mathematics

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Some Generalized Contraction Classes and Common Fixed Points in b-Metric Space Endowed with a Graph †

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† Dedicated to Professor Hemant Kumar Pathak in honour of his retirement from regular teaching job at Pt Ravishankar Shukla University, Raipur, India.

Received: 25 July 2019; Accepted: 15 August 2019; Published: 18 August 2019

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Abstract: We have introduced the new notions of R-weakly graph preserving and R-weakly α-admissible pair of multivalued mappings which includes the class of graph preserving mappings, weak graph preserving mappings as well as α-admissible mappings of type S, α_*-admissible mappings of type S and α_*- orbital admissible mappings of type S respectively. Some generalized contraction and rational contraction classes are also introduced for a pair of multivalued mappings and common fixed point theorems are proved in a b-metric space endowed with a graph. We have also applied our results to obtain common fixed point theorems for R-weakly α-admissible pair of multivalued mappings in a b-metric space which are the proper extension and generalization of many known results. Proper examples are provided in support of our results. Our main results and its consequences improve, generalize and extend many known fixed point results existing in literature.

Keywords: $(\mathcal{C}, \Psi^*, G, \gamma s)$ contractions; directed graph; R-Weakly graph preserving; rational contractions; R-weakly α-admissible

MSC: 47H10; 54H25

1. Introduction

In recent years three important tools have been successfully utilized in fixed point theory to generalize fixed point theorems for single-valued mappings and multi valued mappings. One such tool is the use of control functions. Khan et al. [1] introduced the altering distance function ψ which is a control function where $\psi : [0, \infty) \to [0, \infty)$ is continuous, non decreasing and satisfies the condition $\psi(t) = 0$ if and only if $t = 0$. Recently in [2], the authors considered the condition $\sum_{n=1}^{\infty} \psi(t) < +\infty$ which in turn implies $\psi(t) < t$ and $\psi(0) = 0$ where as George et al. [3] considered the control function ψ which is continuous, non decreasing and satisfies the condition $\psi(t) = 0$ implies $t = 0$ which clearly implies that $\psi(0)$ is not necessarily 0. Doric [4] considered contraction conditions involving two control functions for two single valued self mappings defined on a metric space. The results of Doric [4] extends and generalizes the results of Rhoades [5], Dutta and choudhary [6] and Zhang and Song [7]. The second such tool is endowing the metric space with a directed or undirected graph and using certain concepts such as edge preservness, graph preservness, weak graph preservness of the mappings involved, transitivity of the graph, completeness of the metric space, etc. The first work in this direction was initiated by Jachymski [8] in which the author introduced the concept of a graph preserving mapping and G-contraction for a single valued mapping defined on a metric space endowed with a graph. Later Phonon et al. [9] generalised the concept of a graph preserving mapping by introducing weak graph preserving mapping and proved fixed point theorems for multivalued mappings in a metric space endowed with a graph. More results in this direction were considered wherein Bojor [10] considered Reich type contraction, Mohanta and Patra [11] discussed common fixed point of a hybrid pair of mappings in a b-metric space endowed with a graph, Cholamjiak et al. [12] discussed viscosity approximation method for fixed point problems in a Hilbert space endowed with graph, Sauntai et al. [13] proved the existence of coupled fixed point for θ-ψ contraction mapping whereas Sultan and Vetrivel [14] considered Mizoguci-Takahashi contractions in a metric space endowed with a graph. Further, recently some interesting fixed point results for multivalued mappings in a metric space endowed with a graph were discussed in [15–19]. The third such tool is to use a function such as $\alpha : [0, \infty) \to [0, \infty)$ and use the concepts of α-admissible mappings, weakly α-admissible mappings, α_*-admissible mappings, etc. Initially Samet et al. [20] introduced the concept of α-admissible mappings and proved fixed point theorems for such mappings in a metric space. Later this concept was extended and generalised by many authors. In the sequel Sintunavarat [21] introduced α-admissible mappings of type S and weak α-admissible mappings of type S and Arshad et al. [22] introduced α-orbital admissible mappings in a metric space. Extending these concepts to the case of multivalued mappings, Ameer et al. [23] introduced α_*-admissible and α_*-orbital admissible multivalued mappings whereas Haitham et al. [2] introduced α-admissible multivalued mappings of type S in a b-metric space. In all these works the authors proved fixed point theorems for the corresponding α-admissible type of mappings satisfying various contraction conditions. Very recently the C-class functions were introduced in [19] which later proved to be a powerful tool in fixed point theory. In the present work using C-class functions and some modified versions of control functions as in George et al. [3], we have introduced generalized classes of contractions and rational contractions for a pair of multivalued mappings and proved common fixed point theorems in a b-metric space endowed with a graph. In Section 3.1 we have introduced R-weakly graph preserving pair of mappings and R-weakly α-admissible pair of mappings which are proper extension and generalization of the class of graph preserving mappings, weak graph preserving mappings and α-admissible multivalued mappings of type S, α_*-admissible multivalued mappings of type S and α_*- orbital admissible multivalued mappings of type S respectively. In Section 3.2 we have introduced the classes of $(\mathcal{C}, \Psi^*, G, \gamma s)$ and $(\mathcal{C}, \Psi^*, G, \gamma)$ contractions and rational contractions and proved common fixed point theorems for these class of mappings in a b-metric space endowed with a graph. In Section 3.3 we have applied the results of Section 3.2 to obtain fixed point theorems for R-weakly α-admissible mappings in a b-metric space. Our main results and its consequences are improved, generalized and extended versions of many results appearing in literature.

2. Preliminaries

Let (X, d) be a metric space. For $x \in X$ and $A, B \in CL(X)$, define $d(x, A) = inf\{d(x, y) : y \in A\}$, $D(A, B) = inf\{d(x, B) : x \in A\}$, $P_B(a) = \{b \in B : d(a, b) = d(a, B)\}$ and $H(A, B) = max\{sup_{x \in A} d(x, B), sup_{y \in B} d(y, A)\}$. It is well known that H is the Hausdorff metric induced by d on X.

Let (X_G, d_G, s) be the b-metric space with coefficient $s \geq 1$ and endowed with a directed graph G whose set of vertices $V(G)$ coincides with X_G and the set of edges will be denoted by $E(G)$.

Definition 1. *[24] $\mathcal{C} = \{F : [0, \infty)^2 \to \mathbb{R}\}$ such that F satisfies (1), (2) and (3) below:*

(1) F is continuous,

(2) $F(p, q) \leq p$,

(3) $F(p, q) = p \to p = 0$ or $q = 0$

Definition 2. *[8] (X_G, d_G, s) is said to be G-regular if and only if whenever $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in X_G such that $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$ then $(x_n, x_G) \in E(G)$.*

Definition 3. *[8] T is said to be graph preserving if $(x, y) \in E(G) \Rightarrow (u, v) \in E(G)$ for all $u \in Tx$ and $v \in Ty$.*

Definition 4. *[9] T is said to be weak graph preserving if it satisfies the following:*

- *for each $x, y \in X$, $(x, y) \in E(G) \rightarrow$ for each $u \in Tx$ there exists $v \in P_{Ty}(u)$ such that $(u, v) \in E(G)$.*

Lemma 1. *[25] For any sequence $\{p_n\}$ in a b-metric space (X, d, s), if we can find $\alpha \in [0, 1)$, satisfying $d(p_n, p_{n+1}) \leq \alpha d(p_{n-1}, p_n)$ then $\{p_n\}$ is a Cauchy sequence.*

As introduced in [3], we will use the following class of functions: $\Psi^* = \{\psi : [0, \infty) \to [0, \infty)\}$ such that ψ is continuous, non decreasing and $\psi(t) = 0$ implies $t = 0$ and $\Phi^* = \{\phi : [0, \infty) \to [0, \infty)\}$ such that ϕ is lower semi continuous and $\phi(t) = 0$ implies $t = 0$.

3. Main Results

We begin this section with the following definitions:

Definition 5. *A sequence $\{x_n\}_{n\in\mathbb{N}}$ in (X_G, d_G) is said to be G-convergent if and only if there exist $x_G \in X_G$ such that for all $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d_G(x_n, x_G) < \epsilon$ for all $n \geq n_0$.*

Definition 6. *A sequence $\{x_n\}_{n\in\mathbb{N}}$ in (X_G, d_G) is a G-Cauchy sequence if and only if $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$ and for all $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d_G(x_n, x_m) < \epsilon$ for all $n, m \geq n_0$.*

Definition 7. *A metric space (X_G, d_G) is said to be G-regular if and only if whenever a sequence $\{x_n\}$ with $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$ G-converges to some x_G, then $(x_n, x_G) \in E(G)$ for all $n \in \mathbb{N}$.*

Definition 8. *(X_G, d_G) is said to be G-complete if and only if every G-Cauchy sequence in (X_G, d_G) is G-convergent.*

Note that every complete metric space is G-complete but the converse is not necessarily true as shown in the following example:

Example 1. *Let $X_G = [0, 1)$, $d_G(x_G, y_G) = |x_G - y_G|$, $G = (V(G), E(G))$, with $V(G) = X_G$ and $E(G) = \{(0, 0), (\frac{1}{2^n}, \frac{1}{2^{n+1}}), (\frac{1}{2^n}, 0)\}$. Clearly (X_G, d_G) is not complete. However, we see that the G-Cauchy sequence $\{x_n\}$ in X_G given by $x_n = \frac{1}{2^n}$ is G-convergent and hence (X_G, d_G) is G-complete.*

Let $A, B \in CL(X_G)$. For $a \in A$, define $R_B(a) = \{b \in B : d(a, b) \leq H(A, B)\}$. Then clearly $P_B(a) \subset R_B(a)$ and for any closed subset B of X_G, $R_B(a) \neq \phi$.

