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Article

On Ball Numbers

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Abstract: We first shortly review, in part throwing a new light on, basics of ball numbers for balls having a positively homogeneous Minkowski functional and turn over then to a new particular class of ball numbers of balls having a Minkowski functional being homogeneous with respect to multiplication with a specific diagonal matrix. Applications to crystal breeding, temperature expansion and normalizing density generating functions in big data analysis are indicated and a challenging problem from the inhomogeneity program is stated.

Keywords: norm and antinorm ball; matrix homogeneous ball; star body; star radius; norm and antinorm radius; p-spherical radius parameter; dual surface content measure; density normalizing constant; crystal breeding; temperature expansion

1. Introduction

The circle number π is one of the most famous numbers because of the unique role it plays in mathematics and because of its fascinating properties, which have been explored to a broad readership by many authors (see, for example, [1–5]). Studying two of the most basic properties, the so-called area content and circumference properties, and checking them for circles of more general shape and in higher dimensions is the aim of the present note.

Various approaches to generalizing π by proving different statements on circles under more general assumptions can be found in the literature. The authors of [6] called the circumference-to-diameter ratio of (norm-)circles in a normed space $X = (\mathbb{R}^2, ||.||)$ the curly pi, $\bar{\omega}(X)$, and indicate that the range of $\bar{\omega}(X)$ is the interval $[3, 4]$. It was proved already in [7] that $\pi_p = \bar{\omega}(X)$ satisfies $\pi \leq \bar{\omega}(X) \leq 4$ if X is in particular an l_p-space, and that $\bar{\omega}(X^*) = \bar{\omega}(X)$ where X^* is the dual space of X, that is the space of all linear functionals on X endowed with the dual norm $||.||^*$ of $||.||$.The latter equation was proved for the general norm case already in [8] while some statements in [9] reprove and generalize early results on $\bar{\omega}(X)$ from [10]. For the l_p-case, a series approximation of π_p is studied in [11]. It was used in [12] that the norm length $\mu_B(\partial B)$ of a norm circle ∂B satisfies, in the notation given there, $2 \leq \frac{\mu_B(\partial B)}{\mu_B(B)} \leq \pi$ where B and ∂B are the unit disc and its boundary, the unit circle, respectively, in a two-dimensional Minkowski space, and μ_B is the corresponding Minkowski measure. Additionally motivating and justifying each of these considerations, in [13], the authors referred to Hilbert's fourth problem where he suggested an examination of geometries that "stand next to Euclidean geometry" in some sense, and then surveyed basic results on the geometry of unit discs in normed spaces. A specific type of such results is dealt with in [14] where the following question is answered: Of all closed curves of fixed length, which encloses the largest one? Recalling that our planet sweeps out equal areas in equal times, the question of how to construct curves having this Kepler property is also answered. Three postulates for constructing generalized circle numbers are discussed in [15] where the author aimed such numbers to satisfy as many of these postulates as possible, namely those with respect to the circumference-to-diameter ratio, the area content of the unit circle and the arc length of the upper (lower) half circle.

There are other branches of mathematics where certain modifications or generalizations of π play a role. People dealing with generalized trigonometric functions consider sometimes functions having a period slightly different from π. Solving certain differential equations may also lead to such functions, and eigenvalues of certain differential operators are closely connected with generalized circle numbers. Moreover, there are many papers dealing with the prerequisites of multivariate generalizations of the circle number π where the notion of non-Euclidean circumference of a circle is replaced with that of suitably defined surface content of non-Euclidean spheres or where the notion of Euclidean volume is replaced with another one. For remaining basic in this Introduction, we omit any related details here.

