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A Convergence Theorem for the Nonequilibrium States in the Discrete Thermostatted Kinetic Theory

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Abstract: The existence and reaching of nonequilibrium stationary states are important issues that need to be taken into account in the development of mathematical modeling frameworks for far off equilibrium complex systems. The main result of this paper is the rigorous proof that the solution of the discrete thermostatted kinetic model catches the stationary solutions as time goes to infinity. The approach towards nonequilibrium stationary states is ensured by the presence of a dissipative term (thermostat) that counterbalances the action of an external force field. The main result is obtained by employing the Discrete Fourier Transform (DFT).

Keywords: thermostat; nonequilibrium stationary states; discrete Fourier transform; discrete kinetic theory; nonlinearity

1. Introduction

The modeling of a complex living system requires much attention considering the large number of components or active particles, the multiple nonlinear interactions, and the emerging collective behaviors [1–3]. The evolution of a complex system and the related global collective behaviors is usually driven by an external event (a predator for a swarm, an alert for a crowd of pedestrians, a vaccine for a tumor); see, among others, [4–6] and the references cited therein. Accordingly, a suitable modeling framework needs to take into account the nonequilibrium conditions under which a complex living system operates. Different modeling frameworks coming from the applied sciences have been developed [7–9], and in particular, the tools of nonequilibrium statistical mechanics have been proposed and employed in an attempt to follow the evolution of a complex system from the transient state to the stationary state; see [10–12].

Recently, the discrete thermostatted kinetic theory was proposed in [13,14] for the modeling and analysis of a far from equilibrium complex system; applications to biology [15,16] and crowd dynamics have been developed [17]. According to this theory, the complex system is divided into different functional subsystems composed by particles expressing the same task, which is usually a strategy. The strategy is modeled by introducing a scalar variable called activity; the interactions among the particles, called active particles, is modeled according to the stochastic game theory [18]. The nonequilibrium condition is modeled by introducing an external force field coupled to a dissipative term (called a thermostat, in analogy with the nonequilibrium statistical mechanics [19,20]), which allows the reaching of a nonequilibrium stationary state. Depending on the phenomenon under consideration, the activity variable can have a continuous or a discrete structure. Consequently, the mathematical structure reduces to a system of nonlinear partial integro-differential equations (continuous structure [21]) or nonlinear ordinary differential equations (discrete structure [13]).

The present paper is devoted to a further mathematical analysis of the discrete thermostatted kinetic theory framework proposed in [13]. The conditions of the existence and uniqueness of the nonequilibrium stationary solution were investigated in [14]. This paper provides the mathematical proof that the solutions of the discrete thermostatted kinetic theory framework approach the nonequilibrium stationary solutions when the time goes to infinity. The main result is obtained by employing the Discrete Fourier Transform (DFT).

It is worth noting that to the best of our knowledge, this is the first time that this proof has been presented for the discrete thermostatted kinetic theory proposed in [13]. However similar investigations have been addressed for the thermostatted Kac equation [22–25]. Further applications can be envisaged for biosystems [26,27], in medicine [28] and for complex systems [29].

The present paper is structured into three sections. In particular, Section 2 is devoted to the foundations and the main assumptions of the discrete thermostatted kinetic theory. The Discrete Fourier Transform (DFT) and the statement of the main result, concerning the convergence of the solutions of the discrete thermostatted kinetic theory to the related nonequilibrium stationary solutions, are presented in Section 3. The proof of the main result is detailed in Section 4.

2. The Discrete Thermostatted Framework

Let $I_u = \{u_1, u_2, \dots, u_n\}$ be a discrete subset of $\mathbb{R}$, $F_i(t) \geq 0$ for $i \in \{1, 2, \dots, n\}$ and $t > 0$, and $\mathbf{f}(t) = (f_1(t), f_2(t), \dots, f_n(t))$, where, for $i \in \{1, 2, \dots, n\}$,

$$f_i(t) := f(t, u_i) : [0, +\infty[\times I_u \to \mathbb{R}^+$$

is the solution of the following nonlinear ordinary differential equation:

$$\frac{df_i}{dt}(t) = J_i[\mathbf{f}](t) + F_i(t) - \sum_{i=1}^{n} \left(\frac{u_i^2 \,(J_i[\mathbf{f}] + F_i)}{\mathbb{E}_2[\mathbf{f}]} \right) f_i(t). \tag{1}$$

The operator $J_i[\mathbf{f}](t)$, for $i \in \{1, 2, \dots, n\}$, is given by:

$$\begin{aligned} J_i[\mathbf{f}](t) &= G_i[\mathbf{f}](t) - L_i[\mathbf{f}](t) \\ &= \sum_{h=1}^{n} \sum_{k=1}^{n} \eta B_{hk}^i f_h(t) f_k(t) - f_i(t) \sum_{k=1}^{n} \eta f_k(t), \end{aligned}$$

where $\eta > 0$ and, for $i, h, k \in \{1, 2, \dots, n\}$, $B_{hk}^i : I_u \times I_u \times I_u \to \mathbb{R}^+$; the function $\mathbb{E}_2[\mathbf{f}](t)$ denotes the second order moment of $\mathbf{f}$:

$$\mathbb{E}_2[\mathbf{f}](t) = \sum_{i=1}^{n} u_i^2 f_i(t).$$

Let $\mathbb{E}_2 \in \mathbb{R}^+$ and $\mathcal{R}_{\mathbf{f}}^2$ denote the function space:

$$\mathcal{R}_{\mathbf{f}}^2 = \mathcal{R}_{\mathbf{f}}^2 \left(\mathbb{R}^+; \mathbb{E}_2\right) = \left\{ \mathbf{f} \in C\left([0, +\infty]; \left(\mathbb{R}^+\right)^n\right) : \mathbb{E}_2[\mathbf{f}] = \mathbb{E}_2 \right\}.$$