3.1. R-Weakly Graph Preserving and R-Weakly α-Admissible Mappings

Definition 9. *Mappings $S, T : X \to CL(X)$ are pairwise R-weakly graph preserving if and only if for all $x_G, y_G \in X_G$ with $(x_G, y_G) \in E(G)$, the following holds:*

(9.1) For $u_G \in Sx_G$, there exists $v_G \in R_{Ty_G}(u_G)$ such that $(u_G, v_G) \in E(G)$
(9.2) For $u_G \in Tx_G$, there exists $v_G \in R_{Sy_G}(u_G)$ such that $(u_G, v_G) \in E(G)$

Remark 1. *If $u_G \in Sx_G$ and $v_G \in R_{Ty_G}(u_G)$ then from the definition of $R_B(a)$ it is clear that $d(u_G, v_G) \leq H(Sx_G, Ty_G)$.*

Remark 2. *For $S = T$, the above definition becomes that of a R-weakly graph preserving mapping. Clearly all graph preserving mappings and weak graph preserving mappings are R-weakly graph preserving but the converse is not necessarily true as is clear from the following example:*

Example 2. *Let X_G, d_G and G be as in Example 1. Let $T : X_G \to X_G$ given by*

$$Tx_G = \begin{cases} \{0\}, & \text{if } x_G = 0 \\ \{\frac{1}{2^{n+1}}, \frac{1}{2^{n+2}}, 0\}, & \text{if } x_G = \frac{1}{2^n}. \end{cases}$$

Then clearly T is R-weakly graph preserving as shown below:

For $(\frac{1}{2^n}, \frac{1}{2^{n+1}}) \in E(G)$, $T(\frac{1}{2^n}) = \{\frac{1}{2^{n+1}}, \frac{1}{2^{n+2}}, 0\}$ and $T(\frac{1}{2^{n+1}}) = \{\frac{1}{2^{n+2}}, \frac{1}{2^{n+3}}, 0\}$. $H(T(\frac{1}{2^n}), T(\frac{1}{2^{n+1}})) = \frac{1}{2^{n+2}})$. For $u = \frac{1}{2^{n+1}} \in T(\frac{1}{2^n})$ we have $R_{T_{\frac{1}{2^{n+1}}}}(u) = \{\frac{1}{2^{n+2}}\}$ and $(\frac{1}{2^{n+1}}, \frac{1}{2^{n+2}}) \in E(G)$. For $u = \frac{1}{2^{n+2}} \in T(\frac{1}{2^n})$ we have $R_{T_{\frac{1}{2^{n+1}}}}(u) = \{\frac{1}{2^{n+2}}, \frac{1}{2^{n+3}}, 0\}$ and $(\frac{1}{2^{n+2}}, \frac{1}{2^{n+3}}) \in E(G)$. For $u = 0 \in T(\frac{1}{2^n})$ we have $R_{T_{\frac{1}{2^{n+1}}}}(u) = \{\frac{1}{2^{n+2}}, \frac{1}{2^{n+3}}, 0\}$ and $(0,0) \in E(G)$.

For $(\frac{1}{2^n}, 0) \in E(G)$, $T(\frac{1}{2^n}) = \{\frac{1}{2^{n+1}}, \frac{1}{2^{n+2}}, 0\}$ and $T(0) = \{0\}$. $H(T(\frac{1}{2^n}), T(0)) = \frac{1}{2^{n+1}})$. For $u = \frac{1}{2^{n+1}} \in T(\frac{1}{2^n})$ we have $R_{T(0)}(u) = \{0\}$ and $(\frac{1}{2^{n+1}}, 0) \in E(G)$. For $u = \frac{1}{2^{n+2}} \in T(\frac{1}{2^n})$ we have $R_{T(0)}(u) = \{0\}$ and $(\frac{1}{2^{n+2}}, 0) \in E(G)$. For $u = 0 \in T(\frac{1}{2^n})$ we have $R_{T(0)}(u) = \{0\}$ and $(0,0) \in E(G)$. Similar arguments follow in the case when $(0,0) \in E(G)$.

However, T is not weak graph preserving, as for $(\frac{1}{2^n}, \frac{1}{2^{n+1}}) \in E(G)$, we have $T(\frac{1}{2^n}) = \{\frac{1}{2^{n+1}}, \frac{1}{2^{n+2}}, 0\}$, $T(\frac{1}{2^{n+1}}) = \{\frac{1}{2^{n+2}}, \frac{1}{2^{n+3}}, 0\}$ and for For $u = \frac{1}{2^{n+2}} \in T(\frac{1}{2^n})$ we have $P_{T_{\frac{1}{2^{n+1}}}}(u) = \{\frac{1}{2^{n+2}}\}$ and $(\frac{1}{2^{n+2}}, \frac{1}{2^{n+2}}) \notin E(G)$.

Let (X, d_b, s) be any b-metric space with coefficient $s \geq 1$, $S, T : X \to CL(X)$ and $\alpha : X \times X \to [0, \infty)$.

Definition 10. *A sequence $\{x_n\}_{n\in\mathbb{N}}$ in (X, d_b, s) is a α-Cauchy sequence if and only if $\alpha(x_n, x_{n+1}) \geq s$ for all $n \in \mathbb{N}$ and for all $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d_b(x_n, x_m) < \epsilon$ for all $n, m \geq n_0$.*

Definition 11. *A b-metric space (X, d_b, s) is said to be α-regular if and only if whenever a sequence $\{x_n\}$ with $\alpha(x_n, x_{n+1}) \geq s$ for all $n \in \mathbb{N}$ G-converges to some x, then $\alpha(x_n, x) \geq s$ for all $n \in \mathbb{N}$.*

Definition 12. *(X, d_b, s) is said to be α-complete if and only if every α-Cauchy sequence in (X, d_b, s) is convergent.*

Definition 13. *Mappings $S, T : X \to CL(X)$ are pairwise R-weakly α-admissible of type S if and only if for all $u, v \in X$ with $\alpha(u, v) \geq s$ the following conditions holds:*

(13.1a) *For $x \in Su$, we can find $y \in R_{Tv}(x)$ such that $\alpha(x, y) \geq s$.*
(13.1b) *For $x \in Tu$, we can find $y \in R_{Sv}(x)$ such that $\alpha(x, y) \geq s$.*

Remark 3. *Clearly every pair of R-weakly α-admissible mappings of type S includes α-admissible mappings of type S (see [2]), α_*-admissible mappings of type S (see [23]) and α_*- orbital admissible mappings of type S (see [23]) as shown in the following example: .*

Example 3. *Let $X = [0, 1)$, $d_b(x, y) = |x - y|$, $\alpha : X \times X \to [0, \infty)$ given by*

$$\alpha(x, y) = \begin{cases} 1, & \text{if } (x, y) \in \{(\frac{1}{2^n}, \frac{1}{2^n}), (\frac{1}{2^n}, 0), (0, 0)\} \\ 0, & \text{if otherwise} \end{cases}$$

Let $S, T : X \to CL(X)$ be defined by

$$Sx = \begin{cases} \{\frac{1}{2^n}, 0\}, & \text{if } x = \frac{1}{2^n} \\ \{0\}, & \text{otherwise} \end{cases}$$

and

$$Tx = \begin{cases} \{\frac{1}{2^{n+1}}, 0\}, & \text{if } x = \frac{1}{2^n} \\ \{0\}, & \text{otherwise} \end{cases}$$

Then the pair (S, T) is R-weakly α-admissible but non of α-admissible mappings of type S, α_*-admissible mappings of type S and α_*- orbital admissible mappings of type S.

3.2. Common Fixed Point Theorems in b-Metric Space Endowed with a Graph

In this section we introduce the classes of $(\mathcal{C}, \Psi^*, G, \gamma s)$ and $(\mathcal{C}, \Psi^*, G, \gamma)$ contractions and rational contractions and prove common fixed point theorems in b-metric space endowed with a graph.