Generalized circle numbers and their multivariate counterparts, the ball numbers, can be defined in various ways. For the original approach to generalizing the circle number π in the sense of the present study, we refer to [16,17]. It was shown there that Archimedes' or Ludolph's number π is not alone as a circle number reflecting the two mentioned circle properties. There exists a continuous and monotonously increasing function $p \to \pi(p), p > 0$ with $\lim_{p\to 0} \pi(p) = 0$, $\pi(1) = 2$, $\pi(2) = \pi$, $\lim_{p\to\infty} \pi(p) = 4$ and such that, for each $p > 0$, $\pi(p)$ reflects both the area content and the $l_{2,p}$-generalized circumference properties of the $l_{2,p}$-circle. This means that, if $\mathcal{U}_p(r)$ and $A_p(r)$ denote the $l_{2,p}$-generalized circumference and the common (Euclidean) area content of an $l_{2,p}$-circle $C_p(r) = \{(x,y)^T \in \mathbb{R}^2 : |x|^p + |y|^p = r^p\}$ of p-generalized radius r and its circumscribed disc $K_p(r)$, respectively, then the ratios $\mathcal{U}_p(r)/(2r)$ and $A_p(r)/r^2$ are the same and do not depend on $r > 0$ and their common value is $\pi(p), p > 0$.

If $p \geq 1$, then the notion of the $l_{2,p}$-generalized perimeter coincides with that of the l_{2,p^*}-arc-length AL_{p,p^*} of $C_p(r)$ for the conjugate p^* satisfying $\frac{1}{p} + \frac{1}{p^*} = 1$. Hence, the non-Euclidean, unless for $p = 2$, metric generated by the unit ball of the space l_{2,p^*} turns out to be of special interest for measuring the length of an $l_{2,p}$-circle.

Looking forward to the case $p \in (0,1)$, recognize that the $l_{2,q}$-arc-length of $C_p(r)$ may be represented for arbitrary $q \geq 1$ as

$$AL_{p,q}(r) = \int_0^{2\pi} d_{K_q(1)}(x'(\varphi), y'(\varphi))d\varphi$$

where d_K denotes the Minkowski functional of a star-shaped set K and the function $\varphi \to (x(\varphi), y(\varphi))$ is a differentiable parameter representation of the considered $l_{2,p}$-circle $C_p(r)$.

If $p \in (0,1)$ and p^{**} satisfies the equation $\frac{1}{p} - \frac{1}{p^{**}} = 1$, then the notion of the $l_{2,p}$-generalized circumference is based upon the Minkowski functional of $K(p^{**})$ where

$$K(q) = \{(x,y) \in R^2 : \frac{1}{|x|^q} + \frac{1}{|y|^q} \geq 1\}, q > 0$$

is a non-convex star-shaped set and the $K(q)$-based arc-length is actually defined as

$$AL_{p,q}(r) = \int_0^{2\pi} d_{K(q)}(x'(\varphi), y'(\varphi))d\varphi$$

for arbitrary $q > 0$.

A possible interpretation of the $l_{2,p}$-generalized circumference of an $l_{2,p}$-circle is in the case $p \geq 1$ that among all $l_{2,q}$-arc-lengths $AL_{p,q}(r)$ of $C_p(r)$ with $q \geq 1$ just the l_{2,p^*}-arc-length $AL_{p,p^*}(r)$ coincides with the derivative of the area function, i.e.,

$$AL_{p,q}(r) = \frac{d}{dr}A_p(r), r > 0$$

if and only if $q = p^*$.

If $p \in (0,1)$, then among all $K(q)$-based arc-lengths $AL_{p,q}(r)$ of $C_p(r)$ with $q > 0$ just the $K(q^{**})$-based arc-length $AL_{p,p^{**}}(r)$ satisfies the latter equation, which therefore holds if and only if $q = p^{**}$.

Hence, this geometric-analytical way of defining an arc-length of an $l_{2,p}$-circle is equivalent to considering the derivative of the area content function of the circumscribed disc. This view onto what is a circle's arc-length has been established in [18] within a more general and well motivated multi-dimensional context. It turned out in [16,17] that, as in the case $p = 2$, it holds $\pi(p) = A_p(1)$ in all cases $p > 0$.

