant m-1 \tag{5}$$

By backward substitution in Equation (5), we have

$$\begin{aligned}
L_{m-1,t} &= k_{m-1} L_{1,t-1} + k_m L_{1,t-2} \\
L_{m-2,t} &= k_{m-2} L_{1,t-1} + k_{m-1} L_{1,t-2} + k_m L_{1,t-3} \\
&\vdots \\
L_{i,t} &= k_i L_{1,t-1} + k_{i+1} L_{1,t-2} + \cdots + k_m L_{1,t+i-1-m},
\end{aligned}$$

and hence,

$$L_{i,t} = \sum_{j=i}^{m} k_j L_{1,t+i-1-j} \tag{6}$$

Now, from Equations (4) and (6), we have $L_{1,t} = U_t \ \ \forall t$ and $L_{i,t} = \sum_{j=i}^{m} k_j U_{t+i-1-j}$. Thus,

$$H_n = \sum_{i=1}^{m} L_{i,t} H_{n-i+2-t} = \sum_{i=1}^{m} H_{n-i+2-t} \sum_{j=i}^{m} k_j U_{t+i-1-j}.$$

Now, by replacing t by $t+1$, we get

$$H_n = \sum_{i=1}^{m} \left(H_{n-i+1-t} \sum_{j=i}^{m} k_j U_{t+i-j} \right).$$

Alternately, $H_n = {}^{(n-t)}H_{1\times m} \times K_{m\times m} \times {}^{t}U^T_{1\times m}$. □

3.1. Main Formulas

Lemma 1. *For positive integers z, m, t with $z = 2n$ and $n \geq m+1$, we have*

$$H_{2n+1} = {}^{2n-t}H_{1\times m} \times K_{m\times m} \times A_{m\times m} \times {}^{t}U^T_{1\times m} \tag{7}$$

Proof. By Proposition 1, we have

$$H_{2n+b} = {}^{2n+b-t}H_{1\times m} \times K_{m\times m} \times {}^{t}U^T_{1\times m} \tag{8}$$

In particular, we have

$$H_{2n+1} = {}^{2n+1-t}H_{1\times m} \times K_{m\times m} \times {}^{t}U^{T}_{1\times m}. \tag{9}$$

Thus,

$$\begin{aligned}
H_{2n+1} &= \begin{pmatrix} H_{2n-t+1} \\ H_{2n-t} \\ \vdots \\ H_{2n-t-m+2} \end{pmatrix}^{T} \begin{pmatrix} k_1 & k_2 & \cdots & k_m \\ k_2 & k_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ k_m & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} U_t \\ U_{t-1} \\ \vdots \\ U_{t-m+1} \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ H_{2n-t} \\ \vdots \\ H_{2n-t-m+2} \end{pmatrix}^{T} \begin{pmatrix} 0 & 0 & \cdots & 0 \\ k_2 & k_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ k_m & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} U_t \\ U_{t-1} \\ \vdots \\ U_{t-m+1} \end{pmatrix} \\
&+ \quad H_{2n-t+1} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_m \end{pmatrix}^{T} \begin{pmatrix} U_t \\ U_{t-1} \\ \vdots \\ U_{t-m+1} \end{pmatrix} \\
&= \begin{pmatrix} H_{2n-t} \\ \vdots \\ H_{2n-t-m+2} \\ 0 \end{pmatrix}^{T} \begin{pmatrix} k_2 & k_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ k_m & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} U_t \\ U_{t-1} \\ \vdots \\ U_{t-m+1} \end{pmatrix} \\
&+ \begin{pmatrix} H_{2n-t} \\ \vdots \\ H_{2n-t-m+2} \\ H_{2n-t-m+1} \end{pmatrix}^{T} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_m \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_m \end{pmatrix}^{T} \begin{pmatrix} U_t \\ U_{t-1} \\ \vdots \\ U_{t-m+1} \end{pmatrix} \\
&= \begin{pmatrix} H_{2n-t} \\ \vdots \\ H_{2n-t-m+2} \\ H_{2n-t-m+1} \end{pmatrix}^{T} \begin{pmatrix} k_2 & k_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ k_m & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} U_t \\ U_{t-1} \\ \vdots \\ U_{t-m+1} \end{pmatrix}
\end{aligned}$$

$$+\begin{pmatrix} H_{2n-t} \\ \vdots \\ H_{2n-t-m+2} \\ H_{2n-t-m+1} \end{pmatrix}^T \begin{pmatrix} k_1k_1 & \cdots & k_{m-1}k_1 & k_mk_1 \\ k_1k_2 & \cdots & k_{m-1}k_2 & k_mk_2 \\ \vdots & \ddots & \vdots & \vdots \\ k_1k_m & \cdots & k_{m-1}k_m & k_mk_m \end{pmatrix} \begin{pmatrix} U_t \\ U_{t-1} \\ \vdots \\ U_{t-m+1} \end{pmatrix}$$

