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On Pata–Suzuki-Type Contractions

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Abstract: In this paper, we aim to obtain fixed-point results by merging the interesting fixed-point theorem of Pata and Suzuki in the framework of complete metric space and to extend these results by involving admissible mapping. After introducing two new contractions, we investigate the existence of a (common) fixed point in these new settings. In addition, we shall consider an integral equation as an application of obtained results.

Keywords: pata type contraction; Suzuki type contraction; C-condition; orbital admissible mapping

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1. Introduction and Preliminaries

For the solution of several differential/fractional/integral equations, fixed-point theory plays a significant role. In such investigations, usually well-known Banach fixed-point theorems are sufficient to provide the desired results. In the case of the inadequacy, the researcher in the fixed-point theory proposes some extension of the Banach contraction principle. Among them, we recall one of the significant theorems given by Popescu [1] inspired from the notion of C-condition defined by Suzuki [2].

Definition 1 (See [3])**.** *Let T be a self-mapping on a metric space (X,d). It is called C-condition if*

$$\frac{1}{2}d(\varkappa,T\varkappa) \le d(\varkappa,y) \text{ implies that } d(T\varkappa,Ty) \le d(\varkappa,y), \forall \varkappa,y \in X.$$

Indeed, by using the notion of C-condition, Suzuki [2] extended the famous Edelstein Theorem. More precisely, For a self-mapping T on a compact metric space (X,d), if T is C-condition and the inequality $d(T\varkappa,Ty) < d(\varkappa,y)$, for all $\varkappa \neq y$, then T possesses a unique fixed point.

Popescu [1] considered Bogin-type fixed-point theorem involving the notion of C-condition in a complete metric space as follows:

Theorem 1. *Let a self-mapping T on a complete metric space (X,d) satisfy the following condition:*

$$\frac{1}{2}d(\varkappa,T\varkappa) \le d(\varkappa,y) \tag{1}$$

implies

$$d(T\varkappa,Ty) \le ad(\varkappa,y) + b[d(\varkappa,T\varkappa)+d(y,Ty)] + c[d(\varkappa,Ty)+d(y,T\varkappa)] \tag{2}$$

where $a \ge 0$, $b > 0$, $c > 0$ and $a+2b+2c = 1$. Then T has a unique fixed point.

Another outstanding generalization of Banach mapping principle was given by Pata [4]. Before giving the result of Pata [4], we fix some notations:

For an arbitrary point $\varkappa_0$ in a complete metric space (X, d), we shall consider a functional

$$\|\varkappa\| = d(\varkappa, \varkappa_0), \forall \varkappa \in X,$$

that will be called "the zero of X". In addition, $\psi : [0,1] \to [0,\infty)$ will be a fixed increasing function that is continuous at zero, with $\psi(0) = 0$.

Theorem 2 (See [4])**.** *Let T be a self-mapping on a metric space (X, d). Suppose that $\beta \in [0, \alpha]$ $\Lambda \geq 0$ and $\alpha \geq 1$ are fixed constants. A self-mapping T possesses a unique fixed point if*

$$d(T\varkappa, Ty) \quad \leq (1-\varepsilon)d(\varkappa, y) + \Lambda(\varepsilon)^{\alpha}\psi(\varepsilon)\left[1 + \|\varkappa\| + \|y\|\right]^{\beta},$$

holds for all $\varkappa, y \in X$ and for every $\varepsilon \in [0,1]$.

This theorem has been extended, modified, and generalized by several authors, e.g., [5–16].

The main goal of this paper is to introduce new contractions that are inspired from the results of Suzuki [2], Popescu [1], and Pata [4]. More precisely, our new contraction not only merges these two successful generalization Banach contractions, but also extends the structure by involving α-admissible mappings in it. After that, we aim to investigate the existence and uniqueness of this new contraction in the context of complete metric spaces.

For this purpose, we recall some basic notions and results from recent literature.

Definition 2 ([17])**.** *Let $X \neq \emptyset$ and $\alpha : X \times X \to [0,\infty)$ be an auxiliary function. A self-mapping T on X is called α-orbital admissible if*

$$\alpha(\varkappa, T\varkappa) \geq 1 \text{ implies that } \alpha(T\varkappa, T^2\varkappa) \geq 1, \text{ for any } \varkappa \in X.$$

Lemma 1 (See[18])**.** *Let $\{p_n\}$ be a sequence on a metric space (X, d). Suppose that the sequence $\{d(p_{n+1}, p_n)\}$ is nonincreasing with*

$$\lim_{n\to\infty} d(p_{n+1}, p_n) = 0,$$

If $\{p_n\}$ is not a Cauchy sequence then there exists a $\delta > 0$ and two strictly increasing sequences $\{m_k\}$ and $\{n_k\}$ in $\mathbb{N}$ such that the following sequences tend to δ :

$$d(p_{m_k}, p_{n_k}), d(p_{m_k}, p_{n_{k+1}}), d(p_{m_{k-1}}, p_{n_k}), d(p_{m_{k-1}}, p_{n_{k+1}}), d(p_{m_{k+1}}, p_{n_{k+1}}),$$

as $k \to \infty$.

