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Article

On Pre-Commutative Algebras

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Abstract: In this paper, we introduce the notions of generalized commutative laws in algebras, and investigate their relations by using Smarandache disjointness. Moreover, we show that every pre-commutative BCK-algebra is bounded.

Keywords: abelian; (left-, right-) pre-commutative; commutative; d/BCK-algebra; Smarandache disjoint

1. Introduction

If $(X, *)$ is an algebra (or binary system, groupoid), then one is often interested in observing how properties or axioms are maintained in closely related groupoids. Such is the case when we are dealing with homomorphic images and isotopies, for example [1]. Allen et al. [2] introduced the notion of a deformation function, and obtained several properties related to d/BCK-algebras. Allen et al. [3] introduced the concept of several types of groupoids related to the semigroup by using a deformation function, i.e., twisted semigroups, and obtained several properties of generalized associative laws. We generalize the notion of commutativity in several algebras by using the functions, i.e., commutative, left-pre-commutative, right-pre-commutative and pre-commutative conditions, and we develop the notions with aid of Smarandache disjointness and describe some relations between BCK-algebras and a pre-commutative axiom.

2. Preliminaries

A *d-algebra* [4] is a non-empty set X with a constant 0 and a binary operation " $*$ " satisfying the following axioms: (I) $x * x = 0$, (II) $0 * x = 0$, (III) $x * y = 0$ and $y * x = 0$ imply $x = y$, for all $x, y \in X$. For more detailed information we refer to [5–7].

A BCK-algebra [8] is a d-algebra X satisfying the following additional axioms: (IV) $((x * y) * (x * z)) * (z * y) = 0$, (V) $(x * (x * y)) * y = 0$ for all $x, y, z \in X$.

A d/BCK-algebra $(X, *, 0)$ is said to be *bounded* if there exists an element $1 \in X$ such that $x * 1 = 0$ for any $x \in X$. A d/BCK-algebra $(X, *, 0)$ is said to be *commutative* if $x * (x * y) = y * (y * x)$ for all $x, y \in X$, i.e., $x \wedge y = y \wedge x$. A d/BCK-algebra $(X, *, 0)$ is said to be *positive implicative* if if $(x * z) * (y * z) = (x * y) * z$ for all $x, y, z \in X$. For more detailed information we refer to [8].

We denote d/BCK-algebras or another groupoids (algebras) by $(X, *, 0)$ usually when we need to emphasize the constant 0, but sometimes we denote it by $(X, *)$ in short.

Theorem 1. [8] *For a bounded commutative BCK-algebra* $(X, *, 0)$, *we have*

(i) $NNx = x$ *for all* $x \in X$,
(ii) $Nx \vee Ny = N(x \wedge y), Nx \wedge Ny = N(x \vee y)$ *for all* $x, y \in X$,
(iii) $Nx * Ny = y * x$ *for all* $x, y \in X$.

Given a d/BCK-algebra $(X, *, 0)$, we define a relation "$\leq$" on X by

$$x \leq y \Leftrightarrow x * y = 0$$

In BCK-algebras, the relation $\leq$ is a partial order, and so $(X, \leq)$ forms a partially ordered set with the least element 0, but it need not be a partial order in d-algebras.

Theorem 2. [8] *Let $(X, *, 0)$ be a BCK-algebra. Then*

(i) $x * 0 = x$ *for all* $x \in X$,
(ii) $(x * y) * z = (x * z) * y$ *for all* $x, y, z \in X$.

3. Pre-Commutative Algebras

Given a groupoid (or an algebra) $(X, *)$, it is said to be

(i) *abelian* if $x * y = y * x$;
(ii) *left-pre-commutative* if $x * y = \varphi(y) * x$ for some map $\varphi : X \to X$;
(iii) *right-pre-commutative* if $x * y = y * \varphi(x)$ for some map $\varphi : X \to X$;
(iv) *pre-commutative* if $x * y = \varphi(y) * \varphi(x)$ for some map $\varphi : X \to X$,

for all $x, y \in X$.