Definition 14. *Let $S, T : X_G \to CL(X_G)$. Then the pair (S, T) belongs $(\mathcal{C}, \Psi^*, G, \gamma s)$ contraction class if and only if all $x_G, y_G \in X_G$ with $(x_G, y_G) \in E(G)$, satisfies the following:*

(14.1) there exists $F \in \mathcal{C}$, $\psi \in \Psi^*$ and $\gamma > 1$ such that

$$\psi(\gamma s H(Sx_G, Ty_G)) \leq F(\psi(M(x_G, y_G)), M(x_G, y_G)) \text{ and}$$
$$\psi(\gamma s H(Tx_G, Sy_G)) \leq F(\psi(M(y_G, x_G)), M(y_G, x_G))$$

where

$$M(x_G, y_G) = \max\left\{d_G(x_G, y_G), d_G(Sx_G, x_G), d_G(Ty_G, y_G), \frac{d_G(y_G, Sx_G) + d_G(x_G, Ty_G)}{2s}\right\}$$

Theorem 1. *Let (X_G, d_G, s) be G-complete, G-regular and $S, T : X_G \to CL(X_G)$ satisfy the following:*

(1.1) For some arbitrary $x_{G0} \in X_G$ there exists $x_{G1} \in Tx_{G0} \bigcup Sx_{G0}$ such that $(x_{G0}, x_{G1}) \in E(G)$,
(1.2) S and T are pairwise R-weakly graph preserving,
(1.3) $(S, T) \in (\mathcal{C}, \Psi^*, G, \gamma s)$ for some $F \in \mathcal{C}$ and $\psi \in \Psi^*$.

Then we can find $u_G \in X_G$ such that $u_G \in Su_G \bigcap Tu_G$.

Proof. By (1.1), suppose $x_{G0} \in X_G$, and $x_{G1} \in S(x_{G0})$. By (9.1), we can find $x_{G2} \in R_{T(x_{G1})}(x_{G1})$ with $(x_{G1}, x_{G2}) \in E(G)$ and $d((x_{G1}, x_{G2}) \leq H(S(x_{G0}), T(x_{G1})$. By (9.2), for $x_{G2} \in T(x_{G1})$ there exists $x_{G3} \in R_{S(x_{G2})}(x_{G2})$ with $(x_{G2}, x_{G3}) \in E(G)$ and $d(x_{G2}, x_{G3}) \leq H(T(x_{G1}), S(x_{G2})$. Continuing inductively, we construct the sequence $\{x_{Gn}\}$ such that for $n \geq 0$

$$\begin{aligned} & x_{G2n+1} \in S(x_{G2n}), x_{G2n+2} \in T(x_{G2n+1}) \\ & (x_{Gn}, x_{Gn+1}) \in E(G),\ d_G(x_{G2n+1}, x_{G2n+2}) \leq H(S(x_{G2n}), T(x_{G2n+1})) \\ & \text{and } \ d_G(x_{G2n+2}, x_{G2n+3}) \leq H(T(x_{G2n+1}), S(x_{G2n+2})). \end{aligned} \tag{1}$$

If n is odd we have

$$\begin{aligned} \psi(\gamma d_G(x_{Gn}, x_{Gn+1})) & \leq \psi(\gamma s H(Sx_{Gn-1}, Tx_{Gn})) \\ & \leq F(\psi(M(x_{Gn-1}, x_{Gn})), M(x_{Gn-1}, x_{Gn})) \\ & \leq \psi(M(x_{Gn-1}, x_{Gn})) \end{aligned} \tag{2}$$

where

$$\begin{aligned}
M(x_{Gn-1}, x_{Gn}) &= \max\Big\{d_G(x_{Gn-1}, x_{Gn}), d_G(Sx_{Gn-1}, x_{Gn-1}), d_G(Tx_{Gn}, x_{Gn}), \\
&\quad \frac{d_G(x_{Gn}, Sx_{Gn-1}) + d_G(x_{Gn-1}, Tx_{Gn})}{2s}\Big\} \\
&\le \max\Big\{d_G(x_{Gn-1}, x_{Gn}), d_G(x_{Gn}, x_{Gn-1}), d_G(x_{Gn+1}, x_{Gn}), \\
&\quad \frac{d_G(x_{Gn}, x_{Gn}) + d_G(x_{Gn-1}, x_{Gn+1})}{2s}\Big\} \\
&\le \max\Big\{d_G(x_{Gn-1}, x_{Gn}), d_G(x_{Gn+1}, x_{Gn}), \frac{d_G(x_{Gn-1}, x_{Gn+1})}{2s}\Big\} \\
&\le \max\Big\{d_G(x_{Gn-1}, x_{Gn}), d_G(x_{Gn+1}, x_{Gn}), \\
&\quad \frac{d_G(x_{Gn-1}, x_{Gn}) + d_G(x_{Gn}, x_{Gn+1})}{2}\Big\} \\
&\le \max\{d_G(x_{Gn-1}, x_{Gn}), d_G(x_{Gn}, x_{Gn+1})\}
\end{aligned}$$

If $d_G(x_{Gn+1}, x_{Gn}) > d_G(x_{Gn-1}, x_{Gn})$, then $M(x_{Gn-1}, x_{Gn}) \le d_G(x_{Gn+1}, x_{Gn})$. Then Equation (2) gives

$$\psi(\gamma d_G(x_{Gn}, x_{Gn+1})) \le \psi(d_G(x_{Gn}, x_{Gn+1})).$$

Since ψ is non decreasing we get

$$\gamma d_G(x_{Gn}, x_{Gn+1}) \le d_G(x_{Gn}, x_{Gn+1})$$

a contradiction. So, we have

$$d_G(x_{Gn}, x_{Gn+1}) \le d_G(x_{Gn-1}, x_{Gn}) \tag{3}$$

and so

$$M(x_{Gn-1}, x_{Gn}) \le d_G(x_{Gn-1}, x_{Gn})$$

By Equation (2) we get

$$\psi(\gamma d_G(x_{Gn}, x_{Gn+1})) \le \psi(d_G(x_{Gn-1}, x_{Gn}))$$

and then

$$d_G(x_{Gn}, x_{Gn+1}) \le \frac{1}{\gamma} d_G(x_{Gn-1}, x_{Gn}) \tag{4}$$

If n is even we have

$$\begin{aligned}
\psi(\gamma d_G(x_{Gn}, x_{Gn+1})) &\le \psi(\gamma s H(Tx_{Gn-1}, Sx_{Gn})) \\
&\le F(\psi(M(x_{Gn}, x_{Gn-1})), M(x_{Gn}, x_{Gn-1})) \\
&\le \psi(M(x_{Gn}, x_{Gn-1}))
\end{aligned} \tag{5}$$

where

$$\begin{aligned}M(x_{Gn}, x_{Gn-1}) &= \max\Big\{d_G(x_{Gn}, x_{Gn-1}), d_G(Sx_{Gn}, x_{Gn}), d_G(Tx_{Gn-1}, x_{Gn-1}),\\ &\frac{d_G(x_{Gn-1}, Sx_{Gn}) + d_G(x_{Gn}, Tx_{Gn-1})}{2s}\Big\}\\ &\leq \max\Big\{d_G(x_{Gn}, x_{Gn-1}), d_G(x_{Gn+1}, x_{Gn}), d_G(x_{Gn}, x_{Gn-1}),\\ &\frac{d_G(x_{Gn-1}, x_{Gn+1}) + d_G(x_{Gn}, x_{Gn})}{2s}\Big\}\\ &\leq \max\Big\{d_G(x_{Gn}, x_{Gn-1}), d_G(x_{Gn+1}, x_{Gn}), \frac{d_G(x_{Gn-1}, x_{Gn+1})}{2s}\Big\}\\ &\leq \max\Big\{d_G(x_{Gn}, x_{Gn-1}), d_G(x_{Gn+1}, x_{Gn}),\\ &\frac{d_G(x_{Gn-1}, x_{Gn}) + d_G(x_{Gn}, x_{Gn+1})}{2}\Big\}\\ &\leq \max\{d_G(x_{Gn}, x_{Gn-1}), d_G(x_{Gn}, x_{Gn+1})\}\end{aligned}$$

If $d_G(x_{Gn}, x_{Gn+1}) > d_G(x_{Gn}, x_{Gn-1})$, then $M(x_{Gn}, x_{Gn-1}) \leq d_G(x_{Gn}, x_{Gn+1})$. Then Equation (5) gives

$$\psi(\gamma d_G(x_{Gn}, x_{Gn+1})) \leq \psi(d_G(x_{Gn}, x_{Gn+1})).$$

Since ψ is non decreasing we get

$$\gamma d_G(x_{Gn}, x_{Gn+1}) \leq d_G(x_{Gn}, x_{Gn+1})$$

a contradiction. So, we have

$$d_G(x_{Gn}, x_{Gn+1}) \leq d_G(x_{Gn-1}, x_{Gn}) \tag{6}$$

and so

$$M(x_{Gn-1}, x_{Gn}) \leq d_G(x_{Gn-1}, x_{Gn})$$

By Equation (5) we get

$$\psi(\gamma d_G(x_{Gn}, x_{Gn+1})) \leq \psi(d_G(x_{Gn-1}, x_{Gn}))$$

and then

$$d_G(x_{Gn}, x_{Gn+1}) \leq \frac{1}{\gamma} d_G(x_{Gn-1}, x_{Gn}) \tag{7}$$

Thus by Lemma 1 we conclude that $\{x_{Gn}\}$ is a G-Cauchy sequence. By G-completeness and G-regularity of (X_G, d_G), we can find $u_G \in X_G$ such that $x_{Gn} \to u_G$ as $n \to \infty$ and $(x_{G2n}, u_G) \in E(G)$, $(x_{G2n-1}, u_G) \in E(G)$.