A first related definition of ball numbers can be found in [19] and the particular ball numbers of ellipsoids and platonic bodies are dealt with in [20,21], respectively. In [19], the notion of a circle number was extended to that of a ball number of an arbitrary $l_{n,p}$-ball

$$K_{n,p}(r) = \{(x_1, ..., x_n) \in R^n : \sum_{i=1}^{n} |x_i|^p \le r^p\}, p > 0, n \in \{2, 3, ...\}$$

of p-generalized radius $r > 0$. Let us denote the (usual) volume of $K_{n,p}(r)$ and the suitably defined $l_{n,p}$-generalized surface content of its boundary $S_{n,p}(r)$ by $V_{n,p}(r)$ and $\mathcal{O}_{n,p}(r)$, respectively. The ratio $V_{n,p}(r)/r^n = V_{n,p}(1)$ does not depend on $r > 0$. It was shown that this ratio equals in fact the ratio $\mathcal{O}_{n,p}(r)/(nr^{n-1})$. It is said that the common value which these ratios actually attain reflects the volume and the p-generalized surface content properties of the $l_{n,p}$-ball.

The definition of the $l_{n,p}$-generalized surface content of $S_{n,p}(r)$ was given in two steps. Firstly, the notion of the surface content from Euclidean geometry was extended to the notion of the $l_{n,q}$-surface content $O^{(n)}_{p,q}(r)$ of the $l_{n,p}$-sphere $S_{n,p}(r)$ for $p \ge 1, q \ge 1$. Secondly, it was shown that just the ratio $O^{(n)}_{p,p^*}(r)/(nr^{n-1})$ with the conjugate $p^* \ge 1$ satisfying $\frac{1}{p} + \frac{1}{p^*} = 1$ coincides with the ratio $V_{n,p}(r)/r^n$ and does not depend on $r > 0$. This was motivation to put $\mathcal{O}_{n,p}(r) = O^{(n)}_{p,p^*}(r)$ and to consider $V_{n,p}(1) = \pi_n(p)$ as the $l_{n,p}$-ball number if $p \ge 1$. This generalized the two-dimensional approach in [16].

If $p \in (0,1)$, then the notion of the $l_{n,q}$-surface content $O^{(n)}_{p,q}(r)$ of the sphere $S_{n,p}(r)$ was introduced based upon the (Minkowski functional of the) non-convex star-shaped set

$$K(q) = \{x_{(n)} \in R^n : \sum_{i=1}^{n} \frac{1}{|x_i|^q} \ge 1\}, q > 0.$$

In a similar way as in the case $p \ge 1$, it turns out that it is reasonable to put $V_{n,p}(1) = \pi_n(p)$ in the case $p \in (0,1)$, too.

For two first applications of all mentioned ball numbers, we refer to the normalizing problem of probabilistic distribution theory discussed, e.g., in [20,22,23] and the thin layer property considered in [19,20].

In the present paper, we introduce ball numbers also being originally designed for certain purposes of probability theory. To be more concrete, they turn out to be normalizing multiplicative constants making certain nonnegative integrable functions being probability densities. The structure of probability density level sets gives rise to introduce radius variables of much more general type than in the case of $l_{n,p}$-balls and to each time newly adopt the notion of surface content in the suitable non-Euclidean sense. In Section 2, we shortly review what is known for ball numbers of balls having a positive homogeneous Minkowski functional, and in Section 3 we start considering balls with a diagonal matrix homogeneous Minkowski functional, extending the two-dimensional consideration in [24] to arbitrary finite dimension.

2. Positively Homogeneous Star Balls

While the l_p-world dealt with so far already serves great flexibility in applied situations, sometimes it may be desirable to go further. Here, we allow balls to be generated by arbitrary norms or antinorms or even more general functionals. Throughout this section, let $B \subset \mathbb{R}^n$ be a star body having the origin 0_n in its interior and let its topological boundary S be called a star sphere. A countable collection $\mathcal{F} = \{C_1, C_2, ...\}$ of pairwise disjoint cones C_j with vertex being the origin 0_n and $\mathbb{R}^n = \bigcup_j C_j$ builds a fan. Let A be a Borel subset of S and $S_j = S \cap C_j, j = 1, 2, ...$. According to Assumption 1 in [22], we assume that for every j, the set

$$G(A \cap S_j) = \{\vartheta \in \mathbb{R}^{n-1} : \exists! \eta > 0 \text{ with } \theta = (\vartheta^T, \eta)^T \in A \cap S_j\}$$

