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# New Concepts of Picture Fuzzy Graphs with Application

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**Abstract:** The picture fuzzy set is an efficient mathematical model to deal with uncertain real life problems, in which a intuitionistic fuzzy set may fail to reveal satisfactory results. Picture fuzzy set is an extension of the classical fuzzy set and intuitionistic fuzzy set. It can work very efficiently in uncertain scenarios which involve more answers to these type: yes, no, abstain and refusal. In this paper, we introduce the idea of the picture fuzzy graph based on the picture fuzzy relation. Some types of picture fuzzy graph such as a regular picture fuzzy graph, strong picture fuzzy graph, complete picture fuzzy graph, and complement picture fuzzy graph are introduced and some properties are also described. The idea of an isomorphic picture fuzzy graph is also introduced in this paper. We also define six operations such as Cartesian product, composition, join, direct product, lexicographic and strong product on picture fuzzy graph. Finally, we describe the utility of the picture fuzzy graph and its application in a social network.

**Keywords:** fuzzy graph; picture fuzzy graph; social network; direct product; lexicographic product; strong product

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## 1. Introduction

In 1965, L. A. Zadeh [1] introduced the idea of fuzzy sets which has been successfully applied to solve the various real life decision problems, which are often uncertain. Fuzzy set is a generalization of crisp set, where the elements of the set have varying degrees of membership. Since the crisp set consists of two truth values (0 ('false') and 1 ('true')) which is unable to work with uncertain real life problem. Fuzzy set permits its member to have the value of membership degree between 0 and 1 for better outcome, instead of considering only 0 or 1. The membership degree of a member is not same as probability, rather it indicates the belongingness degree of the element to the fuzzy set, which is a single value within the interval  $[0, 1]$ . However, those single values of membership degree cannot handle the uncertainties due to lack of knowledge of the problem. Atanassov [2] has proposed the intuitionistic fuzzy set to manage this kind of uncertain situation using an additional membership degree, defined as hesitation margin. Intuitionistic fuzzy set is as an extension of Zadeh's fuzzy set. The fuzzy set can consider only the membership degree of a member in a given set and sum of the membership degree and non membership is always equal to 1, while intuitionistic fuzzy set can consider independent membership degree and non membership degree, the only requirement is that sum of membership degree and non membership degree is not greater than 1. It is more flexible and efficient compared to classical fuzzy set to work with uncertainty due to the presence

of hesitation margin. The intuitionistic fuzzy set is used significantly in those real life scenarios, where human perception and knowledge are required, which are generally imprecise and not fully reliable. Over the last few decades, the intuitionistic fuzzy set has been obtaining lots of attention from researchers and scientists. Intuitionistic fuzzy set is used in successfully in different field including image processing [3], decision making [4], social network [5], logic programming [6], market prediction [7], machine learning [8], medical diagnosis recognition [9], robotic systems [10], etc. The concept of neutrality degree cannot be considered in intuitionistic fuzzy set theory. However, neutrality degree needs to consider in many real life scenarios, such as democratic election station. Human beings generally give opinions having more answers of the type: yes, no, abstain and refusal. For example, in a democratic voting system, 1000 people participated in the election. The election commission issues 1000 ballot paper and one person can take only one ballot for giving his/her vote and  $\alpha$  is only one candidate. The results of the election are generally divided into four groups came with the number of ballot papers namely “vote for the candidate (500)”, “abstain in vote (200)”, “vote against candidate (200)” and “refusal of voting (100)”. The “abstain in vote” describes that ballot paper is white which contradicts both “vote for the candidate” and “vote against candidate” but it considers the vote. However, “refusal of voting” means bypassing the vote. This type of real life scenarios cannot be handled by intuitionistic fuzzy set. If we use intuitionistic fuzzy sets to describe the above voting system, the information of voting for non-candidates may be ignored. To solve this problem, Cuong and Kreinovich [11] proposed the concept of picture fuzzy set which is a modified version of fuzzy set and Intuitionistic fuzzy set. Picture fuzzy set (PFS) allows the idea degree of positive membership, degree of neutral membership and degree of negative membership of an element. Numerous papers [12–21] have been published on the picture fuzzy set. Cuong [22] described several properties of PFS and also studied a ranking method based on distance measurement between two PFSs. Phong et al. [23] have proposed several compositions of picture fuzzy relations. Then, Cuong and Hai [24] have extended some fuzzy logic operators such as conjunctions, complements, disjunctions and implications for PFSs. They proposed fuzzy inference processes for picture fuzzy systems. Peng and Dai [19] have proposed an algorithmic approach for PFS and used it in a decision making problem. Garg [20] have introduced some aggregation operations of PFS and their applications in multi-criteria decision making.

The basic idea of fuzzy graph was proposed by Rosenfeld [25], 10 years after Zadeh’s significant paper on fuzzy sets. Fuzzy graphs have numerous applications in modern engineering, science and medicine especially in the areas of telecommunication, information theory, logistics, neural networks, transportation, expert systems, manufacturing, cluster analysis, social network, medical diagnosis, control theory etc. Fuzzy graph is used as an essential tool to model those problems and it gives more efficient, flexibility and compatibility to model any real world problem when compared to the classic graph. The concept of a complement fuzzy graph was introduced by Mordeson and Nair [26] and further described by Sunitha and Kumar [27]. Shannon and Atanassov [28] have proposed the definition of intuitionistic fuzzy relation and intuitionistic fuzzy graphs and described several proprieties in [28]. Parvathi et al. [29–31] described different operations on intuitionistic fuzzy graphs. Rashmanlou et al. [32] introduced several operations such as direct product, lexicographic product and strong product on intuitionistic fuzzy graphs. They have also described the join, union, composition, and Cartesian production between two intuitionistic fuzzy graphs. For more details on intuitionistic fuzzy graphs, please refer to [33–36].

Fuzzy graph has used to model many decision making problem in uncertain environment. A number of generalizations of fuzzy graphs have been introduced to deal the uncertainty of the complex real life problems. As uncertainties are well expressed using picture fuzzy set, picture fuzzy graph would be prominent research direction for modeling the uncertain optimization problems. This motivated us to introduce the idea picture fuzzy graph its related operations. In this study, we propose the definition of picture fuzzy graph based on picture fuzzy relation. We also introduce several types of picture fuzzy graph such as regular picture fuzzy graph, strong picture fuzzy graph, complete

picture fuzzy graph, connected picture fuzzy graph and complement picture fuzzy graph. The idea of isomorphism of picture fuzzy graph is introduced in this paper. We describe the weak and co weak isomorphic picture fuzzy graph. We describe the degree, size and order of a picture fuzzy graph. Busy and free nodes of picture fuzzy graph are also defined in this paper. Six operations on picture fuzzy graphs, viz. Cartesian product, composition, join, direct product, lexicographic and strong product are introduced in this paper. Finally, we describe the utility of picture fuzzy graph and its application in social network.

## 2. Preliminaries

**Definition 1.** A fuzzy set  $A$ , on the universe of discourse of  $X$  can also be described a set of ordered pairs as:

$$A = \{x, \mu_A(x) \mid x \in X\}. \tag{1}$$

In classical set theory,  $\mu_A(x)$  has only the two membership values 0 ('false') and 1 ('true'). Instead of considering only those two truth values, fuzzy set can assign the value of  $\mu_A(x)$  within the interval between 0 ('false') and 1 ('true') for better outcome.

**Definition 2.** Let  $A$  be a intuitionistic fuzzy set.  $A$  in  $X$  is defined by

$$A = \{x, \mu_A(x), \nu_A(x) \mid x \in X\}, \tag{2}$$

where  $\mu_A(x) \in [0, 1]$  and  $\nu_A(x) \in [0, 1]$  follows with the condition  $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ . The  $\mu_A(x)$  is used to represent the degree of membership and  $\nu_A(x)$  represents the degree of non-membership of the element  $x$  in the set  $A$ . For each intuitionistic fuzzy set  $A$  in  $X$ , the intuitionistic fuzzy index is described as  $\pi_A(x) = 1 - (\mu_A(x) + \nu_A(x))$ .