We will now prove that $u_G \in Su_G \bigcap Tu_G$.

$$\begin{aligned}\psi(\gamma d_G(u_G, Tu_G)) &\leq \psi(s[\gamma d_G(u_G, x_{G2n+1})) + \gamma d_G(x_{G2n+1}, Tu_G)])\\ &\leq \psi(s[\gamma d_G(u_G, x_{G2n+1})) + \gamma H(Sx_{G2n}, Tu_G)])\end{aligned}$$

As $n \to \infty$, we get

$$\psi(\gamma d_G(u_G, Tu_G)) \leq lim_{n\to\infty} \psi(s\gamma H(Sx_{G2n}, Tu_G)). \tag{8}$$

Now, using (14.1) we have

$$\psi(s\gamma H(Sx_{G2n}, Tu_G)) \leq F(\psi(M(x_{G2n}, u_G)), M(x_{G2n}, u_G))$$

where

$$\begin{aligned} M(x_{G2n}, u_G) = \max \Big\{ & d_G(x_{G2n}, u_G), d_G(Sx_{G2n}, x_{G2n}), d_G(Tu_G, u_G), \\ & \frac{d_G(x_{G2n}, Tu_G) + d_G(u_G, Sx_{G2n})}{2s} \Big\} \\ \leq \max \Big\{ & d_G(x_{G2n}, u_G), d_G(Sx_{G2n}, x_{G2n}), d_G(Tu_G, u_G), \\ & \frac{s[d_G(x_{G2n}, u_G) + d_G(u_G, Tu_G)] + d_G(u_G, Sx_{G2n})}{2s} \Big\} \end{aligned}$$

Note that as $n \to \infty$, $d_G(Sx_{G2n}, x_{G2n}) \to 0$, $d_G(u_G, Sx_{G2n}) \to 0$ and so $M(x_{G2n}, u_G) \to d_G(Tu_G, u_G)$ Now, if $d_G(Tu_G, u_G) \neq 0$ then from Equation (8) as $n \to \infty$ we have,

$$\psi(\gamma d_G(Tu_G, u_G) \leq \psi(d_G(Tu_G, u_G))$$

again a contradiction and so $d_G(Tu_G, u_G) = 0$ which implies that $u_G \in \overline{Tu_G}$ and since Tu_G is closed we have $u_G \in Tu_G$.

In addition, we have

$$\begin{aligned} \psi(\gamma d_G(u_G, Su_G)) &\leq \psi(s[\gamma d_G(u_G, x_{G2n})) + \gamma d_G(x_{G2n}, Su_G)]) \\ &\leq \psi(s[\gamma d_G(u_G, x_{G2n})) + \gamma H(Tx_{G2n-1}, Su_G)]) \end{aligned}$$

As $n \to \infty$, we get

$$\psi(\gamma d_G(u_G, Su_G)) \leq lim_{n\to\infty} \psi(s\gamma H(Tx_{G2n-1}, Su_G)). \tag{9}$$

Now, using(14.1) we have

$$\psi(s\gamma H(Tx_{G2n-1}, Su_G)) \leq F(\psi(M(u_G, x_{G2n-1})), M(u_G, x_{G2n-1}))$$

where

$$\begin{aligned} M(u_G, x_{G2n-1}) = \max \Big\{ & d_G(x_{G2n-1}, u_G), d_G(Tx_{G2n-1}, x_{G2n-1}), d_G(Su_G, u_G), \\ & \frac{d_G(x_{G2n-1}, Su_G) + d_G(u_G, Tx_{G2n-1})}{2s} \Big\} \\ \leq \max \Big\{ & d_G(x_{G2n-1}, u_G), d_G(Tx_{G2n-1}, x_{G2n-1}), d_G(Su_G, u_G), \\ & \frac{s[d_G(x_{G2n-1}, u_G) + d_G(u_G, Su_G)] + d_G(u_G, Tx_{G2n-1})}{2s} \Big\} \end{aligned}$$

Note that as $n \to \infty$, $d_G(Tx_{G2n}, x_{G2n-1}) \to 0$, $d_G(u_G, Tx_{G2n-1}) \to 0$ and so $M(u_G, x_{G2n-1}) \to d_G(Su_G, u_G)$ Now, if $d_G(Su_G, u_G) \neq 0$ then from Equation (18) as $n \to \infty$ we have,

$$\psi(\gamma d_G(Su_G, u_G) \leq \psi(d_G(Su_G, u_G))$$

again a contradiction and so $d_G(Su_G, u_G) = 0$ which implies that $u_G \in \overline{Su_G}$ and since Su_G is closed we have $u_G \in Su_G$. Hence $u_G \in Su_G \bigcap Tu_G$. □

If the b-metric d_G is continuous then condition (14.1) can be replaced with a much weaker condition as follows:

Definition 15. *The pair (S,T) belongs to $(\mathcal{C},\Psi^*,G,\gamma)$ contraction class if it satisfies the following:*

(15.1) *there exists $F\in\mathcal{C}$, $\psi\in\Psi^*$ and $\gamma>1$ such that*

$$\psi(\gamma H(Sx_G,Ty_G))\leq F(\psi(M(x_G,y_G)),M(x_G,y_G)) \text{ and}$$
$$\psi(\gamma H(Tx_G,Sy_G))\leq F(\psi(M(y_G,x_G)),M(y_G,x_G))$$

Theorem 2. *Let (X_G,d_G,s) be G-complete, G-regular with d_G continuous and $S,T:X_G\to CL(X_G)$ satisfy (1.1), ((1.2)) and the following:*

(2.1) $(S,T)\in(\mathcal{C},\Psi^*,G,\gamma)$ *for some $F\in\mathcal{C}$, $\psi\in\Psi^*$and$\gamma>1$.*

Then we can find $u_G\in X_G$ such that $u_G\in Su_G\bigcap Tu_G$.

Proof. Proceeding as in the proof of Theorem 1, it is easy to see that $\{x_{Gn}\}$ is G-Cauchy. By G-completeness and G-regularity of (X_G,d_G), there exists $u_G\in X_G$ such that $x_{Gn}\to u_G$ as $n\to\infty$ and $(x_{G2n},u_G)\in E(G)$.

We will now prove that $u_G\in Su_G\bigcap Tu_G$. Using (9.1) again, we see that for $x_{G2n+1}\in Sx_{G2n}$ there exists ${y^*}_G\in R_{Tu_G}(x_{G2n+1})$ such that $(x_{G2n+1},{y^*}_G)\in E(G)$. Then we have

$$\begin{aligned}\psi(\gamma d_G(x_{G2n+1},Tu_G)) &\leq\psi(\gamma d_G(x_{G2n+1},{y^*}_G))\\ &\leq\psi(\gamma H(Sx_{G2n},Tu_G))\\ &\leq F(\psi(M(x_{G2n},u_G)),M(x_{G2n},u_G))\end{aligned}\tag{10}$$

where

$$\begin{aligned}M(x_{G2n},u_G)&=\max\Big\{d_G(x_{G2n},u_G),d_G(Sx_{G2n},x_{G2n}),d_G(Tu_G,u_G),\\ &\qquad\frac{d_G(x_{G2n},Tu_G)+d_G(u_G,Sx_{G2n})}{2s}\Big\}\\ &\leq\max\Big\{d_G(x_{G2n},u_G),d_G(Sx_{G2n},x_{G2n}),d_G(Tu_G,u_G),\\ &\qquad\frac{s[d_G(x_{G2n},u_G)+d_G(u_G,Tu_G)]+d_G(u_G,Sx_{G2n})}{2s}\Big\}\end{aligned}$$

Note that as $n\to\infty$, $d_G(Sx_{G2n},x_{G2n})\to 0$, $d_G(u_G,Sx_{G2n})\to 0$ and so $M(x_{G2n},u_G)\to d_G(Tu_G,u_G)$ Now, if $d_G(Tu_G,u_G)\neq 0$ then from Equation (10) as $n\to\infty$ we have,

$$\psi(\gamma d_G(Tu_G,u_G)\leq\psi(d_G(Tu_G,u_G))$$

again a contradiction and so $d_G(Tu_G,u_G)=0$ which implies that $u_G\in\overline{Tu_G}$ and since Tu_G is closed we have $u_G\in Tu_G$.

$u_G\in Su_G$ follows on the same line as in the proof of Theorem 1 and hence $u_G\in Su_G\bigcap Tu_G$. □

If the graph G is transitive then conditions (14.1) and (15.1) can be replaced with a much weaker condition as follows:

Definition 16. *The pair (S,T) belongs to $(\mathcal{C},\Psi^*,G,\gamma)$ contraction class if it satisfies the following:*

(16.1) *there exists $F\in\mathcal{C}$, $\psi\in\Psi^*$ such that*

$$\psi(H(Sx_G,Ty_G))\leq F(\psi(M(x_G,y_G)),M(x_G,y_G)) \text{ and}$$
$$\psi(H(Tx_G,Sy_G))\leq F(\psi(M(y_G,x_G)),M(y_G,x_G))$$

Theorem 3. *Let (X_G,d_G,s) be G-complete, G-regular with d_G continuous and $S,T:X_G\to CL(X_G)$ satisfy (1.1), ((1.2)) and the following:*

(3.1) *G satisfies transitivity property,*
(3.2) $(S, T) \in (\mathcal{C}, \Psi^*, G)$ *for some* $F \in \mathcal{C}$, $\psi \in \Psi^*$.