2. Main Results

We start with the definition of the α-Pata–Suzuki contraction:

Definition 3. *Let (X, d) be a metric space and let $\Lambda \geq 0$, $\alpha \geq 1$ and $\beta \in [0, \alpha]$ be fixed constants. A self-mapping T, defined on X, is called α-Pata–Suzuki contraction if for every $\varepsilon \in [0,1]$ and all $x, y \in X$, satisfies the following condition*

(i) $\mathcal{T}$ *is an α-orbital admissible mapping*

(ii)

$$\frac{1}{2}d(x, \mathcal{T}x) \leq d(x, y)$$

implies

$$\alpha(x, \mathcal{T}x)\alpha(y, \mathcal{T}y)d(\mathcal{T}x, \mathcal{T}y) \leq P(x, y)$$

where

$$P(x,y) = (1-\varepsilon)\max\left\{d(x,y), d(x,\mathcal{T}x), d(y,\mathcal{T}y), \tfrac{1}{2}\left[d(x,\mathcal{T}y)+d(y,\mathcal{T}x)\right]\right\}$$
$$+\Lambda(\varepsilon)^{\alpha}\psi(\varepsilon)\left[1+\|x\|+\|y\|+\|\mathcal{T}x\|+\|\mathcal{T}y\|\right]^{\beta}.$$

This is the first main result of this paper.

Theorem 3. *Let (X,d) be a metric space and $\mathcal{T}$ be a self-mapping on X. If*

(i) *T on X is α-Pata–Suzuki contraction;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0,\mathcal{T}x_0) \geq 1$;*
(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1}) \geq 1$ for all n and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n,x) \geq 1$ for all n, we have $\alpha(x,\mathcal{T}x) \geq 1$;*
(iv) *$\alpha(x^*,\mathcal{T}x^*) \geq 1$ for all $x^* \in Fix(\mathcal{T})$, where $Fix(\mathcal{T}) := \{x \in X : Tx = x\}$, then $\mathcal{T}$ has a fixed point $z \in X$.*

Proof. Due to assumptions of the theorem, there is $x_0 \in X$ so that $\alpha(x_0,\mathcal{T}x_0) \geq 1$. In addition, we set $\|x\| = d(x,x_0), \forall x \in X$. Since $\mathcal{T}$ is an α-orbital admissible mapping, we have

$$\alpha(\mathcal{T}x_0,\mathcal{T}^2x_0) \geq 1.$$

and iteratively, we have

$$\alpha(\mathcal{T}^n x_0,\mathcal{T}^{n+1}x_0) \geq 1 \text{ for each } n \in \mathbb{N}. \tag{3}$$

Starting at this point x_0 we shall construct an iterative sequence $\{x_n\}$ by $x_n = \mathcal{T}^n x_0$ for $n = 1,2,3,\cdots$. Here, we assume that consequent terms are distinct. Indeed, if there exists $k_0 \in \mathbb{N}$ such that

$$\mathcal{T}_0^k x_0 = x_{k_0} = x_{k_0+1} = \mathcal{T}^{k_0+1}x_0 = \mathcal{T}(\mathcal{T}^k x_0) = \mathcal{T}(x_{k_0}),$$

then, x_{k_0} forms a fixed point. To avoid from the trivial case, we suppose that

$$x_n \neq x_{n+1} \text{ for all } n = 1,2,3,\cdots.$$

To prove that the sequence $\{d(x_n,x_{n+1})\}$ is decreasing, suppose on the contrary that

$$d(x_n,x_{n+1}) = \max\{d(x_n,x_{n+1}), d(x_n,x_{n-1})\}.$$

Since $\frac{1}{2}d(x_{n-1},x_n) \leq d(x_{n-1},x_n)$ and since T is a α-Pata–Suzuki contraction, we find that

