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On q-Hermite-Hadamard Inequalities for Differentiable Convex Functions

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Abstract: In this paper, we establish some new results on the left-hand side of the q-Hermite–Hadamard inequality for differentiable convex functions with a critical point. Our work extends the results of Alp et. al (q-Hermite Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions, J. King Saud Univ. Sci., 2018, 30, 193-203), by considering the critical point-type inequalities.

Keywords: Hermite-Hadamard inequalities; q-derivative; q-integral; convex functions

1. Introduction

Quantum calculus (also known as q-calculus) is the study of calculus without limits, where classical mathematical formulas are obtained as $q \to 1$. Firstly introduced by Euler (1707–1783) in the tracks of Newton's infinite series, the study of q-calculus was established in the early Twentieth Century after the work of Jackson (1910) on defining an integral later known as the q-Jackson integral; see [1–4]. In q-calculus, the classical derivative is replaced by the q-difference operator in order to deal with non-differentiable functions; see [5,6] for more details. Applications of q-calculus can be found in various fields of mathematics and physics, and the interested readers are referred to [7–10].

The theory of convex functions has been widely studied and applied to various fields of science. Due to its close relation to the theory of inequalities, a rich literature on inequalities can be found in the study of convex functions; see [11–18]. This includes the Hermite–Hadamard inequality, introduced by Hermite and Hadamard independently, which has been studied extensively in recent years.

Let $J \subseteq \mathbb{R}$ be an interval and $f : J \to \mathbb{R}$ be a function from J to $\mathbb{R}$. Recall that f is said to be a convex function if it satisfies the inequality:

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y), \tag{1}$$

for all $x, y \in J$ and $\lambda \in [0,1]$. In addition, if an equality holds for all $x, y \in J$ and $\lambda \in [0,1]$, then f is said to be affine.

It is also well known that f is convex if and only if it satisfies the Hermite–Hadamard inequality, which is defined by:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{2}$$

for all $a, b \in J$ and $a < b$. One can estimate by the right-hand side of (2) by using Iyengar's inequality, which is defined by:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{M(b-a)}{4} - \frac{1}{4M(b-a)}(f(b)-f(a))^2,$$

where M denotes the Lipschitz constant, that is $M = \sup\left\{\left|\frac{f(x)-f(y)}{x-y}\right|; x \neq y\right\}$.

This fundamental result of Hermite and Hadamard has attracted many mathematicians, and consequently, this inequality has been generalized and extended in many directions; see [19–31] and the references cited therein.

In 2018, Alp et al. [32] studied the q-analogue of Hermite–Hadamard's inequality for increasing functions, that is,

$$f\left(\frac{qa+b}{1+q}\right) \leq \frac{1}{b-a}\int_a^b f(x)_a d_q x \leq \frac{qf(a)+f(b)}{1+q}, \tag{3}$$

where q is a constant with $0 < q < 1$. Moreover, they studied the generalized q-Hermite–Hadamard inequality for differentiable convex functions, that is,

$$\max\{I_1, I_2, I_3\} \leq \frac{1}{b-a}\int_a^b f(x)\ {}_a d_q x \leq \frac{qf(a)+f(b)}{1+q}, \tag{4}$$

where:

$$\begin{aligned} I_1 &= f\left(\frac{qa+b}{1+q}\right), \\ I_2 &= f\left(\frac{a+qb}{1+q}\right) + \frac{(1-q)(b-a)}{1+q} f'\left(\frac{a+qb}{1+q}\right), \\ I_3 &= f\left(\frac{a+b}{2}\right) + \frac{(1-q)(b-a)}{2(1+q)} f'\left(\frac{a+b}{2}\right). \end{aligned}$$

This paper aims to establish the generalized q-Hermite–Hadamard inequality for differentiable convex functions with a critical point.

The paper is organized as follows. Some basic concepts are recalled in Section 2. Section 3 contain the main results, while conclusions are given in Section 4.