Then we can find $u_G \in X_G$ *such that* $u_G \in Su_G \bigcap Tu_G$.

Proof. As in the proof of Theorem 1, we construct the sequence as in Equation (1). Then again proceeding as in the same proof we can show that

$$\lim_{n\to\infty} d_G(x_{Gn}, x_{Gn+1}) = 0. \tag{11}$$

In order to show that sequence $\{x_{Gn}\}$ is G-Cauchy, it is enough to prove that the subsequence $\{x_{G2n}\}$ is G-Cauchy. If $\{x_{G2n}\}$ is not G-Cauchy, then we can find $\epsilon > 0$ and subsequences $\{x_{G2m(k)}\}$ and $\{x_{G2n(k)}\}$, such that $n(k)$ is the smallest integer for which $n(k) > m(k) > k, d_G(x_{G2m(k)}, x_{G2n(k)}) \geq \epsilon$. That is,

$$d_G(x_{G2m(k)}, x_{G2n(k)-2}) < \epsilon \tag{12}$$

Now, we have

$$\begin{aligned}
\epsilon &\leq d_G(x_{G2m(k)}, x_{G2n(k)}) \\
&\leq s d_G(x_{G2m(k)}, x_{G2n(k)-2}) + s^2 d_G(x_{G2n(k)-2}, x_{G2n(k)-1}) + s^2 d_G(x_{G2n(k)-1}, x_{G2n(k)}) \\
&< s\epsilon + s^2 d_G(x_{G2n(k)-2}, x_{G2n(k)-1}) + s^2 d_G(x_{G2n(k)-1}, x_{G2n(k)})
\end{aligned}$$

As $k \to \infty$, using Equation (11) in the above inequality we get

$$\epsilon \leq d_G(x_{G2m(k)}, x_{G2n(k)}) \leq s\epsilon, \quad \text{or}$$

$$\lim_{k\to\infty} d_G(x_{G2m(k)}, x_{G2n(k)}) = \lambda\epsilon, \qquad (1 \leq \lambda \leq s) \tag{13}$$

In addition, we have

$$\begin{aligned}
&|d_G(x_{G2m(k)}, x_{G2n(k)+1}) - d_G(x_{G2m(k)}, x_{G2n(k)})| \leq d_G(x_{G2n(k)}, x_{G2n(k)+1}) \\
\text{and} \quad &|d_G(x_{G2m(k)-1}, x_{G2n(k)}) - d_G(x_{G2m(k)}, x_{G2n(k)})| \leq d_G(x_{G2m(k)}, x_{2m(k)-1}).
\end{aligned}$$

Letting $k \to \infty$ and using Equations (11) and (13), we get

$$\lim_{k\to\infty} d_G(x_{G2m(k)-1}, x_{G2n(k)}) = \lim_{k\to\infty} d_G(x_{G2m(k)}, x_{G2n(k)+1}) = \lambda\epsilon. \tag{14}$$

From

$$|d_G(x_{G2m(k)-1}, x_{G2n(k)+1}) - d_G(x_{G2m(k)-1}, x_{G2n(k)})| \leq d_G(x_{G2n(k)}, x_{G2n(k)+1})$$

and making use of Equations (11) and (14), we get

$$\lim_{k\to\infty} d_G(x_{G2m(k)-1}, x_{G2n(k)+1}) = \lambda\epsilon \tag{15}$$

In addition, from the definition of M and from Equations (11) and (13)–(15), we have

$$\lim_{k\to\infty} M(x_{G2m(k)-1}, x_{G2n(k)}) = \lambda\epsilon \tag{16}$$

Since the graph G is transitive, we have $(x_{G2m(k)-1}, x_{G2n(k)}) \in E(G)$. Then

$$\begin{aligned}\psi(d_G(x_{G2m(k)}, x_{G2n(k)+1})) &\leq \psi(H(Tx_{G2m(k)-1}, Sx_{G2n(k)})) \\ &\leq F(\psi(M(x_{G2m(k)-1}, x_{G2n(k)})), (M(x_{G2m(k)-1}, x_{G2n(k)}))\end{aligned}$$

As $k \to \infty$, using Equations (14) and (15), we get

$$\psi(\lambda\epsilon) < \psi(\lambda\epsilon)$$

which is not possible. Hence, $\{x_{G2n}\}$ is G-Cauchy and subsequently $\{x_{Gn}\}$ is G-Cauchy. By G-completeness and G-regularity of (X_G, d_G), there exists $u_G \in X_G$, such that $x_{Gn} \to u_G$ as $n \to \infty$ and $(x_{G2n+1}, u_G), (x_{G2n}, u_G) \in E(G)$. By (3.2) we have

$$\begin{aligned}\psi(d_G(x_{G2n+1}, Tu_G)) &\leq \psi(H(Sx_{G2n}, Tu_G)) \\ &\leq F(\psi(M(x_{G2n}, u_G)), M(x_{G2n}, u_G))\end{aligned} \tag{17}$$

where

$$M(x_{G2n}, u_G) = \max\Big\{d_G(x_{G2n}, u_G), d_G(Sx_{G2n}, x_{G2n}), d_G(Tu_G, u_G), \frac{d_G(x_{G2n}, Tu_G) + d_G(u_G, Sx_{G2n})}{2s}\Big\}$$

Note that as $n \to \infty$, $d_G(Sx_{G2n}, x_{G2n}) \to 0$, $d_G(u_G, Sx_{G2n}) \to 0$, and so $M(x_{G2n}, u_G) \to d_G(Tu_G, u_G)$ Now, if $d_G(Tu_G, u_G) \neq 0$, then from Equation (17) as $n \to \infty$, we have

$$\psi(d_G(Tu_G, u_G) < \psi(d_G(Tu_G, u_G))$$

again, a contradiction. Thus, $d_G(Tu_G, u_G) = 0$, which implies that $u_G \in \overline{Tu_G}$, and since Tu_G is closed, we have $u_G \in Tu_G$. Similarly we have

$$\begin{aligned}\psi(d_G(x_{G2n+2}, Su_G)) &\leq \psi(H(Tx_{G2n+1}, Su_G)) \\ &\leq F(\psi(M(u_G, x_{G2n+1})), M(u_G, x_{G2n+1}))\end{aligned} \tag{18}$$

where

$$M(u_G, x_{G2n+1}) = \max\Big\{d_G(u_G, x_{G2n+1}), d_G(Su_G, u_G), d_G(Tx_{G2n+1}, x_{G2n+1}), \frac{d_G(u_G, Tx_{G2n+1}) + d_G(x_{G2n+1}, Su_G)}{2s}\Big\}$$

Note that as $n \to \infty$, $d_G(Tx_{G2n+1}, x_{G2n+1}) \to 0$, $d_G(u_G, Tx_{G2n+1}) \to 0$, and so $M(u_G, x_{G2n+1}) \to d_G(Su_G, u_G)$ Now, if $d_G(Su_G, u_G) \neq 0$, then from Equation (18) as $n \to \infty$, we have

$$\psi(d_G(Su_G, u_G) < \psi(d_G(Su_G, u_G))$$

again, a contradiction. Thus, $d_G(Su_G, u_G) = 0$, which implies that $u_G \in \overline{Su_G}$, and since Su_G is closed, we have $u_G \in Su_G$. Thus $u_G \in Su_G \bigcap Tu_G$. $\square$

Now we have the following deductions from Theorems 1 and 2:

Corollary 1. *Let (X_G, d_G, s) be G-complete, G-regular and $S, T : X_G \to CL(X_G)$ satisfy conditions (1.1), (1.2) and the following:*

(1.1) For all $x_G, y_G \in X_G$ with $(x_G, y_G) \in E(G)$

$$\psi(\gamma s H(Sx_G, Ty_G)) \leq \psi(M(x_G, y_G)) - \phi(M(x_G, y_G)) \text{ and}$$
$$\psi(\gamma s H(Tx_G, Sy_G)) \leq \psi(M(y_G, x_G)) - \phi(M(y_G, x_G))$$

ψ, ϕ and $M(x_G, y_G)$ are as in Definition 14. Then we can find $u_G \in X_G$ such that $u_G \in Su_G \bigcap Tu_G$.

Proof. The proof follows by taking $F(r,t) = r - \phi(t)$ in Theorem 1. □

Corollary 2. *Let (X_G, d_G, s) be G-complete, G-regular, d_G be continuous and $S, T : X_G \to CL(X_G)$ satisfy conditions (1.1), (1.2) and the following:*

(2.1) For all $x_G, y_G \in X_G$ with $(x_G, y_G) \in E(G)$

$$H(Sx_G, Ty_G) \leq \theta(M(x_G, y_G))M(x_G, y_G) \text{ and}$$
$$H(Tx_G, Sy_G) \leq \theta(M(y_G, x_G))M(y_G, x_G)$$

where $\theta : [0, \infty) \to [0, \frac{1}{\gamma})$ satisfies $limsup_{n \to \infty}\theta(t_n) \to \frac{1}{\gamma}$ implies $t_n \to 0$, and $M(x_G, y_G)$ is as in Definition 14. Then we can find $u_G \in X_G$ such that $u_G \in Su_G \bigcap Tu_G$.