$$= \begin{pmatrix} H_{2n-t} \\ \vdots \\ H_{2n-t-m+2} \\ H_{2n-t-m+1} \end{pmatrix}^T \begin{pmatrix} k_1k_1+k_2 & \cdots & k_{m-1}k_1+k_m & k_mk_1 \\ k_1k_2+k_3 & \cdots & k_{m-1}k_2 & k_mk_2 \\ \vdots & \ddots & \vdots & \vdots \\ k_1k_m & \cdots & k_{m-1}k_m & k_mk_m \end{pmatrix} \begin{pmatrix} U_t \\ U_{t-1} \\ \vdots \\ U_{t-m+1} \end{pmatrix}$$

$$= \begin{pmatrix} H_{2n-t} \\ \vdots \\ H_{2n-t-m+2} \\ H_{2n-t-m+1} \end{pmatrix}^T \begin{pmatrix} k_1 & k_2 & \cdots & k_m \\ k_2 & k_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ k_m & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} k_1 & k_2 & \cdots & k_m \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix} \begin{pmatrix} U_t \\ U_{t-1} \\ \vdots \\ U_{t-m+1} \end{pmatrix}$$

$$= {}^{2n-t}H_{1\times m} \times K_{m\times m} \times A_{m\times m} \times {}^{t}U^T_{1\times m}.$$

□

The following two results are used to get our main formula of U_{2n} in terms of ${}^{n}U_{1\times m}$ only.

Theorem 1. *For integers m, n, t, b with $n > m > 1$, we have*

$$H_{2n+b} = {}^{2n-t}H_{1\times m} \times K_{m\times m} \times A^b_{m\times m} \times {}^{t}U^T_{1\times m},$$

where ${}^{t}U^T$ denotes the transpose of the matrix ${}^{t}U$.

Proof. Lemma 1 gives the result for $b = 1$. Now, assume the statement is true for $b - 1$, i.e.,

$$H_{2n+b-1} = {}^{2n-t}H_{1\times m} \times K_{m\times m} \times A^{b-1}_{m\times m} \times {}^{t}U^T_{1\times m}. \tag{10}$$

Replacing n by $n + \frac{1}{2}$ in Equation (10), we get

$$H_{2n+b} = {}^{2n-t+1}H_{1\times m} \times K_{m\times m} \times A^{b-1}_{m\times m} \times {}^{t}U^T_{1\times m}.$$

By repeating the same method of Lemma 1, one can easily show that

$$H_{2n+b} = {}^{2n-t}H_{1\times m} \times K_{m\times m} \times A^b_{m\times m} \times {}^{t}U^T_{1\times m}.$$

Now, assume the theorem is true for $b + 1$, i.e.,

$$H_{2n+b+1} = {}^{2n-t}H_{1\times m} \times K_{m\times m} \times A^{b+1}_{m\times m} \times {}^{t}U^T_{1\times m}. \tag{11}$$

Replacing n by $n - \frac{1}{2}$ in Equation (11) and reversing the steps of the proof of Lemma 1, we have

$$H_{2n+b} = {}^{2n-t}H_{1\times m} \times K_{m\times m} \times A^b_{m\times m} \times {}^{t}U^T_{1\times m}.$$

□

Lemma 2. *Let m, n be positive integers. We have that*

$$H_{m+n} = {}^{m}H_{1\times m} \times K_{m\times m} \times {}^{n}U^{T}_{1\times m}$$

where ${}^{n}U^{T}$ denotes the transpose of the matrix ${}^{n}U$.

Proof. By Proposition 1, we have $H_n = {}^{(n-t)}H_{1\times m} \times K_{m\times m} \times {}^{t}U^{T}_{1\times m}$

Now, by replacing t by $n-m$, we get

$$H_n = {}^{m}H_{1\times m} \times K_{m\times m} \times {}^{n-m}U^{T}_{1\times m}.$$

In particular, when n is replaced by $n+m$, we have

$$H_{m+n} = {}^{m}H_{1\times m} \times K_{m\times m} \times {}^{n}U^{T}_{1\times m}.$$

□

3.2. Procedure

In this subsection, we provide a method of finding the nth term of any linear homogeneous recurrence m-step relation using the last few terms of its corresponding Fibonacci's family m-step sequence. To find the nth term of the recurrence relation H_n, $n \geq 3m$, we apply the following steps:

1. Find ${}^{m}U$.
2. Compute $K \times A^{-t}$, where $1 \leq t \leq m-1$.
3. Set $z := \lfloor \log_2(n-m) \rfloor$ and $w := \lfloor log_2(m+b_1) \rfloor$.
4. Rewrite $n-m$ as $(((m+b_1) \times 2 + b_2) \cdots \times 2 + b_{z-w+1})$, where $b_i = 0$ or 1, $2 \leq i \leq z-w+1$, and $0 \leq b_1 \leq 2^{\lfloor \log_2(m) \rfloor + 1} - m$.
5. Find ${}^{m+b_1}U$.
6. Set ${}^{0}S := {}^{m+b_1}U$.
7. Use the following algorithm to find ${}^{z-w}S := {}^{n-m}U$, for $i = 1 : z-w$:

{
    for ($j = 1 : m$):
    {
        ${}^{i}S(1,j) = {}^{i-1}S \times (K \times A^{1-j}) \times {}^{i-1}S^{T}$
    }
    If $b_{i+1} = 1$:
    {
        set $Q = \sum\limits_{j=1}^{m} k_j {}^{i}S(1,j)$
        set ${}^{i}S(1,j) = {}^{i}S(1,j-1), 2 \leq j \leq m$
        set ${}^{i}S(1,1) = Q$
    }
}

8. $H_n = {}^{m}H \times K \times {}^{z-w}S^{T}$.

3.3. Numerical Example

Given $H_n = H_{n-1} - 2H_{n-2} + H_{n-3} - H_{n-4} + H_{n-6} + H_{n-7}$, with initial values

$$H_1 = 1,\ H_2 = 2,\ H_3 = 1,\ H_4 = 4,\ H_5 = -3,\ H_6 = 6,\ H_7 = -4.$$

We use our previous procedure to compute H_{46}.

1. $^0S := {^m}U_{1\times m} = \begin{bmatrix} 2 & 2 & 0 & -2 & -1 & 1 & 1 \end{bmatrix}$
2. $K_{m\times m} = \begin{pmatrix} 1 & -2 & 1 & -1 & 0 & 1 & 1 \\ -2 & 1 & -1 & 0 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, $A_{m\times m} = \begin{pmatrix} 1 & -2 & 1 & -1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$
3. $z = 5, n - m = 39 = (((7+2)\times 2+1)\times 2+1), b_1 = 2, b_2 = 1, b_3 = 1$ and $w = 3$.
4. $^0S := {^9}U_{1\times 7} = \begin{bmatrix} 0 & 2 & 2 & 2 & 0 & -2 & -1 \end{bmatrix}$
5. $^1S = \begin{bmatrix} 0 & -28 & -28 & -12 & 12 & 20 & 5 \end{bmatrix}$
6. $b_2 = 1$, then

   $^1S = \begin{bmatrix} 65 & 0 & -28 & -28 & -12 & 12 & 20 \end{bmatrix}$

   $^2S = \begin{bmatrix} 11409 & 4897 & -2432 & -4792 & -2696 & 664 & 2312 \end{bmatrix}$
7. $b_3 = 1$, then

   $^2S = \begin{bmatrix} 6951 & 11409 & 4897 & -2432 & -4792 & -2696 & 664 \end{bmatrix}$
8. $H_{46} = 30091$.

4. Application

In this section, we present an application that is concerned with computational number theory. By Property 1, we have $\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \frac{1+\sqrt{5}}{2}$, where F_n is the classical Fibonacci sequence. Therefore, one can estimate $\sqrt{5}$ using two consecutive terms in the Fibonacci sequence.

It is well known that the explicit form of the recurrence relation $F_n = (r_1 + r_2)F_{n-1} - (r_1 r_2)F_{n-2}$ is given by

$$F_n = c_1 r_1^n + c_2 r_2^n,$$

where $r_1,\ r_2$ are the roots of the characteristic equation $x^2 - x - 1 = 0$ and c_1, c_2 are constants. We use this fact to approximate imperfect square roots.

Let a be an imperfect square. To approximate $\sqrt{a}$, rewrite a as $(b+i)^2 \times 10^m$, where $b \in Z,\ 0 < i < 1,\ 1 \le b \le 9,$ and $\frac{m}{2} \in Z$. Since $\sqrt{a} = 10^{m/2} \times (b+i)$, we only need to find $b+i$.

Let $H_n = r_1^n + r_2^n$, where $r_1 = b - (b+i)$ and $r_2 = b + (b+i)$. Then $H_n = (r_1+r_2)H_{n-1} - (r_1 r_2)H_{n-2} = (2b)H_{n-1} - (b^2 - (b+i)^2)H_{n-2}$. Since $\lim_{n\to\infty} \frac{H_n}{H_{n-1}} = r_2 = 2b + i$, then $\sqrt{a} = (b+i) \times 10^{m/2} \approx (\frac{H_n}{H_{n-1}} - b) \times 10^{m/2}$.

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