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New Polynomial Bounds for Jordan's and Kober's Inequalities Based on the Interpolation and Approximation Method

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Abstract: In this paper, new refinements and improvements of Jordan's and Kober's inequalities are presented. We give new polynomial bounds for the $sinc(x)$ and $\cos(x)$ functions based on the interpolation and approximation method. The results show that our bounds are tighter than the previous methods.

Keywords: Jordan's inequality; Kober's inequality; polynomial; bounds

1. Introduction

Jordan's inequality:

$$\frac{2}{\pi} \le sinc(x) = \frac{\sin(x)}{x} < 1, \qquad x \in (0, \pi/2], \tag{1}$$

has been studied in a large number of literature works, and many refinements have been presented [1–6].

Zhang et al. [7] gave the polynomial bounds of degree one:

$$\frac{2}{\pi} + \frac{\pi - 2}{\pi^2}(\pi - 2x) \le \frac{\sin(x)}{x} \le \frac{2}{\pi} + \frac{2}{\pi^2}(\pi - 2x), \qquad x \in (0, \pi/2]. \tag{2}$$

Zhang and Ma [8] gave the improvement of Inequality (2):

$$1 + \frac{4 - 2\pi}{\pi^2}x \le \frac{\sin(x)}{x} \le \frac{8\sqrt{2} - \sqrt{2}\pi}{2\pi} + \frac{2\sqrt{2} - 8\sqrt{2}}{\pi^2}x, \qquad x \in (0, \pi/2]. \tag{3}$$

Qi et al. [9] presented the polynomial bounds of degree two:

$$\frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2) \le \frac{\sin(x)}{x} \le \frac{2}{\pi} + \frac{\pi - 2}{\pi^3}(\pi^2 - 4x^2), \qquad x \in (0, \pi/2]. \tag{4}$$

Zhang and Ma [8] gave the improvement of Inequality (4):

$$1 + \frac{12 - 4\pi}{\pi^2}x + \frac{4\pi - 16}{\pi^3}x^2 \le \frac{\sin(x)}{x} \le 1 + \frac{8 - 4\pi}{\pi^3}x^2, \qquad x \in (0, \pi/2]. \tag{5}$$

Deng [10] obtained the polynomial bounds of degree three:

$$\frac{2}{\pi} + \frac{2}{3\pi^4}(\pi^3 - 8x^3) \le \frac{\sin(x)}{x} \le \frac{2}{\pi} + \frac{\pi - 2}{\pi^4}(\pi^3 - 8x^3), \qquad x \in (0, \pi/2], \tag{6}$$

and Jiang and Yun [11] gave the polynomial bounds of degree four:

$$\frac{2}{\pi}+\frac{1}{2\pi^5}(\pi^4-16x^4)\le\frac{\sin(x)}{x}\le\frac{2}{\pi}+\frac{\pi-2}{\pi^5}(\pi^4-16x^4),\qquad x\in(0,\pi/2]. \tag{7}$$

Debnath et al. [12] gave the improvements of Inequality (4) and Inequality (7):

$$g^l_{4,D1}(x)\le\frac{\sin(x)}{x}\le g^u_{4,D1}(x),\qquad x\in(0,\pi/2], \tag{8}$$

and:

$$g^l_{4,D2}(x)\le\frac{\sin(x)}{x}\le g^u_{4,D2}(x),\qquad x\in(0,\pi/2], \tag{9}$$

where

$g^l_{4,D1}(x)=\frac{2}{\pi}+\frac{1}{\pi^3}(\pi^2-4x^2)+(1-\frac{3}{\pi})-(\frac{1}{6}-\frac{4}{\pi^3})x^2,$
$g^u_{4,D1}(x)=\frac{2}{\pi}+\frac{1}{\pi^3}(\pi^2-4x^2)+(1-\frac{3}{\pi})-(\frac{1}{6}-\frac{4}{\pi^3})x^2+\frac{1}{120}x^4,$
$g^l_{4,D2}(x)=\frac{2}{\pi}+\frac{1}{2\pi^5}(\pi^4-16x^4)+(1-\frac{5}{2\pi})-\frac{1}{6}x^2,$
$g^u_{4,D2}(x)=\frac{2}{\pi}+\frac{\pi-2}{2\pi^5}(\pi^4-16x^4)+(1-\frac{5}{2\pi})-\frac{1}{6}x^2+(\frac{8}{\pi^5}+\frac{1}{120})x^4.$