2. Preliminaries

In this section, some basic results are mentioned. Throughout this section, we let $J = [a, b] \subseteq \mathbb{R}$ be an interval and q be a constant with $0 < q < 1$.

Definition 1. *[33] The q-derivative of a continuous function $f : J \to \mathbb{R}$ at x is defined as:*

$$ {}_aD_q f(x) = \frac{f(x) - f(qx + (1-q)a)}{(1-q)(x-a)}, \ \textit{for} \quad x \neq a. \tag{5}$$

For $x = a$, we define ${}_aD_q f(a) = \lim_{x\to a} {}_aD_q f(x)$.

If ${}_aD_q f(x)$ exists for all $x \in J$, then f is q-differentiable on J. Moreover, if $a = 0$, then (5) reduces to ${}_0D_q f = D_q f$, where D_q is the q-derivative of f, which is defined by:

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}.$$

For more details, see [4].

The higher-order q-derivatives of functions on J are also defined.

Definition 2. *[33] For a continuous function $f : J \to \mathbb{R}$, the second-order q-derivative of f on J, if ${}_aD_qf$ is q-differentiable on J, denoted by ${}_aD_q^2f$ and defined by:*

$$ {}_aD_q^2f = {}_aD_q({}_aD_q). $$

Similarly, provided that ${}_aD_q^{n-1}f$ is the q-derivative on J for some integer $n > 2$, the n^{th}-order q-derivative of f on J is the function from $J \to \mathbb{R}$ defined by:

$$ {}_aD_q^nf = {}_aD_q({}_aD_q^{n-1}f). $$

Example 1. *Let $f : J \to \mathbb{R}$ with $f(x) = x^2 + 1$. Let q be a constant with $0 < q < 1$. Then, for $x \neq a$, we have:*

$$ \begin{aligned} {}_aD_q(x^2+1) &= \tfrac{(x^2+1)-\left[(qx+(1-q)a)^2+1\right]}{(1-q)(x-a)} \\ &= \tfrac{(1+q)x^2-2qax-(1-q)a^2}{(x-a)} \\ &= (1+q)x + (1-q)a. \end{aligned} \tag{6} $$

For $x = a$, ${}_aD_qf(a) = \lim_{x \to a} {}_aD_qf(x) = 2a$.

Definition 3. *[33] The q-integral of a continuous function $f : J \to \mathbb{R}$ is defined as:*

$$ \int_a^x f(t)\ {}_ad_qt = (1-q)(x-a)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a), \tag{7} $$

for $x \in J$.

Note that if $a = 0$, then (7) becomes the classical q-integral of f, that is,

$$ \int_0^x f(t)\ {}_0d_qt = (1-q)x\sum_{n=0}^{\infty} q^n f(q^n x) $$

for $x \in [0, \infty)$; see [4] for more details.

Example 2. *Let $f : [a, b] \to \mathbb{R}$ with $f(x) = 2x$. Let q be a constant with $0 < q < 1$. Then, we have:*

$$ \begin{aligned} \int_a^b f(x)\ {}_ad_qx &= \int_a^b 2x\ {}_ad_qx \\ &= 2(1-q)(b-a)\sum_{n=0}^{\infty} q^n(q^n b + (1-q^n)a) \\ &= \frac{2(b-a)(b+qa)}{1+q}. \end{aligned} $$

Note that if $q \to 1$, we obtain the classical integration:

$$ \int_a^b f(x)dx = \int_a^b 2xdx = b^2 - a^2. $$

Theorem 1. *Assume that the function $f : J \to \mathbb{R}$ is continuous. Then, we have the following:*

(i) ${}_aD_q \int_a^x f(t)\ {}_ad_qt = f(x) - f(a)$;
(ii) $\int_c^x {}_aD_qf(t)\ {}_ad_qt = f(x) - f(c)$ *for* $c \in (a, x)$.