$$\begin{aligned}
d(x_n,x_{n+1}) &= d(\mathcal{T}x_{n-1},\mathcal{T}x_n)\\
&\leq \alpha(x_{n-1},\mathcal{T}x_{n-1})\alpha(x_n,\mathcal{T}x_n)d(\mathcal{T}x_{n-1},\mathcal{T}x_n)\\
&\leq (1-\varepsilon)\max\left\{d(x_{n-1},x_n), d(x_{n-1},x_n), d(x_n,x_{n+1}), \tfrac{1}{2}\left[d(x_n,x_n)+d(x_{n-1},x_{n+1})\right]\right\}\\
&\qquad +\Lambda(\varepsilon)^{\alpha}\psi(\varepsilon)\left[1+\|x_{n-1}\|+\|x_n\|+\|\mathcal{T}x_{n-1}\|+\|\mathcal{T}x_n\|\right]^{\beta}\\
&\leq (1-\varepsilon)d(x_n,x_{n+1}) + K(\varepsilon)^{\alpha}\psi(\varepsilon),
\end{aligned}$$

for some $K > 0$. It follows that $d(x_n,x_{n+1}) = 0$ which is a contradiction. Hence, $\{d(x_n,x_{n+1})\}$ is a decreasing sequence, thus tending to some non-negative real number, say, d^*.

As a next step, we shall show that the sequence $\{\|x_n\|\}$ is bounded. For simplicity, let $C_n = \|x_n\|$, and hence, we claim that the sequence $\{C_n\}$ is bounded.

Since the sequence $\{d(x_n, x_{n+1})\}$ is decreasing, from the triangle inequality, we find that

$$\begin{aligned} C_n = d(x_n, x_0) &\leq d(x_n, x_{n+1}) + d(\mathcal{T}x_n, \mathcal{T}x_0) + C_1 \\ &\leq 2C_1 + d(\mathcal{T}x_n, \mathcal{T}x_0). \end{aligned}$$

We assert that

$$\frac{1}{2}d(x_n, x_{n+1}) \leq d(x_n, x_0) \text{ or } \frac{1}{2}d(x_{n-1}, x_n) \leq d(x_{n-1}, x_0).$$

Suppose, on contrary that

$$\frac{1}{2}d(x_n, x_{n+1}) > d(x_n, x_0) \text{ and } \frac{1}{2}d(x_{n-1}, x_n) > d(x_{n-1}, x_0).$$

In this case, we derive that

$$\begin{aligned} d(x_{n-1}, x_n) &\leq d(x_{n-1}, x_0) + d(x_0, x_n) \\ &< \frac{1}{2}\left[d(x_{n-1}, x_n) + d(x_n, x_{n+1})\right] \\ &\leq d(x_{n-1}, x_n), \end{aligned}$$

is a contradiction. Hence, our assertion is held, i.e.,

$$\frac{1}{2}d(x_n, x_{n+1}) \leq d(x_n, x_0) \text{ or } \frac{1}{2}d(x_{n-1}, x_n) \leq d(x_{n-1}, x_0).$$

Also, on account of (3), we have

$$\alpha(x_n, \mathcal{T}x_n)\alpha(x_0, \mathcal{T}x_0) \geq 1.$$

Regarding T is α-Pata–Suzuki contraction, we get

$$\begin{aligned} d(\mathcal{T}x_n, \mathcal{T}x_0) &\leq \alpha(x_n, \mathcal{T}x_n)\alpha(x_0, \mathcal{T}x_0)d(\mathcal{T}x_n, \mathcal{T}x_0) \\ &\leq (1-\varepsilon)\max\left\{ d(x_n, x_0),\quad d(x_0, x_1),\quad d(x_n, x_{n+1}),\quad \tfrac{1}{2}\left[d(x_n, x_1) + d(x_0, x_{n+1})\right] \right\} \\ &\qquad + \Lambda(\varepsilon)^{\alpha}\psi(\varepsilon)\left[1 + \|x_n\| + \|x_0\| + \|x_{n+1}\| + \|x_1\|\right]^{\beta} \\ &\leq (1-\varepsilon)\max\{C_n, C_1, C_1 + C_n\} + \Lambda(\varepsilon)^{\alpha}\psi(\varepsilon)\left[1 + C_n + C_1 + C_1 + C_n\right]^{\beta} \\ &\leq (1-\varepsilon)(C_1 + C_n) + \Lambda(\varepsilon)^{\alpha}\psi(\varepsilon)\left[1 + 2C_n + 2C_1\right]^{\beta}. \end{aligned}$$

Consequently, we derive from the above inequality that

$$\begin{aligned} C_n = d(x_n, x_0) &\leq d(x_n, x_{n+1}) + d(fx_n, fx_0) + C_1 \\ &\leq 2C_1 + (1-\varepsilon)(C_1 + C_n) + a(\varepsilon)^{\alpha}\psi(\varepsilon). \end{aligned}$$

A simple calculation yields that

$$\varepsilon C_n \leq a(\varepsilon)^{\alpha}\psi(\varepsilon) + b,$$

for some constants $a, b > 0$. By verbatim of the proof of ([18], Lemma 1.5) it follows that the sequence $\{C_n\}$ is bounded.