Agarwal et al. [13] and Chen et al. [14] presented the further improvements of the polynomials bounds of degree three and four:

$$g^l_{3,A}(x)\le\frac{\sin(x)}{x}\le g^u_{3,A}(x),\qquad x\in(0,\pi/2], \tag{10}$$

$$g^l_{3,C}(x)\le\frac{\sin(x)}{x}\le g^u_{3,C}(x),\qquad x\in(0,\pi/2], \tag{11}$$

$$g^l_{4,C}(x)\le\frac{\sin(x)}{x}\le g^u_{4,C}(x),\qquad x\in(0,\pi/2], \tag{12}$$

where

$g^l_{3,A}(x)=1+\frac{4(66-43\pi+7\pi^2)}{\pi^2}x-\frac{4(124-83\pi+14\pi^2)}{\pi^3}x^2-\frac{4(12-4\pi)}{\pi^4}x^3,$
$g^u_{3,A}(x)=1+\frac{4(75-49\pi+8\pi^2)}{\pi^2}x-\frac{4(142-95\pi+16\pi^2)}{\pi^3}x^2-\frac{4(12-4\pi)}{\pi^4}x^3,$
$g^l_{3,C}(x)=1-\frac{4(3\pi-8)}{\pi^3}x^2+\frac{16(\pi-3)}{\pi^4}x^3,$
$g^u_{3,C}(x)=1-\frac{2(5\pi-2-16\sqrt{2}+2\sqrt{2}\pi)}{\pi^2}x+\frac{8(4\pi-4-16\sqrt{2}+3\sqrt{2}\pi)}{\pi^3}x^2-\frac{32(\pi-2-4\sqrt{2}+\sqrt{2}\pi)}{\pi^4}x^3,$
$g^l_{4,C}(x)=1-\frac{4(-48\sqrt{2}-2+17\pi+4\sqrt{2}\pi)}{\pi^3}x^2+\frac{32(-28\sqrt{2}-2+9\pi+3\sqrt{2}\pi)}{\pi^4}x^3-\frac{64(-16\sqrt{2}-2+5\pi+2\sqrt{2}\pi)}{\pi^5}x^4,$
$g^u_{4,C}(x)=1-\frac{4(-8\sqrt{2}-7+3\pi+2\sqrt{2}\pi)}{\pi^2}x+\frac{4(-32\sqrt{2}-68+13\pi+16\sqrt{2}\pi)}{\pi^3}x^2-\frac{32(-4\sqrt{2}-26+3\pi+5\sqrt{2}\pi)}{\pi^4}x^3+\frac{64(-12+\pi+2\sqrt{2}\pi)}{\pi^5}x^4.$

Zhang and Ma [8] gave the polynomial bounds of degree five:

$$g^l_5(x)\le\frac{\sin(x)}{x}\le g^u_5(x), \tag{13}$$

where

$g^l_5(x)=1+\frac{32-2048\sqrt{2}+2187\sqrt{3}-(113+128\sqrt{2})\pi}{2\pi^2}x+\frac{-448+26{,}624\sqrt{2}-27{,}702\sqrt{3}+(1255+1536\sqrt{2})\pi}{2\pi^3}x^2$
$+\frac{1168-62{,}464\sqrt{2}+64{,}152\sqrt{3}-(2825+3392\sqrt{2})\pi}{\pi^4}x^3+\frac{-2688+125{,}952\sqrt{2}-128{,}304\sqrt{3}+(5664+6528\sqrt{2})\pi}{\pi^5}x^4$
$+\frac{2304-92{,}160\sqrt{2}+93{,}312\sqrt{3}-(4176+4608\sqrt{2})\pi}{\pi^6}x^5,$

$g_5^u(x) = 1 + \frac{64+256\sqrt{2}-(92+32\sqrt{2})\pi}{\pi^3}x^2 + \frac{-624-1536\sqrt{2}+(528+256\sqrt{2})\pi}{\pi^4}x^3$
$+ \frac{1920+3072\sqrt{2}-(1088+640\sqrt{2})\pi}{\pi^5}x^4 + \frac{-1792-2048\sqrt{2}+(768+512\sqrt{2})\pi}{\pi^6}x^5.$

