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Article

Differential Equations Arising from the Generating Function of the (r, β)-Bell Polynomials and Distribution of Zeros of Equations

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Abstract: In this paper, we study differential equations arising from the generating function of the (r, β)-Bell polynomials. We give explicit identities for the (r, β)-Bell polynomials. Finally, we find the zeros of the (r, β)-Bell equations with numerical experiments.

Keywords: differential equations; Bell polynomials; r-Bell polynomials; (r, β)-Bell polynomials; zeros

MSC: 05A19; 11B83; 34A30; 65L99

1. Introduction

The moments of the Poisson distribution are a well-known connecting tool between Bell numbers and Stirling numbers. As we know, the Bell numbers B_n are those using generating function

$$e^{(e^t-1)} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

The Bell polynomials $B_n(\lambda)$ are this formula using the generating function

$$e^{\lambda(e^t-1)} = \sum_{n=0}^{\infty} B_n(\lambda) \frac{t^n}{n!}, \tag{1}$$

(see [1,2]).

Observe that

$$B_n(\lambda) = \sum_{i=0}^{n} \lambda^i S_2(n,i),$$

where $S_2(n,i) = \frac{1}{i!} \sum_{l=0}^{i} (-1)^{i-l} \binom{i}{l} l^n$ denotes the second kind Stirling number.

The generalized Bell polynomials $B_n(x,\lambda)$ are these formula using the generating function:

$$\sum_{n=0}^{\infty} B_n(x,\lambda) \frac{t^n}{n!} = e^{xt-\lambda(e^t-t-1)}, \text{ (see [2]).}$$

In particular, the generalized Bell polynomials $B_n(x,-\lambda) = E_\lambda[(Z+x-\lambda)^n], \lambda, x \in \mathbb{R}, n \in \mathbb{N}$, where Z is a Poission random variable with parameter $\lambda > 0$ (see [1–3]). The (r, β)-Bell polynomials $G_n(x, r, \beta)$ are this formula using the generating function:

$$F(t,x,r,\beta)=\sum_{n=0}^{\infty}G_n(x,r,\beta)\frac{t^n}{n!}=e^{rt+(e^{\beta t}-1)\frac{x}{\beta}}, \tag{2}$$

(see [3]), where, β and r are real or complex numbers and $(r,\beta)\neq(0,0)$. Note that $B_n(x+r,-x)=G_n(x,r,1)$ and $B_n(x)=G_n(x,0,1)$. The first few examples of (r,β)-Bell polynomials $G_n(x,r,\beta)$ are

$$\begin{aligned}
G_0(x,r,\beta)&=1,\\
G_1(x,r,\beta)&=r+x,\\
G_2(x,r,\beta)&=r^2+\beta x+2rx+x^2,\\
G_3(x,r,\beta)&=r^3+\beta^2x+3\beta rx+3r^2x+3\beta x^2+3rx^2+x^3,\\
G_4(x,r,\beta)&=r^4+\beta^3x+4\beta^2rx+6\beta r^2x+4r^3x+7\beta^2x^2+12\beta rx^2\\
&\quad+6r^2x^2+6\beta x^3+4rx^3+x^4,\\
G_5(x,r,\beta)&=r^5+\beta^4x+5\beta^3rx+10\beta^2r^2x+10\beta r^3x+5r^4x+15\beta^3x^2+35\beta^2rx^2\\
&\quad+30\beta r^2x^2+10r^3x^2+25\beta^2x^3+30\beta rx^3+10r^2x^3+10\beta x^4+5rx^4+x^5.
\end{aligned}$$

From (1) and (2), we see that

$$\begin{aligned}
\sum_{n=0}^{\infty}G_n(x,r,\beta)\frac{t^n}{n!}&=e^{(e^{\beta t}-1)\frac{x}{\beta}}e^{rt}\\
&=\left(\sum_{k=0}^{\infty}B_k(x/\beta)\beta^k\frac{t^k}{k!}\right)\left(\sum_{m=0}^{\infty}r^m\frac{t^m}{m!}\right)\\
&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}B_k(x/\beta)\beta^kr^{n-k}\right)\frac{t^n}{n!}.
\end{aligned}\tag{3}$$

Compare the coefficients in Formula (3). We can get

$$G_n(x,r,\beta)=\sum_{k=0}^{n}\binom{n}{k}\beta^kB_k(x/\beta)r^{n-k},\quad (n\geq 0).$$