Theorem 2. *Assume that the functions $f, g : J \to \mathbb{R}$ are continuous and $\alpha \in \mathbb{R}$. Then, we have the following:*

(i) $\int_a^x [f(t) + g(t)]\ {}_ad_qt = \int_a^x f(t)\ {}_ad_qt + \int_a^x g(t)\ {}_ad_qt$;
(ii) $\int_a^x (\alpha f)(t)\ {}_ad_qt = \alpha \int_a^x f(t)\ {}_ad_qt$;

(iii) $\int_c^x f(t)\ {}_aD_q g(t)\ {}_ad_q t = (fg)|_c^x - \int_c^x g(qt + (1-q)a)\,{}_aD_q f(t)\ {}_ad_q t$ *for* $c \in (a, x)$.

For the proofs of the properties in Theorems 1 and 2, see [34].

3. Main Results

In this section, we present our main results on the left-hand side of the q-Hermite–Hadamard inequality for differentiable convex functions with a critical point.

Theorem 3. *Suppose that* $f : [a, b] \to \mathbb{R}$ *is a differentiable convex function on* (a, b) *such that* $f'(c) = 0$ *for* $c \in (a, b)$*, and let* q *be a constant with* $0 < q < 1$*. Then, we have:*

$$\begin{aligned} f\left(\tfrac{q(a+c)+(1-q)b}{1+q}\right) + \quad & f'\left(\tfrac{q(a+c)+(1-q)b}{1+q}\right)\left(\tfrac{q(b-c)}{1+q}\right) \\ & \le \tfrac{1}{b-a}\int_a^b f(x)_a d_q x \\ & \le \tfrac{qf(a)+f(b)}{1+q}. \end{aligned} \tag{8}$$

Proof. Since the function f is differentiable on (a, b), there exists a tangent line at the point $\frac{q(a+c)+(1-q)b}{1+q} \in (a, b)$, given by:

$$h(x) = f\left(\frac{q(a+c)+(1-q)b}{1+q}\right) + f'\left(\frac{q(a+c)+(1-q)b}{1+q}\right)\left(x - \frac{q(a+c)+(1-q)b}{1+q}\right). \tag{9}$$

Since f is a convex function on $[a, b]$, it follows that $h(x) \le f(x)$ for all $x \in [a, b]$. After q-integrating of (9) on $[a, b]$, we have:

$$\begin{aligned} & \int_a^b h(x)\ {}_ad_q x \\ & = \int_a^b \Big[f\left(\tfrac{q(a+c)+(1-q)b}{1+q}\right) + f'\left(\tfrac{q(a+c)+(1-q)b}{1+q}\right) \\ & \quad \times \left(x - \tfrac{q(a+c)+(1-q)b}{1+q}\right)\Big]\ {}_ad_q x \\ & = (b-a)f\left(\tfrac{q(a+c)+(1-q)b}{1+q}\right) \\ & \quad + f'\left(\tfrac{q(a+c)+(1-q)b}{1+q}\right)\left(\int_a^b x\ {}_ad_q x - (b-a)\tfrac{q(a+c)+(1-q)b}{1+q}\right) \\ & = (b-a)f\left(\tfrac{q(a+c)+(1-q)b}{1+q}\right) \\ & \quad + f'\left(\tfrac{q(a+c)+(1-q)b}{1+q}\right)\left((b-a)\tfrac{(qa+b)}{1+q} - (b-a)\tfrac{q(a+c)+(1-q)b}{1+q}\right) \\ & = (b-a)\left[f\left(\tfrac{q(a+c)+(1-q)b}{1+q}\right) + f'\left(\tfrac{q(a+c)+(1-q)b}{1+q}\right)\left(\tfrac{q(b-c)}{1+q}\right)\right] \\ & \le \int_a^b f(x)\ {}_ad_q x. \end{aligned}$$