Zeng and Wu [15] obtained the polynomial bounds of degree $m(m \geq 2)$ for $sinc(x)$:

$$\frac{2}{\pi} + \frac{2}{m\pi^{m+1}}(\pi^m - 2^m x^m) \leq \frac{\sin(x)}{x} \leq \frac{2}{\pi} + \frac{\pi-2}{\pi^{m+1}}(\pi^m - 2^m x^m), \qquad x \in (0, \pi/2]. \tag{14}$$

Another famous inequality,

$$\cos(x) \geq 1 - \frac{2}{\pi}x, \qquad x \in [0, \pi/2], \tag{15}$$

is called Kober's inequality. Some improvements for Kober's inequality have been proven [16,17]. Sándor [18] presented the polynomial bounds of degree one and two for $\cos(x)$:

$$1 - \frac{2}{\pi}x \leq \cos(x) \leq 1 - \frac{2}{\pi}x + \frac{2}{\pi^2}x(\pi - 2x), \qquad x \in [0, \pi/2], \tag{16}$$

$$1 - \frac{x^2}{2} \leq \cos(x) \leq 1 - \frac{4x^2}{\pi^2}, \qquad x \in [0, \pi/2]. \tag{17}$$

Zhang et al. [7] gave the refinement of Kober's inequality:

$$1 - \frac{4-\pi}{\pi}x - \frac{2(\pi-2)}{\pi^2}x^2 \leq \cos(x) \leq 1 - \frac{4}{\pi^2}x^2, \qquad x \in [0, \pi/2]. \tag{18}$$

Bhayo and Sándor [19] further proved that:

$$1 - \frac{x^2/2}{1 + x^2/12} \leq \cos(x) \leq 1 - \frac{24x^2/(5\pi^2)}{1 + 4x^2/(5\pi^2)}, \qquad x \in [0, \pi/2]. \tag{19}$$

It is very obvious that the right sides of Inequality (16), Inequality (17), and Inequality (18) are the same. Recently, Bercu [20] provided a Padé-approximant-based method and obtained the following inequalities:

$$\frac{-7x^2 + 60}{3x^2 + 60} < \frac{\sin(x)}{x} < \frac{11x^4 - 360^2 + 2520}{60x^2 + 2520}, \qquad x \in (0, \pi/2]. \tag{20}$$

$$\frac{17x^4 - 480x^2 + 1080}{2x^4 + 60x^2 + 1080} < \cos(x) < \frac{3x^4 - 56^2 + 120}{4x^2 + 120}, \qquad x \in [0, \pi/2]. \tag{21}$$

Zhang et al. [21] gave the improvements of Inequality (20) and Inequality (21):

$$\frac{60{,}480 - 9240x^2 + 364x^4 - 5x^6}{840(72 + x^2)} < \frac{\sin(x)}{x} < \frac{166{,}320 - 22{,}260x^2 + 551x^4}{15(11{,}088 + 364x^2 + 5x^4)}, \qquad x \in (0, \pi/2]. \tag{22}$$

$$\frac{20{,}160 - 9720x^2 + 660x^4 - 13x^6}{360(x^2 + 56)} < \cos(x) < \frac{15{,}120 - 6900x^2 + 313x^4}{15{,}120 + 660x^2 + 13x^4}, \qquad x \in [0, \pi/2]. \tag{23}$$

In this paper, we present new refinements and improvements for Jordan's and Kober's inequalities based on the interpolation and approximation method. New two-sided polynomial bounds of both inequalities are given. The results show that our bounds are tighter than the previous conclusions.

2. Main Results

Firstly, we introduce a theorem of interpolation and approximation, which is very useful for our proof [22].

Theorem 1. *Let $w_0, w_1, \cdots, w_r$ be $r+1$ distinct points in $[a,b]$ and $n_0, n_1, \cdots n_r$ be $r+1$ integers ≥ 0. Let $N = n_0 + \cdots + n_r + r$. Suppose that $g(t)$ is a polynomial of degree N such that:*

$$g^{(i)}(w_j) = f^{(i)}(w_j), \qquad i = 0, \cdots, n_j, j = 0, \cdots, r.$$

Then, there exists $\xi(t) \in [a,b]$ such that:

$$f(t) - g(t) = \frac{f^{(N+1)}(\xi(x))}{(N+1)!} \prod_{i=0}^{r} (t - w_i)^{n_i+1}.$$

Next, we give new polynomial bounds of $sinc(x)$ and $cos(x)$ based on the above theorem of interpolation and approximation.