Similarly we also have

$$G_n(x+y,r,\beta)=\sum_{k=0}^{n}\binom{n}{k}G_k(x,r,\beta)B_{n-k}(y/\beta)\beta^{n-l}.$$

Recently, many mathematicians have studied the differential equations arising from the generating functions of special polynomials (see [4–8]). Inspired by their work, we give a differential equations by generation of (r,β)-Bell polynomials $G_n(x,r,\beta)$ as follows. Let D denote differentiation with respect to t, D^2 denote differentiation twice with respect to t, and so on; that is, for positive integer N,

$$D^NF=\left(\frac{\partial}{\partial t}\right)^NF(t,x,r,\beta).$$

We find differential equations with coefficients $a_i(N,x,r,\beta)$, which are satisfied by

$$\left(\frac{\partial}{\partial t}\right)^NF(t,x,r,\beta)-a_0(N,x,r,\beta)F(t,x,r,\beta)-\cdots-a_N(N,x,r,\beta)e^{\beta tN}F(t,x,r,\beta)=0.$$

Using the coefficients of this differential equation, we give explicit identities for the (r,β)-Bell polynomials. In addition, we investigate the zeros of the (r,β)-Bell equations with numerical methods. Finally, we observe an interesting phenomena of 'scattering' of the zeros of (r,β)-Bell equations. Conjectures are also presented through numerical experiments.

2. Differential Equations Related to (R, β)-Bell Polynomials

Differential equations arising from the generating functions of special polynomials are studied by many authors to give explicit identities for special polynomials (see [4–8]). In this section, we study differential equations arising from the generating functions of (r, β)-Bell polynomials.

Let

$$F = F(t, x, r, \beta) = \sum_{n=0}^{\infty} G_n(x, r, \beta) \frac{t^n}{n!} = e^{rt+(e^{\beta t}-1)\frac{x}{\beta}}, \quad x, r, \beta \in \mathbb{C}. \tag{4}$$

Then, by (4), we have

$$\begin{aligned} DF = \frac{\partial}{\partial t} F(t, x, r, \beta) &= \frac{\partial}{\partial t} \left(e^{rt+(e^{\beta t}-1)\frac{x}{\beta}} \right) \\ &= e^{rt+(e^{\beta t}-1)\frac{x}{\beta}} (r + xe^{\beta t}) \\ &= re^{rt+(e^{\beta t}-1)\frac{x}{\beta}} + xe^{(r+\beta)t+(e^{\beta t}-1)\frac{x}{\beta}} \\ &= rF(t, x, r, \beta) + xF(t, x, r + \beta, \beta), \end{aligned} \tag{5}$$

$$\begin{aligned} D^2F &= rDF(t, x, r, \beta) + xDF(t, x, r + \beta, \beta) \\ &= r^2F(t, x, r, \beta) + x(2r + \beta)F(t, x, r + \beta, \beta) + x^2F(t, x, r + 2\beta, \beta), \end{aligned} \tag{6}$$

and

$$\begin{aligned} D^3F &= r^2DF(t, x, r, \beta) + x(2r + \beta)DF(t, x, r + \beta, \beta) + x^2DF(t, x, r + 2\beta, \beta) \\ &= r^3F(t, x, r, \beta) + x \left(r^2 + (2r + \beta)(r + \beta) \right) F(t, x, r + \beta, \beta) \\ &\quad + x^2(3r + 3\beta)F(t, x, r + 2\beta, \beta) + x^3F(t, x, r + 3\beta, \beta). \end{aligned}$$

We prove this process by induction. Suppose that

$$D^N F = \sum_{i=0}^{N} a_i(N, x, r, \beta)F(t, x, r + i\beta, \beta), (N = 0, 1, 2, \ldots). \tag{7}$$

is true for N. From (7), we get

$$\begin{aligned} D^{N+1}F &= \sum_{i=0}^{N} a_i(N, x, r, \beta)DF(t, x, r + i\beta, \beta) \\ &= \sum_{i=0}^{N} a_i(N, x, r, \beta) \left\{ (r + i\beta)F(t, x, r + i\beta, \beta) + xF(t, x, r + (i + 1)\beta, \beta) \right\} \\ &= \sum_{i=0}^{N} a_i(N, x, r, \beta)(r + i\beta)F(t, x, r + i\beta, \beta) \\ &\qquad + x \sum_{i=0}^{N} a_i(N, x, r, \beta)F(t, x, r + (i + 1)\beta, \beta) \\ &= \sum_{i=0}^{N} (r + i\beta)a_i(N, x, r, \beta)F(t, x, r + i\beta, \beta) \\ &\qquad + x \sum_{i=1}^{N+1} a_{i-1}(N, x, r, \beta)F(t, x, r + i\beta, \beta). \end{aligned} \tag{8}$$