The existence and uniqueness of the solution of the related Cauchy problem has been proven in [13] under the following assumptions:

H1 The function B_{hk}^i is normalized with respect to i, namely for all $h, k \in \{1, 2, \dots, n\}$, one has:

$$\sum_{i=1}^{n} B_{hk}^i = 1;$$

H2 $u_i \geq 1$, for all $i \in \{1, 2, \ldots, n\}$.

A nonequilibrium stationary solution of Equation (1) is a function f_i, for $i \in \{1, 2, \ldots, n\}$, which is the solution of the following equation:

$$J_i[\mathbf{f}] + F_i - \sum_{i=1}^{n} \left(\frac{u_i^2 \left(J_i[\mathbf{f}] + F_i\right)}{\mathbb{E}_2} \right) f_i = 0. \tag{2}$$

Let $\tilde{\mathcal{R}}_{\mathbf{f}}^2$ be the following function space:

$$\tilde{\mathcal{R}}_{\mathbf{f}}^2 \left(\mathbb{R}^+; \mathbb{E}_2\right) = \left\{ \mathbf{f} \in \left(\mathbb{R}^+\right)^n : \mathbb{E}_2[\mathbf{f}] = \mathbb{E}_2 \right\}.$$

The existence of nonequilibrium stationary solutions $\mathbf{g} \in \tilde{\mathcal{R}}_{\mathbf{f}}^2$ was proven in [14], under Assumptions **H1–H2** and:

H3 $\sum_{i=1}^{n} u_i B_{hk}^i = 0$, for all $h, k \in \{1, 2, \ldots, n\}$;

H4 $\sum_{i=1}^{n} u_i^2 B_{hk}^i = u_h^2$, for all $h, k \in \{1, 2, \ldots, n\}$;

H5 $\mathbb{E}_0[\mathbf{f}] := \sum_{i=1}^{n} f_i = \mathbb{E}_2[\mathbf{f}] = 1$.

In particular, in [14], it was proven that the nonequilibrium stationary solution is unique if the following constraint on the force field $\mathbf{F}(t) = (F, F, \ldots, F)$ holds true:

$$F \geq 2\eta \mathbb{E}_2^2 \left(1 + \frac{1}{\|u\|_2^2} \right).$$

Proposition 1 ([14])**.** *Assume that Assumptions **H1–H5** hold true. Then:*

1. *The evolution equation of* $\mathbb{E}_1[\mathbf{f}](t) = \sum_{i=1}^{n} u_i f_i(t)$ *reads:*

$$\mathbb{E}_1'[\mathbf{f}](t) + \left(\eta + \sum_{i=1}^{n} u_i^2 f_i \right) \mathbb{E}_1[\mathbf{f}](t) - \sum_{i=1}^{n} u_i F_i = 0; \tag{3}$$

2. *The first-order moment converges to a constant, which depends on the parameters of the system, as t goes to infinity, i.e.,*

$$\mathbb{E}_1[\mathbf{f}](t) \to K := \frac{\sum_{i=1}^{n} u_i F_i}{\eta + \sum_{i=1}^{n} u_i^2 F_i}; \tag{4}$$

3. *Let* $\mathbf{f}_0$ *be the initial data of the Cauchy problem related to* (1)*, then:*

$$|\mathbb{E}_1[\mathbf{f}](t) - K| \leq c\, e^{-\left(\eta + \sum_{i=1}^{n} u_i^2 F_i \right) t}, \tag{5}$$

where c is a constant that depends on the system.

Remark 1. *Equation* (1) *was proposed in [13] for the modeling of a complex system, which is assumed to be composed of n subsystems (called* functional subsystems*). In particular:*

- *The function* $f_i(t)$*, for* $i \in \{1, 2, \ldots, n\}$*, denotes the distribution function of the ith functional subsystem;*

- *The function* $\mathbf{F}(t) = (F_1(t), F_2(t), \ldots, F_n(t))$ *is the external force field acting on the whole system;*
- *The term:*

$$\alpha := \sum_{i=1}^{n} \left(\frac{u_i^2 \left(J_i[\mathbf{f}] + F_i\right)}{\mathbb{E}_2} \right)$$

represents the thermostat term, which allows keeping the quantity $\mathbb{E}_2[\mathbf{f}](t)$ *constant;*
- *The term* η_{hk} *is the* interaction rate *related to the encounters between the functional subsystem h and the functional subsystem k, for* $h, k \in \{1, 2, \ldots, n\}$*;*
- *The function* B_{hk}^i *denotes the* transition probability density *that the functional subsystem h falls into i after interacting with the functional subsystem k, for* $i, h, k \in \{1, 2, \ldots, n\}$*;*
- *The operator* $J_i[\mathbf{f}](t)$*, for* $i \in \{1, 2, \ldots, n\}$*, models the net flux related to the ith functional subsystem;* $G_i[\mathbf{f}](t)$ *denotes the gain term operator and* $L_i[\mathbf{f}](t)$ *the loss term operator.*