Theorem 2. *For $x \in (0, \pi/2]$, we have that:*

$$\begin{aligned} 1 + c_1x^2 + d_1x^3 + e_1x^4 + f_1x^5 + g_1x^6 + h_1x^7 &\leq sinc(x) \\ &\leq 1 + b_2x + c_2x^2 + d_2x^3 + e_2x^4 + f_2x^5 + g_2x^6 + h_2x^7, \end{aligned} \tag{24}$$

where

$$\begin{aligned}
c_1 &= \frac{448-8192\sqrt{2}+8748\sqrt{3}-(1111/2+512\sqrt{2})\pi}{\pi^3}, \\
d_1 &= \frac{-7104+122{,}880\sqrt{2}-255{,}879\sqrt{3}/2+(14{,}691/2+7168\sqrt{2})\pi}{\pi^4}, \\
e_1 &= \frac{44{,}352-712{,}704\sqrt{2}+730{,}458\sqrt{3}-(40{,}256+39{,}424\sqrt{2})\pi}{\pi^5}, \\
f_1 &= \frac{-136{,}000+2{,}007{,}040\sqrt{2}-2{,}033{,}910\sqrt{3}+(110{,}550+106{,}496\sqrt{2})\pi}{\pi^6}, \\
g_1 &= \frac{204{,}288-2{,}752{,}512\sqrt{2}+2{,}764{,}368\sqrt{3}-(150{,}192+141{,}312\sqrt{2})\pi}{\pi^7}, \\
h_1 &= \frac{-119{,}808+1{,}474{,}560\sqrt{2}-1469{,}664\sqrt{3}+(80{,}352+73{,}728\sqrt{2})\pi}{\pi^8}, \\
b_2 &= \frac{-3398+2048\sqrt{2}+3159\sqrt{3}/2-(137/2+256\sqrt{2}+162\sqrt{3})\pi}{\pi^2}, \\
c_2 &= \frac{80{,}572-45{,}056\sqrt{2}-39{,}123\sqrt{3}+(2683/2+6144\sqrt{2}+3564\sqrt{3})\pi}{\pi^3}, \\
d_2 &= \frac{-762{,}398+395{,}264\sqrt{2}+393{,}174\sqrt{3}-(12{,}389+59{,}648\sqrt{2}+31{,}914\sqrt{3})\pi}{\pi^4}, \\
e_2 &= \frac{3{,}712{,}680-1{,}769{,}472\sqrt{2}-2{,}048{,}004\sqrt{3}+(62{,}154+299{,}520\sqrt{2}+149{,}040\sqrt{3})\pi}{\pi^5}, \\
f_2 &= \frac{-9{,}854{,}424+4{,}276{,}224\sqrt{2}+5{,}820{,}336\sqrt{3}-(173{,}844+820{,}224\sqrt{2}+382{,}968\sqrt{3})\pi}{\pi^6}, \\
g_2 &= \frac{13{,}545{,}792-5{,}308{,}416\sqrt{2}-8{,}538{,}048\sqrt{3}+(254{,}016+1{,}161{,}216\sqrt{2}+513{,}216\sqrt{3})\pi}{\pi^7}, \\
h_2 &= \frac{-7{,}537{,}536+2{,}654{,}208\sqrt{2}+5{,}038{,}848\sqrt{3}-(150{,}336+663{,}552\sqrt{2}+279{,}936\sqrt{3})\pi}{\pi^8}.
\end{aligned}$$

Proof. Let $e_{sinc,l}(x) = sinc(x) - 1 - c_1x^2 - d_1x^3 - e_1x^4 - f_1x^5 - g_1x^6 - h_1x^7$, $e_{sinc,u}(x) = sinc(x) - 1 - b_2x - c_2x^2 - d_2x^3 - e_2x^4 - f_2x^5 - g_2x^6 - h_2x^7$, then we have $e^{(8)}_{sinc,l}(x) = e^{(8)}_{sinc,u}(x) = sinc^{(8)}(x)$.