From (8), we get

$$D^{N+1}F = \sum_{i=0}^{N+1} a_i(N + 1, x, r, \beta)F(t, x, r + i\beta, \beta). \tag{9}$$

We prove that

$$D^{k+1}F = \sum_{i=0}^{k+1} a_i(k+1, x, r, \beta) F(t, x, r+i\beta, \beta).$$

If we compare the coefficients on both sides of (8) and (9), then we get

$$a_0(N+1, x, r, \beta) = r a_0(N, x, r, \beta), \quad a_{N+1}(N+1, x, r, \beta) = x a_N(N, x, r, \beta), \tag{10}$$

and

$$a_i(N+1, x, r, \beta) = (r+i\beta) a_{i-1}(N, x, r, \beta) + x a_{i-1}(N, x, r, \beta), (1 \leq i \leq N). \tag{11}$$

In addition, we get

$$F(t, x, r, \beta) = a_0(0, x, r, \beta) F(t, x, r, \beta). \tag{12}$$

Now, by (10), (11) and (12), we can obtain the coefficients $a_i(j, x, r, \beta)_{0 \leq i,j \leq N+1}$ as follows. By (12), we get

$$a_0(0, x, r, \beta) = 1. \tag{13}$$

It is not difficult to show that

$$\begin{aligned} & rF(t, x, r, \beta) + xF(t, x, r+\beta, \beta) \\ &= DF(t, x, r, \beta) \\ &= \sum_{i=0}^{1} a_i(1, x, r, \beta) F(t, x, r+\beta, \beta) \\ &= a_0(1, x, r, \beta) F(t, x, r, \beta) + a_1(1, x, r, \beta) F(t, x, r+\beta, \beta). \end{aligned} \tag{14}$$

Thus, by (14), we also get

$$a_0(1, x, r, \beta) = r, \quad a_1(1, x, r, \beta) = x. \tag{15}$$

From (10), we have that

$$a_0(N+1, x, r, \beta) = r a_0(N, x, r, \beta) = \cdots = r^N a_0(1, x, r, \beta) = r^{N+1}, \tag{16}$$

and

$$a_{N+1}(N+1, x, r, \beta) = x a_N(N, x, r, \beta) = \cdots = x^N a_1(1, x, r, \beta) = x^{N+1}. \tag{17}$$

For $i = 1, 2, 3$ in (11), we have

$$a_1(N+1, x, r, \beta) = x \sum_{k=0}^{N} (r+\beta)^k a_0(N-k, x, r, \beta), \tag{18}$$

$$a_2(N+1, x, r, \beta) = x \sum_{k=0}^{N-1} (r+2\beta)^k a_1(N-k, x, r, \beta), \tag{19}$$

and

$$a_3(N+1, x, r, \beta) = x \sum_{k=0}^{N-2} (r+3\beta)^k a_2(N-k, x, r, \beta). \tag{20}$$

By induction on i, we can easily prove that, for $1 \leq i \leq N$,

$$a_i(N+1, x, r, \beta) = x \sum_{k=0}^{N-i+1} (r+i\beta)^k a_{i-1}(N-k, x, r, \beta). \tag{21}$$

Here, we note that the matrix $a_i(j, x, r, \beta)_{0 \le i,j \le N+1}$ is given by

$$\begin{pmatrix} 1 & r & r^2 & r^3 & \cdots & r^{N+1} \\ 0 & x & x(2r+\beta) & x(3r^2+3r\beta+\beta^2) & \cdots & \cdot \\ 0 & 0 & x^2 & x^2(3r+3\beta) & \cdots & \cdot \\ 0 & 0 & 0 & x^3 & \cdots & \cdot \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & x^{N+1} \end{pmatrix}$$