Remark 2. *Let* $p \in \mathbb{N}$*. Equation* (1) *can be further generalized as follows:*

$$\frac{df_i}{dt}(t) = J_i[\mathbf{f}](t) + F_i(t) - \sum_{j=1}^{n} \left(\frac{u_j^p \left(J_j[\mathbf{f}] + F_j\right)}{\mathbb{E}_p[\mathbf{f}]} \right) f_i(t).$$

The above framework allows keeping the following pth order moment constant:

$$\mathbb{E}_p[\mathbf{f}](t) = \sum_{i=1}^{n} u_i^p f_i(t).$$

3. Convergence to the Stationary State

The main result of this paper is the proof that the function $\mathbf{f}(t)$, the solution of Equation (1), converges to the nonequilibrium stationary solution $\mathbf{g}$ of (2). The proof is based on the Discrete Fourier Transform (DFT).

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$; the DFT is defined as follows:

$$\hat{x}_m = \sum_{l=1}^{n} x_l e^{-\frac{2\pi\iota}{n} m(l-1)}, \qquad m \in \{1, 2, \ldots, n\},$$

where ι denotes the imaginary unit.

Let $\hat{f}_m(t)$ and $\hat{g}_m$, the DFT of the solution $\mathbf{f}(t)$ of (1) and of the solution $\mathbf{g}$ of (2), respectively, be defined as follows:

$$\hat{f}_m(t) = \sum_{l=1}^{n} f_l(t) e^{-\frac{2\pi\iota}{n} m(l-1)}, \qquad \hat{g}_m = \sum_{l=1}^{n} g_l e^{-\frac{2\pi\iota}{n} m(l-1)}, \tag{6}$$

for $m \in \{1, 2, \ldots, n\}$.

Theorem 1. *Let* $\mathbf{f}(t)$ *be the solution of Equation* (1) *and* $\mathbf{g}$ *the solution of Equation* (2)*. If Assumptions* ***H1–H4*** *hold true, then* $\mathbf{f}(t)$ *converges to* $\mathbf{g}$*, as* $t \to +\infty$*.*

4. Proof of the Main Result

Proof of Theorem 1. The first step is the derivation of the DFT of the discrete thermostatted Equation (1).

Multiplying both sides of (1) by $e^{-\frac{2\pi\iota}{n} m(l-1)}$ and summing for l from 1–n, one has:

$$\sum_{l=1}^{n} \frac{df_l}{dt}(t) e^{-\frac{2\pi\iota}{n} m(l-1)} = \sum_{l=1}^{n} \left(J_l[\mathbf{f}](t) + F_l(t) - \alpha f_l(t)\right) e^{-\frac{2\pi\iota}{n} m(l-1)}, \tag{7}$$

for $m \in \{1, 2, \ldots, n\}$.

The left-hand side of (7) is written as:

$$\begin{aligned} \sum_{l=1}^{n} \frac{df_l}{dt}(t) e^{-\frac{2\pi\iota}{n} m(l-1)} &= \frac{d}{dt}\left(\sum_{l=1}^{n} f_l(t) e^{-\frac{2\pi\iota}{n} m(l-1)}\right) \\ &= \frac{d\hat{f}_m}{dt}(t). \end{aligned} \tag{8}$$

The first term in the right-hand side of the (7), using the property of operators $G_i[\mathbf{f}](t)$ and $L_i[\mathbf{f}](t)$, is written as:

$$\begin{aligned} \sum_{l=1}^{n} J_l[\mathbf{f}](t) e^{-\frac{-2\pi\iota}{n} m(l-1)} &= \sum_{l=1}^{n} \left(G_l[\mathbf{f}](t) - L_l[\mathbf{f}](t)\right) e^{-\frac{2\pi\iota}{n} m(l-1)} \\ &= \sum_{l=1}^{n} \left(\sum_{h=1}^{n}\sum_{k=1}^{n} \eta B_{hk}^{l} f_h(t) f_k(t)\right) e^{-\frac{2\pi\iota}{n} m(l-1)} \\ &- \sum_{l=1}^{n} \left(f_l(t) \sum_{k=1}^{n} \eta f_k(t)\right) e^{-\frac{2\pi\iota}{n} m(l-1)}, \end{aligned} \tag{9}$$

where:

$$\begin{aligned} &\sum_{l=1}^{n} \left(\sum_{h=1}^{n}\sum_{k=1}^{n} \eta B_{hk}^{l} f_h(t) f_k(t)\right) e^{-\frac{2\pi\iota}{n} m(l-1)} \\ &= \eta \sum_{h=1}^{n}\sum_{k=1}^{n} f_h(t) f_k(t) \left(\sum_{l=1}^{n} B_{hk}^{l} e^{-\frac{2\pi\iota}{n} m(l-1)}\right). \end{aligned} \tag{10}$$