is well defined where $\exists! \eta > 0$ means that there is an $\eta > 0$ which is uniquely determined. The latter quantity is denoted $\eta_j, j = 1, 2, ...$. The Minkowski functional of B is defined by $h_B(x) = \inf\{\lambda > 0 : x \in \lambda B\}$, $x \in \mathbb{R}^n$ where $\lambda M = \{\lambda x, x \in M\}$ for $M \subset \mathbb{R}^n$ and $\lambda x = (\lambda x_1, ..., \lambda x_n)^T$ for $x = (x_1, ..., x_n)^T$. In the present section, we assume that the functional h_B is positive homogeneous of degree one, that is $h_B(\lambda x) = \lambda h_B(x), x \in \mathbb{R}^n, \lambda > 0$ and call then B positively homogeneous. Among the typical examples of such B are norm and antinorm unit balls $\{x \in \mathbb{R}^n : ||x|| \leq 1\}$, respectively. In these cases, one can choose $\mathcal{F} = \{\mathbb{R}^{n,+}, \mathbb{R}^{n,-}\}$ where $\mathbb{R}^{n,+(-)} = \{x \in \mathbb{R}^n : x_n > (<) 0\}$ is the upper (lower) half space. For the notion of antinorm, we refer to [25]. The variable r may be considered in the general case as the star radius of the ball $B(r) = rB$ and in the particular cases of norm or antinorm balls as norm or antinorm radius, respectively. For every $j = 1, 2, ...$, the following star-spherical coordinate transformation $StSph_j : [0, \infty) \times G(S_j) \to C_j$ has been introduced in the mentioned paper putting

$$x_i = r\vartheta_i, i = 1, ..., n-1 \text{ and } x_n = y_j(\vartheta) \text{ where } \vartheta = (\vartheta_1, ..., \vartheta_{n-1})^T \text{ and } y_j = r\eta_j(\vartheta).$$

It follows from Lemma 1 in [22] and the proof of Theorem 2 in [23] that, if $h_B = ||.||$ is a norm, and $N(\vartheta)$ is the outer normal vector to S at $(\vartheta^T, \eta)^T \in S$ then

$$dx = r^{n-1} ||N(\vartheta)||^* d\vartheta dr \tag{1}$$

where $||.||^*$ is the dual of the norm $||.||$. Similarly, if $h_B = ||.||$ is an antinorm, then the star body B is radially concave with respect to a fan $\mathcal{F} = \{C_1, C_2, ...\}$. Let

$$h_B^{\mathcal{F}}(u) = \sum_i I_{C_i}(u) \inf\{u^T y : y \in S \cap C_i\}$$

be the antisupport function of B with respect to $\mathcal{F}$ and

$$B^o = \{\lambda(u)u : 0 \leq \lambda(u) h_B^{\mathcal{F}}(u) \leq 1, u \in S_E^{n-1}\}$$

the antipolar set of B where $0 \leq \lambda(u) \leq \infty$ if $0 \leq h_B^{\mathcal{F}}(u) \leq \infty$ and S_E^{n-1} is the Euclidean unit sphere in $\mathbb{R}^n$. Let further $N(\vartheta)$ denote the inner normal vector to S at $(\vartheta^T, \eta)^T \in S$. Again, changing variables according to the transformations $StSph_j$ proves that

$$dx = r^{n-1} h_{B^0}(N(\vartheta)) d\vartheta dr = r^{n-1} h_B^{\mathcal{F}}(N(\vartheta)) d\vartheta dr. \tag{2}$$

Let S^- and S^+ denote the lower and upper half of the star sphere S, respectively. We consider a surface measure defined for any Borel subset A of S by

$$O(A) = \int_{G(A \cap S^+)} ||N(\vartheta)||^* d\vartheta + \int_{G(A \cap S^-)} ||N(\vartheta)||^* d\vartheta$$

or

$$O(A) = \int\limits_{G(A\cap S^+)} h_B^{\mathcal{F}}(N(\vartheta))d\vartheta + \int\limits_{A\cap G(S^-)} h_B^{\mathcal{F}}(N(\vartheta))d\vartheta$$

if B is a norm or antinorm ball, respectively. If we define

$$\pi(B) = \frac{O(S)}{n} \tag{3}$$

then $\pi(B)$ satisfies the equations

$$\frac{\mu(B(r))}{r^n} = \pi(B) = \frac{O(S(r))}{nr^{n-1}} \text{ for all } r > 0 \tag{4}$$

where μ denotes the Lebesgue measure or volume in $\mathbb{R}^n$. The following definition is now well motivated.