**Definition 3.** Let  $A$  be a picture fuzzy set.  $A$  in  $X$  is defined by

$$A = \{x, \mu_A(x), \eta_A(x), \nu_A(x) \mid x \in X\}, \tag{3}$$

where  $\mu_A(x)$ ,  $\eta_A(x)$  and  $\nu_A(x)$  follow the condition  $0 \leq \mu_A(x) + \eta_A(x) + \nu_A(x) \leq 1$ . The  $\mu_A(x), \eta_A(x), \nu_A(x) \in [0, 1]$ , denote respectively the positive membership degree, neutral membership degree and negative membership degree of the element  $x$  in the set  $A$ . For each picture fuzzy set  $A$  in  $X$ , the refusal membership degree is described as  $\pi_A(x) = 1 - (\mu_A(x) + \eta_A(x) + \nu_A(x))$ .

**Definition 4.** A picture fuzzy relation  $R$  is a picture fuzzy subset of  $X \times Y$  described by

$$R = \{(x, y), \mu_R(x, y), \eta_R(x, y), \nu_R(x, y) \mid x \in X, y \in Y\}. \tag{4}$$

Here,  $\mu_R : X \times Y \rightarrow [0, 1]$ ,  $\eta_R : X \times Y \rightarrow [0, 1]$  and  $\nu_R : X \times Y \rightarrow [0, 1]$  must satisfy the condition  $0 \leq \mu_R(x, y) + \eta_R(x, y) + \nu_R(x, y) \leq 1$  for every  $(x, y) \in X \times Y$ .

**Definition 5.** Let  $A$  and  $B$  be two picture fuzzy sets. The union, intersection and complement are described as follows.

1.  $A \cup B = \{x, \max(\mu_A(x), \mu_B(x)), \min(\eta_A(x), \eta_B(x)), \min(\nu_A(x), \nu_B(x)) \mid x \in X, y \in Y\}$
2.  $A \cap B = \{x, \min(\mu_A(x), \mu_B(x)), \min(\eta_A(x), \eta_B(x)), \max(\nu_A(x), \nu_B(x)) \mid x \in X, y \in Y\}$
3.  $A^c = \{\nu_A(x), \eta_A(x), \mu_A(x) \mid x \in X\}$

**Definition 6.** Let  $A$  and  $B$  be two picture fuzzy sets on two universes  $X_1$  and  $X_2$ . The Cartesian product of  $A$  and  $B$  is described as follows.

$$A \times B = \{(x, y), \mu_A(x) \wedge \mu_B(y), \eta_A(x) \wedge \eta_B(y), \nu_A(x) \vee \nu_B(y) \mid x \in X_1, y \in X_2\}. \tag{5}$$

**Definition 7.** Let  $V$  be a finite nonempty set of vertices. A fuzzy graph  $G$  is described by an order pair of functions  $\sigma$  and  $\mu$ ,  $G = (\sigma, \mu)$ .  $\sigma$  represents a fuzzy subset of  $V$  and  $\mu$  represents a symmetric fuzzy relation on  $\sigma$ , i.e.,  $\sigma: V \rightarrow [0, 1]$  and  $\mu: V \times V \rightarrow [0, 1]$  such that

$$\mu(u, v) \leq (\sigma(u) \cap \sigma(v)), \forall u, \forall v \in V. \tag{6}$$

**Definition 8.** A fuzzy graph  $G = (\sigma, \mu)$  is called complete fuzzy graph if

$$\mu(x, y) = (\sigma(x) \cap \sigma(y)), \forall x, \forall y \in \sigma. \tag{7}$$

**Definition 9.** A fuzzy graph  $G = (\sigma, \mu)$  is called strong fuzzy graph if

$$\mu(x, y) = (\sigma(x) \cap \sigma(y)), \forall x, y \in \mu. \tag{8}$$

**Definition 10.** The complement of a fuzzy graph  $G = (\sigma, \mu)$  is a fuzzy graph and it is represented as  $G^c = (\sigma^c, \mu^c)$ , where  $\sigma^c = \sigma$  and

$$\mu^c(u, v) = \sigma(u) \wedge \sigma(v) - \mu(u, v). \tag{9}$$

**Definition 11.** Let  $V$  be a finite nonempty set of vertices. An intuitionistic fuzzy graph is described by an order pair  $G = (A, B)$ , where  $A$  is an intuitionistic fuzzy set on  $V$  and  $B$  is an intuitionistic fuzzy set on  $V^2$ .  $A$  and  $B$  satisfy  $\mu_B(x, y) \leq \mu_A(x) \wedge \mu_B(x, y) \leq \nu_A(x) \wedge \nu_B(x), \forall (x, y) \in V^2$  and  $\mu_B(x, y) = \nu_B(x, y) = 0 \forall (x, y) \in V^2 - E$ . Here,  $\mu_A(x)$  and  $\nu_A(x)$  are used to denote the degree of membership and non membership of the node  $x$  and  $\mu_B(x, y)$  and  $\nu_B(x, y)$  are used to denote the degree of membership and non membership of the arcs  $(x, y)$ .

**Definition 12.** Let  $G_1^* = (V_1, E_1)$  and  $G_2^* = (V_2, E_2)$  be two classical (crisp) graph. The Cartesian product of two graph  $G_1^*$  and  $G_2^*$  is an another graph  $G_3^*$  (written as  $G_3^* = G_1^* \times G_2^* = (V_3, E_3)$ ), where  $V_3 = V_1 \times V_2$  and  $E_3 = \{(x, x_2), (x, y_2) | x \in V_1, x_2 y_2 \in E_2\} \cup \{(x_1, z), (y_1, z) | z \in V_2, x_1 y_1 \in E_1\}$ .

**Definition 13.** Let  $G_1^* = (V_1, E_1)$  and  $G_2^* = (V_2, E_2)$  be two classical (crisp) graph. The direct product of two graph  $G_1^*$  and  $G_2^*$  is an another graph  $G_3^*$  (written as  $G_3^* = G_1^* \times G_2^* = (V_3, E_3)$ ), where  $V_3 = V_1 \times V_2$  and  $E_3 = \{(x_1, x_2), (y_1, y_2) | x_1 y_1 \in E_1, x_2 y_2 \in E_2\}$ .

**Definition 14.** Let  $G_1^* = (V_1, E_1)$  and  $G_2^* = (V_2, E_2)$  be two classical (crisp) graph. The lexicographic product of two graph  $G_1^*$  and  $G_2^*$  is an another graph  $G_3^*$  (written as  $G_3^* = G_1^* \cdot G_2^* = (V_3, E_3)$ ), where  $V_3 = V_1 \times V_2$  and  $E_3 = \{(x, x_2), (x, y_2) | x \in V_1, x_2 y_2 \in E_2\} \cup \{(x_1, x_2), (y_1, y_2) | x_1 y_1 \in E_1, x_2 y_2 \in E_2\}$ .

**Definition 15.** Let  $G_1^* = (V_1, E_1)$  and  $G_2^* = (V_2, E_2)$  be two classical (crisp) graph. The strong product of two graph  $G_1^*$  and  $G_2^*$  is an another graph  $G_3^*$  (written as  $G_3^* = G_1^* \cdot G_2^* = (V_3, E_3)$ ), where  $V_3 = V_1 \times V_2$  and  $E_3 = \{(x, x_2), (x, y_2) | x \in V_1, x_2 y_2 \in E_2\} \cup \{(x_1, z), (y_1, z) | z \in V_2, x_1 y_1 \in E_1 \cup (x_1, x_2), (y_1, y_2) | x_1 y_1 \in E_1, x_2 y_2 \in E_2\}$ .

**Definition 16.** Let  $G_1^* = (V_1, E_1)$  and  $G_2^* = (V_2, E_2)$  be two classical (crisp) graph. The union of two graphs  $G_1^*$  and  $G_2^*$  is an another graph  $G_3^*$  (written as  $G_3^* = G_1^* \cup G_2^* = (V_3, E_3)$ ), where  $V_3 = V_1 \cup V_2$  and  $E_3 = E_1 \cup E_2$ .

**Definition 17.** Let  $G_1^* = (V_1, E_1)$  and  $G_2^* = (V_2, E_2)$  be two classical (crisp) graph. The joining of two graphs  $G_1^*$  and  $G_2^*$  is an another graph  $G_3^*$  (written as  $G_3^* = G_1^* \cup G_2^* = (V_3, E_3)$ ), where  $V_3 = V_1 \cup V_2$  and  $E_3 = E_1 \cup E_2 \cup E'$ , where  $E'$  represents the set of all arcs joining the nodes  $V_1$  and  $V_2$ .

### 3. Picture Fuzzy Graph

In this section, we define the picture fuzzy graph and introduce different types of regular picture fuzzy graphs, strong picture fuzzy graphs, complete picture fuzzy graphs, and complement picture fuzzy graph.

