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Weak Partial b-Metric Spaces and Nadler's Theorem

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Abstract: The purpose of this paper is to define the notions of weak partial b-metric spaces and weak partial Hausdorff b-metric spaces along with the topology of weak partial b-metric space. Moreover, we present a generalization of Nadler's theorem by using weak partial Hausdorff b-metric spaces in the context of a weak partial b-metric space. We present a non-trivial example which show the validity of our result and an application to nonlinear Volterra integral inclusion for the applicability purpose.

Keywords: multivalued mappings; Hausdorff metric space; Nadler's theorem

MSC: 55M20; 47H10

1. Introduction

The famous Banach contraction principle has been generalized in many directions, whether by generalizing the contractive condition or by extending the domain of the function. Bakhtin [1] and Czerwik [2] introduced b-metric spaces generalizing the ordinary metric space and considering the problem of convergence of measurable functions with respect to measure; Czerwik [2] proved the variant of Banach contraction in b-metric spaces. Later on, many authors proved fixed point results for both single and multivalued mapping in the context of b-metric spaces (see also [2–13]).

Matthews [14] established the notion of a partial metric space and proved an analogue of Banach's principle in such spaces. The concept of partial Hausdorff metric was given by Aydi et al. [6] and they established a fixed point theorem for multivalued mappings in partial metric spaces. Excluding the idea of small self-distance, Heckmann [15] generalized the partial metric space to weak partial metric spaces (see more [16–22]).

Shukla [23] introduced the concept of the partial b-metric and proved some fixed point results. Beg [7] presented the idea of the almost partial Hausdorff metric and extended Nadler's theorem (Nadler [19]) to weak partial metric spaces.

The aim of this paper is to introduce the notion of the weak partial b-metric space, the $\mathcal{H}^+$-type partial Hausdorff b-metric and prove Nadler's theorem to weak partial b-metric spaces. An example and application to Volterra type integral inclusion to support our result will be given.

2. Preliminaries

Consistent with Beg [7], notion of weak partial metric and related concepts are as follows:

Definition 1. *[7] Let M be a nonempty set. A function $\varrho : M \times M \to \mathbb{R}^+$ is called weak partial metric if for all $s, t, z \in M$, following assertions hold:*

(WP1) $\varrho(s,s) = \varrho(s,t)$ *iff* $s = t$;
(WP2) $\varrho(s,s) \leq \varrho(s,t)$;
(WP3) $\varrho(s,t) = \varrho(t,s)$;
(WP4) $\varrho(s,t) \leq \varrho(s,z) + \varrho(z,t)$.

The pair (M, ϱ) is called weak partial metric space.

We refer [7] to readers for detail work in weak partial metric space.

Let $CB^{\varrho}(M)$ be the family of nonempty, closed and bounded subsets of a weak partial metric space (M, ϱ). Define

$$\varrho(x,U) = \inf\{\varrho(x,u), u \in U\}, \;\; \delta_{\varrho}(U,V) = \sup\{\varrho(u,V) : u \in U\}$$

and

$$\delta_{\varrho}(V,U) = \sup\{\varrho(v,U) : v \in V\},$$

where $U, V \in CB^{\varrho}(M)$ and $s \in M$. Also

$$\varrho(x,U) = 0 \Rightarrow \varrho^s(x,U) = 0,$$

where $\varrho^s(x,U) = \inf\{\varrho^s(x,u), u \in U\}$.

Remark 1. *[7] If $\phi \neq U \subseteq M$, then*

$$u \in \overline{U} \textit{ if and only if } \varrho(u,U) = \varrho(u,u).$$

Definition 2. *[7] Let (M, ϱ) be a weak partial metric space. For $U, V \in CB^{\varrho}(M)$, define*

$$\mathcal{H}_{\varrho}^{+}(U,V) = \frac{1}{2}\{\delta_{\varrho}(U,V) + \delta_{\varrho}(V,U)\}.$$

The mapping $\mathcal{H}_{\varrho}^{+} : CB^{\varrho}(M) \times CB^{\varrho}(M) \to [0,\infty)$, is called $\mathcal{H}_{\varrho}^{+}$-type Hausdorff metric induced by ϱ.

Proposition 1. *[7] Let (M, ϱ) be a weak partial metric space. For any $U, V, Y \in CB^{\varrho}(M)$, we have:*

(wh1) $\mathcal{H}_{\varrho}^{+}(U,U) \leq \mathcal{H}_{\varrho}^{+}(U,V)$;
(wh2) $\mathcal{H}_{\varrho}^{+}(U,V) = \mathcal{H}_{\varrho}^{+}(V,U)$;
(wh3) $\mathcal{H}_{\varrho}^{+}(U,V) \leq \mathcal{H}_{\varrho}^{+}(U,Y) + \mathcal{H}_{\varrho}^{+}(Y,V)$.