It is very obvious that:

$$sinc^{(8)}(x) = \frac{(x^8 - 56x^6 + 1680x^4 - 20{,}160x^2 + 40{,}320)\sin(x) + (8x^7 - 336x^5 + 6720x^3 - 40{,}320x)\cos(x)}{x^9}.$$

Let $h(x) = (x^8 - 56x^6 + 1680x^4 - 20{,}160x^2 + 40{,}320)\sin(x) + (8x^7 - 336x^5 + 6720x^3 - 40{,}320x)\cos(x)$; we have:

$$h'(x) = x^8 cos(x) > 0, x \in (0, \pi/2).$$

Therefore, $h(x)$ is an incremental function in $(0, \pi/2)$, and we have $h(x) \geq h(0) = 0$; and then, $sinc^{(8)}(x) \geq 0$, for $x \in (0, \pi/2)$.

By the definition of $e_{sinc,l}(x)$ and $e_{sinc,u}(x)$, we have:

$$e_{sinc,l}(0) = e_{sinc,l}(\frac{\pi}{4}) = e_{sinc,l}(\frac{\pi}{3}) = e_{sinc,l}(\frac{\pi}{2}) = e'_{sinc,l}(0) = e'_{sinc,l}(\frac{\pi}{4}) = e'_{sinc,l}(\frac{\pi}{3}) = e'_{sinc,l}(\frac{\pi}{2}) = 0,$$

$$e_{sinc,u}(0) = e_{sinc,u}(\frac{\pi}{6}) = e_{sinc,u}(\frac{\pi}{4}) = e_{sinc,u}(\frac{\pi}{3}) = e_{sinc,u}(\frac{\pi}{2}) = e'_{sinc,u}(\frac{\pi}{6}) = e'_{sinc,u}(\frac{\pi}{4}) = e'_{sinc,u}(\frac{\pi}{3}) = 0.$$

By Theorem 1, there exit $\zeta_j(x) \in (0, \pi/2), j = 1, 2$, such that:

$$e_{sinc,l}(x) = \frac{e^{(8)}_{sinc,l}(\zeta_1(x))}{8!} x^2 (x - \frac{\pi}{4})^2 (x - \frac{\pi}{3})^2 (x - \frac{\pi}{2})^2 \geq 0,$$

$$e_{sinc,u}(x) = \frac{e^{(8)}_{sinc,u}(\zeta_2(x))}{8!} x (x - \frac{\pi}{6})^2 (x - \frac{\pi}{4})^2 (x - \frac{\pi}{3})^2 (x - \frac{\pi}{2}) \leq 0,$$

which means the conclusion is valid.

The proof of Theorem 2 is completed. □

Theorem 3. *For $x \in [0, \pi/2]$, we have that:*

$$\begin{aligned} 1 + \gamma_1 x^2 + \delta_1 x^3 + \xi_1 x^4 + \eta_1 x^5 + \lambda_1 x^6 + \theta_1 x^7 &\leq \cos(x) \\ &\leq 1 + \beta_2 x + \gamma_2 x^2 + \delta_2 x^3 + \xi_2 x^4 + \eta_2 x^5 + \lambda_2 x^6 + \theta_2 x^7, \end{aligned} \tag{25}$$