Example 1. (i). *Define a binary operation "$*$" on a set* $\mathbf{N} := \{0, 1, 2, 3, \cdots\}$ *by the following table.*

*	0	1	2	3	4	5	6	7	8	⋯⋯
0	0	1	2	3	4	5	6	7	8	⋯⋯
1	0	0	1	2	3	4	5	6	7	⋯⋯
2	1	0	0	1	2	3	4	5	6	⋯⋯
3	2	1	0	0	1	2	3	4	5	⋯⋯
4	3	2	1	0	0	1	2	3	4	⋯⋯
5	4	3	2	1	0	0	1	2	3	⋯⋯
6	5	4	3	2	1	0	0	1	2	⋯⋯
7	6	5	4	3	2	1	0	0	1	⋯⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮⋮

Then it is easy to see that $m * n = (n+1) * m$ for any $m, n \in \mathbf{N}$. If we let $\varphi(n) := n + 1$, then $m * n = n * (n+1) * m = \varphi(n) * m$ for any $m, n \in \mathbf{N}$. Hence $(\mathbf{N}, *)$ is a left-pre-commutative groupoid.
(ii). *If we define $m \oplus n := n * m$ in (i), then $(\mathbf{N}, \oplus)$ is a right-pre-commutative groupoid, since $m \oplus n = n * m = \varphi(m) * n = n \oplus \varphi(m)$ for any $m, n \in \mathbf{N}$.*
(iii). *Define a binary operation "$*$" on a set $X := \{0, 1, 2, 3, 4, 5\}$ by the following table.*

*	0	1	2	3	4	5
0	0	1	2	3	4	5
1	0	0	1	2	3	4
2	1	0	0	1	2	3
3	2	1	0	0	1	2
4	3	2	1	0	0	1
5	4	3	2	1	0	0

Then it is easy to see that $m * n = (5 - n) * (5 - m)$ for all $m, n \in X$. If we define a map $\varphi : X \to X$ by $\varphi(0) = 5, \varphi(1) = 4, \varphi(2) = 3, \varphi(3) = 2, \varphi(4) = 1$ and $\varphi(5) = 0$, then $m * n = (5 - n) * (5 - m) = \varphi(n) * \varphi(m)$ for all $m, n \in X$. Hence $(X, *)$ is a pre-commutative groupoid.

Remark 1. *In Example 1-(i), $(\mathbf{N}, *)$ is not a right-pre-commutative groupoid. In fact, if there exists a map $\psi : (\mathbf{N}, *) \to (\mathbf{N}, *)$ such that $m * n = n * \psi(m)$ for any $m, n \in \mathbf{N}$, then $0 = 0 * 0 = 0 * \psi(0)$. From the table of Example 1-(i), we obtain $\psi(0) = 0$. It follows that $1 = 0 * 1 = 1 * \psi(0) = 1 * 0 = 0$, a contradiction. This shows that Example 1-(i) is a left-pre-commutative groupoid which is not a right-pre-commutative groupoid. Similarly, Example 1-(ii) is a right-pre-commutative groupoid which is not a left-pre-commutative groupoid.*

Remark 2. *In Example 1-(i), $(\mathbf{N}, *)$ is not a pre-commutative groupoid. In fact, if there exists a map $\psi : (\mathbf{N}, *) \to (\mathbf{N}, *)$ such that $m * n = n * \psi(m)$ for any $m, n \in \mathbf{N}$, then $0 = 1 * 0 = \psi(0) * \psi(1)$. From the table of Example 1-(i), we have either $\psi(0) = \psi(1)$ or $\psi(0) = \psi(1) + 1$. If $\psi(0) = \psi(1)$, then $1 = 0 * 1 = \psi(1) * \psi(0) = 0$, a contradiction. Hence we have $\psi(0) = \psi(1) + 1$. Now, since $1 = 2 * 0 = \psi(0) * \psi(2)$, from the table of Example 1-(i), we have either $\psi(0) = \psi(2) + 1$ or $\psi(2) = \psi(0) + 2$. If we take $\psi(2) = \psi(0) + 2$, then $2 = 0 * 2 = \psi(2) * \psi(0) = [\psi(0) + 2] * \psi(0) = 1$ from the table, a contradiction. Hence we have $\psi(0) = \psi(2) + 1$. Since $3 = 0 * 3 = \psi(3) * \psi(0)$, from the table, we obtain either $\psi(0) = \psi(3) + 3$ or $\psi(3) = \psi(0) + 4$. If $\psi(3) = \psi(0) + 4$, then $2 = 3 * 0 = \psi(0) * \psi(3) = \psi(0) * [\psi(0) + 4] = 4$, a contradiction. Hence we have $\psi(0) = \psi(3) + 3$. In this fashion, we obtain $\psi(0) = \psi(1) + 1 = \psi(2) + 1 = \psi(3) + 3 = \psi(4) + 4 = \cdots$. It follows that $\psi(0) \geq n$ for all $n \in \mathbf{N}$, a contradiction. This shows that Example 1-(i) is not a pre-commutative groupoid.*