Definition 1. *The surface measure $A \to O(A)$ is called dual surface content measure on the Borel σ-field on S and, because of Equation (4), the number $\pi(B)$ is called the ball number of the star body B. The left and right hand side equations in Equation (4) are called the volume and the surface content property of $\pi(B)$, respectively.*

In the two-dimensional case, $n = 2$, the volume and surface content properties in Equation (4) are called the area content and circumference properties, respectively. The dual surface content measure is just the same as the well known notion of Euclidean surface content if $||.||$ is the Euclidean norm. Several properties of ball numbers are discussed and specific examples can be found in [19,20]. Clearly, $\pi(B) = \mu(B)$ may be evaluated and can be interpreted in different ways. One of the simplest, nevertheless even in the case of Euclidean balls often overlooked in the literature, properties of the dual surface content is

$$O(A(r)) = f'(r) \text{ where } f(r) = \mu(sector(A,r))$$

and where

$$sector(A,r) = \{x \in \mathbb{R}^n : \frac{x}{h_B(x)} \in A,\ h_B(x) \leq r\}.$$

3. A Class of Diagonal Matrix Homogeneous Star Balls

In this section, we consider a particular case of generalized balls which are not positively homogeneous but are homogeneous with respect to multiplication with certain diagonal matrices. Let $p = (p_1, ..., p_n)^T$ where $p_i, i = 1, ..., n$ are pairwise different positive real numbers, and call

$$B_p(r) = \{x \in \mathbb{R}^n : \frac{|x_1|^{p_1}}{p_1} + ... + \frac{|x_n|^{p_n}}{p_n} \leq r\}$$

the p-ball with p-spherical radius parameter $r > 0$. Moreover, we call $B_p = B_p(1)$ and its topological boundary $\partial B_p = S_p$ the p-unit ball and p-unit sphere, respectively, and emphasize again that differently from Section 2 p is a vector here. The p-sphere having p-spherical radius parameter $r > 0$,

$$S_p(r) = \{x \in \mathbb{R}^n : \frac{|x_1|^{p_1}}{p_1} + ... + \frac{|x_n|^{p_n}}{p_n} = r\}$$

can be generated from the p-unit sphere by the matrix multiplication

$$S_p(r) = D_p(r)S_p,\ r > 0$$

where

$$D_p(r) = diag(r^{\frac{1}{p_1}}, ..., r^{\frac{1}{p_n}})$$

is a specific diagonal matrix. The sphere $S_p(r)$ and the ball $B_p(r)$ are called diagonal matrix homogeneous star sphere and ball, respectively. We consider now the coordinate transformation

$$x_i = r^{\frac{1}{p_i}}\theta_i,\ i = 1,...,n-1 \text{ and } x_n = r^{\frac{1}{p_n}}\eta(\theta_1,...,\theta_{n-1})$$

where η describes the upper or lower half sphere, and put $\eta_i = \frac{\partial}{\partial\theta_i}\eta$, $i = 1,...,n-1$. The absolute value of the Jacobian of this transformation is

$$D = |det\frac{D(x_1,...,x_n)}{D(r,\theta_1,...,\theta_{n-1})}| = |\begin{vmatrix} \frac{1}{p_1}r^{\frac{1}{p_1}-1}\theta_1 & r^{\frac{1}{p_1}} & 0 & \dots & \dots & 0 \\ \frac{1}{p_2}r^{\frac{1}{p_2}-1}\theta_2 & 0 & r^{\frac{1}{p_2}} & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ \frac{1}{p_{n-2}}r^{\frac{1}{p_{n-2}}-1}\theta_{n-2} & 0 & 0 & & & 0 \\ \frac{1}{p_{n-1}}r^{\frac{1}{p_{n-1}}-1}\theta_{n-1} & 0 & 0 & & & r^{\frac{1}{p_{n-1}}} \\ \frac{1}{p_n}r^{\frac{1}{p_n}-1}\eta & r^{\frac{1}{p_n}}\eta_1 & r^{\frac{1}{p_n}}\eta_2 & \dots & & r^{\frac{1}{p_n}}\eta_{n-1} \end{vmatrix}|.$$