On the other hand, since f is a convex function, we obtain:

$$\begin{aligned} \tfrac{1}{(b-a)}\int_a^b f(x)_a d_q x & = \tfrac{1}{(b-a)}\left[(1-q)(b-a)\sum_{n=0}^{\infty} q^n f(q^n b + (1-q^n)a)\right] \\ & = (1-q)\sum_{n=0}^{\infty} q^n f(q^n b + (1-q^n)a) \\ & \le (1-q)\sum_{n=0}^{\infty} q^n [q^n f(b) + (1-q^n)f(a)] \\ & = (1-q)\left[\tfrac{f(b)}{1-q^2} + \tfrac{f(a)}{1-q} - \tfrac{f(a)}{1-q^2}\right] \\ & = \tfrac{qf(a)+f(b)}{1+q}. \end{aligned}$$

The proof is complete. □

Remark 1. *In Theorem 3, if* $q \in (0, \frac{c-b}{a-b}]$*, then* $\frac{q(a+c)+(1-q)b}{1+q} \in [c, b)$*. We can reduce the left-hand side of Theorem 3 as:*

$$f\left(\frac{q(a+c)+(1-q)b}{1+q}\right) \leq \frac{1}{(b-a)} \int_a^b f(x)\ {}_a d_q x \leq \frac{qf(a)+f(b)}{1+q},$$

since $f'\left(\frac{q(a+c)+(1-q)b}{1+q}\right)\left(\frac{q(b-c)}{1+q}\right) \geq 0.$

Remark 2. *In Remark 1, if* $c \to a^+$*, then* $\frac{c-b}{a-b} \to 1^-$*. Since* $q \in (0,1)$*, we have* $\frac{q(a+c)+(1-q)b}{1+q} \in (a, b)$*. We can reduce the left-hand side of Theorem 3 as:*

$$f\left(qa+(1-q)b\right) \leq \frac{1}{(b-a)} \int_a^b f(x)\ {}_a d_q x \leq \frac{qf(a)+f(b)}{1+q},$$

since $f'\left(\frac{q(a+c)+(1-q)b}{1+q}\right)\left(\frac{q(b-c)}{1+q}\right) \geq 0.$

Corollary 1. *Assume that* $f : [a,b] \to \mathbb{R}$ *is a differentiable convex function on* (a,b) *such that* $f'\left(\frac{a+b}{2}\right) = 0$*, for* $0 < q < 1$*. Then, we have:*

$$\begin{aligned} f\left(\frac{q(a+\frac{a+b}{2})+(1-q)b}{1+q}\right) & +f'\left(\frac{q(a+\frac{a+b}{2})+(1-q)b}{1+q}\right)\frac{q(b-a)}{2(1+q)} \\ & \leq \frac{1}{b-a}\int_a^b f(x)_a d_q x \\ & \leq \frac{qf(a)+f(b)}{1+q}. \end{aligned} \tag{10}$$

Corollary 2. *Assume that* $f : [a,b] \to \mathbb{R}$ *is a differentiable convex function on* (a,b) *such that* $f'(0) = 0$*, for* $0 \in (a,b)$ *and* $0 < q < 1$*. Then, we have:*

$$\begin{aligned} f\left(\frac{qa+(1-q)b}{1+q}\right) & +f'\left(\frac{qa+(1-q)b}{1+q}\right)\frac{qb}{(1+q)} \\ & \leq \frac{1}{b-a}\int_a^b f(x)_a d_q x \leq \frac{qf(a)+f(b)}{1+q}. \end{aligned} \tag{11}$$

Theorem 4. *Let* $f : [a,b] \to \mathbb{R}$ *be a differentiable convex function on* (a,b) *such that* $f'(c) = 0$ *for* $c \in (a,b)$ *and* $0 < q < 1$*. Then, we have:*

$$\begin{aligned} f\left(\frac{(1-q)a+q(c+b)}{1+q}\right) & +f'\left(\frac{(1-q)a+q(c+b)}{1+q}\right)\left(\frac{q(2a-b-c)+b-a}{1+q}\right) \\ & \leq \frac{1}{b-a}\int_a^b f(x)_a d_q x \\ & \leq \frac{qf(a)+f(b)}{1+q}. \end{aligned} \tag{12}$$