In what follows we prove that $d^* = 0$ by employing the fact that $\{C_n\}$ is bounded. Indeed, we have that

$$\begin{aligned} d(x_{n+1}, x_n) &= d(\mathcal{T}x_n, \mathcal{T}x_{n-1}) \\ &\leq \alpha(x_{n-1}, \mathcal{T}x_{n-1})\alpha(x_n, \mathcal{T}x_n)d(\mathcal{T}x_n, \mathcal{T}x_{n-1}) \\ &\leq (1-\varepsilon)d(x_n, x_{n-1}) + \Lambda(\varepsilon)^\alpha \psi(\varepsilon)\left[1 + \|x_n\| + \|x_{n-1}\| + \|x_n\| + \|x_{n+1}\|\right]^\beta \\ &\leq (1-\varepsilon)d(x_n, x_{n-1}) + \Lambda(\varepsilon)^\alpha \psi(\varepsilon)\left[1 + 2\|x_n\| + \|x_{n-1}\| + \|x_{n+1}\|\right]^\beta \\ &\leq (1-\varepsilon)d(x_n, x_{n-1}) + K(\varepsilon)^\alpha \psi(\varepsilon), \end{aligned}$$

for some $K > 0$. As $n \to \infty$ in the inequality above, it follows that $d^* = 0$.

As a next step, we shall indicate that $\{x_n\}$ is a Cauchy sequence by using the method of *Reductio ad Absurdum*. Assume, on the contrary, that the sequence $\{x_n\}$ is not Cauchy. Accordingly, regarding on Lemma 1, there exists $\delta > 0$ and two increasing sequences $\{m_k\}$ and $\{n_k\}$, with $n_k > m_k > k$ such that the sequences $d(x_{m_k}, x_{n_k})$,$d(x_{m_k}, x_{n_{k+1}})$,$d(x_{m_{k-1}}, x_{n_k})$,$d(x_{m_{k-1}}, x_{n_{k+1}})$,$d(x_{m_{k+1}}, x_{n_{k+1}})$ tends to δ as $n \to \infty$.

We claim that $\frac{1}{2}d(x_{m_{k-1}}, x_{m_k}) \leq d(x_{m_{k-1}}, x_{n_k})$. Indeed, if the inequality above is not held, that is, if $\frac{1}{2}d(x_{m_{k-1}}, x_{m_k}) > d(x_{m_{k-1}}, x_{n_k})$ then we get a contradiction. More precisely, by letting $k \to \infty$ in the previous inequality, we get $\delta \leq 0$, a contradiction.

Hence, our claim is valid, i.e., $\frac{1}{2}d(x_{m_{k-1}}, x_{m_k}) \leq d(x_{m_{k-1}}, x_{n_k})$. Notice also that $\alpha(x_{m_{k-1}}, f(x_{m_{k-1}}))\alpha(x_{n_k}, fx_{n_k}) \geq 1\ \forall k \geq N$. Since T is α-Pata–Suzuki contraction, we deduce that

$$\begin{aligned} d(x_{m_k}, x_{n_{k+1}}) &= d(\mathcal{T}x_{m_{k-1}}, \mathcal{T}x_{n_k}) \\ &\leq \alpha(x_{m_{k-1}}, \mathcal{T}(x_{m_{k-1}}))\alpha(x_{n_k}, \mathcal{T}, x_{n_k})d(\mathcal{T}x_{m_{k-1}}, \mathcal{T}x_{n_k}) \\ &\leq (1-\varepsilon)\max\left\{\begin{array}{c} d(x_{m_{k-1}}, x_{n_k}), d(x_{m_{k-1}}, x_{m_k}), d(x_{n_k}, x_{n_{k+1}}), \\ \frac{1}{2}\left[d(x_{n_k}, x_{m_k}) + d(x_{m_{k-1}}, x_{n_{k+1}})\right] \end{array}\right\} \\ &\qquad + \Lambda(\varepsilon)^\alpha \psi(\varepsilon)\left[1 + \|x_{m_{k-1}}\| + \|x_{n_k}\| + \|x_{m_k}\| + \|x_{n_{k+1}}\|\right]^\beta \\ &\leq (1-\varepsilon)\max\left\{\begin{array}{c} d(x_{m_{k-1}}, x_{n_k}), d(x_{m_{k-1}}, x_{m_k}), d(x_{n_k}, x_{n_{k+1}}), \\ \frac{1}{2}\left[d(x_{n_k}, x_{m_k}) + d(x_{m_{k-1}}, x_{n_{k+1}})\right] \end{array}\right\} \\ &\qquad + K(\varepsilon)^\alpha \psi(\varepsilon), \end{aligned}$$