Successively multiplying the column with number $1+i$ by $\frac{\theta_i}{p_i r}$ and subtracting the result from the first column, $i = 1,...,n-1$, leads to

$$D = \prod_{i=1}^{n-1}\frac{p_i r}{\theta_i}|\begin{vmatrix} 0 & \frac{1}{p_1}r^{\frac{1}{p_1}-1}\theta_1 & 0 & \dots & \dots & 0 \\ 0 & 0 & \frac{1}{p_2}r^{\frac{1}{p_2}-1}\theta_2 & \ddots & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & & \ddots & 0 \\ 0 & 0 & 0 & \dots & \dots & \frac{1}{p_{n-1}}r^{\frac{1}{p_{n-1}}-1}\theta_{n-1} \\ \xi & \frac{1}{p_1}r^{\frac{1}{p_n}-1}\theta_1\eta_1 & \frac{1}{p_2}r^{\frac{1}{p_n}-1}\theta_1\eta_2 & \dots & \dots & \frac{1}{p_{n-1}}r^{\frac{1}{p_n}-1}\theta_{n-1}\eta_{n-1} \end{vmatrix}|$$

where $\xi = r^{\frac{1}{p_n}-1}(\frac{1}{p_n}\eta - \sum_{i=1}^{n-1}\frac{\theta_i\eta_i}{p_i})$. It follows that $D = r^{\frac{1}{p_1}+...+\frac{1}{p_{n-1}}}|\xi|$. If $N(\theta) = (\nabla\eta(\theta)^T, -1)^T$ denotes the normal vector to S at $(\theta^T, \eta(\theta))^T$, then

$$D = r^{\frac{1}{p_1}+...+\frac{1}{p_n}-1}f_p(\theta) \quad \text{where} \quad f_p(\theta) = |\left(diag(\frac{1}{p_1},...,\frac{1}{p_n})(\theta^T,\eta)^T, N(\theta)\right)| \tag{5}$$

and $(.,.)$ means Euclidean scalar product. The quantity $\pi(B_p) = \frac{1}{n}O_p(S_p)$ satisfies the equations

$$\frac{\mu(B_p(r))}{r^{\frac{1}{p_1}+...+\frac{1}{p_1}}} = \pi(B_p) = \frac{O_p(S_p(r))}{nr^{\frac{1}{p_1}+...+\frac{1}{p_1}-1}} \tag{6}$$

where, for Borel measurable subset A of S_p,

$$O_p(A) = \int\limits_{\{\vartheta\in B_p(1):(\vartheta^T,\eta(\vartheta))\in A\cap\{\eta(\vartheta)>0\}\}} f_p(\theta)d\theta + \int\limits_{\{\vartheta\in B_p(1):(\vartheta^T,\eta(\vartheta))\in A\cap\{\eta(\vartheta)<0\}\}} f_p(\theta)d\theta$$

defines the surface content measure on the Borel σ-field on S_p.

Remark 1. *Due to the structure of function f_p, it remains an open question here in which differential-geometric sense the notion of dual surface content measure is generalized this way. This problem was stated first for dimension 2 in [24].*

The following definition is now well defined.

Definition 2. *The number $\pi(B_p)$ is called the ball number of the diagonal matrix homogeneous ball B_p.*

Ball numbers of homogeneous star-balls are proved in [22] to be one of two factorially normalizing constants of density generating functions. A similar result was derived in [24] for (p,q)-circle numbers where $p \neq q$. The following example deals with a corresponding multivariate generalization.