Proof. Since the function f is differentiable on (a,b), there exists a tangent line at the point $\frac{(1-q)a+q(c+b)}{1+q} \in (a,b)$, which is given by:

$$k(x) = f\left(\frac{(1-q)a+q(c+b)}{1+q}\right) + f'\left(\frac{(1-q)a+q(c+b)}{1+q}\right)\left(x - \frac{(1-q)a+q(c+b)}{1+q}\right). \tag{13}$$

Since f is convex on $[a,b]$, it follows that $k(x) \leq f(x)$ for all $x \in [a,b]$. After q-integrating (13), we obtain:

$$\begin{aligned}
&\int_a^b k(x)\ {}_a d_q x \\
&= \int_a^b \left[f\left(\tfrac{(1-q)a+q(c+b)}{1+q}\right) \right. \\
&\quad \left. + f'\left(\tfrac{(1-q)a+q(c+b)}{1+q}\right)\left(x - \tfrac{(1-q)a+q(c+b)}{1+q}\right)\right]\ {}_a d_q x \\
&= (b-a) f\left(\tfrac{(1-q)a+q(c+b)}{1+q}\right) \\
&\quad + f'\left(\tfrac{1-q)a+q(c+b)}{1+q}\right)\left(\int_a^b x\ {}_a d_q x - (b-a)\tfrac{(1-q)a+q(c+b)}{1+q}\right) \\
&= (b-a) f\left(\tfrac{(1-q)a+q(c+b)}{1+q}\right) \\
&\quad + f'\left(\tfrac{(1-q)a+q(c+b)}{1+q}\right)\left[(b-a)\left(\left(\tfrac{aq+b}{1+q}\right) - \tfrac{(1-q)a+q(c+b)}{1+q}\right)\right] \\
&= (b-a)\left[f\left(\tfrac{(1-q)a+q(c+b)}{1+q}\right) \right. \\
&\quad \left. + f'\left(\tfrac{(1-q)a+q(c+b)}{1+q}\right)\left(\tfrac{q(2a-b-c)+b-a}{1+q}\right)\right] \\
&\leq \int_a^b f(x)_a d_q x.
\end{aligned} \tag{14}$$

The proof is complete. □

Remark 3. *In Theorem 4, if* $q \in (\frac{1}{2}, \frac{c-a}{b-a}]$, *then* $f'\left(\frac{(1-q)a+q(c+b)}{1+q}\right) \leq 0$ *and* $\frac{q(2a-b-c)+b-a}{1+q} < 0$. *We can reduce the left-hand side of Theorem 4 as:*

$$f\left(\frac{(1-q)a+q(c+b)}{1+q}\right) \leq \frac{1}{(b-a)} \int_a^b f(x)\ {}_a d_q x \leq \frac{qf(a)+f(b)}{1+q},$$

since $f'\left(\frac{(1-q)a+q(c+b)}{1+q}\right)\left(\frac{q(2a-b-c)+b-a}{1+q}\right) \geq 0.$

Remark 4. *In Remark 3, if* $c \to b^-$, *then* $q \to 1^-$. *Since* $q \in (\frac{1}{2}, 1)$, *we have* $\frac{(1-q)a+q(b+c)}{1+q} \in (\frac{a+2b}{3}, b)$. *We can reduce the left-hand side of Theorem 4 as:*

$$f\left((1-q)(\frac{2a+b}{3}) + qb)\right) \leq \frac{1}{(b-a)} \int_a^b f(x)\ {}_a d_q x \leq \frac{qf(a)+f(b)}{1+q},$$

since $f'\left(\frac{(1-q)a+q(c+b)}{1+q}\right)\left(\frac{q(2a-b-c)+b-a}{1+q}\right) \geq 0.$