Proof. The proof follows by taking $F(r,t) = \gamma\theta(t).r$ and $\psi(t) = t$ in Theorem 2. □

Example 4. *Let $X_G = [0,1)$, $d_G(x_G, y_G) = |x_G - y_G|^2$, $G = (V, E)$, with $V(G) = X_G$ and $E(G) = \{(0,0), (\frac{1}{2^n}, \frac{1}{2^{n+1}}), (\frac{1}{2^n}, 0)\}$ and $S, T : X_G \to CB(X_G)$ be defined by*

$$Sx_G = \begin{cases} \{0\}, & \text{if } x_G = 0 \\ \{\frac{1}{2^{n+1}}, 0\}, & \text{if } x_G = \frac{1}{2^n}. \end{cases}$$

and

$$Tx_G = \begin{cases} \{0\}, & \text{if } x_G = 0 \\ \{\frac{1}{2^{n+2}}, 0\}, & \text{if } x_G = \frac{1}{2^n}. \end{cases}$$

Then clearly d_G is a continuous b-metric and (X_G, d_G) is a G-complete b-metric space endowed with graph G, but not complete. Let $\psi(t) = 2t + 1$ and $\phi(t) = \frac{t}{4}$ for all $t \in [0, \infty)$. Then $\psi(t) \in \Psi^*$, $\psi \notin \Psi$ and $\phi(t) \in \Phi^*$. Choose $\gamma = \frac{3}{2}$.

- For $(x_G, y_G) = (\frac{1}{2^n}, 0) \in E(G)$, we have $Sx_G = \{\frac{1}{2^{n+1}}, 0\}, Sy_G = \{0\}, Tx_G = \{\frac{1}{2^{n+2}}, 0\}$, $Ty_G = \{0\}$, $d_G(x_G, y_G) = \frac{1}{2^{2n}}$, $H(Sx_G, Ty_G) = \frac{1}{2^{2n+2}}, \psi(\gamma H(Sx_G, Ty_G)) = \frac{3}{2^{2n+2}} + 1, M(x_G, y_G) = \frac{1}{2^{2n}}, \psi(M(x_G, y_G)) = \frac{1}{2^{2n-1}} + 1, \phi(M(x_G, y_G)) = \frac{1}{2^{2n+2}}$ and

$$\begin{aligned} \psi(\gamma H(Sx_G, Ty_G)) &= \frac{3}{2^{2n+2}} + 1 < \frac{7}{2^{2n+2}} + 1 \\ &= \frac{1}{2^{2n-1}} + 1 - \frac{1}{2^{2n+2}} = \psi(M(x_G, y_G)) - \phi(M(x_G, y_G)). \end{aligned}$$

- For $(x_G, y_G) = (0,0) \in E(G)$, we have $Sx_G = \{0\} = Sy_G = Tx_G = Ty_G, 0 = d_G(x_G, y_G) = H(Sx_G, Ty_G) = M(x_G, y_G), \psi(H(Sx_G, Ty_G)) = \psi(M(x_G, y_G)) = 1, \phi(M(x_G, y_G)) = 0$ and

$$\psi(H(Sx_G, Ty_G)) = 1 = \psi(M(x_G, y_G)) - \phi(M(x_G, y_G)).$$

- In addition, for $(x_G, y_G) = (\frac{1}{2^n}, \frac{1}{2^{n+1}}) \in E(G)$, we have $Sx_G = \{\frac{1}{2^{n+1}}, 0\}$, $Sy_G = \{\frac{1}{2^{n+2}}, 0\}$, $Tx_G = \{\frac{1}{2^{n+2}}, 0\}$, $Ty_G = \{\frac{1}{2^{n+3}}, 0\}$. Simple calculations shows that $d_G(x_G, y_G) = \frac{1}{2^{2n+2}}$,

$H(Sx_G, Ty_G) = \frac{9}{2^{2n+6}}$, $\psi(\gamma H(Sx_G, Ty_G)) = \frac{27}{2^{2n+6}} + 1$, $M(x_G, y_G) = \frac{1}{2^{2n+2}}$, $\psi(M(x_G, y_G)) = \frac{1}{2^{2n+1}} + 1$, $\phi(M(x_G, y_G)) = \frac{1}{2^{2n+4}}$ and

$$\begin{aligned}\psi(\gamma H(Sx_G, Ty_G)) &= \frac{27}{2^{2n+6}} + 1 < \frac{28}{2^{2n+6}} + 1 \\ &< \frac{1}{2^{2n+1}} + 1 - \frac{1}{2^{2n+4}} = \psi(M(x_G, y_G)) - \phi(M(x_G, y_G)) \text{ for all } n \in \mathbb{N}.\end{aligned}$$

Thus, we see that for all $x_G, y_G \in X_G$ with $(x_G, y_G) \in E(G)$

$$\psi(H(Sx_G, Ty_G)) \leq \psi(M(x_G, y_G)) - \phi(M(x_G, y_G))$$

Simple calculations also show that S and T are pairwise R-weakly graph preserving. Thus taking $F(r, t) = r - \phi(t)$, we see that all conditions of Theorem 2 are satisfied and $0 \in S(0) \bigcap T(0)$.

Next we introduce the class of generalized rational contractions.

Definition 17. *Let $S, T : X_G \to CL(X_G)$. Then the pair (S, T) belongs to $(\mathcal{C}, \Psi^*, G, \gamma s)$ rational contractions class if and only if for all $x_G, y_G \in X_G$ with $(x_G, y_G) \in E(G)$, the following holds:*

(17.1) there exists $F \in \mathcal{C}$, $\psi \in \Psi^*$ and $\gamma > 1$ such that

$$\psi(\gamma s H(Sx_G, Ty_G)) \leq F(\psi(M^{RC}(x_G, y_G)), M^{RC}(x_G, y_G)) \text{ and}$$
$$\psi(\gamma s H(Tx_G, Sy_G)) \leq F(\psi(M^{RC}(y_G, x_G)), M^{RC}(x_G, y_G))$$

where

$$M^{RC}(x_G, y_G) = \max\Big\{d_G(x_G, y_G), d_G(x_G, Ty_G)d_G(y_G, Sx_G), \frac{d_G(x_G, Sx_G)d_G(y_G, Ty_G)}{1 + d_G(x_G, y_G)}, \frac{d_G(x_G, Sx_G)d_G(x_G, Ty_G) + d_G(y_G, Ty_G)d_G(y_G, Sx_G}{2s[1 + d(x_G, y_G)]}\Big\}$$

Theorem 4. *Let (X_G, d_G, s) be G-complete and $S, T : X_G \to CL(X_G)$ satisfy (1.1), (1.2) and the following:*

(4.1) $(S, T) \in (\mathcal{C}, \Psi^*, G)$ rational contractions.

Then there exists $u_G \in X_G$ such that $u_G \in Su_G \bigcap Tu_G$.

Proof. Proceeding as in Theorem 1, we construct the sequence $\{x_{Gn}\}$ as given in Equation (1).

For an odd integer n we have

$$\begin{aligned}\psi(\gamma d_G(x_{Gn}, x_{Gn+1})) &\leq \psi(\gamma s H(Sx_{Gn-1}, Tx_n)) \\ &\leq F(\psi(M^{RC}(x_{Gn-1}, x_{Gn})), M^{RC}(x_{Gn-1}, x_{Gn})) \\ &\leq \psi(M^{RC}(x_{Gn-1}, x_{Gn}))\end{aligned} \tag{19}$$

where

$$\begin{aligned} M^{RC}(x_{Gn-1}, x_{Gn}) = \max \Big\{ & d_G(x_{Gn-1}, x_{Gn}), d_G(x_{Gn-1}, Tx_{Gn}) d_G(x_{Gn}, Sx_{Gn-1}) \\ & \frac{d_G(Sx_{Gn-1}, x_{Gn-1}) d_G(Tx_{Gn}, x_{Gn})}{1 + d_G(x_{Gn-1}, x_{Gn})}, \\ & \frac{d_G(x_{Gn-1}, Sx_{Gn-1}) d_G(x_{Gn-1}, Tx_{Gn}) + d_G(x_{Gn}, Tx_{Gn}) d_G(x_{Gn}, Sx_{Gn-1}}{2s[1 + d(x_{Gn-1}, x_{Gn})]} \Big\} \\ \leq \max \Big\{ & d_G(x_{Gn-1}, x_{Gn}), d_G(x_{Gn-1}, x_{Gn+1}) d_G(x_{Gn}, x_{Gn}) \\ & \frac{d_G(x_{Gn}, x_{Gn-1}) d_G(x_{Gn+1}, x_{Gn})}{1 + d_G(x_{Gn-1}, x_{Gn})}, \\ & \frac{d_G(x_{Gn-1}, x_{Gn}) d_G(x_{Gn-1}, x_{Gn+1}) + d_G(x_{Gn}, x_{Gn+1}) d_G(x_{Gn}, x_{Gn})}{2s[1 + d(x_{Gn-1}, x_{Gn})]} \Big\} \\ \leq \max \Big\{ & d_G(x_{Gn-1}, x_{Gn}), d_G(x_{Gn+1}, x_{Gn}), \\ & \frac{d_G(x_{Gn-1}, x_{Gn}) + d_G(x_{Gn}, x_{Gn+1})}{2} \Big\} \\ \leq \max & \{ d_G(x_{Gn-1}, x_{Gn}), d_G(x_{Gn}, x_{Gn+1}) \} \end{aligned}$$