where $K > 0$. By letting $k \to \infty$ in the obtained inequality above, we get that $\delta = 0$, a contradiction.

Hence, $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $z^* \in X$ such that $x_n \to z^*$ and by (v) and $\alpha(z^*, \mathcal{T}z^*) \geq 1$.

Now, we shall prove that $z^* = \mathcal{T}z^*$. Suppose, on the contrary, that $z^* \neq \mathcal{T}z^*$. For this purpose, we need to prove the claim: For each $n \geq 1$, at least one of the following assertions holds.

$$\frac{1}{2}d(x_{n-1}, x_n) \leq d(x_{n-1}, z^*) \text{ or } \frac{1}{2}d(x_n, x_{n+1}) \leq d(x_n, z^*).$$

Again, we use the method of Reductio ad Absurdum and assume it does not hold, i.e.,

$$\frac{1}{2}d(x_{n-1}, x_n) > d(x_{n-1}, z^*) \text{ and } \frac{1}{2}d(x_n, x_{n+1}) > d(x_n, z^*),$$

for some $n \geq 1$. Then, keeping in mind that $\{d(x_n, x_{n+1})\}$ is a decreasing sequence, the triangle inequality infers

$$\begin{aligned} d(x_{n-1}, x_n) &\leq d(x_{n-1}, z^*) + d(z^*, x_n) \\ &< \tfrac{1}{2}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \\ &< d(x_{n-1}, x_n), \end{aligned}$$

which is a contradiction, and so the claim holds.

Due to the assumption (v) and the observation (3), we have

$$\alpha(x_n, \mathcal{T}x_n)\alpha(z^*, \mathcal{T}z^*) \geq 1, \text{ holds for all } n \in N.$$

Taking $\frac{1}{2}d(x_n, \mathcal{T}x_n) \leq d(x_n, z^*)$ into account, the assumption (i) yields that

$$\begin{aligned} d(\mathcal{T}x_n, \mathcal{T}z^*) \leq{}& (1-\varepsilon)\max\left\{ d(x_n, z^*), \quad d(z^*, \mathcal{T}z^*), \quad d(x_n, x_{n+1}), \quad \tfrac{1}{2}\left[d(x_n, \mathcal{T}z^*) + d(z^*, \mathcal{T}x_n)\right] \right\} \\ &+\Lambda(\varepsilon)^{\alpha}\psi(\varepsilon)\left[1 + \|x_n\| + \|z^*\| + \|\mathcal{T}z^*\| + \|\mathcal{T}x_n\|\right]^{\beta} \\ ={}& (1-\varepsilon)\max\left\{ d(x_n, z^*), \quad d(z^*, \mathcal{T}z^*), \quad d(x_n, x_{n+1}), \quad \tfrac{1}{2}\left[d(x_n, \mathcal{T}z^*) + d(z^*, \mathcal{T}x_n)\right] \right\} \\ &+K(\varepsilon)^{\alpha}\psi(\varepsilon), \end{aligned}$$

for some $K > 0$. By letting $n \to \infty$ in the inequality above, we find that

$$\begin{aligned} d(z^*, fz_1) &\leq (1-\varepsilon)\max\left\{0, d(z^*, \mathcal{T}z^*), 0, \tfrac{d(z^*, \mathcal{T}z^*)}{2}\right\} + K(\varepsilon)^{\alpha}\psi(\varepsilon) \\ &< (1-\varepsilon)d(z^*, \mathcal{T}z^*) + K(\varepsilon)^{\alpha}\psi(\varepsilon) \end{aligned}$$

for some $K > 0$. It implies that $d(z^*, \mathcal{T}z^*) = 0$, a contradiction. Hence $z^* = \mathcal{T}z^*$.