Example 2. *Let $X := \{0, 1, 2, 3\}$ be a set. If we define a map $\varphi : X \to X$ by $\varphi(0) = 0, \varphi(1) = 2, \varphi(2) = 3$ and $\varphi(3) = 1$, then the algebra $(X, *)$ with the following table is left-pre-commutative, but not abelian.*

$*$	0	1	2	3
0	α	β	β	β
1	β	γ	δ	γ
2	β	γ	γ	δ
3	β	δ	γ	γ

where $\alpha, \beta, \gamma, \delta$ are distinct elements of X. For example, if we let $\alpha = 0, \beta = 2, \gamma = 1, \delta = 3$, then the algebra $(X, *)$ with the following table is left-pre-commutative, but not abelian.

$*$	0	1	2	3
0	0	2	2	2
1	2	1	3	1
2	2	1	1	3
3	2	3	1	1

Example 3. *Given a set $X := \{0, 1, 2, 3\}$ with a map φ defined in Example 2, the algebra $(X, *)$ with the following table is right-pre-commutative, but not abelian, since $1 * 2 = \gamma$ and $\varphi(2) * \varphi(1) = 3 * 2 = \delta$.*

$*$	0	1	2	3
0	α	β	β	β
1	β	γ	γ	δ
2	β	δ	γ	γ
3	β	γ	δ	γ

where $\alpha, \beta, \gamma, \delta$ are distinct elements of X. For example, the algebra $(X, *)$ with the following table is right-pre-commutative, but not abelian.

$*$	0	1	2	3
0	0	1	1	1
1	1	2	2	3
2	1	3	2	2
3	1	2	3	2

Proposition 1. *If $(X, *)$ is abelian, then it is (left-, right-) pre-commutative.*

Proof. Take $\varphi := id_X$. $\square$

Theorem 3. *If the groupoid $(X, *, e)$ is a group, then (left-, right-) pre-commutativity implies abelian property.*

Proof. Let $(X, *, e)$ be a group which is left-pre-commutative. Then there exists a map $\psi : X \to X$ such that $x * y = \psi(y) * x$ for any $x, y \in X$. It follows that $x = x * e = \psi(e) * x$. By cancelation law we obtain $\psi(e) = e$. Also, $e = x * x^{-1} = \psi(x^{-1}) * x$ and hence $\psi(x^{-1}) = x^{-1}$ for all $x \in X$, i.e., $\psi(x) = x$ for all $x \in X$. Hence $x * y = \psi(y) * x = y * x$ for all $x, y \in X$, i.e., $(X, *, e)$ is abelian.

If $(X, *, e)$ is a group which is right-pre-commutative, then $x * y = y * \varphi(x)$ for some $\varphi : X \to X$. It follows that $x = x * e = e * \varphi(x) = \varphi(x)$ and hence $\varphi(x) = x$ for all $x \in X$. Hence $x * y = y * \varphi(x) = y * x$ for all $x, y \in X$, i.e., $(X, *, e)$ is abelian.

If $(X, *, e)$ is a group which is pre-commutative, then

$$e = x * x^{-1} = \xi(x^{-1}) * \xi(x)$$

for some $\xi : X \to X$. Since $(X, *, e)$ is a group, we have $\xi(x)^{-1} = \xi(x^{-1})$. It follows that $(\xi(e))^{-1} = \xi(e^{-1}) = \xi(e)$. Since $x = x * e = \xi(e) * \xi(x) = (\xi(e))^{-1} * \xi(x)$, we obtain

$$\begin{aligned} \xi(e) * x &= \xi(e) * [(\xi(e)^{-1} * \xi(x)] \\ &= [\xi(e) * (\xi(e))^{-1}] * \xi(x) \\ &= e * \xi(x) \\ &= \xi(x). \end{aligned}$$