where

$\gamma_1 = \frac{4721/2 - 2560\sqrt{2} + (8 + 128\sqrt{2} + 243\sqrt{3}/2)\pi}{\pi^2}$,
$\delta_1 = \frac{-35{,}301 + 37{,}888\sqrt{2} - (128 + 1792\sqrt{2} + 3645\sqrt{3}/2)\pi}{\pi^3}$,
$\xi_1 = \frac{203{,}230 - 217{,}600\sqrt{2} + (808 + 9856\sqrt{2} + 10{,}692\sqrt{3})\pi)}{\pi^4}$,
$\eta_1 = \frac{-567{,}420 + 608{,}256\sqrt{2} - (2512 + 26{,}624\sqrt{2} + 30{,}618\sqrt{3})\pi}{\pi^5}$,
$\lambda_1 = \frac{771{,}264 - 829{,}440\sqrt{2} + (3840 + 35{,}328\sqrt{2} + 42{,}768\sqrt{3})\pi}{\pi^6}$,
$\theta_1 = \frac{-409{,}536 + 442{,}368\sqrt{2} - (2304 + 18{,}432\sqrt{2} + 23{,}328\sqrt{3})\pi}{\pi^7}$,
$\beta_2 = \frac{458 + 256\sqrt{2} - 729\sqrt{3} + (27 + 64\sqrt{2} + 27\sqrt{3}/2)\pi}{\pi}$,
$\gamma_2 = \frac{-23{,}399/2 - 5120\sqrt{2} + 17{,}010\sqrt{3} - (594 + 1536\sqrt{2} + 675\sqrt{3}/2)\pi}{\pi^2}$,
$\delta_2 = \frac{118{,}669 + 39{,}168\sqrt{2} - 159{,}165\sqrt{3} + (5319 + 14{,}912\sqrt{2} + 3429\sqrt{3})\pi}{\pi^3}$,
$\xi_2 = \frac{-62{,}0514 - 142{,}848\sqrt{2} + 768{,}852\sqrt{3} - (24{,}840 + 74{,}880\sqrt{2} + 18{,}090\sqrt{3})\pi}{\pi^4}$,
$\eta_2 = \frac{1{,}766{,}268 + 248{,}832\sqrt{2} - 2{,}028{,}564\sqrt{3} + (63{,}828 + 205{,}056\sqrt{2} + 52{,}164\sqrt{3})\pi}{\pi^5}$,
$\lambda_2 = \frac{-2{,}592{,}000 - 165{,}888\sqrt{2} + 2{,}776{,}032\sqrt{3} - (85{,}536 + 290{,}304\sqrt{2} + 77{,}760\sqrt{3})\pi}{\pi^6}$,
$\theta_2 = \frac{1{,}529{,}280 - 1{,}539{,}648\sqrt{3} + (46{,}656 + 165{,}888\sqrt{2} + 46{,}656\sqrt{3})\pi}{\pi^7}$.

Proof. Let $e_{cos,l}(x) = \cos(x) - \alpha_1 - \gamma_1 x^2 - \delta_1 x^3 - \xi_1 x^4 - \eta_1 x^5 - \lambda_1 x^6 - \theta_1 x^7$, $e_{cos,u}(x) = \cos(x) - \alpha_2 - \beta_2 x - \gamma_2 x^2 - \delta_2 x^3 - \xi_2 x^4 - \eta_2 x^5 - \lambda_2 x^6 - \theta_2 x^7$; then, we have $e^{(8)}_{cos,l}(x) = e^{(8)}_{cos,u}(x) = \cos^{(8)}(x)$.

It is easy to see that $\cos^{(8)}(x) = \cos(x)$ and $\cos^{(8)}(x) \geq 0$, for $x \in (0, \pi/2)$.

By the definition of $e_{cos,l}(x)$ and $e_{cos,u}(x)$, we have:

$$e_{cos,l}(0) = e_{cos,l}(\frac{\pi}{4}) = e_{cos,l}(\frac{\pi}{3}) = e_{cos,l}(\frac{\pi}{2}) = e'_{cos,l}(0) = e'_{cos,l}(\frac{\pi}{4}) = e'_{cos,l}(\frac{\pi}{3}) = e'_{cos,l}(\frac{\pi}{2}) = 0,$$

$$e_{cos,u}(0) = e_{cos,u}(\frac{\pi}{6}) = e_{cos,u}(\frac{\pi}{4}) = e_{cos,u}(\frac{\pi}{3}) = e_{cos,u}(\frac{\pi}{2}) = e'_{cos,u}(\frac{\pi}{6}) = e'_{cos,u}(\frac{\pi}{4}) = e'_{cos,u}(\frac{\pi}{3}) = 0.$$

By Theorem 1, there exist $\zeta_j(x) \in (0, \pi/2), j = 3, 4$, such that

$$e_{cos,l}(x) = \frac{e^{(8)}_{cos,l}(\zeta_3(x))}{8!} x^2 (x - \frac{\pi}{4})^2 (x - \frac{\pi}{3})^2 (x - \frac{\pi}{2})^2 \geq 0,$$

$$e_{cos,u}(x) = \frac{e^{(8)}_{cos,u}(\zeta_4(x))}{8!} x (x - \frac{\pi}{6})^2 (x - \frac{\pi}{4})^2 (x - \frac{\pi}{3})^2 (x - \frac{\pi}{2}) \leq 0,$$

which means the conclusion is valid.