As a final step, we examine the uniqueness of the found fixed point z^*. Suppose that v^* is another fixed point of $\mathcal{T}$ that is distinct from z^*. $\mathcal{T}z^* = z^*$ and $\mathcal{T}v^* = v^*$. By (v) we have

$$\alpha(z^*, \mathcal{T}z^*) \geq 1 \text{ and } \alpha(v, \mathcal{T}v^*) \geq 1.$$

Since $\frac{1}{2}d(z^*, \mathcal{T}z^*) \leq d(z^*, v^*)$ the assumption (i) yields that

$$\begin{aligned} d(\mathcal{T}z^*, \mathcal{T}v^*) &\leq (1-\varepsilon)\max\left\{d(z^*, v^*), d(z, \mathcal{T}z^*), d(v^*, \mathcal{T}v^*), \tfrac{1}{2}\left[d(z^*, \mathcal{T}v) + d(v^*, \mathcal{T}z^*)\right]\right\} \\ &\quad +\Lambda(\varepsilon)^{\alpha}\psi(\varepsilon)\left[1 + 2\|z^*\| + 2\|v^*\|\right]^{\beta} \\ &< (1-\varepsilon)d(z^*, v^*) + K(\varepsilon)^{\alpha}\psi(\varepsilon) \end{aligned}$$

for some $K > 0$ that yields that $d(z^*, v^*) = 0$, a contradiction. Hence $z^* = v^*$. □

Example 1. *Let $X = [0, \infty)$ and let $d(x, y) = |x - y|$ for all $x, y \in X$. Let $\Lambda = \frac{1}{2}$, $\alpha = 1$, $\beta = 1$ and $\psi(\epsilon) = \epsilon^{\frac{1}{2}}$ for every $\epsilon \in [0, 1]$ and a mapping $T : X \to X$ be defined by*

$$Tx = \begin{cases} \frac{1}{2}x & if\ 0 \leq x \leq 1, \\ 2x & if\ x > 1. \end{cases},$$

Also, we define a function $\alpha : X \times X \to [0, \infty)$ in the following way

$$\alpha(x, y) = \begin{cases} 1 & if\ 0 \leq x, y \leq 1, \\ 0 & otherwise \end{cases}$$

Also, we have

$$\frac{1}{2} - 1 + \epsilon \leq \frac{1}{2}(1 - 2 + \frac{\epsilon}{2}) \leq \frac{1}{2}(\epsilon)^{\frac{1}{2}}.$$

Now

$$\frac{1}{2}d(x, Tx) = \frac{1}{2}|x - \frac{x}{2}| \leq d(x, y)$$

implies

$$\begin{aligned}
d(Tx, Ty) & \\
&=\leq |Tx - Ty| \\
&= |\tfrac{x}{2} - \tfrac{y}{2}| \\
&= \tfrac{1}{2}|x - y| \\
&\leq \tfrac{1}{2}P(x, y) \\
&= (1 - \epsilon)P(x, y) + (\tfrac{1}{2} - 1 + \epsilon)P(x, y) \\
&\leq (1 - \epsilon)P(x, y) + (\tfrac{1}{2} - 1 + \epsilon)\,[1 + \|x\| + \|y\| + \|Tx\| + \|Ty\|] \\
&\leq (1 - \epsilon)P(x, y) + (\tfrac{1}{2}\epsilon\epsilon^{\frac{1}{2}})\,[1 + \|x\| + \|y\| + \|Tx\| + \|Ty\|]
\end{aligned}$$

Hence, T satisfies all the conditions of theorem and T has a unique fixed point.

Immediate Consequences

In this subsection, we list a few consequences of our main result. These corollaries also indicate how we can conclude further consequences.

If we let $\alpha(x, Tx) = 1$ for all $x \in X$, we get the following results:

Theorem 4. *Let T be a self-mapping on a metric space (X, d). Suppose that $\beta \in [0, \alpha]$ $\Lambda \geq 0$ and $\alpha \geq 1$ are fixed constants. A self-mapping T possesses a unique fixed point if $\frac{1}{2}d(\varkappa, \mathcal{T}\varkappa) \leq d(\varkappa, y)$ implies*

$$d(\mathcal{T}\varkappa, \mathcal{T}y) \leq P(\varkappa, y)$$

where

$$\begin{aligned}
P(\varkappa, y) &= (1 - \varepsilon)\max\left\{d(\varkappa, y), d(\varkappa, \mathcal{T}\varkappa), d(y, \mathcal{T}y), \tfrac{1}{2}\left[d(\varkappa, \mathcal{T}y) + d(y, \mathcal{T}\varkappa)\right]\right\} \\
&\quad + \Lambda(\varepsilon)^{\alpha}\psi(\varepsilon)\left[1 + \|\varkappa\| + \|y\| + \|\mathcal{T}\varkappa\| + \|\mathcal{T}y\|\right]^{\beta}.
\end{aligned}$$

for all $\varkappa, y \in X$ and for every $\varepsilon \in [0, 1]$.