Example 1 (Normalizing density generating functions). *Let a function $g : [0,\infty) \to [0,\infty)$ satisfy the assumption*

$$0 < I(g;p) = \int\limits_0^\infty r^{\frac{1}{p_1}+\ldots+\frac{1}{p_n}-1} g(r)dr < \infty.$$

If

$$\varphi_{g;p}(x) = C(g;p)g(\sum_{i=1}^n \frac{|x_i|^{p_i}}{p_i}), x = (x_1,\ldots,x_n)^T \in \mathbb{R}^n$$

is aimed to be a probability density, then it follows from [26] that the normalizing constant allows the representation

$$C(g;p) = \frac{1}{nI(g;p)\pi(B_p)}.$$

To be more specific, let a density generating function be defined by

$$g(r) = \begin{cases} (1-r)^\nu & \text{if } 0 \leq r \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

where $\nu > 0$ is a parameter. Then,

$$I(g;p) = B(\frac{1}{p_1}+\ldots+\frac{1}{p_n}, \nu+1).$$

Here, $B(.,.)$ denotes the well known Beta function. It follows from Equation (6) that $\pi(B_p) = \frac{O_p(S_p)}{n}$. Making use of the multi Beta function $B(.,\ldots,.)$, it has been shown in [26] that

$$O_p(S_p) = 2^n B(\frac{1}{p_1},\ldots,\frac{1}{p_n}) \prod_{i=1}^n p_i^{\frac{1}{p_i}-1}.$$

Thus,

$$C(g;p) = \frac{\Gamma(\frac{1}{p_1}+\ldots+\frac{1}{p_n}+\nu+1)}{\Gamma(\nu+1)} \prod_{i=1}^n \frac{p_i^{1-\frac{1}{p_i}}}{2\Gamma(\frac{1}{p_i})}$$

and the resulting probability density is

$$\phi_{g;p}(x) = \begin{cases} C(g;p)(1-\sum\limits_{i=1}^n \frac{|x_i|^{p_i}}{p_i})^\nu & \text{if } \sum\limits_{i=1}^n \frac{|x_i|^{p_i}}{p_i} \leq 1 \\ 0 & \text{otherwise} \end{cases}.$$

This particular density is called the p-spherical Pearson Type II density with parameter $\nu > 0$. For two more particular cases, the p-spherical Pearson Type VII and Kotz type densities, we refer to [26]. Let us recall that there and in the present example p is a vector having different positive components. For the $l_{n,p}$-symmetric Pearson Type II density where $p > 0$ is just a real number, see [27].

Proposition 1 (From uniqueness to unlimited universality). *Let Λ be a nonempty subset of the real line and $\mathcal{B}(\Lambda) = \{B_\lambda, \lambda \in \Lambda\}$ a family of star bodies having the origin in its interior and satisfying the condition*

$$B_{\lambda_1} \subset B_{\lambda_2} \quad \textit{for all} \quad \lambda_1 \le \lambda_2 \quad \textit{from} \;\; \Lambda$$

as well as the basic assumption from the beginning of Section 2. It follows then from the properties of the volume measure that the ball number function

$$\lambda \to \pi(B_\lambda), \lambda \in \Lambda$$

is nondecreasing. Under suitable further assumptions with regard to the family $\mathcal{B}(\Lambda)$, the ball number function becomes a continuous function. As a consequence, keeping in mind the properties of the volume measure, any positive real number can be represented in infinitely many ways as a ball number regardless the unique role $\pi = 3,14159...$ plays.

Proposition 2 (Thin layer property). *Let us call*

$$\mathcal{L}_p(r,\varepsilon) = \{x \in \mathbb{R}^n : r \le \sum_{i=1}^{n} \frac{|x_i|^{p_i}}{p_i} \le r + \varepsilon\}$$

a thin layer around the boundary of $B_p(r)$. Its volume

$$\mu(\mathcal{L}_p(r,\varepsilon)) = \int\limits_{\mathcal{L}_p(r,\varepsilon)} dx$$

can be evaluated by changing Cartesian with $(p,\wp)$-spherical coordinates in the sense of Definition 1 in [26] where the formal parameter is chosen as $\wp = 1$. According to Equation (6) in [26], where $\pi_p^* = \pi_p$, we have similarly to Theorem 3 in [19] that