Now, we give explicit expressions for $a_i(N+1, x, r, \beta)$. By (18), (19), and (20), we get

$$\begin{aligned} a_1(N+1, x, r, \beta) &= x \sum_{k_1=0}^{N} (r+\beta)^{k_1} a_0(N-k_1, x, r, \beta) \\ &= \sum_{k_1=0}^{N} (r+\beta)^{k_1} r^{N-k_1}, \end{aligned}$$

$$\begin{aligned} a_2(N+1, x, r, \beta) &= x \sum_{k_2=0}^{N-1} (r+2\beta)^{k_2} a_1(N-k_2, x, r, \beta) \\ &= x^2 \sum_{k_2=0}^{N-1} \sum_{k_1=0}^{N-1-k_2} (r+\beta)^{k_1} (r+2\beta)^{k_2} r^{N-k_2-k_1-1}, \end{aligned}$$

and

$$\begin{aligned} &a_3(N+1, x, r, \beta) \\ &= x \sum_{k_3=0}^{N-2} (r+3\beta)^{k_3} a_2(N-k_3, x, r, \beta) \\ &= x^3 \sum_{k_3=0}^{N-2} \sum_{k_2=0}^{N-2-k_3} \sum_{k_1=0}^{N-2-k_3-k_2} (r+3\beta)^{k_3} (r+2\beta)^{k_2} (r+\beta)^{k_1} r^{N-k_3-k_2-k_1-2}. \end{aligned}$$

By induction on i, we have

$$\begin{aligned} &a_i(N+1, x, r, \beta) \\ &= x^i \sum_{k_i=0}^{N-i+1} \sum_{k_{i-1}=0}^{N-i+1-k_i} \cdots \sum_{k_1=0}^{N-i+1-k_i-\cdots-k_2} \left(\prod_{l=1}^{i} (r+l\beta)^{k_l} \right) r^{N-i+1-\sum_{l=1}^{i} k_l}. \end{aligned} \tag{22}$$

Finally, by (22), we can derive a differential equations with coefficients $a_i(N, x, r, \beta)$, which is satisfied by

$$\left(\frac{\partial}{\partial t} \right)^N F(t, x, r, \beta) - a_0(N, x, r, \beta) F(t, x, r, \beta) - \cdots - a_N(N, x, r, \beta) e^{\beta t N} F(t, x, r, \beta) = 0.$$

Theorem 1. *For same as below $N = 0, 1, 2, \ldots$, the differential equation*

$$D^N F = \sum_{i=0}^{N} a_i(N, x, r, \beta) e^{i\beta t} F(t, x, r, \beta)$$

has a solution

$$F = F(t, x, r, \beta) = e^{rt + (e^{\beta t} - 1)\frac{x}{\beta}},$$

where

$$
\begin{aligned}
&a_0(N,x,r,\beta) = r^N,\\
&a_N(N,x,r,\beta) = x^N,\\
&a_i(N,x,r,\beta) = x^i \sum_{k_i=0}^{N-i} \sum_{k_{i-1}=0}^{N-i-k_i} \cdots \sum_{k_1=0}^{N-i-k_i-\cdots-k_2} \left(\prod_{l=1}^{i} (r+l\beta)^{k_l} \right) r^{N-i-\sum_{l=1}^{i} k_l},\\
&\quad (1 \le i \le N).
\end{aligned}
$$

From (4), we have this

$$D^N F = \left(\frac{\partial}{\partial t}\right)^N F(t,x,r,\beta) = \sum_{k=0}^{\infty} G_{k+N}(x,r,\beta) \frac{t^k}{k!}. \tag{23}$$

By using Theorem 1 and (23), we can get this equation:

$$
\begin{aligned}
\sum_{k=0}^{\infty} G_{k+N}(x,r,\beta)\frac{t^k}{k!} &= D^N F\\
&= \left(\sum_{i=0}^{N} a_i(N,x,r,\beta) e^{i\beta t} \right) F(t,x,r,\beta)\\
&= \sum_{i=0}^{N} a_i(N,x,r,\beta) \left(\sum_{l=0}^{\infty} (i\beta)^l \frac{t^l}{l!} \right) \left(\sum_{m=0}^{\infty} G_m(x,r,\beta) \frac{t^m}{m!} \right)\\
&= \sum_{i=0}^{N} a_i(N,x,r,\beta) \left(\sum_{k=0}^{\infty} \sum_{m=0}^{k} \binom{k}{m} (i\beta)^{k-m} G_m(x,r,\beta) \frac{t^k}{k!} \right)\\
&= \sum_{k=0}^{\infty} \left(\sum_{i=0}^{N} \sum_{m=0}^{k} \binom{k}{m} (i\beta)^{k-m} a_i(N,x,r,\beta) G_m(x,r,\beta) \right) \frac{t^k}{k!}.
\end{aligned} \tag{24}
$$

Compare coefficients in (24). We get the below theorem.