If $d_G(x_{Gn+1}, x_{Gn}) > d_G(x_{Gn-1}, x_{Gn})$, then $M^{RC}(x_{Gn-1}, x_{Gn}) \leq d_G(x_{Gn+1}, x_{Gn})$. Then Equation (19) gives

$$\psi(\gamma d_G(x_{Gn}, x_{Gn+1})) < \psi(d_G(x_{Gn}, x_{Gn+1}))$$

a contradiction. So, we have

$$d_G(x_{Gn}, x_{Gn+1}) \leq d_G(x_{Gn-1}, x_{Gn}) \tag{20}$$

and so

$$M^{RC}(x_{Gn-1}, x_{Gn}) \leq d_G(x_{Gn-1}, x_{Gn})$$

By Equation (19) we get

$$\psi(\gamma d_G(x_{Gn}, x_{Gn+1})) \leq \psi(d_G(x_{Gn-1}, x_{Gn}))$$

and then

$$d_G(x_{Gn}, x_{Gn+1}) \leq \frac{1}{\gamma} d_G(x_{Gn-1}, x_{Gn}) \tag{21}$$

In case of an even integer n, a similar proof will give inequality Equation (21). By Lemma 1 we conclude that $\{x_{Gn}\}$ is a Cauchy sequence. By G-completeness and G-regularity of (X_G, d_G, s), we can find $u_G \in X_G$ such that $x_{Gn} \to u_G$ as $n \to \infty$ and $(x_{G2n}, u_G) \in E(G)$.

We will show that $u_G \in Su_G \bigcap Tu_G$. We have

$$\begin{aligned} \psi(\gamma d_G(u_G, Tu_G)) &\leq \psi(s[\gamma d_G(u_G, x_{G2n+1})) + \gamma d_G(x_{G2n+1}, Tu_G)]) \\ &\leq \psi(s[\gamma d_G(u_G, x_{G2n+1})) + \gamma H(Sx_{G2n}, Tu_G)]) \end{aligned}$$

As $n \to \infty$, we get

$$\psi(\gamma d_G(u_G, Tu_G)) \leq lim_{n\to\infty} \psi(s\gamma H(Sx_{G2n}, Tu_G)) \tag{22}$$

Now,

$$\psi(s\gamma H(Sx_{G2n}, Tu_G)) \leq F(\psi(M^{RC}(x_{G2n}, u_G)), M(x_{G2n}, u_G))$$

where

$$\begin{aligned} M^{RC}(x_{G2n}, u_G) = \max \Big\{ & d_G(x_{G2n}, u_G), d_G(x_{G2n}, Tu_G) d_G(u_G, Sx_{G2n}) \\ & \frac{d_G(Sx_{G2n}, x_{G2n}) d_G(Tu_G, u_G)}{1 + d_G(x_{G2n}, u_G)}, \\ & \frac{d_G(x_{G2n}, Sx_{G2n}) d_G(x_{G2n}, Tu_G) + d_G(u_G, Tu_G) d_G(u_G, Sx_{G2n}}{2s[1 + d(x_{Gn-1}, u_G)]} \Big\} \end{aligned}$$

Note that as $n \to \infty$, $d_G(Sx_{G2n}, x_{G2n}) \to 0$, $d_G(u_G, Sx_{G2n}) \to 0$ and so $M^{RC}(x_{G2n}, u_G) \to 0$. Then from Equation (22) as $n \to \infty$ we have,

$$\psi(\gamma d_G(Tu_G, u_G) \leq 0$$

and so $d_G(Tu_G, u_G) = 0$ which implies that $u_G \in \overline{Tu_G}$ and since Tu_G is closed we have $u_G \in Tu_G$.

Again we have

$$\begin{aligned} \psi(\gamma d_G(u_G, Su_G)) &\leq \psi(s[\gamma d_G(u_G, x_{G2n})) + \gamma d_G(x_{G2n}, Su_G)]) \\ &\leq \psi(s[\gamma d_G(u_G, x_{G2n})) + \gamma H(Tx_{G2n-1}, Su_G)]) \end{aligned}$$

As $n \to \infty$, we get

$$\psi(\gamma d_G(u_G, Su_G)) \leq lim_{n\to\infty} \psi(s\gamma H(Tx_{G2n-1}, Su_G)). \tag{23}$$

Now, using (17.1) we have

$$\psi(s\gamma H(Tx_{G2n-1}, Su_G)) \leq F(\psi(M^{RC}(u_G, x_{G2n-1})), M(u_G, x_{G2n-1}))$$

where

$$\begin{aligned} M(u_G, x_{G2n-1}) = \max \Big\{ & d_G(x_{G2n}, u_G), d_G(u_G, Tx_{G2n-1}) d_G(x_{G2n-1}, Su_G) \\ & \frac{d_G(Su_G, u_G) d_G(Tx_{G2n-1}, x_{G2n=1})}{1 + d_G(x_{G2n-1}, u_G)}, \\ & \frac{d_G(u_G, Su_G) d_G(u_G, Tx_{G2n-1}) + d_G(x_{G2n-1}, Tx_{G2n-1}) d_G(x_{G2n-1}, Su_G}{2s[1 + d(x_{G2n-1}, u_G)]} \Big\} \end{aligned}$$

Note that as $n \to \infty$, $d_G(Tx_{G2n-1}, x_{G2n-1}) \to 0$, $d_G(u_G, Tx_{G2n-1}) \to 0$ and so $M(u_G, x_{G2n-1}) \to 0$. Then from Equation (23) as $n \to \infty$ we have,

$$\psi(\gamma d_G(u_G, Su_G) \leq 0$$

and so $d_G(u_G, Su_G) = 0$ which implies that $u_G \in \overline{Su_G}$ and since Su_G is closed we have $u_G \in Su_G$. Hence $COFIX\{S, T\} \neq \phi$. □

Definition 18. *The pair (S, T) belongs to $(\mathcal{C}, \Psi^*, G, \gamma)$ rational contractions class if it satisfies the following:*

(18.1) there exists $F \in \mathcal{C}$, $\psi \in \Psi^*$ and $\gamma > 1$ such that

$$\begin{aligned} &\psi(\gamma H(Sx_G, Ty_G)) \leq F(\psi(M^{RC}(x_G, y_G)), M^{RC}(x_G, y_G)) \ \textit{and} \\ &\psi(\gamma H(Tx_G, Sy_G)) \leq F(\psi(M^{RC}(y_G, x_G)), M^{RC}(y_G, x_G)) \end{aligned}$$

Proceeding on the same lines as in the proof of Theorems 2 and 4 we can prove the following

Theorem 5. *Let (X_G, d_G, s) be G-complete, G-regular, d_G continuous and $S, T : X_G \to CL(X_G)$ satisfy (1.1), (1.2) and the following:*

(5.1) $(S, T) \in (\mathcal{C}, \Psi^*, G, \gamma)$ *rational contractions class.*

Then we can find $u_G \in X_G$ such that $u_G \in Su_G \bigcap Tu_G$.

3.3. Common Fixed Point Theorems for R-Weakly α-Admissible Mappings in a b-Metric Space

This section deals with common fixed point theorems for R-weakly α-admissible mappings in a b-metric space which are obtained as direct application of our results of Section 3.2.

Theorem 6. *Let (X, d_b, s) be α-complete, α-regular and the following conditions holds:*

(6.1) *There exists $x_0, x_1 \in X$ such that $x_1 \in Tx_0 \bigcup Sx_0$ and $\alpha(x_0, x_1) \geq s$,*
(6.2) *The pair (S, T) is R-weakly α-admissible of type S,*
(6.3) *for some $F \in \mathcal{C}$, $\psi \in \Psi^*$ and for all $u, v \in X$ with $\alpha(u, v) \geq s$*

$$\psi(\gamma s H(Su, Tv)) \leq F(\psi(M^\alpha(u, v)), M^\alpha(u, v)) \text{ and}$$
$$\psi(\gamma s H(Tu, Sv)) \leq F(\psi(M^\alpha(v, u)), M^\alpha(v, u))$$

where

$$M^\alpha(u, v) = \max\left\{d_b(u, v), d_b(Su, u), d_b(Tv, v), \frac{d_b(v, Su) + d_b(u, Tv)}{2s}\right\}$$

Then there exist $z \in X$ such that $z \in Sz \bigcap Tz$.