Theorem 5. *[Generalized q-Hermite–Hadamard inequality for convex differentiable functions].* Let $f : [a,b] \to \mathbb{R}$ be a differentiable convex function on (a,b) such that $f'(c) = 0$ for $c \in (a,b)$ and $0 < q < 1$. Then, we have:

$$\max\{I_1, I_2, \} \leq \frac{1}{b-a} \int_a^b f(x)\ {}_a d_q x \leq \frac{qf(a)+f(b)}{1+q}, \tag{15}$$

where

$$\begin{aligned}
I_1 &= f\left(\tfrac{q(a+c)+(1-q)b}{1+q}\right) + f'\left(\tfrac{q(a+c)+(1-q)b}{1+q}\right)\left(\tfrac{q(b-c)}{1+q}\right), \\
I_2 &= f\left(\tfrac{(1-q)a+q(c+b)}{1+q}\right) + f'\left(\tfrac{(1-q)a+q(c+b)}{1+q}\right)\left(\tfrac{q(2a-b-c)+b-a}{1+q}\right).
\end{aligned}$$

Proof. A combination of (8) and (12) yields (15). Thus, the proof is complete. □

Example 3. *Define the function $f(x) = x^2$ on $[-1, 3]$, and let $q \in (0, 1)$. Applying Theorem 3 with $a = -1$, $b = 3$, and $c = 0$, the left-hand side becomes:*

$$
\begin{aligned}
& f\left(\frac{q(a+c)+(1-q)b}{1+q}\right) + f'\left(\frac{q(a+c)+(1-q)b}{1+q}\right)\frac{q(b-c)}{1+q} - \frac{1}{b-a}\int_a^b f(x)_a d_q x \\
&= f\left(\frac{3-4q}{1+q}\right) + f'\left(\frac{3-4q}{1+q}\right)\left(\frac{3q}{1+q}\right) - \frac{1}{4}\left[4(1-q)\sum_{n=0}^{\infty} q^n f(3q^n - (1-q^n))\right] \\
&= \frac{-9q^4 - 9q^3 - 9q^2 - 16q}{(1+q)^2(1+q+q^2)} \leq 0.
\end{aligned}
$$

For the right-hand side, we have:

$$
\frac{1}{3-(-1)}\int_{-1}^{3} x^2{}_a d_q x - \frac{qf(-1)+f(3)}{1+q} = \frac{16}{1+q+q^2} - \frac{8}{1+q} + 1 - \frac{9+q}{1+q} \leq 0.
$$

Example 4. *Define function $f(x) = x^2$ on $[-1, 1]$, and let $q \in (0, 1)$. Applying Corollary 2 with $a, b = -1$ and $c = 0$, the left hand-side becomes:*

$$
\begin{aligned}
& f\left(\frac{qa+(1-q)b}{1+q}\right) + f'\left(\frac{qa+(1-q)b}{1+q}\right)\frac{(qb)}{1+q} - \frac{1}{b-a}\int_a^b f(x)_a d_q x \\
&= f\left(\frac{1-2q}{1+q}\right) + f'\left(\frac{1-2q}{1+q}\right)\left(\frac{q}{1+q}\right) - (1-q)\sum_{n=0}^{\infty} q^n f(2q^n - 1) \\
&= \frac{4q^2 - 4q + 1}{(1+q)^2} + \frac{2q(1-2q)}{(1+q)^2} - \frac{1+2q-2q^2+q^3}{(1+q+q^2)(1+q)} \leq 0.
\end{aligned}
$$

For the right-hand side, we have:

$$
\begin{aligned}
& \frac{1}{1-(-1)}\int_{-1}^{1} x^2{}_a d_q x - \frac{qf(-1)+f(1)}{1+q} \\
&= \frac{1}{2}\left[(1-q)(2)\sum_{n=0}^{\infty} q^n f(q^n - (1-q^n))\right] - \frac{1+q}{1+q} \\
&= \frac{4}{1+q+q^2} - \frac{4}{1+q} + 1 - 1 \leq 0.
\end{aligned}
$$