Let $a := \xi(e)$. Then $a^2 = e$ and

$$\begin{aligned} (a * y) * (a * x) &= (\xi(e) * y) * (\xi(e) * x) \\ &= \xi(y) * \xi(x) \\ &= x * y. \end{aligned}$$

Thus

$$\begin{aligned} a * x &= (a * y)^{-1} * (x * y) \\ &= y^{-1} * a^{-1} * x * y \\ &= y^{-1} * (a * x) * y. \end{aligned}$$

Since x is arbitrary, $a * x$ is arbitrary. If we let $u := a * x$, then $u = a * x = y^{-1} * (a * x) * y = y^{-1} * u * y$, i.e., $y * u = u * y$, i.e., $(X, *, e)$ is abelian. $\square$

Let K be a field. A groupoid $(K, *)$ is said to be *linear* if $x * y := A + Bx + Cy$ for any $x, y \in K$, where A, B, C are (fixed) elements of K.

Proposition 2. *If a linear groupoid $(K, *)$ is left-pre-commutative, then it is abelian.*

Proof. Since $(K, *)$ is left-pre-commutative, there exists a map $\varphi : X \to X$ such that $x * y = \varphi(y) * x$ for any $x, y \in K$, i.e.,

$$A + Bx + Cy = A + B\varphi(y) + Cx. \tag{1}$$

If we let $x := 0, y := 1$ in (1) consequently, then $A + Cy = A + B\varphi(y)$ and $C = B\varphi(1)$. If $B = C = 0$, then $(K, *)$ is trivially abelian. If $B \neq 0$, then $\varphi(y) = \frac{C}{B}y$. By (1) we have $A + Bx + Cy = A + Cy + Cx$, proving $B = C$. Thus, $x * y = A + B(x + y) = y * x$. Similarly, if $C \neq 0$, then $B = C$, proving $(K, *)$ is abelian. □

Proposition 3. *If a linear groupoid* $(K, *)$ *is right-pre- commutative, then it is abelian.*

Proof. The proof is similar to Proposition 2, and we omit it. □

Note that, in a linear groupoid $(K, *)$, the "pre-commutativity" does not imply the "abelian". In fact, if we define $x * y := A + B(x - y)$ for any $x, y \in K$, where $A, B \neq 0$ in K, then it is pre-commutative, but not abelian.

Proposition 4. *Let K be a field. Define a binary operation "$*$" on K by*

$$x * y := x(x - y)$$

for all $x, y \in K$*. If* $(K, *)$ *is pre-commutative, then* $K \cong Z_2 = \{0, 1\}$*.*

Proof. It is easy to show that $(K, *, 0)$ is a d-algebra. Assume that $(K, *)$ is pre-commutative. Then there exists a map $\varphi : K \to K$ such that $x * y = \varphi(y) * \varphi(x)$ for any $x, y \in K$. It follows that

$$x * y = \varphi(y) * \varphi(x) = \varphi(y)[\varphi(y) - \varphi(x)] \tag{2}$$

If we let $x := 0$ in (2), then $0 = 0 * y = \varphi(y)[\varphi(y) - \varphi(0)]$ for all $y \in K$. Then either $\varphi(y) = 0$ or $\varphi(y) - \varphi(0) = 0$ for all $y \in K$. Assume that $\varphi(y_0) \neq 0$ for some $y_0 \in K$. Then $\varphi(y_0) - \varphi(0) = 0$, i.e., $\varphi(y_0) = \varphi(0)$. Let $u := \varphi(0)$. It follows that $\varphi(y) \in \{0, u\}$ for any $y \in X$. This means that $|\varphi(K)| \leq 2$ and $|\varphi(K) \times \varphi(K)| \leq 4$. Since $(K, *)$ is pre-commutative, $x * y = \varphi(y) * \varphi(x)$ has at most 4 elements, i.e., $|K \times K| \leq 4$. Thus $K \cong Z_2 = \{0, 1\}$ is a possible field. If we let $\varphi(0) := 1$ and $\varphi(1) := 0$, then $(K, *) \cong Z_2 = \{0, 1\}$. □