Let $(X, \preceq)$ be a partially ordered set and d be a metric on X. We say that $(X, \preceq, d)$ is regular if for every nondecreasing sequence $\{\varkappa_n\} \subset X$ such that $\varkappa_n \to x \in X$ as $n \to \infty$, there exists a subsequence $\{\varkappa_{n(k)}\}$ of $\{\varkappa_n\}$ such that $\varkappa_{n(k)} \preceq x$ for all k.

Theorem 5. *Let $(X, \preceq)$ be a partially ordered set and d be a metric on X such that (X, d) is complete. Let $T : X \to X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that $\beta \in [0, \alpha]$ $\Lambda \geq 0$ and $\alpha \geq 1$ are fixed constants such that the self-mapping T satisfies the following condition: $\frac{1}{2}d(\varkappa, \mathcal{T}\varkappa) \leq d(\varkappa, y)$ implies*

$$d(\mathcal{T}\varkappa, \mathcal{T}y) \leq P(\varkappa, y)$$

where

$$\begin{aligned}
P(\varkappa, y) &= (1 - \varepsilon)\max\left\{d(\varkappa, y), d(\varkappa, \mathcal{T}\varkappa), d(y, \mathcal{T}y), \tfrac{1}{2}\left[d(\varkappa, \mathcal{T}y) + d(y, \mathcal{T}\varkappa)\right]\right\} \\
&\quad + \Lambda(\varepsilon)^{\alpha}\psi(\varepsilon)\left[1 + \|\varkappa\| + \|y\| + \|\mathcal{T}\varkappa\| + \|\mathcal{T}y\|\right]^{\beta}.
\end{aligned}$$

for all $\varkappa, y \in X$ with $\varkappa \preceq y$ and for every $\varepsilon \in [0, 1]$. Suppose also that the following conditions hold:

(i) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;*

(ii) $(X, \preceq, d)$ *is regular.*

(iii) *T is nondecreasing with respect to* $\preceq$ *(that is,* $\varkappa, y \in X$, $\varkappa \preceq y \Longrightarrow T\varkappa \preceq Ty$.*)*

Then T has a fixed point.

Moreover, if for all $\varkappa, y \in X$ *there exists* $z \in X$ *such that* $\varkappa, \preceq z$ *and* $y \preceq z$*, we have uniqueness of the fixed point.*

Proof. Set $\alpha : X \times X \to [0, \infty)$ in a way that

$$\alpha(x, y) = \begin{cases} 1 \text{ if } \varkappa \preceq y \text{ or } \varkappa \succeq y, \\ 0 \text{ otherwise.} \end{cases}$$

It is apparent that T is an α-Suzuki-Pata contractive mapping, i.e.,

$$\alpha(\varkappa, y)d(T\varkappa, Ty) \leq P(\varkappa, y),$$

for all $\varkappa, y \in X$. By assumption, the inequality $\alpha(\varkappa_0, T\varkappa_0) \geq 1$ is observed. In addition, for all $\varkappa, y \in X$, due to the fact that T is nondecreasing, we find

$$\alpha(x, y) \geq 1 \Longrightarrow x \succeq y \text{ or } x \preceq y \Longrightarrow Tx \succeq Ty \text{ or } Tx \preceq Ty \Longrightarrow \alpha(Tx, Ty) \geq 1.$$

Consequently, we note that T is $\alpha-$admissible. Now, assume that $(X, \preceq, d)$ is regular. Let $\{\varkappa_n\}$ be a sequence in X such that $\alpha(\varkappa_n, \varkappa_{n+1}) \geq 1$ for all n and $\varkappa_n \to x \in X$ as $n \to \infty$. From the regularity hypothesis, there exists a subsequence $\{\varkappa_{n(k)}\}$ of $\{x_n\}$ such that $\varkappa_{n(k)} \preceq x$ for all k. On account of α we derive that $\alpha(\varkappa_{n(k)}, \varkappa) \geq 1$ for all k. Consequently, the existence and uniqueness of the fixed point is derived by Theorem 3. $\square$

3. Application

In this section, we shall consider an application for our main result. Let $X = C[0, 1]$ be the space of all continuous functions defined on interval $[0, 1]$ with the metric

$$d(x, y) = \sup_{t \in [0,1]} |x(t) - y(t)|.$$

In what follows we shall use Theorem 5 to show that there is a solution to the following integral equation:

$$x(t) = y(t) + \int_0^1 k(t, s, x(s))ds, t \in [0, 1] \tag{4}$$

Assume that $k(t, s, x)$ is continuous. Let $y \in C[0, 1]$.