**Definition 18.** Let  $G^* = (V, E)$  be a graph. A pair  $G = (A, B)$  is called a picture fuzzy graph on  $G^*$  where  $A = (\mu_A, \eta_A, \nu_A)$  is a picture fuzzy set on  $V$  and  $B = (\mu_B, \eta_B, \nu_B)$  is a picture fuzzy set on  $E \subseteq V \times V$  such that for each arc  $uv \in E$ .

$$\mu_B(u, v) \leq \min(\mu_A(u), \mu_A(v)), \eta_B(u, v) \leq \min(\eta_A(u), \eta_A(v)), \nu_B(u, v) \geq \max(\nu_A(u), \nu_A(v)) \quad (10)$$

An example of picture fuzzy graph is shown in Figure 1.

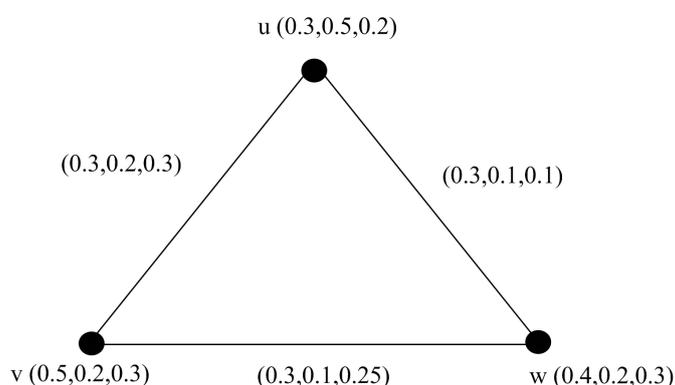


Figure 1. A picture fuzzy graph.

**Definition 19.** A picture fuzzy graph  $G = (A, B)$  is said to be regular picture fuzzy graph if

$$\sum_{\substack{u \\ u \neq v}} \mu_B(u, v) = \text{constant}, \sum_{\substack{u \\ u \neq v}} \eta_B(u, v) = \text{constant}, \sum_{\substack{u \\ u \neq v}} \nu_B(u, v) = \text{constant}, \forall u, v \in E \quad (11)$$

Figure 1 is also an example of regular picture fuzzy graph.

**Definition 20.** A picture fuzzy graph  $G = (A, B)$  is defined as strong picture fuzzy graph if

$$\mu_B(u, v) = \mu_A(u) \wedge \mu_A(v), \eta_B(u, v) = \eta_A(u) \wedge \eta_A(v), \nu_B(u, v) = \nu_A(u) \vee \nu_A(v) \quad \forall u, v \in E. \quad (12)$$

An example of a strong picture fuzzy graph is shown in Figure 2.

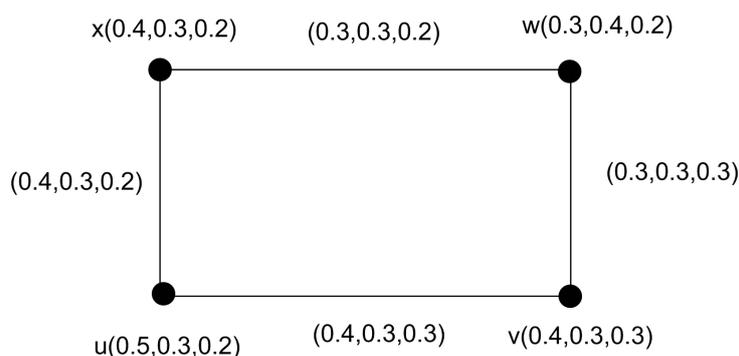
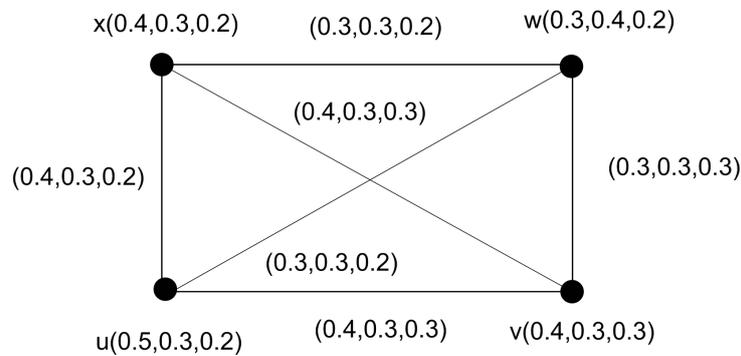


Figure 2. A strong picture fuzzy graph.

**Definition 21.** A picture fuzzy graph  $G = (A, B)$  is defined as complete picture fuzzy graph if

$$\mu_B(u, v) = \mu_A(u) \wedge \mu_A(v), \eta_B(u, v) = \eta_A(u) \wedge \eta_A(v), \nu_B(u, v) = \nu_A(u) \vee \nu_A(v) \quad \forall u, v \in V. \quad (13)$$

An example of a complete picture fuzzy graph is shown in Figure 3. Note that every complete picture fuzzy graph is a strong picture fuzzy graph but not conversely.



**Figure 3.** A complete picture fuzzy graph.

**Definition 22.** A path  $p$  in a picture fuzzy graph  $G = (A, B)$  is a sequence of different vertices  $v_0, v_1, v_2, \dots, v_k$  such that

$$(\mu_B(v_{i-1}v_i) \eta_B(v_{i-1}v_i) \nu_B(v_{i-1}v_i)) > 0, i = 1, 2, \dots, k. \quad (14)$$

Here,  $k$  represents the length of path.

**Definition 23.** Let  $G = (A, B)$  be a picture fuzzy graph. If two nodes  $u$  and  $v$  are connected by a path of length  $k$  in  $G$  such as  $p: u_0, u_1, u_2, \dots, u_{k-1}, u_k$  then  $\mu_B(u, v)$ ,  $\eta_B(u, v)$  and  $\nu_B(u, v)$  are described as follows.

$$\begin{aligned} \mu_B^k(u, v) &= \mu_B(u, u_1) \wedge \mu_B(u_1, u_2) \wedge \dots \wedge \mu_B(u_{k-1}, v) \\ \eta_B^k(u, v) &= \eta_B(u, u_1) \wedge \eta_B(u_1, u_2) \wedge \dots \wedge \eta_B(u_{k-1}, v) \\ \nu_B^k(u, v) &= \nu_B(u, u_1) \vee \nu_B(u_1, u_2) \vee \dots \vee \nu_B(u_{k-1}, v). \end{aligned}$$

Let  $(\mu_B^\infty(u, v), \eta_B^\infty(u, v), \nu_B^\infty(u, v))$  be the strength of connectedness between the two nodes  $u$  and  $v$  of a picture fuzzy graph  $G$ . Then  $\mu_B^\infty(u, v)$ ,  $\eta_B^\infty(u, v)$ , and  $\nu_B^\infty(u, v)$  are defined as follows.

$$\begin{aligned} \mu_B^\infty(u, v) &= \sup\{\mu_B^k(u, v) | k = 1, 2, \dots\}, \\ \eta_B^\infty(u, v) &= \sup\{\eta_B^k(u, v) | k = 1, 2, \dots\} \\ \nu_B^\infty(u, v) &= \inf\{\nu_B^k(u, v) | k = 1, 2, \dots\}, \end{aligned}$$

where  $\inf$  is used to find the minimum and  $\sup$  is used to find maximum membership value. Consider a connected picture fuzzy graph as shown in the Figure 3 and the  $x - u$  paths are

- $P_1$  :  $x - u$  with membership value  $(0.4, 0.3, 0.2)$ ,
- $P_2$  :  $x - w - u$  with membership value  $(0.3, 0.3, 0.2)$ ,
- $P_3$  :  $x - v - u$  with membership value  $(0.4, 0.3, 0.3)$ ,
- $P_4$  :  $x - w - v - u$  with membership value  $(0.3, 0.3, 0.3)$ ,
- $P_5$  :  $x - v - w - u$  with membership value  $(0.3, 0.3, 0.3)$ .

By routine computation, we have

$$\begin{aligned} \mu_B^\infty(x, u) &= \sup\{0.4, 0.3, 0.4, 0.3, 0.3\} = 0.4, \\ \eta_B^\infty(x, u) &= \sup\{0.3, 0.3, 0.3, 0.3, 0.3\} = 0.3, \\ \nu_B^\infty(x, u) &= \inf\{0.2, 0.2, 0.3, 0.3, 0.3\} = 0.2. \end{aligned}$$

The strength of connectedness between the two nodes  $x$  and  $u$  of a picture fuzzy graph  $G$  (Figure 3) is  $(0.4, 0.3, 0.2)$ .