For Assumption **H5**, one has:

$$\begin{aligned} \sum_{l=1}^{n} \left(f_l(t) \sum_{k=1}^{n} \eta f_k(t)\right) e^{-\frac{2\pi\iota}{n} m(l-1)} &= \eta \sum_{l=1}^{n} f_l(t) e^{-\frac{2\pi\iota}{n} m(l-1)} \\ &= \eta \hat{f}_m(t). \end{aligned} \tag{11}$$

The second term of (7) reads:

$$\sum_{l=1}^{n} F_l e^{-\frac{2\pi\iota}{n} m(l-1)} = \hat{F}_m. \tag{12}$$

For Assumptions **H1**, **H3**, **H4**, and **H5**, the third term of the right-hand side of the (7) is written as:

$$\begin{aligned} &\sum_{l=1}^{n} \left(\sum_{l=1}^{n} u_l^2 \left(J_l[\mathbf{f}](t) + F_l\right)\right) f_l(t) e^{-\frac{2\pi\iota}{n} m(l-1)} \\ &= \sum_{l=1}^{n} \left(\sum_{l=1} u_l^2 F_l\right) f_l(t) e^{-\frac{2\pi\iota}{n} m(l-1)} \\ &= \left(\sum_{l=1}^{n} u_l^2 F_l\right) \hat{f}_m(t). \end{aligned} \tag{13}$$

By (8)–(13), (7) is rewritten, for $m \in \{1, 2, \ldots, n\}$, as follows:

$$\begin{aligned}\frac{d\hat{f}_m}{dt}(t) + \hat{f}_m(t)\left(\sum_{l=1}^{n} u_l^2 F_l\right) + \eta \hat{f}_m(t) - \hat{F}_m \\ = \eta \sum_{h=1}^{n}\sum_{k=1}^{n} f_h(t) f_k(t) \left(\sum_{l=1}^{n} B_{hk}^{l} e^{-\frac{2\pi\iota}{n} m(l-1)}\right).\end{aligned} \tag{14}$$

Let $\mathbf{f}_1(t) = (f_{1l}(t))_l$ and $\mathbf{f}_2(t) = (f_{2l}(t))_l$, for $l \in \{1, 2, \ldots, n\}$, two different solutions of the framework (1), and:

$$v_m(t) := \left(\hat{f}_{1m}(t) - \hat{f}_{2m}(t)\right) + m\left(\mathbb{E}_1[\mathbf{f}_1](t) - \mathbb{E}_1[\mathbf{f}_2](t)\right), \quad m \in \{1, 2, \ldots, n\}. \tag{15}$$

By (15) and (14) and straightforward calculations, one has:

$$\begin{aligned}\frac{dv_m}{dt}(t) + \eta v_m(t) &= \frac{d\hat{f}_{1m}}{dt}(t) - \frac{d\hat{f}_{2m}}{dt}(t) + \eta \hat{f}_{1m} - \eta \hat{f}_{2m} \\ &+ \eta m\left(\mathbb{E}_1[\mathbf{f}_1](t) - \mathbb{E}_1[\mathbf{f}_2](t)\right) + m\left(\mathbb{E}_1'[\mathbf{f}_1](t) - \mathbb{E}_1'[\mathbf{f}_2](t)\right) \\ &= -\hat{f}_{1m}\left(\sum_{l=1}^{n} u_l^2 F_l\right) + \hat{f}_{2m}\left(\sum_{l=1}^{n} u_l^2 F_l\right) \\ &+ \eta\left(\sum_{h=1}^{n}\sum_{k=1}^{n} f_{1h}(t) f_{1k}(t)\left(\sum_{l=1}^{n} B_{hk}^{l} e^{-\frac{2\pi\iota}{n} m(l-1)}\right)\right) \\ &- \eta\left(\sum_{h=1}^{n}\sum_{k=1}^{n} f_{2h}(t) f_{2k}(t)\left(\sum_{l=1}^{n} B_{hk}^{l} e^{-\frac{2\pi\iota}{n} m(l-1)}\right)\right) \\ &+ \eta m\left(\mathbb{E}_1[\mathbf{f}_1](t) - \mathbb{E}_1[\mathbf{f}_2](t)\right) + m\left(\mathbb{E}_1'[\mathbf{f}_1](t) - \mathbb{E}_1'[\mathbf{f}_2](t)\right).\end{aligned} \tag{16}$$

By using the (3), one has:

$$\begin{aligned}\mathbb{E}_1'[\mathbf{f}_1](t) - \mathbb{E}_1'[\mathbf{f}_2](t) &= -\mathbb{E}_1[\mathbf{f}_1](t)\left(\eta + \sum_{l=1}^{n} u_l^2 f_l\right) + \mathbb{E}_1[\mathbf{f}_2](t)\left(\eta + \sum_{l=1}^{n} u_l^2 f_l\right) \\ &= \left(\eta + \sum_{l=1}^{n} u_l^2 F_l\right)\left(\mathbb{E}_1[\mathbf{f}_2](t) - \mathbb{E}_1[\mathbf{f}_1](t)\right).\end{aligned} \tag{17}$$