We consider the following conditions:

(a) $k : [0, 1] \times [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous;

(b) there exists a continuous function $\gamma : [0, \infty] \times \mathbb{R} \to \mathbb{R}$ such that

$$\sup_{t \in [0,1]} \int_0^1 \gamma(t, s) \leq 1;$$

(c) there exists $\varepsilon \in [0,1]$ such that

$$\frac{1}{2}\left|x(s)-y(s)-\int_0^1 k(t,s,x(s))ds\right| \leq |x(s)-y(s)|$$

implies

$$|k(t,s,x(s))-k(t,s,y(s))| \leq (1-\varepsilon)\,|x(s)-y(s)|\,,$$

for all $x, y \in X$;

(d) there exists $x_0 \in C([0,1])$ such that for all $t \in [0,1]$, we have

$$\zeta(x_0(t), \int_0^1 k(t,s,x(s))ds) \geq 0,$$

where $\zeta : X \times X \to [0,\infty)$;

(e) For all $t \in [0,1]$, $x, y \in C[0,1]$,

$$\zeta(x(t),y(t)) \geq 0 \Rightarrow \zeta(\int_0^1 k(t,s,x(s))ds, \int_0^1 k(t,s,y(s))ds) \geq 0;$$

(f) If x_n is a sequence in C[0,1] such that $x_n \to x \in C[0,1]$ and $\zeta(x_n, x_{n+1}) \geq 0$ for all n, then $\zeta(x_n, x) \geq 0$ for all n.

Theorem 6. *Suppose that the conditions (a)–(f) are satisfied. Then, the integral Equation (4) has solution in $C[0,1]$.*

Proof. Since k and the function y are continuous, now define an operator

$$\mathcal{T} : C[0,1] \to C[0,1]$$

write the integral Equation (4) in the form $x = \mathcal{T}x$, where

$$\mathcal{T}x(t) = y(t) + \int_0^1 k(t,s,x(s))ds. \tag{5}$$

It follows that

$$\frac{1}{2}|x(s)-y(s)-\int_0^1 k(t,s,x(s))ds| \leq (1-\varepsilon)|x(s)-y(s)|$$

implies

$$
\begin{aligned}
d(\mathcal{T}x,\mathcal{T}y) &= \sup_{t\in[0,1]} |\mathcal{T}x(t)-\mathcal{T}y(t)| \\
&\leq \sup_{t\in[0,1]} \int_0^1 |k(t,s,x(s))-k(t,s,y(s))|ds \\
&\leq \sup_{s\in[0,1]} \int_0^1 \gamma(t,s)ds|k(t,s,x)-k(t,s,y)| \\
&\leq (1-\varepsilon)|x(s)-y(s)| \\
&\leq (1-\varepsilon)\max\left\{d(x,y),d(x,\mathcal{T}x),d(y,\mathcal{T}y),\tfrac{1}{2}\left[d(x,\mathcal{T}y)+d(y,\mathcal{T}x)\right]\right\} \\
&\quad +\Lambda(\varepsilon)^{\alpha}\psi(\varepsilon)\left[1+\|x\|+\|y\|+\|\mathcal{T}x\|+\|\mathcal{T}y\|\right]^{\beta} \quad \lambda\geq 0 \quad \alpha\geq 1 \text{ and } \beta\in[0,\alpha].
\end{aligned}
$$

Define the function $\alpha : C[0,1]\times C[0,1]\to[0,+\infty)$ by

$$
\alpha(x,y)=\begin{cases} 1 & if\ \zeta(x(t),y(t))\geq 0, t\in[0,1], \\ 0 & otherwise. \end{cases}
$$

For all $x,y\in C[0,1]$, we have

Therefore, all the conditions of Theorem 5 are satisfied. Consequently, the mapping $\mathcal{T}$ has a unique fixed point in X, which is a solution of integral equation. □

4. Conclusions

In this paper, we combine and extend significant fixed-point results, namely Suzuki [2], Popescu [1], and Pata [4] by involving the admissible mappings. As in [3] (see also [19]), by proper choice of the auxiliary admissible mapping α and replacing the set $P(\varkappa,y)$ with some concrete subset, we can derive several more consequences. Since the techniques are the same in [3], we skip the details and we avoid listing all possible corollaries. Indeed, Theorem 4 and Theorem 5 are the basic examples of this consideration. Notice also that the given example and an integral equation can be improved according to choice of α.

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