$$\mu(\mathcal{L}_p(r,\varepsilon)) = n\pi_p(B_p)\frac{1}{\frac{1}{p_1} + ... + \frac{1}{p_n}} r^{\frac{1}{p_1}+...+\frac{1}{p_n}}[(1+\Delta)^\kappa - 1]$$

where $\Delta = \frac{\varepsilon}{r}$ and $\kappa = \frac{1}{p_1} + ... + \frac{1}{p_n}$. Note that, as $\Delta \to +0$,

$$(1+\Delta)^\kappa - 1 = \exp\{\kappa \ln(1+\Delta)\} - 1 = \exp\{\kappa[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} \mp \ldots]\} - 1$$

$$= \exp\{\kappa\Delta - \frac{\kappa\Delta^2}{2} \pm \ldots]\} - 1 = 1 + [\kappa\Delta - \frac{\kappa\Delta^2}{2} + 0(\Delta^3)] + \frac{1}{2}[\kappa^2\Delta^2 + 0(\Delta^3)] - 1 = \kappa\Delta(1 + 0(\Delta))$$

where 0 means Landau's big O symbol from asymptotic analysis. Thus,

$$\mu(\mathcal{L}_p(r,\varepsilon)) \sim \varepsilon n \pi_p(B_p) r^{\frac{1}{p_1}+...+\frac{1}{p_n}-1}, \varepsilon \to +0$$

where $f_1(\varepsilon) \sim f_2(\varepsilon), \varepsilon \to +0$ means asymptotic equivalence, that is $f_1(\varepsilon)/f_2(\varepsilon) \to 1$ as $\varepsilon \to +0$.

The thermal expansion of a body of length l_0 into a certain direction is commonly described by $\Delta l = l_0 \alpha \Delta T$ where ΔT is the temperature difference and α an expansion coefficient. If l_0 depends on the direction then a small ΔT causes a thin layer whose volume may be of interest for various reasons. Similarly, the volume growth of the (thin) layer of a crystal is of interest in crystal breeding.

4. Concluding Remarks

Because of its fascinating properties, the Archimedes or Ludolph number π has been studied for millennia. Strong mathematical tools and deep mathematical results were produced in this process.

The interested reader can study this process starting with the literature mentioned at the very beginning of this paper and the numerous references given there. The second and third paragraphs of Section 1 introduce the history and presence of the process of generalizing π and give a short review on related parts of the literature. In the subsequent paragraphs, we summarize basic facts from the author's work on generalized circle numbers and on ball numbers in the two- and multidimensional l_p-world. Closely related results can be derived in the world of positively homogeneous star balls. Partly known facts from this field are presented here in a new light and with new conclusions in Section 2 where an emphasis is on Equations (1) and (2) serving as the main mathematical tool of a new geometric disintegration method of the Lebesgue measure. It follows from the results in Section 3 that geometric disintegration is based there upon the formula

$$dx = r^{\frac{1}{p_1}+\ldots+\frac{1}{p_n}-1} f_p(\theta_1, \ldots, \theta_{n-1}) d\theta_1 \ldots d\theta_{n-1} dr \tag{7}$$

where $f_p(\theta) = f_p(\theta_1, \ldots, \theta_{n-1})$ is given in Equation (5). As to finally summarize the results of Section 3, we derive ball numbers in cases where a ball's Minkowski functional is not positively homogeneous as in cases considered earlier but is homogeneous with respect to multiplication with the diagonal matrices $D_p(r), r > 0$. One might speak about such ball numbers as dynamic ball numbers because the shape of the balls changes when their radius variable does. In big data analysis, these ball numbers play a role as normalizing constants of density generating functions of multivariate probability distributions having different marginal distributions, and in crystal breeding and material's temperature expansion they are needed to exactly describe volume change, called the thin layer property here.

Getting a look ahead, let us call the case where a ball's Minkowski functional is not positively homogeneous the inhomogeneity case. Clearly, other cases of inhomogeneity than the one considered here will play a role in future data analysis, and possibly other applications to (material) science and technique. To start filling this gap means to start the challenging inhomogeneity project. The reader may feel it more or less challenging and is invited to solve the problem stated in Remark 1 and all similar problems appearing in other cases of inhomogeneity.

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