Bearing (17) in mind, Equation (16) reads:

$$\begin{aligned}\frac{dv_m}{dt}(t) + \eta v_m(t) &= \left(\hat{f}_{2m}(t) - \hat{f}_{1m}(t)\right)\left(\sum_{l=1}^{n} u_l^2 F_l\right) \\ &+ \eta m\left(\mathbb{E}_1[\mathbf{f}_1](t) - \mathbb{E}_1[\mathbf{f}_2](t)\right) \\ &+ \left(\eta + \sum_{l=1}^{n} u_l^2 F_l\right) m\left(\mathbb{E}_1[\mathbf{f}_2](t) - \mathbb{E}_1[\mathbf{f}_1](t)\right) \\ &+ \eta\left(\sum_{h=1}^{n}\sum_{k=1}^{n} f_{1h}(t) f_{1k}(t)\left(\sum_{l=1}^{n} B_{hk}^{l} e^{-\frac{2\pi\iota}{n} m(l-1)}\right)\right) \\ &- \eta\left(\sum_{h=1}^{n}\sum_{k=1}^{n} f_{2h}(t) f_{2k}(t)\left(\sum_{l=1}^{n} B_{hk}^{l} e^{-\frac{2\pi\iota}{n} m(l-1)}\right)\right).\end{aligned} \tag{18}$$

By straightforward calculations, (18) is rewritten as:

$$
\begin{aligned}
\frac{dv_m}{dt}(t)+\eta v_m(t) = & -\left(\hat{f}_{1m}(t)-\hat{f}_{2m}(t)\right)\left(\sum_{l=1}^{n} u_l^2 F_l\right) \\
& +\left(\sum_{l=1}^{n} u_l^2 F_l\right) m\left(\mathbb{E}_1[\mathbf{f}_1](t)-\mathbb{E}_1[\mathbf{f}_2](t)\right) \\
& -\left(\sum_{l=1}^{n} u_l^2 F_l\right) m\left(\mathbb{E}_1[\mathbf{f}_1](t)-\mathbb{E}_1[\mathbf{f}_2](t)\right) \\
& +\eta\, m\left(\mathbb{E}_1[\mathbf{f}_1](t)-\mathbb{E}_2[\mathbf{f}_2](t)\right) \\
& +\left(\eta+\sum_{l=1}^{n} u_l^2 F_l\right) m\left(\mathbb{E}_1[\mathbf{f}_2](t)-\mathbb{E}_1[\mathbf{f}_1](t)\right) \\
& +\eta\left(\sum_{h=1}^{n}\sum_{k=1}^{n} f_{1h}(t) f_{1k}(t)\left(\sum_{l=1}^{n} B_{hk}^{l} e^{-\frac{2\pi\iota}{n} m(l-1)}\right)\right) \\
& -\eta\left(\sum_{h=1}^{n}\sum_{k=1}^{n} f_{2h}(t) f_{2k}(t)\left(\sum_{l=1}^{n} B_{hk}^{l} e^{-\frac{2\pi\iota}{n} m(l-1)}\right)\right) \\
= & -\left(\sum_{l=1}^{n} u_l^2 F_l\right) v_m(t) \\
& +\eta\left(\sum_{h=1}^{n}\sum_{k=1}^{n} f_{1h}(t) f_{1k}(t)\left(\sum_{l=1}^{n} B_{hk}^{l} e^{-\frac{2\pi\iota}{n} m(l-1)}\right)\right) \\
& -\eta\left(\sum_{h=1}^{n}\sum_{k=1}^{n} f_{2h}(t) f_{2k}(t)\left(\sum_{l=1}^{n} B_{hk}^{l} e^{-\frac{2\pi\iota}{n} m(l-1)}\right)\right).
\end{aligned}
\tag{19}
$$

By straightforward calculations, the second and the third terms of the right-hand side of (19) are written as:

$$
\begin{aligned}
& \eta\left(\sum_{h=1}^{n}\sum_{k=1}^{n} f_{1h}(t) f_{1k}(t)\left(\sum_{l=1}^{n} B_{hk}^{l} e^{-\frac{2\pi\iota}{n} m(l-1)}\right)\right) \\
& -\eta\left(\sum_{h=1}^{n}\sum_{k=1}^{n} f_{2h}(t) f_{2k}(t)\left(\sum_{l=1}^{n} B_{hk}^{l} e^{-\frac{2\pi\iota}{n} m(l-1)}\right)\right) \\
& =\eta\sum_{h=1}^{n}\sum_{k=1}^{n}\left(f_{1h}(t) f_{1k}(t)-f_{2h}(t) f_{2k}(t)\right)\left(\sum_{l=1}^{n} B_{hk}^{l} e^{-\frac{2\pi\iota}{n} m(l-1)}\right) \\
& =\eta\sum_{h=1}^{n}\sum_{k=1}^{n}\left(f_{1h}(t) f_{1k}(t)-f_{2h}(t) f_{1k}(t)+f_{2h}(t) f_{1k}(t)-f_{2h}(t) f_{2k}(t)\right) \\
& \cdot\left(\sum_{l=1}^{n} B_{hk}^{l} e^{-\frac{2\pi\iota}{n} m(l-1)}\right) \\
& =\eta\sum_{h=1}^{n}\sum_{k=1}^{n}\left(f_{1k}(t)\left(f_{1h}(t)-f_{2h}(t)\right)+f_{2h}(t)\left(f_{1k}(t)-f_{2k}(t)\right)\right) \\
& \cdot\left(\sum_{l=1}^{n} B_{hk}^{l} e^{-\frac{2\pi\iota}{n} m(l-1)}\right).
\end{aligned}
\tag{20}
$$