The proof of Theorem 3 is completed. □

3. Conclusions and Analysis

In this paper, we presented new refinements and improvements of Jordan's and Kober's inequalities based on the interpolation and approximation method. Theorems 2 and 3 gave new polynomial bounds of the $sinc(x)$ and $\cos(x)$ functions. Table 1 gives the comparison of the maximum errors between $sinc(x)$ and the bounds for different methods. $MaxError_{sinc\_low}$ and $MaxError_{sinc\_upp}$ denote the maximum errors between $sinc(x)$ and the lower and upper bounds. It is obvious that our results are superior to the previous conclusions. Similarly, $MaxError_{cos\_low}$ and $MaxError_{cos\_upp}$ denote the maximum errors between $\cos(x)$ and the lower and upper bounds. Table 2 gives the comparison of the maximum errors of $\cos(x)$. The maximum errors of Inequality (25) in Theorem 3 are less than those of the previous methods.

Table 1. Comparison of the maximum errors between $sinc(x)$ and the bounds for different methods.

Method	Error	
	$MaxError_{sinc\_low}$	$MaxError_{sinc\_upp}$
Zhang [7] (Inequality (2))	8.2396×10^{-2}	2.7320×10^{-1}
Zhang [8] (Inequality (3))	8.2396×10^{-2}	9.3440×10^{-2}
Qi [9] (Inequality (4))	4.5070×10^{-2}	1.1612×10^{-2}
Zhang [8] (Inequality (5))	1.5412×10^{-2}	1.1612×10^{-2}
Deng [10] (Inequality (6))	1.5117×10^{-1}	6.5359×10^{-2}
Jiang [11] (Inequality (7))	2.0423×10^{-1}	1.0245×10^{-1}
Debnath [12] (Inequality (8))	4.7771×10^{-2}	2.8730×10^{-3}
Debnath [12] (Inequality (9))	2.0664×10^{-1}	2.0423×10^{-1}
Agarwal [13] (Inequality (10))	2.6315×10^{-3}	9.8638×10^{-4}
Chen [14] (Inequality (11))	2.4322×10^{-3}	6.5652×10^{-4}
Chen [14] (Inequality (12))	1.0492×10^{-4}	1.1278×10^{-4}
Zeng [15] (Inequality (14) (m = 5))	2.3606×10^{-1}	1.2987×10^{-1}
Zeng [15] (Inequality (14) (m = 10))	2.9972×10^{-1}	2.0465×10^{-1}
Zeng [15] (Inequality (14) (m = 15))	3.2094×10^{-1}	2.4001×10^{-1}
Bercu [20] (Inequality (20))	2.6834×10^{-3}	6.5239×10^{-5}
Zhang [8] (Inequality (13))	1.0600×10^{-5}	5.4563×10^{-6}
Zhang [21] (Inequality (22))	1.1234×10^{-6}	1.9032×10^{-6}
Results of this paper (Inequality (24))	$\mathbf{4.1030 \times 10^{-8}}$	$\mathbf{2.4379 \times 10^{-8}}$

Table 2. Comparison of the maximum errors between $\cos(x)$ and the bounds for different methods.

Method	Error	
	$MaxError_{cos\_low}$	$MaxError_{cos\_upp}$
Sándor [18] (Inequality (16))	2.1051×10^{-1}	5.6010×10^{-2}
Sándor [18] (Inequality (17))	2.3325×10^{-1}	5.6010×10^{-2}
Zhang [7] (Inequality (18))	7.2818×10^{-2}	5.6010×10^{-2}
Bhayo [19] (Inequality (19))	2.3230×10^{-2}	1.0599×10^{-2}
Zhang [21] (Inequality (23))	1.3987×10^{-5}	2.9435×10^{-5}
Results of this paper (Inequality (25))	$\mathbf{3.4330 \times 10^{-7}}$	$\mathbf{2.0736 \times 10^{-7}}$

The same conclusions can be found in Figures 1 and 2. We can see that Inequality (13), Inequality (22), and Inequality (24) have similar results in Table 1. In order to better compare three results, Figure 1 presents the error curves of three methods. Here, the error of the bound is equal to the value of the bound minus the value of the function. Therefore, the error curve of the lower bound is below the x-axis. The error of Inequality (24) is obviously less than the errors of Inequality (13) and Inequality (22). For the same reason, Figure 2 shows the comparison of the errors of Inequality (23) and Inequality (25). It is easy to find that the errors of Inequality (25) are less than those of Inequality (23).