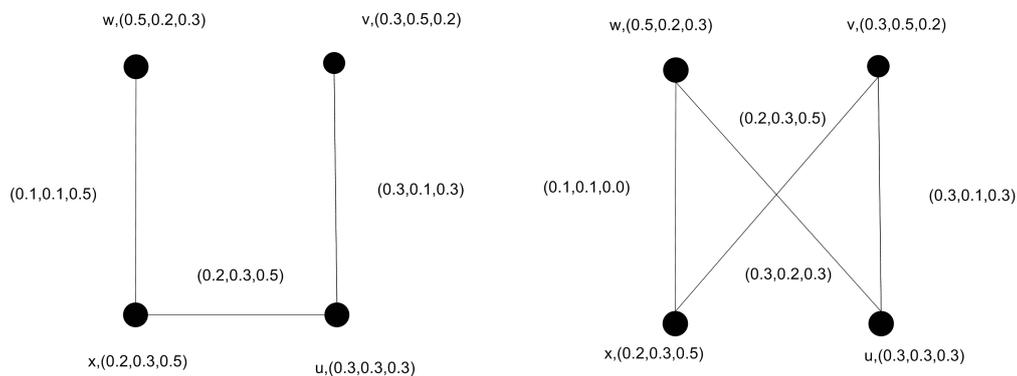
**Definition 24.** Let  $G = (A, B)$  be a picture fuzzy graph. Then,  $G$  is said to be connected picture fuzzy graph if for every vertices  $u, v \in V, \mu_B^\infty(u, v) > 0$  or  $\eta_B^\infty(u, v) > 0$  or  $\nu_B^\infty(u, v) < 1$ .

Figure 3 is an example of connected picture fuzzy graph.

**Definition 25.** The complement of a picture fuzzy graph  $G = (A, B)$  is a picture fuzzy graph  $G^c = (A^c, B^c)$  if and only if follows the (15).

$$\begin{aligned} \mu_A^c &= \mu_A, \eta_A^c = \eta_A, \nu_A^c = \nu_A, \mu_B^c(u, v) = \mu_A(u) \wedge \mu_A(v) - \mu_B(u, v), \\ \eta_B^c(u, v) &= \eta_A(u) \wedge \eta_A(v) - \eta_B(u, v), \nu_B^c(u, v) = \nu_A(u) \vee \nu_A(v) - \nu_B(u, v) \quad \forall u, v \in V. \end{aligned} \quad (15)$$

An example of complement picture fuzzy graph is shown in Figure 4.



**Figure 4.** A picture fuzzy graph and its complement picture fuzzy graph.

We have

$$\mu_A^{cc} = \mu_A^c = \mu_A, \eta_A^{cc} = \eta_A^c = \eta_A, \nu_A^{cc} = \nu_A^c = \nu_A \quad (16)$$

$$\mu_1^{cc}(u, v) = \sigma_1^c(u) \wedge \sigma_1^c(v) - \mu_1^c(u, v) = \sigma_A(u) \wedge \sigma_A(v) - \mu_1^c(u, v). \quad (17)$$

Similarly, we can prove that  $\mu_2^{cc}(u, v) = \mu_2(u, v)$  and  $\mu_3^{cc}(u, v) = \mu_2(u, v) \forall u, v \in V$ . Hence,  $G^{cc} = G$ .

**Definition 26.** Let  $G_1 = (A_1, B_1)$  and  $G_2 = (A_2, B_2)$  be two picture fuzzy graphs.

1. A homomorphism  $h$  from picture fuzzy graph  $G_1$  to  $G_2$  is a mapping function  $h : V_1 \rightarrow V_2$  which always satisfy followings:

- (i)  $\forall x_1 \in V_1, x_1 y_1 \in E_1$ 
  - (a)  $\mu_{A_1}(x_1) \leq \mu_{A_2}(h(x_1)), \eta_{A_1}(x_1) \leq \eta_{A_2}(h(x_1)), \nu_{A_1}(x_1) \geq \nu_{A_2}(h(x_1)),$
  - (b)  $\mu_{B_1}(x_1 y_1) \leq \mu_{B_2}(h(x_1) h(x_2)), \eta_{B_1}(x_1 y_1) \leq \eta_{B_2}(h(x_1) h(x_2)),$   
 $\nu_{B_1}(x_1 y_1) \geq \nu_{B_2}(h(x_1) h(x_2)).$

A isomorphism  $h$  from a picture fuzzy graph  $G_1$  to  $G_2$  is a bijective mapping function  $h : V_1 \rightarrow V_2$  which always satisfy followings:

- 2. (i)  $\forall x_1 \in V_1, x_1 y_1 \in E_1$ 
  - (c)  $\mu_{A_1}(x_1) = \mu_{A_2}(h(x_1)), \eta_{A_1}(x_1) = \eta_{A_2}(h(x_1)), \nu_{A_1}(x_1) = \nu_{A_2}(h(x_1)),$
  - (d)  $\mu_{B_1}(x_1 y_1) = \mu_{B_2}(h(x_1) h(x_2)), \eta_{B_1}(x_1 y_1) = \eta_{B_2}(h(x_1) h(x_2)),$   
 $\nu_{B_1}(x_1 y_1) = \nu_{B_2}(h(x_1) h(x_2)).$

A weak isomorphism  $h$  from picture fuzzy graph  $G_1$  to  $G_2$  is a bijective mapping function  $h : V_1 \rightarrow V_2$  which always satisfy followings:

3. (i)  $\forall x_1 \in V_1$   
 (e)  $h$  is homomorphism.  
 (f)  $\mu_{A_1}(x_1) = \mu_{A_2}(h(x_1)), \eta_{A_1}(x_1) = \eta_{A_2}(h(x_1)), \nu_{A_1}(x_1) = \nu_{A_2}(h(x_1))$ . Thus, a weak isomorphism maintains the costs of the vertices but not necessarily the costs of the edges.
4. A co weak isomorphism  $h$  from picture fuzzy graph  $G_1$  to  $G_2$  is a bijective mapping function  $h : V_1 \rightarrow V_2$  which always satisfy followings:  
 (i)  $\forall x_1 y_1 \in E_1$   
 (g)  $h$  is homomorphism.  
 (h)  $\mu_{A_1}(x_1 y_1) = \mu_{A_2}(h(x_1 y_1)), \eta_{A_1}(x_1 y_1) = \eta_{A_2}(h(x_1 y_1)), \nu_{A_1}(x_1 y_1) = \nu_{A_2}(h(x_1 y_1))$ .

**Definition 27.** A picture fuzzy graph  $G$  is said to be

1. Self complementary picture fuzzy graph if and only if  $G = G^c$ .
2. Self weak complement picture fuzzy graph if  $G$  is weak isomorphic to  $G^c$ .

#### 4. Busy and Free Nodes on Picture Fuzzy Graphs

In this section, we propose open degree, busy values, free node, busy node in a picture fuzzy graph.

**Definition 28.** Let  $G = (A, B)$  be picture fuzzy graph on  $G^*$ . The open degree of a vertex  $u$  is defined as  $deg(u) = (d_\mu(u), d_\eta(u), d_\nu(u))$

$$d_\mu(u) = \sum_{\substack{v \in V \\ u \neq v}} \mu_B(u, v), d_\eta(u) = \sum_{\substack{v \in V \\ u \neq v}} \eta_B(u, v), d_\nu(u) = \sum_{\substack{v \in V \\ u \neq v}} \nu_B(u, v), \forall u, v \in E. \tag{18}$$

The order of  $G$  is defined and denoted as

$$O(G) = \sum_{v \in V} \mu_B(u), \sum_{v \in V} \eta_B(u), \sum_{v \in V} \nu_B(u). \tag{19}$$

The size of picture fuzzy graph is defined as  $S(G) = (S_\mu(G), S_\eta(G), S_\nu(G))$  where

$$S_\mu(G) = \sum_{\substack{v \in V \\ u \neq v}} \mu_B(u, v), S_\eta(G) = \sum_{\substack{v \in V \\ u \neq v}} \eta_B(u, v), S_\nu(G) = \sum_{\substack{v \in V \\ u \neq v}} \nu_B(u, v). \tag{20}$$

**Definition 29.** Let  $G = (A, B)$  be a picture fuzzy graph. Then the busy value of a node  $v$  is defined by  $D(v) = (D_\mu(v), D_\eta(v), D_\nu(v))$ , where

$$D_\mu(v) = \sum \mu_A(v) \wedge \mu_A(v_i), D_\eta(v) = \sum \eta_A(v) \wedge \eta_A(v_i), D_\nu(v) = \sum \nu_A(v) \vee \nu_A(v_i).$$

Here,  $v_i$  are adjacent vertices of  $v$ . The busy value of picture fuzzy graph  $G$  is the sum of all busy values of vertices of  $G$ .