By summing and subtracting $m f_{1k}(t)\left(\mathbb{E}_1[\mathbf{f}_1](t)-\mathbb{E}_1[\mathbf{f}_2](t)\right)$ and $m f_{2h}(t)\left(\mathbb{E}_1[\mathbf{f}_1](t)-\mathbb{E}_1[\mathbf{f}_2](t)\right)$, one has:

$$\begin{aligned}
&\eta \sum_{h=1}^{n}\sum_{k=1}^{n} \left(f_{1h}(t) - f_{2h}(t)\right) f_{1k}(t) \sum_{l=1}^{n} B_{hk}^{l} e^{-\frac{2\pi\iota}{n} m(l-1)} \\
&- \eta \sum_{h=1}^{n}\sum_{k=1}^{n} \left(f_{1k}(t) - f_{2k}(t)\right) f_{2h}(t) \sum_{l=1}^{n} B_{hk}^{l} e^{-\frac{2\pi\iota}{n} m(l-1)} \\
&= \eta \sum_{h=1}^{n}\sum_{k=1}^{n} \Big(f_{1k}(t) \left(f_{1h}(t) - f_{2h}(t)\right) + m f_{1k}(t) \left(\mathbb{E}_1[\mathbf{f}_1](t) - \mathbb{E}_1[\mathbf{f}_2](t)\right) \\
&- m f_{1k}(t) \left(\mathbb{E}_1[\mathbf{f}_1](t) - \mathbb{E}_1[\mathbf{f}_2](t)\right)\Big) \sum_{l=1}^{n} B_{hk}^{l} e^{-\frac{2\pi\iota}{n} m(l-1)} \\
&+ \eta \sum_{h=1}^{n}\sum_{k=1}^{n} \Big(f_{2h}(t) \left(f_{1k}(t) - f_{2k}(t)\right) + m f_{2h}(t) \left(\mathbb{E}_1[\mathbf{f}_1](t) - \mathbb{E}_1[\mathbf{f}_2](t)\right) \\
&- m f_{2h}(t) \left(\mathbb{E}_1[\mathbf{f}_1](t) - \mathbb{E}_1[\mathbf{f}_2](t)\right)\Big) \sum_{l=1}^{n} B_{hk}^{l} e^{-\frac{2\pi\iota}{n} m(l-1)}.
\end{aligned} \tag{21}$$

Rearranging into (21), one has:

$$\begin{aligned}
&\eta \left(\sum_{h=1}^{n}\sum_{k=1}^{n} f_{1h}(t) f_{1k}(t) \left(\sum_{l=1}^{n} B_{hk}^{l} e^{-\frac{2\pi\iota}{n} m(l-1)}\right)\right) \\
&- \eta \left(\sum_{h=1}^{n}\sum_{k=1}^{n} f_{2h}(t) f_{2k}(t) \left(\sum_{l=1}^{n} B_{hk}^{l} e^{-\frac{2\pi\iota}{n} m(l-1)}\right)\right) \\
&= \eta \sum_{h=1}^{n}\sum_{k=1}^{n} \left(\left(f_{1h}(t) - f_{2h}(t)\right) + m\left(\mathbb{E}_1[\mathbf{f}_1](t) - \mathbb{E}_1[\mathbf{f}_2](t)\right)\right) \\
&\cdot f_{1k}(t) \sum_{l=1}^{n} B_{hk}^{l} e^{-\frac{2\pi\iota}{n} m(l-1)} \\
&- \eta m \left(\mathbb{E}_1[\mathbf{f}_1](t) - \mathbb{E}_1[\mathbf{f}_2](t)\right) \sum_{h=1}^{n}\sum_{k=1}^{n} f_{1k}(t) \sum_{l=1}^{n} B_{hk}^{l} e^{-\frac{2\pi\iota}{n} m(l-1)} \\
&- \eta \sum_{h=1}^{n}\sum_{k=1}^{n} \left(\left(f_{1k}(t) - f_{2k}(t)\right) + m\left(\mathbb{E}_1[\mathbf{f}_1](t) - \mathbb{E}_1[\mathbf{f}_2](t)\right)\right) \\
&\cdot f_{2h}(t) \sum_{l=1}^{n} B_{hk}^{l} e^{-\frac{2\pi\iota}{n} m(l-1)} \\
&+ \eta m \left(\mathbb{E}_1[\mathbf{f}_1](t) - \mathbb{E}_1[\mathbf{f}_2](t)\right) \sum_{h=1}^{n}\sum_{k=1}^{n} f_{2h}(t) \sum_{l=1}^{n} B_{hk}^{l} e^{-\frac{2\pi\iota}{n} m(l-1)}.
\end{aligned} \tag{22}$$