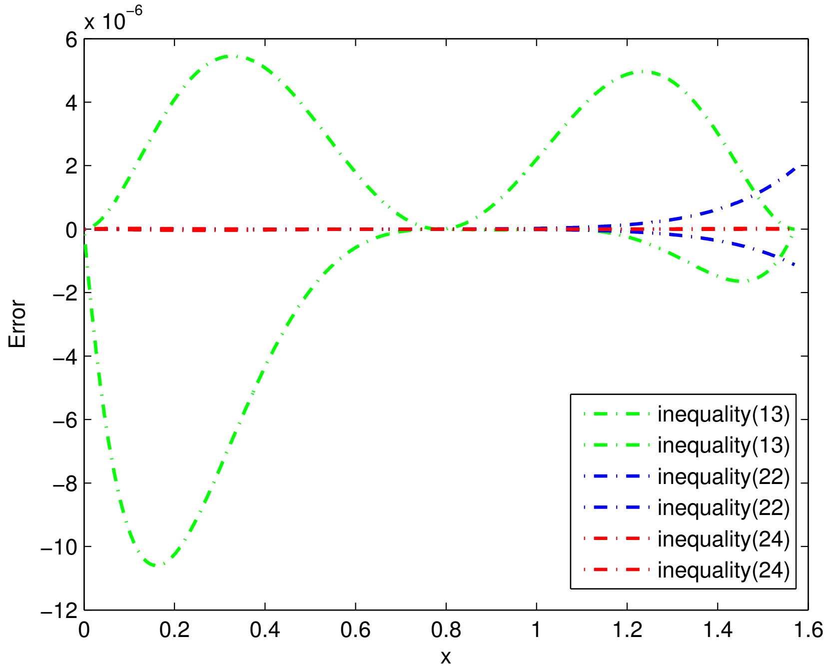

Figure 1. Error plots between $sinc(x)$ and the bounds of Inequality (13), Inequality (22), and Inequality (24).

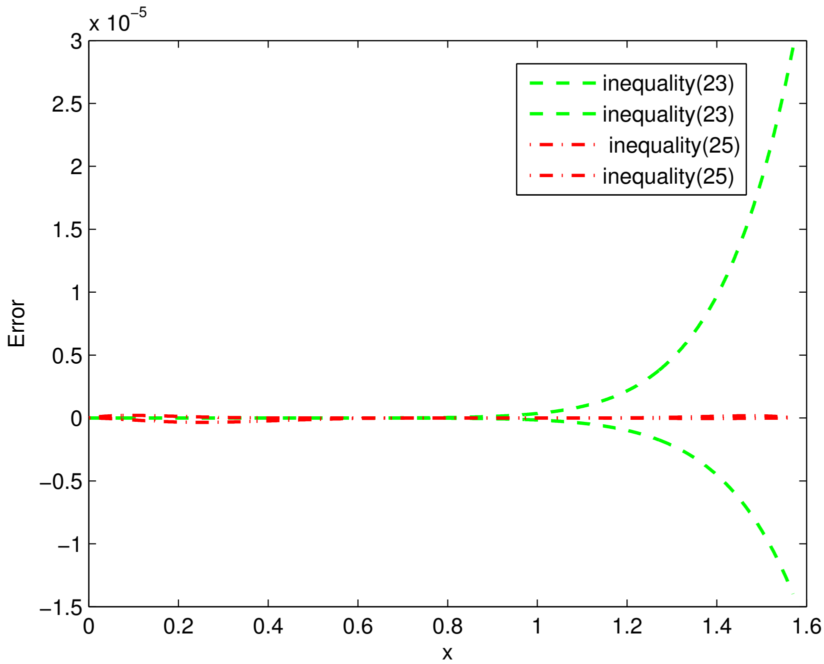

Figure 2. Error plots between $\cos(x)$ and the bounds of Inequality (23) and Inequality (25).

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