Definition 3. *[7] Let (M, ϱ) be a weak partial metric space. A multivalued mapping $\mathcal{T} : M \to CB^{\varrho}(M)$ is called $\mathcal{H}_{\varrho}^{+}$-contraction if*

(1^o) *$\exists\, k \in (0,1)$ such that*

$$\mathcal{H}_{\varrho}^{+}(\mathcal{T}s \backslash \{s\}, \mathcal{T}t \backslash \{t\}) \leq k\varrho(s,t) \textit{ for every } s, t \in M,$$

(2^o) *for every $s \in M$, t in $\mathcal{T}s$ and $\epsilon > 0$, there exists z in $\mathcal{T}t$ such that*

$$\varrho(t,z) \leq \mathcal{H}_{\varrho}^{+}(\mathcal{T}s, \mathcal{T}t) + \epsilon.$$

Beg [7] gave the following variant of Nadler's fixed point theorem.

Theorem 1. *[7] Every $\mathcal{H}_{\varrho}^{+}$-type multivalued contraction on a complete weak partial metric space (M, ϱ) has a fixed point.*

3. Weak Partial *b*-Metric Space

We now define weak partial *b*-metric space and related concepts:

Definition 4. *Let $M \neq \phi$ and $s \geq 1$, a function $\varrho_b : M \times M \to \mathbb{R}^+$ is called weak partial b-metric on M if for all $s, t, z \in M$, following conditions are satisfied:*

(WPB1) $\varrho_b(s, s) = \varrho_b(s, t) \Leftrightarrow s = t$;
(WPB2) $\varrho_b(s, s) \leq \varrho_b(s, t)$;
(WPB3) $\varrho_b(s, t) = \varrho_b(t, s)$;
(WPB4) $\varrho_b(s, t) \leq s[\varrho_b(s, z) + \varrho_b(z, t)]$.

The pair (M, ϱ_b) is a weak partial b-metric space.

Example 1. **(i)** $(\mathbb{R}^+, \varrho_b)$, *where $\varrho_b : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is defined as*

$$\varrho_b(s, t) = |s - t|^2 + 1 \text{ for all } s, t \in \mathbb{R}^+.$$

(ii) $(\mathbb{R}^+, \varrho_b)$, *where $\varrho_b : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is defined as*

$$\varrho_b(s, t) = \frac{1}{2}|s - t|^2 + \max\{s, t\} \text{ for all } s, t \in \mathbb{R}^+.$$

Definition 5. *A sequence $\{s_n\}$ in (M, ϱ_b) is said to converges a point $s \in X$, if and only if*

$$\varrho_b(s, s) = \lim_{n \to \infty} \varrho_b(s, s_n).$$

Remark 2. *If ϱ_b is a weak partial b-metric on M, the function ${\varrho_b}^s : M \times M \to \mathbb{R}^+$ given by ${\varrho_b}^s(s, t) = \varrho_b(s, t) - \frac{1}{2}[\varrho_b(s, s) + \varrho_b(t, t)]$, defines a b-metric on M. Further, a sequence $\{s_n\}$ in $(M, {\varrho_b}^s)$ converges to a point $s \in M$, iff*

$$\lim_{n,m \to \infty} \varrho_b(s_n, s_m) = \lim_{n \to \infty} \varrho_b(s_n, s) = \varrho_b(s, s). \tag{1}$$

Definition 6. *Let (M, ϱ_b) be a weak partial b-metric space. Then*

(1) *A Cauchy sequence in metric space (M, ϱ_b^s) is Cauchy in M.*
(2) *If the metric space (M, ϱ_b^s) is complete, so is weak partial b-metric space (M, ϱ_b).*

Let (M, ϱ_b) be a weak partial *b*-metric space and $CB^{\varrho_b}(M)$ be class of all nonempty, closed and bounded subsets of (M, ϱ_b). For $U, V \in CB^{\varrho_b}(M)$ and $s \in M$, define

$$\varrho_b(s, U) = \inf\{\varrho_b(s, u), u \in U\}, \;\; \delta_{\varrho_b}(U, V) = \sup\{\varrho_b(u, V) : u \in U\}$$

and

$$\delta_{\varrho_b}(V, U) = \sup\{\varrho_b(v, U) : v \in V\}.$$

Now $\varrho_b(s, U) = 0 \Rightarrow {\varrho_b}^s(s, U) = 0$, where ${\varrho_b}^s(s, U) = \inf\{{\varrho_b}^s(s, u), u \in U\}$.

Remark 3. *Let (M, ϱ_b) be a weak partial b-metric space and U a nonempty subset of M, then*

$$u \in \overline{U} \Leftrightarrow \varrho_b(u, U) = \varrho_b(u, u).$$

Proposition 2. *Let (M, ϱ_b) be a weak partial b-metric space. For any $U, V, Y \in CB^{\varrho_b}(M)$, we have the following:*

(i) $\delta_{\varrho_b}(U, U) = \sup\{\varrho_b(u, u) : u \in U\}$;
(ii) $\delta_{\varrho_b}(U, U) \leq \delta_{\varrho_b}(U, V)$;
(iii) $\delta_{\varrho_b}(U, V) = 0 \Rightarrow U \subseteq V$;
(iv) $\delta_{\varrho_b}(U, V) \leq s[\delta_{\varrho_b}(U, Y) + \delta_{\varrho_b}(Y, V)]$.