4. Smarandache -Disjoint and Pre-Commutativity

Allen et al. [9] introduced the notion of a Smarandache disjointness in algebras. We restate the notion of a Smarandache disjointness for its clarification. Simply, two algebras $(X, *)$ and $(X, \circ)$ are said to be *Smarandache disjoint* [9] if we add some axioms of an algebra $(X, *)$ to an algebra $(X, \circ)$, then the algebra $(X, \circ)$ becomes a trivial algebra, i.e., $|X| = 1$, or if we add some axioms of an algebra $(X, \circ)$ to an algebra $(X, *)$, then the algebra $(X, \circ)$ becomes a trivial algebra, i.e., $|X| = 1$. Note that if we add an axiom (A) of an algebra $(X, *)$ to another algebra $(X, \circ)$, then we replace the binary operation "$\circ$" in (A) by the binary operation "$*$". A groupoid $(X, *)$ is said to be a *left-zero-semigroup* [10] if $x * y = x$ for any $x, y \in X$.

Proposition 5. *Left-zero-semigroups and (left-, right-) pre-commutative algebras are Smarandache disjoint.*

Proof. Assume a left-zero-semigroup $(X, *)$ is left-pre- commutative. Then $x = x * y = \varphi(y) * x = \varphi(y)$ for any $x, y \in X$ for some map $\varphi : X \to X$, which implies $\varphi(y) = x$ for all $x, y \in X$. This shows that $|X| = 1$, a contradiction.

Assume a left-zero-semigroup $(X, *)$ is right-pre-commutative. Then $x = x * y = y * \varphi(x) = y$ for any $x, y \in X$ for some map $\varphi : X \to X$. This shows that $|X| = 1$, a contradiction.

Assume a left-zero-semigroup $(X, *)$ is pre-commutative. Then $x = x * y = \varphi(y) * \varphi(x) = \varphi(y)$ for any $x, y \in X$ for some map $\varphi : X \to X$, which implies $\varphi(y) = x$ for all $x, y \in X$. This shows that $|X| = 1$, a contradiction. □

Proposition 6. *Left-pre-commutative algebras and d-algebras are Smarandache disjoint.*

Proof. Let $(X, *, 0)$ be a left-pre-commutative d-algebra. Then $x * y = \varphi(y) * x$ for any $x, y \in X$ for some map $\varphi : X \to X$. If we let $x := 0$, then $0 = 0 * y = \varphi(y) * 0$. Since $(X, *, 0)$ is d-algebra, by applying (I), (II), we obtain $\varphi(y) = 0$ for any $y \in X$. Hence $x * y = \varphi(y) * x = 0 * x = 0$ for any $x, y \in X$, which implies $|X| = 1$ by (III), proving the proposition. □

Theorem 4. *Non-bounded d-algebras and right-pre-commutative algebras are Smarandache disjoint.*

Proof. Let $(X, *, 0)$ be a non-bounded right-pre-commutative d-algebras. Then there exists a map $\varphi : X \to X$ such that

$$x * y = y * \varphi(x) \tag{3}$$

for any $x, y \in X$. If we let $x := 0$ in (3), then $0 = 0 * y = y * \varphi(0)$ for any $y \in X$. We claim that $\varphi(0) = 0$. In fact, if $\varphi(0) \neq 0$, then $y \leq \varphi(0)$ for any $y \in X$, i.e., X is bounded, a contradiction. Hence $0 = 0 * y = y * \varphi(0) = y * 0$ for any $y \in X$, which shows that $y = 0$ for any $y \in X$, i.e., $|X| = 1$. □

5. *BCK*-Algebras and a Pre-Commutativity

Theorem 5. *If a BCK-algebra $(X, *, 0)$ is pre-commutative, then it is bounded.*