By (22), the (19) is rewritten, for $m \in \{1, 2, \ldots, n\}$, as follows:

$$
\begin{aligned}
&\frac{dv_m}{dt}(t) + \eta v_m(t) = -v_m(t)\left(\sum_{l=1}^{n} u_l^2 F_l\right) \\
&- \eta m\left(\mathbb{E}_1[\mathbf{f}_1](t) - \mathbb{E}_1[\mathbf{f}_2](t)\right)\sum_{h=1}^{n}\sum_{k=1}^{n} f_{1k}(t)\sum_{l=1}^{n} B_{hk}^{l} e^{-\frac{2\pi\iota}{n}m(l-1)} \\
&+ \eta m\left(\mathbb{E}_1[\mathbf{f}_1](t) - \mathbb{E}_1[\mathbf{f}_2](t)\right)\sum_{h=1}^{n}\sum_{k=1}^{n} f_{2h}(t)\sum_{l=1}^{n} B_{hk}^{l} e^{-\frac{2\pi\iota}{n}m(l-1)} \\
&+ \eta \sum_{h=1}^{n}\sum_{k=1}^{n}\left((f_{1h}(t) - f_{2h}(t)) + m\left(\mathbb{E}_1[\mathbf{f}_1](t) - \mathbb{E}_1[\mathbf{f}_2](t)\right)\right) \\
&\cdot f_{1k}(t)\sum_{l=1}^{n} B_{hk}^{l} e^{-\frac{2\pi\iota}{n}m(l-1)} \\
&- \eta \sum_{h=1}^{n}\sum_{k=1}^{n}\left((f_{1k}(t) - f_{2k}(t)) + m\left(\mathbb{E}_1[\mathbf{f}_1](t) - \mathbb{E}_1[\mathbf{f}_2](t)\right)\right) \\
&\cdot f_{2h}(t)\sum_{l=1}^{n} B_{hk}^{l} e^{-\frac{2\pi\iota}{n}m(l-1)}.
\end{aligned} \tag{23}
$$

Finally, (23) can be written as follows:

$$
\begin{aligned}
&\frac{dv_m}{dt}(t) + \eta v_m(t) + \left(\sum_{i=1}^{n} u_i^2 F_i\right) v_m(t) = \\
&\left(\mathbb{E}_1[\mathbf{f}_1](t) - \mathbb{E}_1[\mathbf{f}_2](t)\right)\Big(- \eta m \sum_{h=1}^{n}\sum_{k=1}^{n} f_{1k}(t)\sum_{l=1}^{n} B_{hk}^{l} e^{-\frac{2\pi\iota}{n}m(l-1)} \\
&+ \eta m \sum_{h=1}^{n}\sum_{k=1}^{n} f_{2h}(t)\sum_{l=1}^{n} B_{hk}^{l} e^{-\frac{2\pi\iota}{n}m(l-1)}\Big) \\
&+ \eta \sum_{h=1}^{n}\sum_{k=1}^{n}\left((f_{1h}(t) - f_{2h}(t)) + m\left(\mathbb{E}_1[\mathbf{f}_1](t) - \mathbb{E}_1[\mathbf{f}_2](t)\right)\right) \\
&\cdot f_{1k}(t)\sum_{l=1}^{n} B_{hk}^{l} e^{-\frac{2\pi\iota}{n}m(l-1)} \\
&- \eta \sum_{h=1}^{n}\sum_{k=1}^{n}\left((f_{1k}(t) - f_{2k}(t)) + m\left(\mathbb{E}_1[\mathbf{f}_1](t) - \mathbb{E}_1[\mathbf{f}_2](t)\right)\right) \\
&\cdot f_{2h}(t)\sum_{l=1}^{n} B_{hk}^{l} e^{-\frac{2\pi\iota}{n}m(l-1)}.
\end{aligned} \tag{24}
$$

Let $\mathbf{f}_1(t) = \mathbf{f}(t)$ and $\mathbf{f}_2(t) = \mathbf{g}$. By (4), (24), for $m \in \{1, 2, \ldots, n\}$, reads:

$$
\begin{aligned}
&\frac{dv_m}{dt}(t) + \eta v_m(t) + \left(\sum_{i=1}^{n} u_i^2 F_i\right) v_m(t) = \\
&(\mathbb{E}_1[\mathbf{f}](t) - K) \Big(-\eta m \sum_{h=1}^{n} \sum_{k=1}^{n} f_k(t) \sum_{l=1}^{n} B_{hk}^l e^{-\frac{2\pi\iota}{n} m(l-1)} \\
&+ \eta m \sum_{h=1}^{n} \sum_{k=1}^{n} g_h \sum_{l=1}^{n} B_{hk}^l e^{-\frac{2\pi\iota}{n} m(l-1)} \Big) \\
&+ \eta \sum_{h=1}^{n} \sum_{k=1}^{n} \left((f_h(t) - g_h(t)) + m\left(\mathbb{E}_1[\mathbf{f}](t) - K\right) \right) \\
&\cdot f_k(t) \sum_{l=1}^{n} B_{hk}^l e^{-\frac{2\pi\iota}{n} m(l-1)} \\
&- \eta \sum_{h=1}^{n} \sum_{k=1}^{n} \left((f_k(t) - g_k) + m\left(\mathbb{E}_1[\mathbf{f}_1](t) - K\right) \right) \\
&\cdot g_h(t) \sum_{l=1}^{n} B_{hk}^l e^{-\frac{2\pi\iota}{n} m(l-1)}.
\end{aligned}
\tag{25}
$$