Example 5. *Define functions $f(x) = x^2$ on $[-3, 1]$, and let $q \in (0, 1)$. Applying Theorem 4 with $a = -3$, $b = 1$, and $c = 0$, the left-hand side becomes:*

$$
\begin{aligned}
& f\left(\frac{(1-q)a+q(c+b)}{1+q}\right) + f'\left(\frac{(1-q)a+q(c+b)}{1+q}\right)\left(\frac{q(2a-b-c)+b-a}{1+q}\right) \\
& - \frac{1}{b-a}\int_a^b f(x)_a d_q x \\
&= f\left(\frac{4q-3}{1+q}\right) + f'\left(\frac{4q-3}{1+q}\right)\left(\frac{4-7q}{1+q}\right) - \frac{1}{4}\left[4(1-q)\sum_{n=0}^{\infty} q^n f(4q^n - 3)\right] \\
&= \frac{16q^2 - 24q + 9}{(1+q)^2} + \frac{-56q^2 + 74q - 24}{(1+q)^2} - \frac{16}{1+q+q^2} + \frac{24}{1+q} - 9 \leq 0.
\end{aligned}
$$

For the right-hand side, we have:

$$\frac{1}{3-(-1)}\int_{-1}^{3} x^2{}_a d_q x - \frac{qf(-3)+f(1)}{1+q} = \frac{16}{1+q+q^2} - \frac{24}{1+q} + 9 - \frac{9q+1}{1+q} \leq 0.$$

4. Conclusions

In this paper, we considered and investigated the class of differentiable convex functions, which has a critical point in the setting of q-calculus. We used the approach of q-calculus to derive some new results on the left-hand side of q-Hermite–Hadamard inequalities. It is expected that the ideas and techniques presented in this paper will stimulate further research in this field.

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References

1. Brito, A.M.C.; Cruz, D.A. Symmetric Quantum Calculus. Ph.D. Thesis, Aveiro University, Aveiro, Portugal, 2012.
2. Ernst, T. *The History of q-Calculus and a New Method*; UUDM Report 2000:16; Department of Mathematics, Uppsala University: Uppsala, Sweden, 2000.
3. Jackson, F.H. On a q-definite integrals. *Q. J. Pure Appl. Math.* **1910**, *41*, 193–203.
4. Kac, V.; Cheung, P. *Quantum Calculus*; Springer: New York, NY, USA, 2002.
5. Almeida, R.; Torres, D.F.M. Non-differentiable variational principles in terms of a quantum operator. *Math. Methods Appl. Sci.* **2011**, *34*, 2231–2241.
6. Cresson, J.; Frederico, G.S.F.; Torres, D.F.M. Constants of motion for Non-differentiable quantum variational problems. *Topol. Methods Nonlinear Anal.* **2009**, *33*, 217–231. [CrossRef]
7. Askey, R.; Wilson, J. Some basic hypergeometric orthogonal polynomials that generalize the Jacobi polynomails. *Mem. Am. Math. Soc.* **1985**, *54*, 1–55.
8. Ismail, M.E.H.; Simeonov, P. q-Difference operators for orthogonal polynomials. *J. Comput. Appl. Math.* **2009**, *233*, 749–761. [CrossRef]
9. Page, D.N. Information in black hole radiation. *Phys. Rev. Lett.* **1993**, *71*, 3743–3746. [CrossRef] [PubMed]
10. Youm, D. q-deformed conformal quantum mechanics. *Phys. Rev. D* **2000**, *62*, 095009. [CrossRef]
11. Taf, S.; Brahim, K.; Riahi, L. Some results for Hadamard-type inequalities in quantum calculus. *Le Matematiche* **2014**, *69*, 243–258.
12. Dragomir, S.S. Two refinements of Hadamard's inequalities. *Coll. Sci. Pap. Fac. Kragujevac* **1990**, *11*, 23–26.
13. Dragomir, S.S. Some integral inequalities for differentiable convex functions. *Contrib. Sec. Math. Tech. Sci.* **1992**, *13*, 13–17.
14. Noor, M.A.; Noor, K.I.; Awan, M.U. Generalized convexity and integral inequalities. *Appl. Math. Inf. Sci.* **2005**, *9*, 233–243. [CrossRef]
15. Noor, M.A.; Noor, K.I.; Awan, M.U. Some quantum estimates for Hermite–Hadamard inequalities. *Appl. Math. Comput.* **2015**, *251*, 675–679. [CrossRef]
16. Noor, M.A.; Noor, K.I.; Awan, M.U. Quantum ostrowski inequalities for q-differentiable convex functions. *J. Math. Inequal.* **2016**, *10*, 1013–1018. [CrossRef]
17. Wang, W.; Qi, J. Some new estimates of Hermite–Hadamard convex functions with applications. *Int. J. Anal. Appl.* **2017**, *13*, 15–21.
18. Prabseang, J.; Nonlaopon, K.; Tariboon, J. Quantum Hermite–Hadamard inequalities for double integral and q-differentiable convex functions. *J. Math. Inequal.* **2019**, *13*, 675–686.
19. Cristescu, G. Improved integral inequalities for products of convex functions. *J. Inequal. Pure Appl. Math.* **2005**, *6*, Article 35.