Proof. If a BCK-algebra $(X, *, 0)$ is pre-commutative, then $x * y = \varphi(y) * \varphi(x)$ for any $x, y \in X$ for some map $\varphi : X \to X$. It means that the mapping φ is order-reversing, i.e., $x * y = 0$ implies $\varphi(y) * \varphi(x) = 0$. Now, since $(X, *, 0)$ is a BCK-algebra, we have $0 * x = 0$ for all $x \in X$. It follows that $\varphi(x) * \varphi(0) = 0$ for all $x \in X$. This means that $\varphi(0)$ is the greatest element of the poset $(\varphi(X), \leq)$. We claim that $\varphi(0)$ is the greatest element of the poset $(X, \leq)$. Let $\alpha := \varphi(0)$. Since $x * 0 = x$ for all $x \in X$ and $(X, *, 0)$ is pre-commutative, we obtain $x = x * 0 = \varphi(0) * \varphi(x) = \alpha * \varphi(x)$ for any $x \in X$. It follows from Theorem 2-(ii) and (II) that $x * \alpha = (\alpha * \varphi(x)) * \alpha = (\alpha * \alpha) * \varphi(x) = 0 * \varphi(x) = 0$ for any $x \in X$. This shows that $\alpha = \varphi(0)$ is the greatest element of $(X, \leq)$, proving that $(X, *, 0)$ is bounded by $\varphi(0)$. □

The converse of Theorem 5 need not be true in general. See the following example.

Example 4. *Consider a BCK-algebra $X := \{0, 1, 2\}$ with the following table:*

$*$	0	1	2
0	0	0	0
1	1	0	0
2	2	2	0

Then $(X, *, 0)$ is a bounded (positive implicative) BCK-algebra (see [8], p. 243). By routine calculations we see that there is no map $\varphi : X \to X$ satisfying the condition of a pre-commutativity, i.e., $(X, *, 0)$ is a bounded BCK-algebra having no pre-commutative property.

The converse of Theorem 5 is also true if we add the condition of "abelian", i.e., $x * y = y * x$ for all $x, y \in X$.

Proposition 7. *Every bounded commutative BCK-algebra is pre-commutative.*

Proof. By Theorem 1-(iii), we have $x * y = Ny * Nx$ for all $x, y \in X$, which shows that $(X, *, 0)$ is pre-commutative. □

Proposition 8. *If $(X, *, 0)$ is a pre-commutative BCK-algebra, then*

(i) $x = 1 * \varphi(x)$ *for any* $x \in X$,
(ii) $\varphi(x) * \varphi(y) = (1 * \varphi(y)) * (1 * \varphi(x))$.

for some map $\varphi : X \to X$.

Proof. Since X is pre-commutative, by Theorem 5, X is bounded and $\varphi(0)$ is the greatest element of X, say $1 := \varphi(0)$. Since $x * 0 = x$ for all $x \in X$, we obtain $x = x * 0 = \varphi(0) * \varphi(x) = 1 * \varphi(x)$, for all $x \in X$, proving (i). Since X is pre-commutative, we obtain $\varphi(x) * \varphi(y) = y * x = (1 * \varphi(y)) * (1 * \varphi(x))$ by (i). □

Theorem 6. *Let $(X, *, 0)$ be a bounded commutative BCK-algebra. If a map $\varphi : X \to X$ satisfies the condition $x * y = \varphi(y) * \varphi(x)$ for all $x, y \in X$, then $\varphi(x) = Nx$ for all $x \in X$.*

Proof. By Proposition 7, if $(X, *, 0)$ is a bounded commutative BCK-algebra, then it is pre-commutative, i.e., $x * y = \varphi(y) * \varphi(x)$ for all $x, y \in X$. By Proposition 8-(i), we have $x = 1 * \varphi(x)$. Since $(X, *, 0)$ is commutative and bounded, by Theorem 2-(i), we have $Nx = 1 * x = 1 * (1 * \varphi(x)) = \varphi(x) * (\varphi(x) * 1) = \varphi(x) * 0 = \varphi(x)$ for all $x \in X$. This proves the theorem. □

6. Conclusions

We introduced the notions of generalized commutative laws in algebras, and investigated their roles in algebras, and found their interrelationships by using Smarandache disjointness. The notion of pre-commutative law applied to BCK-algebras, and obtained that if a BCK-algebra $(X, *, 0)$ is pre-commutative, then it is bounded. Moreover, we proved that, in a bounded commutative BCK-algebra $(X, *, 0)$, if a map $\varphi : X \to X$ satisfies the condition $x * y = \varphi(y) * \varphi(x)$ for all $x, y \in X$, then $\varphi(x) = Nx$ for all $x \in X$.

Allen et al. [3] have developed the generalization of an associative law, and we discussed the generalization of a commutative law by using (deformed) functions. This idea may apply to several axioms appeared in general algebraic structures, and will generalize several algebraic structures.

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