Let $\underline{v}_m(t) := e^{\eta t} v_m(t)$, then:

$$
\frac{d\underline{v}_m}{dt}(t) = \eta e^{\eta t} v_m(t) + e^{\eta t} \frac{dv_m}{dt}(t). \tag{26}
$$

Bearing (26) in mind and multiplying by $e^{\eta t}$ both sides of (25), one has:

$$
\begin{aligned}
&\frac{d\underline{v}_m}{dt}(t) + \left(\sum_{i=1}^{n} u_i^2 F_i\right) \underline{v}_m(t) = e^{\eta t} \left(\mathbb{E}_1[\mathbf{f}] - K\right) \\
&\cdot \Big(-\eta m \sum_{h=1}^{n} \sum_{k=1}^{n} f_k(t) \sum_{l=1}^{n} B_{hk}^l e^{-\frac{2\pi\iota}{n} m(l-1)} \\
&+ \eta m \sum_{h=1}^{n} \sum_{k=1}^{n} g_h \sum_{l=1}^{n} B_{hk}^l e^{-\frac{2\pi\iota}{n} m(l-1)} \Big) \\
&+ \eta \sum_{h=1}^{n} e^{\eta t} \left((f_h(t) - g_h) + m\left(\mathbb{E}_1[\mathbf{f}] - K\right) \right) \sum_{k=1}^{n} f_k(t) \\
&\cdot \sum_{l=1}^{n} B_{hk}^l e^{-\frac{2\pi\iota}{n} m(l-1)} \\
&+ \eta \sum_{k=1}^{n} e^{\eta t} \left((f_k(t) - g_k) + m\left(\mathbb{E}_1[\mathbf{f}] - K\right) \right) \sum_{h=1}^{n} g_h \\
&\cdot \sum_{l=1}^{n} B_{hk}^l e^{-\frac{2\pi\iota}{n} m(l-1)}.
\end{aligned}
\tag{27}
$$

By (5) and straightforward calculations, one has:

$$\begin{aligned}&\Bigg| e^{\eta t}\left(\mathbb{E}_1[\mathbf{f}]-K\right)\cdot\Big(-\eta m\sum_{h=1}^{n}\sum_{k=1}^{n}f_k(t)\sum_{l=1}^{n}B_{hk}^{l}e^{-\frac{2\pi\iota}{n}m(l-1)}\\&+\eta m\sum_{h=1}^{n}\sum_{k=1}^{n}g_h\sum_{l=1}^{n}B_{hk}^{l}e^{-\frac{2\pi\iota}{n}m(l-1)}\Big)\Bigg| \le ce^{\eta t}e^{-\left(\eta+\sum\limits_{i=1}^{n}u_i^2F_i\right)t},\end{aligned} \tag{28}$$

where c is a constant that depends on the parameters of the system. By using (27) and (28), one has:

$$\frac{d|\underline{v}|_m(t)}{dt}(t)\le ce^{\eta t}e^{-\left(\eta+\sum\limits_{i=1}^{n}u_i^2F_i\right)t}+\lambda\underline{v}_m(t),\quad m\in\{1,2,\ldots,n\}. \tag{29}$$

By integrating (29) between zero and t and using (29), one has:

$$\begin{aligned}|\underline{v}_m(t)|&\le|\underline{v}_m(0)|+c\left|\int_0^t e^{\eta\tau}e^{-\left(\eta+\sum\limits_{i=1}^{n}u_i^2F_i\right)\tau}\,d\tau\right|+\lambda\left|\int_0^t\underline{v}_m(\tau)\,d\tau\right|\\&\le|\underline{v}_m(0)|+c\int_0^t e^{-\sum\limits_{i=1}^{n}u_i^2F_i\tau}\,d\tau+\lambda\int_0^t|\underline{v}_m(0)|\,d\tau.\end{aligned} \tag{30}$$

By the integral Gronwall inequality [30] and (30), one has:

$$|\underline{v}_m(t)|\le|\underline{v}_m(0)|\,e^{\lambda t}+ce^{-\sum\limits_{i=1}^{n}u_i^2F_it}. \tag{31}$$

By dividing (31) by $e^{\eta t}$ and bearing (26) in mind, one has:

$$|\underline{v}_m(t)|\le|\underline{v}_m(0)|\,e^{(\lambda-\eta)t}+ce^{-\left(\eta+\sum\limits_{i=1}^{n}u_i^2F_it\right)}. \tag{32}$$

Finally, by (32) and (15), for $m\in\{1,2,\ldots,n\}$, one has:

$$\left|\hat{f}_m(t)-\hat{g}_m\right|\le(c+c_0)e^{-\left(\eta+\sum\limits_{i=1}^{n}u_i^2F_it\right)}, \tag{33}$$

where c is a constant that depends on the parameters of the system and c_0 on the initial data of the related problem.

It is possible to conclude that:

$$\hat{f}_m(t)\xrightarrow{t\to+\infty}\hat{g}_m,$$

for $m\in\{1,2,\ldots,n\}$. The proof is thus gained. □

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