Theorem 2. *For $k, N = 0, 1, 2, \ldots$, we have*

$$G_{k+N}(x,r,\beta) = \sum_{i=0}^{N} \sum_{m=0}^{k} \binom{k}{m} i^{k-m} \beta^{k-m} a_i(N,x,r,\beta) G_m(x,r,\beta), \tag{25}$$

where

$$
\begin{aligned}
&a_0(N,x,r,\beta) = r^N,\\
&a_N(N,x,r,\beta) = x^N,\\
&a_i(N,x,r,\beta) = x^i \sum_{k_i=0}^{N-i} \sum_{k_{i-1}=0}^{N-i-k_i} \cdots \sum_{k_1=0}^{N-i-k_i-\cdots-k_2} \left(\prod_{l=1}^{i} (r+l\beta)^{k_l} \right) r^{N-i-\sum_{l=1}^{i} k_l},\\
&\quad (1 \le i \le N).
\end{aligned}
$$

By using the coefficients of this differential equation, we give explicit identities for the (r,β)-Bell polynomials. That is, in (25) if $k = 0$, we have corollary.

Corollary 1. *For $N = 0, 1, 2, \ldots$, we have*

$$G_N(x,r,\beta) = \sum_{i=0}^{N} a_i(N,x,r,\beta).$$

For $N = 0, 1, 2, \ldots$, it follows that equation

$$D^N F - \sum_{i=0}^{N} a_i(N, x, r, \beta) e^{i\beta t} F(t, x, r, \beta) = 0$$

has a solution

$$F = F(t, x, r, \beta) = e^{rt + (e^{\beta t} - 1)\frac{x}{\beta}}.$$

In Figure 1, we have a sketch of the surface about the solution F of this differential equation. On the left of Figure 1, we give $-3 \leq x \leq 3, -1 \leq t \leq 1$, and $r = 2, \beta = 5$. On the right of Figure 1, we give $-3 \leq x \leq 3, -1 \leq t \leq 1$, and $r = -3, \beta = 2$.

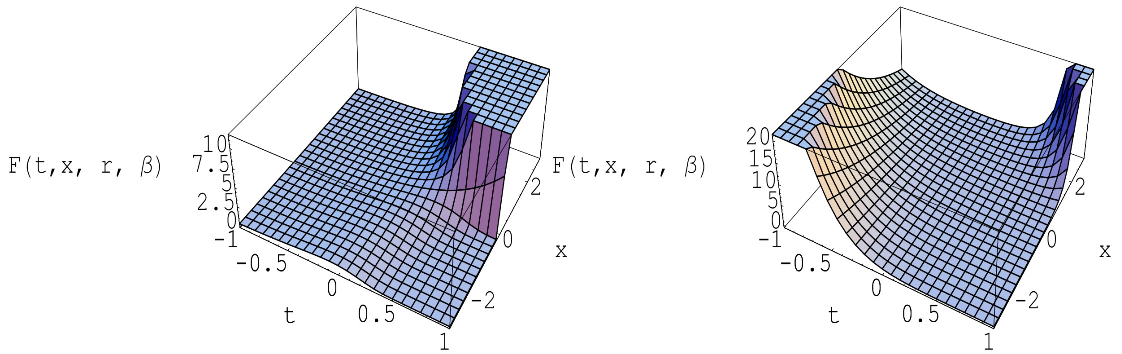

Figure 1. The surface for the solution $F(t, x, r, \beta)$.

Making N-times derivative for (4) with respect to t, we obtain

$$\left(\frac{\partial}{\partial t}\right)^N F(t, x, r, \beta) = \left(\frac{\partial}{\partial t}\right)^N e^{rt + (e^{\beta t} - 1)\frac{x}{\beta}} = \sum_{m=0}^{\infty} G_{m+N}(x, r, \beta) \frac{t^m}{m!}. \tag{26}$$

By multiplying the exponential series $e^{xt} = \sum_{m=0}^{\infty} x^m \frac{t^m}{m!}$ in both sides of (26) and Cauchy product, we derive

$$\begin{aligned} e^{-nt} \left(\frac{\partial}{\partial t}\right)^N F(t, x, r, \beta) &= \left(\sum_{m=0}^{\infty} (-n)^m \frac{t^m}{m!}\right) \left(\sum_{m=0}^{\infty} G_{m+N}(x, r, \beta) \frac{t^m}{m!}\right) \\ &= \sum_{m=0}^{\infty} \left(\sum_{k=0}^{m} \binom{m}{k} (-n)^{m-k} G_{N+k}(x, r, \beta)\right) \frac{t^m}{m!}. \end{aligned} \tag{27}$$