Proof. Endow the b-metric space (X, d_b, s) with a graph G whose set of vertces is given by $V(G) = X$ and the set of edges is given by $E(G) = \{(u, v) \in X \times X : \alpha(u, v) \geq s\}$. Then S and T satisfies the conditions of Theorem 1 and hence by Theorem 1 there exists $z \in X$ such that $z \in Sz \bigcap Tz$. □

Similarly using Theorems 2–5 respectively, we have the following results:

Theorem 7. *Let (X, d_b, s) be α-complete and α-regular, d_b be continuous and let conditions (6.1), (6.2) and the the following holds:*

(7.1) *for some $F \in \mathcal{C}$, $\psi \in \Psi^*$ and for all $u, v \in X$ with $\alpha(u, v) \geq s$*

$$\psi(\gamma H(Su, Tv)) \leq F(\psi(M^\alpha(u, v)), M^\alpha(u, v)) \text{ and}$$
$$\psi(\gamma H(Tu, Sv)) \leq F(\psi(M^\alpha(v, u)), M^\alpha(v, u))$$

Then there exist $z \in X$ such that $z \in Sz \bigcap Tz$.

Theorem 8. *Let (X, d_b, s) be α-complete and α-regular, d_b be continuous and let conditions (6.1), (6.2) and the the following holds:*

(8.1) *α is a triangular function, that is if $\alpha(u, v) \geq s$ and $\alpha(v, w) \geq s$ then $\alpha(u, w) \geq s$,*
(8.2) *for some $F \in \mathcal{C}$, $\psi \in \Psi^*$ and for all $u, v \in X$ with $\alpha(u, v) \geq s$*

$$\psi(H(Su, Tv)) \leq F(\psi(M^\alpha(u, v)), M^\alpha(u, v)) \text{ and}$$
$$\psi(H(Tu, Sv)) \leq F(\psi(M^\alpha(v, u)), M^\alpha(v, u))$$

Then there exist $z \in X$ such that $z \in Sz \bigcap Tz$.

Theorem 9. *Let (X, d_b, s) be α-complete and α-regular, and let conditions (6.1), (6.2) and the the following holds:*

(9.1) *for some* $F \in \mathcal{C}$, $\psi \in \Psi^*$ *and for all* $u, v \in X$ *with* $\alpha(u,v) \geq s$

$$\psi(\gamma s H(Su,Tv)) \leq F(\psi(M^{RC}(u,v)), M^{\alpha}(u,v)) \text{ and}$$
$$\psi(\gamma s H(Tu,Sv)) \leq F(\psi(M^{RC}(v,u)), M^{\alpha}(v,u))$$

where

$$M^{RC}(u,v) = \max\Big\{d_b(u,v), d_b(u,Tv)d_b(v,Su), \frac{d_b(u,Su)d_b(v,Tv)}{1+d_b(u,v)}, \frac{d_b(u,Su)d_b(u,Tv)+d_b(v,Tv)d_b(v,Su}{2s[1+d_b(u,v)]}\Big\}$$

Then there exists $z \in X$ *such that* $z \in Sz \bigcap Tz$.

Theorem 10. *Let* (X, d_b, s) *be* α*-complete and* α*-regular,* d_b *be continuous and let conditions (6.1), (6.2) and the the following holds:*

(10.1) *for some* $F \in \mathcal{C}$, $\psi \in \Psi^*$ *and for all* $u, v \in X$ *with* $\alpha(u,v) \geq s$

$$\psi(\gamma H(Su,Tv)) \leq F(\psi(M^{RC}(u,v)), M^{\alpha}(u,v)) \text{ and}$$
$$\psi(\gamma H(Tu,Sv)) \leq F(\psi(M^{RC}(v,u)), M^{\alpha}(v,u))$$

Then there exists $z \in X$ *such that* $z \in Sz \bigcap Tz$.

Corollary 3. *Let* (X, d_b, s) *be* α*-complete and* α*-regular,* d_b *be continuous and let conditions (6.1), (6.2) and the the following holds:*

(3.1) *For all* $u, v \in X$ *with* $\alpha(u,v) \geq s$

$$H(Su,Tv) \leq \theta(M^{RC}(u,v))M^{RC}(u,v) \text{ and}$$
$$H(Tu,Sv) \leq \theta(M^{RC}(v,u))M^{RC}(v,u)$$

where $\theta : [0,\infty) \to [0, \frac{1}{\gamma})$ *satisfies* $limsup_{n\to\infty}\theta(t_n) \to \frac{1}{\gamma}$ *implies* $t_n \to 0$. *Then there exists* $z \in X$ *such that* $z \in Sz \bigcap Tz$.

Proof. Take $F(r,t) = \gamma\theta(t).r$ and $\psi(t) = t$ in Theorem 10. □

4. Discussions

Remark 4. *Corollary 1 and hence Theorems 1, 2 and 4 are proper extension and generalization of the results of [4–7].*

Remark 5. *Corollary 2 and hence Theorem 2 is a proper extension and generalization of Theorem 3.1 of [9] and Theorem 1.13 of [15] and some of the references therein.*

Remark 6. *For* $s = 1$ *Theorem 3 reduces to Theorem 2.2 of [3] wherein we do not require the metric space* (X_G, d_G) *to be complete and also we do not require the condition* $\Delta = \{(x_G, x_G) : x_G \in X_G\} \subset E(G)$. *Hence Theorem 3 is a substantial improvement and generalisation of Theorem 2.2 of [3].*

Remark 7. *In Example 4 above, note that the mappings* S *and* T *satisfy condition (3.2) also but Theorem 3 (which is a proper improvement of Theorem 3.1 and Theorem 3.2 of [3]) is not applicable as* (X_G, d_G) *is not complete,* Δ *is not a subset of* $E(G)$ *and* $E(G)$ *is not transitive.*

Remark 8. *In view of Remark 3, Theorems 6 and 7 are proper generalisations of Theorems 3.1 and 3.2 of [26] in the sense that our results are valid even for* $s = 1$ *and also we do not require the mappings* (S, T) *to be triangular* α_* *orbital admissible.*

Remark 9. *In [2], the authors used function $\psi : [0, \infty) \to [0, \infty)$ which satisfies the condition $\sum_{n=1}^{+\infty} s^n \psi^n(t) < +\infty$, which in turn implies $\psi(t) < t$ for $t > 0$ and $\psi(0) = 0$. The Ψ^* class of functions used in our Theorem 8 is more general than the one used in [2]. Moreover if $\psi(t) < t$, then condition (3) of Definition 9 in [2] implies (8.2) (for $S = T$). Clearly Theorem 8 is a proper extension and generalisation of the results in [2].*

Remark 10. *In [27], the authors introduced α-Geraghty contractions of type I, II and III for a single valued mapping in a b-metric space. Condition (3.1) above defines α-Geraghty contractions of type R for a pair of multivalued mappings in a b-metric space.*

Remark 11. *(Open problem) In [28] the authors proved fixed point theorems for Ciric type quasi contractions in a b-metric space with Qt-functions and in [29] the authors proved Suzuki type fixed point theorem in a p-metric space. There is further scope for extending and generalising the results in [28] and [29] to R-weakly graph preserving pair of multivalued mappings in b-metric space with Qt-functions endowed with a graph and p-metric space endowed with a graph respectively.*

5. Conclusions

In the known literature of fixed point theory many results have been generalized from the metric space to a metric space endowed with some binary relation such as the partial order or the graph. The same is the case as well with other generalized classes of usual metric spaces such as: b-metric, partial metric, partial b-metric, cone metric, cone b-metric, G-metric, G_b-metric and others. In this paper we introduced the concepts of R-weakly graph preserving and R-weakly $\alpha-$admissible pair of multivalued mappings and considered the question of common fixed point results for R-weakly graph preserving and R-weakly $\alpha-$admissible pair of multivalued mappings in a b-metric space by defining $(\mathcal{C}, \Psi^*, G, \gamma s)$ and $(\mathcal{C}, \Psi^*, G, \gamma)$ contractions and rational contractions. Our results and its consequences generalize, improve, compliment, unify, enrich and extend many known fixed point results in existing literature. Further, we see a wide scope for extension and generalization of other fixed point results existing in literature by using R-weakly graph preserving and R-weakly $\alpha-$admissible pair of multivalued mappings.

Author Contributions: Formal analysis, R.G., H.A.N. and S.R.; Methodology, R.G.; Project administration, R.G.; Writing—original and draft, R.R.; Writing—review and editing, R.G., H.A.N. and S.R.

Funding: This research was funded by Deanship of Scientific Research at Prince Sattam bin Abdulaziz University, Al kharj, Kingdom of Saudi Arabia under International Project Grant No. 2016/01/6714.

Acknowledgments: 1. This research is supported by Deanship of Scientific Research at Prince Sattam bin Abdulaziz University, Al kharj, Kingdom of Saudi Arabia under International Project Grant No. 2016/01/6714. 2. The authors are thankful to the anonymous reviewers and the editor for their valuable suggestions which helped in improving this paper to its present form.

Conflicts of Interest: Authors declare there is no conflict of interest.

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