**Definition 30.** Let  $G = (A, B)$  be a picture fuzzy graph. A vertex  $v$  of  $G$  is said to be busy vertex if

$$\mu_A(v) \leq d_\mu(u), \eta_A(u) \leq d_\eta(u), \nu_A(u) \geq d_\nu(u). \tag{21}$$

Otherwise, it is said to be a free vertex.

**Definition 31.** Let  $G = (A, B)$  be a picture fuzzy graph. Then an edge  $u, v$  is defined as an effective edge if and only if

$$\mu_B(u, v) = \mu_A(u) \wedge \mu_A(v), \eta_B(u, v) = \eta_A(u) \wedge \eta_A(v), \nu_B(u, v) = \nu_A(u) \vee \nu_A(v).$$

**Definition 32.** A connected picture fuzzy graph  $G$  is said to be a highly irregular picture fuzzy graph if and only if every node of  $G$  is adjacent to nodes with different degrees.

### 5. Operations on Picture Fuzzy Graph

In this section, we introduce six operations on picture fuzzy graph, viz., Cartesian product, composition, join, direct product, lexicographic and strong product.

**Definition 33.** Let  $G_1 = (A_1, B_1)$  and  $G_2 = (A_2, B_2)$  are two picture fuzzy graph of  $G_1^* = (V_1, E_1)$  and  $G_2^* = (V_2, E_2)$  respectively. The Cartesian product  $G_1 \times G_2$  of picture fuzzy graph  $G_1$  and  $G_2$  is defined by  $(A, B)$ , where  $A = (\mu_A, \eta_A, \nu_A)$  and  $B = (\mu_B, \eta_B, \nu_B)$  are two picture fuzzy sets on  $V=(V_1 \times V_2)$ , and  $E = \{(x, x_2), (x, y_2) | x \in V_1, x_2y_2 \in E_2\} \cup \{(x_1, z), (y_1, z) | z \in V_2, x_1y_1 \in E_1\}$  respectively which satisfies the followings:

- (i)  $\forall (x_1, x_2) \in V_1 \times V_2,$ 
  - (a)  $\mu_A((x_1, x_2)) = \mu_{A_1}(x_1) \wedge \mu_{A_2}(x_2),$
  - (b)  $\eta_A((x_1, x_2)) = \eta_{A_1}(x_1) \wedge \eta_{A_2}(x_2)$
  - (c)  $\nu_A((x_1, x_2)) = \nu_{A_1}(x_1) \vee \nu_{A_2}(x_2),$
- (ii)  $\forall x \in V_1$  and  $\forall (x_2, y_2) \in E_2,$ 
  - (a)  $\mu_B((x, x_2)(x, y_2)) = \mu_{A_1}(x_1) \wedge \mu_{B_2}(x_2y_2),$
  - (b)  $\eta_B((x, x_2)(x, y_2)) = \eta_{A_1}(x_1) \wedge \eta_{B_2}(x_2y_2),$
  - (c)  $\nu_B((x, x_2)(x, y_2)) = \nu_{A_1}(x_1) \vee \nu_{B_2}(x_2y_2).$
- (iii)  $\forall z \in V_2$  and  $\forall (x_1, x_2) \in E_1,$ 
  - (a)  $\mu_B((x_1, z)(y_1, z)) = \mu_{B_1}(x_1y_1) \wedge \mu_{A_2}(z),$
  - (b)  $\eta_B((x_1, z)(y_1, z)) = \eta_{B_1}(x_1y_1) \wedge \eta_{A_2}(z),$
  - (c)  $\nu_B((x_1, z)(y_1, z)) = \nu_{B_1}(x_1y_1) \vee \nu_{A_2}(z).$

**Definition 34.** The composition  $G_1G_2$  of two picture fuzzy graph  $G_1 = (A_1, B_1)$  and  $G_2 = (A_2, B_2)$  defined as a pair  $(A, B)$ , where  $A = (\mu_A, \eta_A, \nu_A)$  and  $B = (\mu_B, \eta_B, \nu_B)$  are two picture fuzzy sets on  $V=(V_1 \times V_2)$ , and  $E = \{(x, x_2), (x, y_2) | x \in V_1, x_2y_2 \in E_2\} \cup \{(x_1, z), (y_1, z) | z \in V_2, x_1y_1 \in E_1\} \cup \{(x_1, x_2), (y_1, y_2) | x_2y_2 \in V_2 \neq y_2, x_1y_1 \in E_1\}$  respectively, which satisfies the followings:

- (i)  $\forall (x_1, x_2) \in V_1 \times V_2,$ 
  - (a)  $\mu_A((x_1, x_2)) = \mu_{A_1}(x_1) \wedge \mu_{A_2}(x_2),$
  - (b)  $\eta_A((x_1, x_2)) = \eta_{A_1}(x_1) \wedge \eta_{A_2}(x_2),$
  - (c)  $\nu_A((x_1, x_2)) = \nu_{A_1}(x_1) \vee \nu_{A_2}(x_2).$
- (ii)  $\forall x \in V_1$  and  $\forall (x_2, y_2) \in E_2,$ 
  - (a)  $\mu_B((x, x_2)(x, y_2)) = \mu_{A_1}(x_1) \wedge \mu_{B_2}(x_2y_2),$

- (b)  $\eta_B((x, x_2)(x, y_2)) = \eta_{A_1}(x_1) \wedge \eta_{B_2}(x_2y_2),$
- (c)  $\nu_B((x, x_2)(x, y_2)) = \nu_{A_1}(x_1) \vee \nu_{B_2}(x_2y_2).$
- (iii)  $\forall z \in V_2$  and  $\forall (x_1, x_2) \in E_1,$ 
  - (a)  $\mu_B((x_1, z)(y_1, z)) = \mu_{B_1}(x_1y_1) \wedge \mu_{A_2}(z),$
  - (b)  $\eta_B((x_1, z)(y_1, z)) = \eta_{B_1}(x_1y_1) \wedge \eta_{A_2}(z),$
  - (c)  $\nu_B((x_1, z)(y_1, z)) = \nu_{B_1}(x_1y_1) \vee \nu_{A_2}(z).$
- (iv)  $\forall x_2y_2 \in V_2, x_2 \neq y_2$  and  $\forall (x_1y_1) \in E_1,$ 
  - (a)  $\mu_B((x_1, x_2)(y_1, y_2)) = \mu_{A_2}(x_2) \wedge \mu_{A_2}(y_2) \wedge \mu_{B_1}(x_1y_1),$
  - (b)  $\eta_B((x_1, x_2)(y_1, y_2)) = \eta_{A_2}(x_2) \wedge \eta_{A_2}(y_2) \wedge \eta_{B_1}(x_1y_1),$
  - (c)  $\nu_B((x_1, x_2)(y_1, y_2)) = \nu_{A_2}(x_2) \wedge \nu_{A_2}(y_2) \wedge \nu_{B_1}(x_1y_1).$

**Definition 35.** The union  $G_1 \cup G_2$  of two picture fuzzy graph  $G_1 = (A_1, B_1)$  and  $G_2 = (A_2, B_2)$  is defined as  $(A, B)$ , where  $A = (\mu_A, \eta_A, \nu_A)$  is a picture fuzzy set on  $V = V_1 \cup V_2$  and  $B = (\mu_B, \eta_B, \nu_B)$  is an another picture fuzzy set on  $E = E_1 \cup E_2$ , which satisfies the following:

- (i) (a)  $\mu_A(x) = \mu_{A_1}(x)$  if  $x \in V_1$  and  $x \notin V_2,$
- (b)  $\mu_A(x) = \mu_{A_2}(x)$  if  $x \in V_2$  and  $x \notin V_1,$
- (c)  $\mu_A(x) = \mu_{A_1}(x) \wedge \mu_{A_2}(x)$  if  $x \in V_1 \cap V_2,$
- (ii) (a)  $\eta_A(x) = \eta_{A_1}(x)$  if  $x \in V_1$  and  $x \notin V_2,$
- (b)  $\eta_A(x) = \eta_{A_2}(x)$  if  $x \in V_2$  and  $x \notin V_1,$
- (c)  $\eta_A(x) = \eta_{A_1}(x) \wedge \eta_{A_2}(x)$  if  $x \in V_1 \cap V_2,$
- (iii) (a)  $\nu_A(x) = \nu_{A_1}(x)$  if  $x \in V_1$  and  $x \notin V_2,$
- (b)  $\nu_A(x) = \nu_{A_2}(x)$  if  $x \in V_2$  and  $x \notin V_1,$
- (c)  $\nu_A(x) = \nu_{A_1}(x) \wedge \nu_{A_2}(x)$  if  $x \in V_1 \cap V_2,$
- (iv) (a)  $\mu_B(xy) = \mu_{B_1}(xy)$  if  $xy \in E_1$  and  $xy \notin E_2,$
- (b)  $\mu_B(xy) = \mu_{B_2}(xy)$  if  $xy \in E_2$  and  $xy \notin E_1,$
- (c)  $\mu_B(xy) = \mu_{B_1}(xy) \wedge \mu_{B_2}(xy)$  if  $xy \in E_1 \cap E_2,$
- (v) (a)  $\eta_B(xy) = \eta_{B_1}(xy)$  if  $xy \in E_1$  and  $xy \notin E_2,$
- (b)  $\eta_B(xy) = \eta_{B_2}(xy)$  if  $xy \in E_2$  and  $xy \notin E_1,$
- (c)  $\eta_B(xy) = \eta_{B_1}(xy) \wedge \eta_{B_2}(xy)$  if  $xy \in E_1 \cap E_2,$
- (vi) (a)  $\nu_B(xy) = \nu_{B_1}(xy)$  if  $xy \in E_1$  and  $xy \notin E_2,$
- (b)  $\nu_B(xy) = \nu_{B_2}(xy)$  if  $xy \in E_2$  and  $xy \notin E_1,$

$$(c) \quad \nu_B(xy) = \nu_{B_1}(xy) \wedge \nu_{B_2}(xy) \text{ if } xy \in E_1 \cap E_2.$$

**Definition 36.** The join  $G_1 + G_2$  of two picture fuzzy graph  $G_1 = (A_1, B_1)$  and  $G_2 = (A_2, B_2)$  is defined as  $(A, B)$ , where  $A = (\mu_A, \eta_A, \nu_A)$  is a picture fuzzy set on  $V = V_1 \cup V_2$  and  $B = (\mu_B, \eta_B, \nu_B)$  is an another picture fuzzy set on  $E = E_1 \cup E_2 \cup E'$  ( $E'$  represents all edges joining the vertex of  $V_1$  and  $V_2$ ), which satisfies the following:

- (i) (a)  $\mu_A(x) = \mu_{A_1}(x)$  if  $x \in V_1$  and  $x \notin V_2$
- (b)  $\mu_A(x) = \mu_{A_2}(x)$  if  $x \in V_2$  and  $x \notin V_1$
- (c)  $\mu_A(x) = \mu_{A_1}(x) \wedge \mu_{A_2}(x)$  if  $x \in V_1 \cap V_2$
- (ii) (a)  $\eta_A(x) = \eta_{A_1}(x)$  if  $x \in V_1$  and  $x \notin V_2$
- (b)  $\eta_A(x) = \eta_{A_2}(x)$  if  $x \in V_2$  and  $x \notin V_1$
- (c)  $\eta_A(x) = \eta_{A_1}(x) \wedge \eta_{A_2}(x)$  if  $x \in V_1 \cap V_2$
- (iii) (a)  $\nu_A(x) = \nu_{A_1}(x)$  if  $x \in V_1$  and  $x \notin V_2$
- (b)  $\nu_A(x) = \nu_{A_2}(x)$  if  $x \in V_2$  and  $x \notin V_1$
- (c)  $\nu_A(x) = \nu_{A_1}(x) \wedge \nu_{A_2}(x)$  if  $x \in V_1 \cap V_2$
- (iv) (a)  $\mu_B(xy) = \mu_{B_1}(xy)$  if  $xy \in E_1$  and  $xy \notin E_2$
- (b)  $\mu_B(xy) = \mu_{B_2}(xy)$  if  $xy \in E_2$  and  $xy \notin E_1$
- (c)  $\mu_B(xy) = \mu_{B_1}(xy) \wedge \mu_{B_2}(xy)$  if  $xy \in E_1 \cap E_2$
- (v) (a)  $\eta_B(xy) = \eta_{B_1}(xy)$  if  $xy \in E_1$  and  $xy \notin E_2$
- (b)  $\eta_B(xy) = \eta_{B_2}(xy)$  if  $xy \in E_2$  and  $xy \notin E_1$
- (c)  $\eta_B(xy) = \eta_{B_1}(xy) \wedge \eta_{B_2}(xy)$  if  $xy \in E_1 \cap E_2$
- (vi) (a)  $\nu_B(xy) = \nu_{B_1}(xy)$  if  $xy \in E_1$  and  $xy \notin E_2$
- (b)  $\nu_B(xy) = \nu_{B_2}(xy)$  if  $xy \in E_2$  and  $xy \notin E_1$
- (c)  $\nu_B(xy) = \nu_{B_1}(xy) \wedge \nu_{B_2}(xy)$  if  $xy \in E_1 \cap E_2$
- (vii) (a)  $\mu_B(xy) = \mu_{B_1}(x) \vee \mu_{B_2}(y)$  if  $xy \in E'$
- (b)  $\eta_B(xy) = \eta_{B_1}(x) \vee \eta_{B_2}(y)$  if  $xy \in E'$
- (c)  $\nu_B(xy) = \nu_{B_1}(x) \wedge \nu_{B_2}(y)$  if  $xy \in E'$

**Definition 37.** The direct product  $G_1 * G_2$  of two picture fuzzy graph  $G_1$  and  $G_2$  is defined as a pair  $(A, B)$ , where  $A = (\mu_A, \eta_A, \nu_A)$  is a picture fuzzy set on  $V = V_1 \times V_2$  and  $B = (\mu_B, \eta_B, \nu_B)$  is an another picture fuzzy set on  $E = \{(x_1, x_2)(y_1, y_2) | x_1y_1 \in E_1, x_2y_2 \in E_2\}$ , which satisfies the followings:

- (i)  $\forall (x_1, x_2) \in V_1 \times V_2$

- (a)  $\mu_A(x_1, x_2) = \mu_{A_1}(x_1) \vee \mu_{A_2}(x_2)$
- (b)  $\eta_A(x_1, x_2) = \eta_{A_1}(x_1) \vee \eta_{A_2}(x_2)$
- (c)  $\nu_A(x_1, x_2) = \nu_{A_1}(x_1) \wedge \nu_{A_2}(x_2)$

(ii)  $\forall (x_1y_1) \in E_1, \forall (x_2y_2) \in E_2$

- (a)  $\mu_B(x_1, x_2)(y_1, y_2) = \mu_{B_1}(x_1y_1) \vee \mu_{B_2}(x_2y_2)$
- (b)  $\eta_B(x_1, x_2)(y_1, y_2) = \eta_{B_1}(x_1y_1) \vee \eta_{B_2}(x_2y_2)$
- (c)  $\nu_B(x_1, x_2)(y_1, y_2) = \nu_{B_1}(x_1y_1) \wedge \nu_{B_2}(x_2y_2)$ .