By using the Leibniz rule and inverse relation, we obtain

$$\begin{aligned} e^{-nt} \left(\frac{\partial}{\partial t}\right)^N F(t, x, y) &= \sum_{k=0}^{N} \binom{N}{k} n^{N-k} \left(\frac{\partial}{\partial t}\right)^k \left(e^{-nt} F(t, x, r, \beta)\right) \\ &= \sum_{m=0}^{\infty} \left(\sum_{k=0}^{N} \binom{N}{k} n^{N-k} G_{m+k}(x - n, r, \beta)\right) \frac{t^m}{m!}. \end{aligned} \tag{28}$$

So using (27) and (28), and using the coefficients of $\frac{t^m}{m!}$ gives the below theorem.

Theorem 3. *Let m, n, N be nonnegative integers. Then*

$$\sum_{k=0}^{m} \binom{m}{k} (-n)^{m-k} G_{N+k}(x, r, \beta) = \sum_{k=0}^{N} \binom{N}{k} n^{N-k} G_{m+k}(x - n, r, \beta). \tag{29}$$

When we give $m = 0$ in (29), then we get corollary.

Corollary 2. *For* $N = 0, 1, 2, \ldots,$ *we have*

$$G_N(x, r, \beta) = \sum_{k=0}^{N} \binom{N}{k} n^{N-k} G_k(x - n, r, \beta).$$

3. Distribution of Zeros of the (R, β)-Bell Equations

This section aims to demonstrate the benefit of using numerical investigation to support theoretical prediction and to discover new interesting patterns of the zeros of the (r, β)-Bell equations $G_n(x, r, \beta) = 0$. We investigate the zeros of the (r, β)-Bell equations $G_n(x, r, \beta) = 0$ with numerical experiments. We plot the zeros of the $B_n(x, \lambda) = 0$ for $n = 16, r = -5, -3, 3, 5, \beta = 2, 3$ and $x \in \mathbb{C}$ (Figure 2).

In top-left of Figure 2, we choose $n = 16$ and $r = -5, \beta = 2$. In top-right of Figure 2, we choose $n = 16$ and $r = -3, \beta = 3$. In bottom-left of Figure 2, we choose $n = 16$ and $r = 3, \beta = 2$. In bottom-right of Figure 2, we choose $n = 16$ and $r = 5, \beta = 3$.

Prove that $G_n(x, r, \beta), x \in \mathbb{C}$, has $Im(x) = 0$ reflection symmetry analytic complex functions (see Figure 3). Stacks of zeros of the (r, β)-Bell equations $G_n(x, r, \beta) = 0$ for $1 \leq n \leq 20$ from a 3-D structure are presented (Figure 3).

On the left of Figure 3, we choose $r = -5$ and $\beta = 2$. On the right of Figure 3, we choose $r = 5$ and $\beta = 2$. In Figure 3, the same color has the same degree n of (r, β)-Bell polynomials $G_n(x, r, \beta)$. For example, if $n = 20$, zeros of the (r, β)-Bell equations $G_n(x, r, \beta) = 0$ is red.