20. Cristescu, G.; Noor, M.A.; Awan, M.U. Bounds of the second degree cumulative frontier gaps of functions with generalized convexity. *Carpath. J. Math.* **2015**, *31*, 173–180.
21. Dragomir, S.S.; Agarwal, R.P. Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula. *Appl. Math. Lett.* **1998**, *11*, 91–95. [CrossRef]
22. Dragomir, S.S.; Pearce, C.E.M. *Selected Topics on Hermite–Hadamard Inequalities and Applications*; Victoria University: Footscray, Australia, 2000.
23. Ion, D.A. Some estimates on the Hermite–Hadamard inequality through quasi-convex functions. *Ann. Univ. Craiova, Math. Comput. Sci. Ser.* **2007**, *34*, 82–87.
24. Khattri, S.K. Three proofs of the inequality $e < \left(1 + \frac{1}{2}\right)^{n+0.5}$. *Am. Math. Mon.* **2010**, *17*, 273–277.
25. Niculescu, C.P.; Persson, L.-E. *Convex Functions and Their Applications*; Springer: New York, NY, USA, 2006.
26. Noor, M.A. Hermite–Hadamard integral inequalities for log -preinvex functions. *J. Math. Anal. Approx.* **2007**, *2*, 126–131.
27. Noor, M.A. On Hermite–Hadamard integral inequalities for involving two log -preinvex functions. *J. Inequal. Pure Appl. Math.* **2007**, *3*, 75–81.
28. Noor, M.A. Hadamard integral inequalities for product of two preinvex functions. *Nonlinear Anal.* **2009**, *14*, 167–173.
29. Noor, M.A.; Awan, M.U.; Noor, K.I. On some inequalities for relative semi-convex functions. *J. Inequal. Appl.* **2013**, *2013*, 332. [CrossRef]
30. Noor, M.A.; Noor, K.I.; Awan, M.U. Quantum analogues of Hermite–Hadamard type inequalities for generalized convexity. In *Computation, Cryptography and Network Security*; Daras, N., Rassias, M.T., Eds.; Springer: Cham, Switzerland, 2015.
31. Noor, M.A.; Noor, K.I.; Awan, M.U.; Li, J. On Hermite–Hadamard inequalities for h-preinvex functions. *Filomat* **2014**, *28*, 1463–1474. [CrossRef]
32. Alp, N.; Sarıkaya, M.Z.; Kunt, M.; İşcan, İ. q-Hermite–Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions. *J. King Saud. Univ. Sci.* **2018**, *30*, 193–203. [CrossRef]
33. Tariboon, J.; Ntouyas, S.K. Quantum integral inequalities on finite intervals. *J. Inequal. Appl.* **2014**, *2014*, 121. [CrossRef]
34. Tariboon, J.; Ntouyas, S.K. Quantum calculus on finite intervals and applications to impulsive difference equations. *Adv. Difference Equ.* **2013**, *2013*, 282. [CrossRef]