**Definition 38.** The lexicographic product  $G_1 \cdot G_2$  of two picture fuzzy graph  $G_1 = (A_1, B_1)$  and  $G_2 = (A_2, B_2)$  is defined as a pair  $(A, B)$ , where  $A = (\mu_A, \eta_A, \nu_A)$  is a picture fuzzy set on  $V = V_1 \times V_2$  and  $B = (\mu_B, \eta_B, \nu_B)$  is an another picture fuzzy set on  $E = \{(x, x_2) (x, y_2) | x \in V_1, x_2y_2 \in E_2\} \cup \{(x_1, x_2) (y_1, y_2) | x_1y_1 \in E_1, x_2y_2 \in E_2\}$  which satisfies the followings:

(i)  $\forall (x_1, x_2)$

- (a)  $\mu_A(x_1, x_2) = \mu_{A_1}(x_1) \vee \mu_{A_2}(x_2) = \eta_{A_1}(x_1) \vee \eta_{A_2}(x_2) = \nu_{A_1}(x_1) \wedge \nu_{A_2}(x_2)$

(ii)  $\forall x \in V_1, \forall (x_2y_2) \in E_2$

- (a)  $\mu_B(x, x_2)(x, y_2) = \mu_{A_1}(x) \vee \mu_{B_2}(x_2y_2)$
- (b)  $\eta_B(x, x_2)(x, y_2) = \eta_{A_1}(x) \vee \eta_{B_2}(x_2y_2)$
- (c)  $\nu_B(x, x_2)(x, y_2) = \nu_{A_1}(x) \wedge \nu_{B_2}(x_2y_2)$

(iii)  $\forall x_1y_1 \in E_1, \forall (x_2y_2) \in E_2$

- (a)  $\mu_B(x_1, x_2)(y_1, y_2) = \mu_{B_1}(x_1y_1) \vee \mu_{B_2}(x_2y_2)$
- (b)  $\eta_B(x_1, x_2)(y_1, y_2) = \eta_{B_1}(x_1y_1) \vee \eta_{B_2}(x_2y_2)$
- (c)  $\nu_B(x_1, x_2)(y_1, y_2) = \nu_{B_1}(x_1y_1) \wedge \nu_{B_2}(x_2y_2)$

**Definition 39.** The strong product  $G_1G_2$  of two picture fuzzy graph  $G_1 = (A_1, B_1)$  and  $G_2 = (A_2, B_2)$  is defined as a pair  $(A, B)$ , where  $A = (\mu_A, \eta_A, \nu_A)$  is a picture fuzzy set on  $V = V_1 \times V_2$  and  $B = (\mu_B, \eta_B, \nu_B)$  is an another picture fuzzy set on  $E = \{(x, x_2) (x, y_2) | x \in V_1, x_2y_2 \in E_2\} \cup \{(x_1, z) (y_1, z) | z \in V_2, x_1y_1 \in E_1\} \cup \{(x_1, x_2) (y_1, y_2) | x_1y_1 \in E_1, x_2y_2 \in E_2\}$  which satisfies the followings:

(i)  $\forall (x_1, x_2) \in V_1 \times V_2$

- (a)  $\mu_A(x_1, x_2) = \mu_{A_1}(x_1) \vee \mu_{A_2}(x_2)$
- (b)  $\eta_A(x_1, x_2) = \eta_{A_1}(x_1) \vee \eta_{A_2}(x_2)$
- (c)  $\nu_A(x_1, x_2) = \nu_{A_1}(x_1) \wedge \nu_{A_2}(x_2)$

(ii)  $\forall x \in V_1, \forall (x_2y_2) \in E_2$

- (a)  $\mu_B(x, x_2)(x, y_2) = \mu_{A_1}(x) \vee \mu_{B_2}(x_2y_2)$
- (b)  $\eta_B(x, x_2)(x, y_2) = \eta_{A_1}(x) \vee \eta_{B_2}(x_2y_2)$
- (c)  $\nu_B(x, x_2)(x, y_2) = \nu_{A_1}(x) \wedge \nu_{B_2}(x_2y_2)$

(iii)  $\forall x_1y_1 \in E_1, \forall (x_2y_2) \in E_2$

- (a)  $\mu_B(x_1, x_2)(y_1, y_2) = \mu_{B_1}(x_1y_1) \vee \mu_{B_2}(x_2y_2)$
- (b)  $\eta_B(x_1, x_2)(y_1, y_2) = \eta_{B_1}(x_1y_1) \vee \eta_{B_2}(x_2y_2)$
- (c)  $\nu_B(x_1, x_2)(y_1, y_2) = \nu_{B_1}(x_1y_1) \wedge \nu_{B_2}(x_2y_2)$

(iv)  $\forall x_1y_1 \in E_1, z \in V_2$

- (a)  $\mu_B(x_1, z)(y_1, z) = \mu_{B_1}(x_1y_1) \vee \mu_{A_2}(z)$
- (b)  $\eta_B(x_1, z)(y_1, z) = \eta_{B_1}(x_1y_1) \vee \eta_{A_2}(z)$
- (c)  $\nu_B(x_1, z)(y_1, z) = \nu_{B_1}(x_1y_1) \wedge \nu_{A_2}(z)$

## 6. Application of Picture Fuzzy Graph in Social Network

The popularity of social networks like Facebook, Twitter, WhatsApp, ResearchGate, Instagram, and LinkedIn is increasing day by day. They are well known platforms for connecting huge number of people in everywhere in the world. We generally exchange various types of information and issues in social network. It helps us in online marketing (e-commerce and e-business), efficient social and political campaigns, future events, connecting to clients. Social networks are also used as important tools for public awareness by sending information quickly to a wide audience about any natural disaster and terrorist/criminal attack.

A social network is a collection of nodes and links. The nodes are used to represent individual, groups, country, organizations, places, enterprises, etc., and links are used to describe the relationships between nodes. We generally use classical graph to represent the social network where actors are represented by vertices and the relations/flows between vertices are represented as edges. Several manuscripts have been published on the social networks. However, social network cannot be modeled properly by classical graph. Since, in a classical graph, all vertices have equal importance. Due to this reason, all social units (individual or organization) in present social networks are considered of equal importance. But in real life, all social units do not have same the importance. Similarly, all edges (relationships) have equal strength in classical graph. In all existing social networks, it is considered that the strength of relationship between two social units are equal, but in real life it may not be true. Samanta and Pal [37] have introduced an idea to model the social network using a type 1 fuzzy graph. Many researchers [38–40] believed that these uncertainties phenomena can be modeled by a fuzzy graph. But type 1 fuzzy graph are unable to capture more complex relational states between nodes. Since the membership degree of node and edges are determined by human perception. This motivate us to introduce a new social network model using picture fuzzy graph. We define this social network as a picture fuzzy social network.

In a picture fuzzy social network, an account of a person or organization, i.e., a social unit is represented by node and if there is a relationship or flow between two social units then they are connected by one arc. In reality, each node, i.e., a social unit (person or individual) has some good, neutral and bad activities. We represent the good, neutral and bad activities using the good, neutral and bad membership values of the node and similarly the good, neutral and bad membership degree of the edge can be used to describe the strength of relationship between two nodes. For example, three persons have good knowledge of some activities such as educational activity and teaching methodology. However, they have no knowledge in some activities such as administration and finance and they also have very bad knowledge in some activities such as health condition and food habit. We can easily represent this three types of node and edge membership degree using picture fuzzy set which consists each element has three membership values. This type of social network is a real life example of a picture fuzzy graph. Centrality is one of the most key ideas in social networking which finds the node effect on the social network. The centrality of a node is more central from other node. Central person is more close than other person and he can send or access more information. Freeman [41] introduced three different type of measures: degree, closeness, and betweenness of any node centrality. The degree of centrality finds the linkage of a social unit between the others social units. It basically determines the involvement of the social unit (person) in the social network. We can find this degree value any picture node as defined (28). The betweenness determines the number of paths of communication between any two social units through a unit and closeness of any node is as the inverse sum of shortest path length [42,43] to all other social nodes from a specified node. Let  $I(i, j)$  represents the shortest path length from node  $i$  to  $j$ . The diameter of a social network is defined as the longest distance between two nodes in the network. It is described as follows.

$$diameter = \max_{i,j} I(i, j). \quad (22)$$

In this study, we use picture fuzzy set to represent the arc length of a social network. The problem of finding shortest path between two social units is a fundamental and important criteria to find the closeness, betweenness and diameter of a social network. This picture social network model is more flexible and efficient than classical social network model.

The online social network can be represented by a directed or undirected weighted picture fuzzy graph. Let  $\tilde{G}_{un} = (V, \tilde{E}_{un})$  be an undirected picture fuzzy graph. We can define an undirected picture fuzzy social network as a picture fuzzy relational structure  $\tilde{G}_{un} = (V, \tilde{E}_{un})$ , where  $V = \{v_1, v_2, \dots, v_n\}$

represents a non-empty set of picture vertices or actors or nodes, and  $\tilde{E}_{un} = \begin{pmatrix} \tilde{e}_{11} & \dots & \tilde{e}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{e}_{n1} & \dots & \tilde{e}_{nn} \end{pmatrix}$

represents an undirected picture fuzzy relation on  $V$ . A small example of undirected picture fuzzy social network is shown in Figure 5. For undirected picture fuzzy social network where arcs are simply an absent or present undirected picture fuzzy relation with no other information attached.