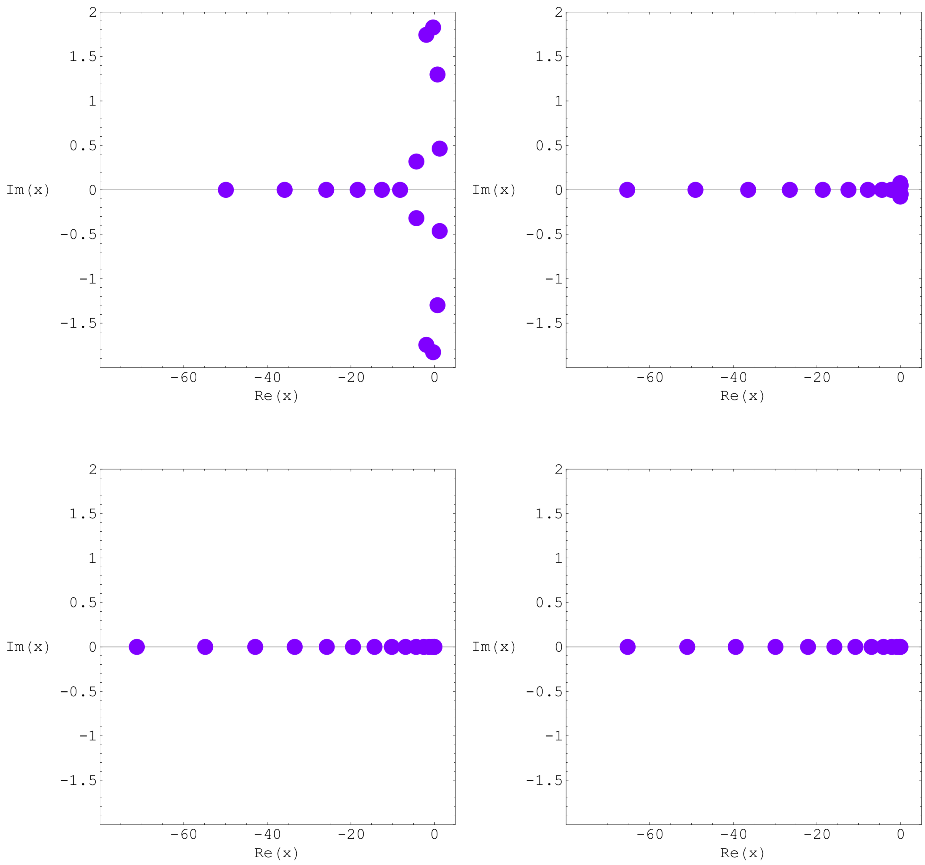

Figure 2. Zeros of $G_n(x, r, \beta) = 0$.

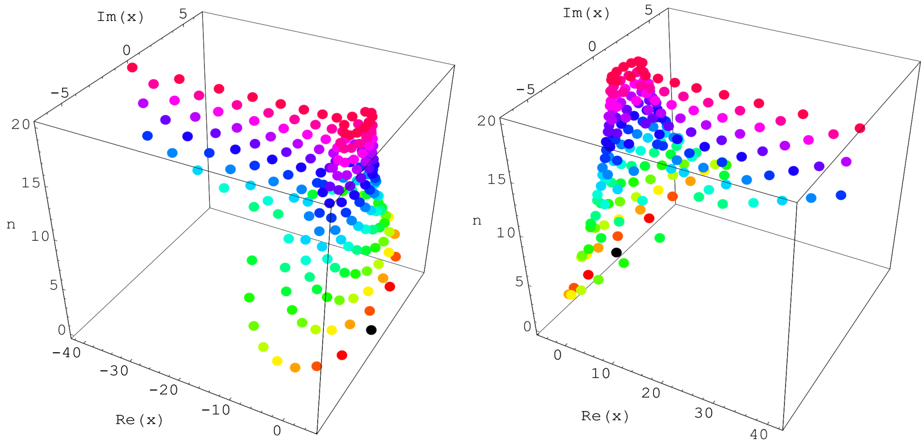

Figure 3. Stacks of zeros of $G_n(x, r, \beta) = 0, 1 \leq n \leq 20$.

Our numerical results for approximate solutions of real zeros of the (r, β)-Bell equations $G_n(x, r, \beta) = 0$ are displayed (Tables 1 and 2).

Table 1. Numbers of real and complex zeros of $G_n(x, r, \beta) = 0$

Degree n	$r = -5, \beta = 2$		$r = 5, \beta = 2$	
	Real Zeros	**Complex Zeros**	**Real Zeros**	**Xomplex Zeros**
1	1	0	1	0
2	0	2	2	0
3	1	2	3	0
4	0	4	4	0
5	1	4	5	0
6	0	6	6	0
7	1	6	7	0
8	0	8	8	0
9	1	8	9	0
10	2	8	10	0

Table 2. Approximate solutions of $G_n(x, r, \beta) = 0, x \in \mathbb{R}$.

Degree n	**x**
1	-5.000
2	-9.317, -2.683
3	-13.72, -5.68, -1.605
4	-18.21, -9.01, -3.77, -1.010
5	-22.8, -12.6, -6.4, -2.61, -0.655
6	-27.4, -16.3, -9.3, -4.7, -1.85, -0.434
7	-32.0, -20.0, -12.0, -7.1, -3.5, -1.34, -0.291

Plot of real zeros of $G_n(x, r, \beta) = 0$ for $1 \leq n \leq 20$ structure are presented (Figure 4).

In Figure 4 (left), we choose $r = 5$ and $\beta = -2$. In Figure 4 (right), we choose $r = 5$ and $\beta = 2$. In Figure 4, the same color has the same degree n of (r, β)-Bell polynomials $G_n(x, r, \beta)$. For example, if $n = 20$, real zeros of the (r, β)-Bell equations $G_n(x, r, \beta) = 0$ is blue.