Let  $\tilde{G}_{dn} = (V, \tilde{E}_{dn})$  be a directed picture fuzzy graph. We can describe a directed picture fuzzy social network as a picture fuzzy relational structure  $\tilde{G}_{dn} = (V, \tilde{E}_{dn})$ , where  $V = \{v_1, v_2, \dots, v_n\}$

denotes a non-empty set of picture fuzzy nodes, and  $\tilde{E}_{dn} = \begin{pmatrix} \tilde{e}_{11} & \dots & \tilde{e}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{e}_{n1} & \dots & \tilde{e}_{nn} \end{pmatrix}$  denotes an undirected picture fuzzy relation on  $V$ .

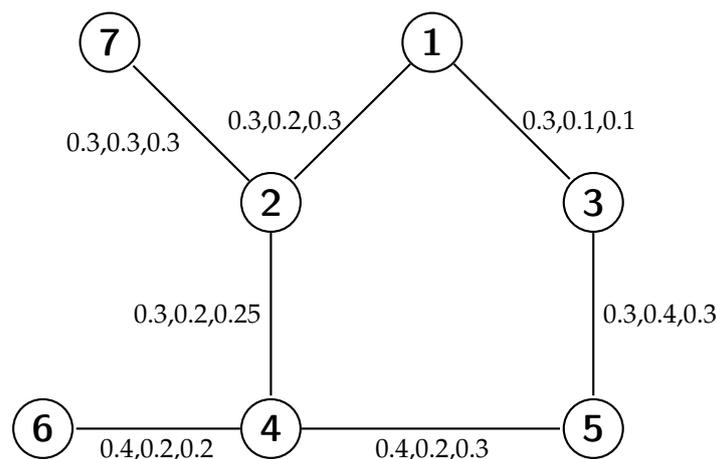


Figure 5. Undirected picture fuzzy social network.

The directed picture fuzzy relation is considered in directed picture fuzzy social network. Directed picture fuzzy graph is more efficient to model the social network, because where arcs are considered with directed picture fuzzy relation would contain more information. The values of  $\tilde{e}_{ij}$  and  $\tilde{e}_{ji}$  are equal in undirected picture fuzzy social network. However,  $\tilde{e}_{ij}$  and  $\tilde{e}_{ji}$  are not equal in directed picture fuzzy social network. A small example of a directed picture fuzzy social network is shown in Figure 6. Let  $\tilde{G}_{dn} = (V, \tilde{E}_{dn})$  be a directed picture fuzzy social network and the picture fuzzy sets are used to describe the arc lengths of  $\tilde{G}_{dn}$ . The sum of the lengths of the arcs that are adjacent to a social node  $v_i$  is called the picture fuzzy in degree centrality of the node  $v_i$ . The picture in degree centrality of node  $v_i$ ,  $\tilde{d}_I(v_i)$ , is described as follows.

$$\tilde{d}_I(v_i) = \sum_{j=1, j \neq i}^n \tilde{e}_{ji}. \tag{23}$$

The sum of the lengths of the arcs that that are adjacent from social node  $v_i$  is called the picture out degree centrality of  $v_i$ . The picture out degree centrality of node  $v_i$ ,  $\tilde{d}_o(v_i)$ , is described as follows.

$$\tilde{d}_o(v_i) = \sum_{j=1, j \neq i}^n \tilde{e}_{ji}. \tag{24}$$

Here, the symbol  $\Sigma$  is an addition operation of picture fuzzy set and  $\tilde{e}_{ji}$  is a picture fuzzy set associated with the arc  $(i, j)$ . The sum of picture in degree centrality and picture out of degree centrality of node  $v_i$  is called picture degree centrality (PDC) of  $v_i$ .

$$\tilde{d}(v_i) = \tilde{d}_I(v_i) \oplus \tilde{d}_o(v_i), \tag{25}$$

$\oplus$  is an addition operation of picture fuzzy set.

Let  $\tilde{G}_{d7} = (V, \tilde{E}_{d7})$  be a directed picture fuzzy social network of the a research Team, where  $V = \{v_1, v_2, \dots, v_7\}$  represents a collection of seven researchers,  $\tilde{E}_{d7}$  represents directed picture fuzzy relation between the seven researchers. This social network is shown in Figure 6. We determine the picture fuzzy in degree, picture fuzzy in degree and picture fuzzy degree centrality of the research term using (23)–(25). This three degree values are shown in Table 1. We use the raking method [44,45] of picture fuzzy set to compare the different degree values. Based on the ranking of picture fuzzy set, research (node) 4 has got highest score value of picture fuzzy in degree centrality. It means that researcher 4 has higher acceptance and good interpersonal relationship in network. The research 2 has got the highest score value of picture fuzzy out degree centrality. It describes that node 2 can nominate many other researchers.

Table 1. The picture fuzzy degree centrality of research team.

Researchers	Picture Fuzzy in Degree Centrality	Picture Fuzzy out Degree Centrality	Picture Fuzzy Degree Centrality
1	(0,0,0)	(0.3,0.2,0.3)	(0.3,0.2,0.3)
2	(0.3,0.2,0.3)	(0.3,0.3,0.3)	(0.3,0.3,0.3)
3	(0.3,0.1,0.1)	(0.3,0.4,0.3)	(0.3,0.4,0.3)
4	(0.3,0.2,0.3)	(0,0,0)	(0.3,0.2,0.3)
5	(0.3,0.4,0.3)	(0.4,0.2,0.3)	(0.3,0.4,0.3)
6	(0,0,0)	(0.4,0.2,0.2)	(0.4,0.2,0.2)
7	(0.3,0.3,0.3)	(0,0,0)	(0.3,0.3,0.3)

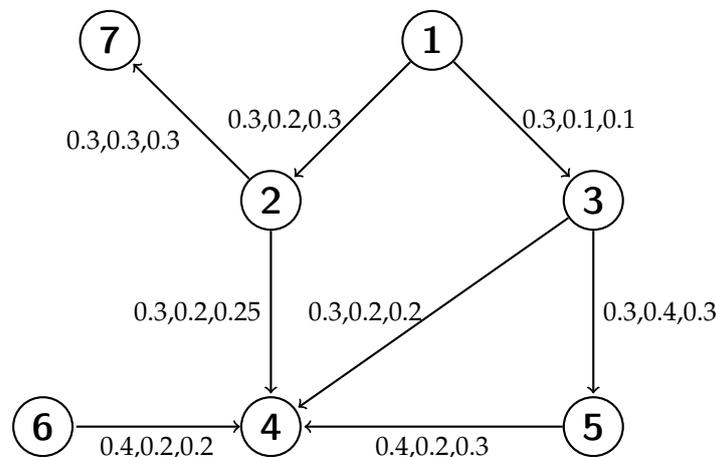


Figure 6. Directed picture fuzzy social network.

In this study, we have provided one simple numerical examples of picture fuzzy graph to represent one small social network. The small sized examples are very much helpful to understand the advantage of our proposed model. Social networks are based on millions of users and big data paradigm. Therefore, as future work, we need to model a large scale practical social network using the picture fuzzy graph and to compute the closeness, betweenness and diameter of the social network. Furthermore, we wish to introduce some heuristic algorithms to find those measurements of any large scale practical social network. Despite the need for future study, the proposed model discussed in this study is an important initial contributions to picture fuzzy graph and social network analysis under uncertain environment.

## 7. Conclusions

Fuzzy graph is an important tool for modeling many uncertain real world decision making problems in different field, e.g., operations research, computer science, optimization problems, number theory, algebra, geometry, etc. To deal with the complex real life problems, a number of generalizations of fuzzy graph have been studied. The picture fuzzy set is a direct extension of fuzzy set and intuitionistic fuzzy set. The main objective of this paper is to introduce the idea of picture fuzzy graph and its different operation. In this paper, first we propose the definition of a picture fuzzy graph based on the picture fuzzy relation. We also describe some different types of picture fuzzy graphs such as regular picture fuzzy graph, strong picture fuzzy graph, complete picture fuzzy graph, and complement picture fuzzy graph. We describe some different types of isomorphic picture fuzzy graph. Six different operations on picture fuzzy graph, viz. Cartesian product, composition, join, direct product, lexicographic and strong product are described in this paper. The picture fuzzy graph can increase flexibility, efficiency, precision and comparability to model the complex real life scenarios system compared to the classical fuzzy graph models. We have also described a model to represent the social network using picture fuzzy graph and we also introduce some centrality measure called picture fuzzy in degree centrality, picture fuzzy out degree centrality and picture fuzzy out degree centrality which are applicable to the picture fuzzy social network. The idea of picture fuzzy graph can be applied in several areas of database system, computer network, social network, transportation network and image processing.

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