Next, we calculated an approximate solution satisfying $G_n(x, r, \beta) = 0, r = 5, \beta = 2, x \in \mathbb{R}$. The results are given in Table 2.

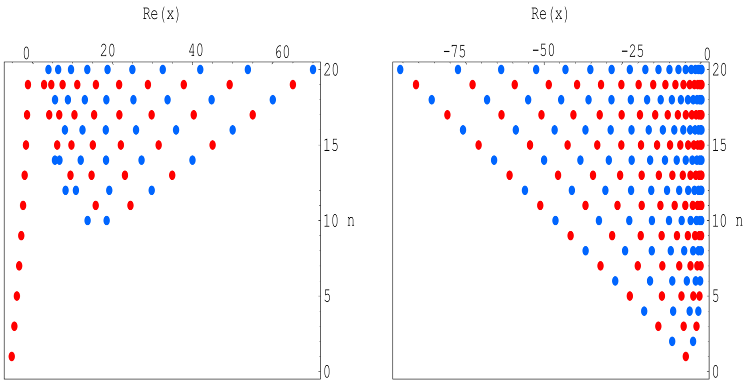

Figure 4. Stacks of zeros of $G_n(x, r, \beta) = 0, 1 \leq n \leq 20$.

4. Conclusions

We constructed differential equations arising from the generating function of the (r, β)-Bell polynomials. This study obtained the some explicit identities for (r, β)-Bell polynomials $G_n(x, r, \beta)$ using the coefficients of this differential equation. The distribution and symmetry of the roots of the (r, β)-Bell equations $G_n(x, r, \beta) = 0$ were investigated. We investigated the symmetry of the zeros of the (r, β)-Bell equations $G_n(x, r, \beta) = 0$ for various variables r and β, but, unfortunately, we could not find a regular pattern. We make the following series of conjectures with numerical experiments:

Let us use the following notations. $R_{G_n(x,r,\beta)}$ denotes the number of real zeros of $G_n(x, r, \beta) = 0$ lying on the real plane $Im(x) = 0$ and $C_{G_n(x,r,\beta)}$ denotes the number of complex zeros of $G_n(x, r, \beta) = 0$. Since n is the degree of the polynomial $G_n(x, r, \beta)$, we have $R_{G_n(x,r,\beta)} = n - C_{G_n(x,r,\beta)}$(see Table 1).

We can see a good regular pattern of the complex roots of the (r, β)-Bell equations $G_n(x, r, \beta) = 0$ for $r > 0$ and $\beta > 0$. Therefore, the following conjecture is possible.

Conjecture 1. *For $r > 0$ and $\beta > 0$, prove or disprove that*

$$C_{H_n(x,y)} = 0.$$

As a result of investigating more $r > 0$ and $\beta > 0$ variables, it is still unknown whether the conjecture 1 is true or false for all variables $r > 0$ and $\beta > 0$ (see Figure 1 and Table 1).

We observe that solutions of (r, β)-Bell equations $G_n(x, r, \beta) = 0$ has $Im(x) = 0$, reflecting symmetry analytic complex functions. It is expected that solutions of (r, β)-Bell equations $G_n(x, r, \beta) = 0$, has not $Re(x) = a$ reflection symmetry for $a \in \mathbb{R}$ (see Figures 2–4).

Conjecture 2. *Prove or disprove that solutions of (r, β)-Bell equations $G_n(x, r, \beta) = 0$, has not $Re(x) = a$ reflection symmetry for $a \in \mathbb{R}$.*

Finally, how many zeros do $G_n(x, r, \beta) = 0$ have? We are not able to decide if $G_n(x, r, \beta) = 0$ has n distinct solutions (see Tables 1 and 2). We would like to know the number of complex zeros $C_{G_n(x,r,\beta)}$ of $G_n(x, r, \beta) = 0, Im(x) \neq 0$.

Conjecture 3. *Prove or disprove that $G_n(x, r, \beta) = 0$ has n distinct solutions.*

As a result of investigating more n variables, it is still unknown whether the conjecture is true or false for all variables n (see Tables 1 and 2). We expect that research in these directions will make a new approach using the numerical method related to the research of the (r, β)-Bell numbers and polynomials which appear in mathematics, applied mathematics, statistics, and mathematical physics. The reader may refer to [5–10] for the details.

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