Proof. **(i)** If $U \in CB^{\varrho_b}(M)$, then for all $u \in U$, we have $\varrho_b(u, U) = \varrho_b(u, u)$ as $\overline{U} = U$. This implies that $\delta_{\varrho_b}(U, U) = \sup\{\varrho_b(u, U) : u \in U\} = \sup\{\varrho_b(u, u) : u \in U\}$.

(ii) Let $u \in U$. Since $\varrho_b(u, u) \leq \varrho_b(u, w)$ for all $w \in U$, therefore we have $\varrho_b(u, u) \leq \inf\{\varrho_b(u, v) : v \in V\} = \varrho_b(u, V) \leq \sup\{\varrho_b(u, V) : u \in U\} = \delta_{\varrho_b}(U, V)$.

(iii) If $\delta_{\varrho_b}(U, V) = 0$, then $\varrho_b(u, V) = 0$ for all $u \in U$. From (i) and (ii), it follows that $\varrho_b(u, u) \leq \delta_{\varrho_b}(U, V) = 0$ for all $u \in U$. Hence $\varrho_b(u, V) = \varrho_b(u, u)$ for all $u \in U$. By Remark 3, we have $u \in \overline{V} = V$, so $U \subseteq V$.

(iv) Let $u \in U, v \in V$ and $t \in Y$. By (WPB4), we have $\varrho_b(u, v) \leq s[\varrho_b(u, t) + \varrho_b(t, v)]$. Since $v \in V$ is arbitrary, therefore $\varrho_b(u, V) \leq s[\varrho_b(u, t) + \varrho_b(t, V)]$ and $\varrho_b(u, V) \leq s[\varrho_b(u, t) + \sup_{t \in Y} \varrho_b(t, V)]$, so that $\varrho_b(u, V) \leq s[\varrho_b(u, t) + \delta_{\varrho_b}(Y, V)]$. Since $t \in Y$ is arbitrary, therefore $\varrho_b(u, V) \leq s$ $[\varrho_b(u, Y) + \delta_{\varrho_b}(Y, V)]$. Since $u \in U$ is arbitrary, we have $\delta_{\varrho_b}(U, V) \leq s[\delta_{\varrho_b}(U, Y) + \delta_{\varrho_b}(Y, V)]$.
□

Definition 7. *Let (M, ϱ_b) be a weak partial b-metric space. For $U, V \in CB^{\varrho_b}(M)$, the mapping $\mathcal{H}^+_{\varrho_b} : CB^{\varrho_b}(M) \times CB^{\varrho_b}(M) \to [0, \infty)$ define by*

$$\mathcal{H}^+_{\varrho_b}(U, V) = \frac{1}{2}\{\delta_{\varrho_b}(U, V) + \delta_{\varrho_b}(V, U)\}$$

is called $\mathcal{H}^+_{\varrho_b}$-type Hausdorff metric induced by ϱ_b.

Proposition 3. *Let (M, ϱ_b) be a weak partial b-metric space. For any $U, V, Y \in CB^{\varrho_b}(M)$, we have:*

(whb1) $\mathcal{H}^+_{\varrho_b}(U, U) \leq \mathcal{H}^+_{\varrho_b}(U, V)$;
(whb2) $\mathcal{H}^+_{\varrho_b}(U, V) = \mathcal{H}^+_{\varrho_b}(V, U)$;
(whb3) $\mathcal{H}^+_{\varrho_b}(U, V) \leq s[\mathcal{H}^+_{\varrho_b}(U, Y) + \mathcal{H}^+_{\varrho_b}(Y, V)]$.

Proof. From (ii) of Proposition 2, we have

$$\mathcal{H}^+_{\varrho_b}(U, U) = \delta_{\varrho_b}(U, U) \leq \delta_{\varrho_b}(U, V) \leq \mathcal{H}^+_{\varrho_b}(U, V).$$

Also (whb2) obviously holds by definition. Now for (whb3), from (iv) of Proposition 2, we have

$$\begin{aligned} \mathcal{H}^+_{\varrho_b}(U, V) &= \frac{1}{2}\{\delta_{\varrho_b}(U, V) + \delta_{\varrho_b}(V, U)\} \\ &\leq \frac{1}{2}\{s[\delta_{\varrho_b}(U, Y) + \delta_{\varrho_b}(Y, V)] + s[\delta_{\varrho_b}(V, Y) + \delta_{\varrho_b}(Y, U)]\} \\ &= s[\frac{1}{2}\{\delta_{\varrho_b}(U, Y) + \delta_{\varrho_b}(Y, U)\} + \frac{1}{2}\{\delta_{\varrho_b}(Y, V) + \delta_{\varrho_b}(V, Y)\}] \\ &= s[\mathcal{H}^+_{\varrho_b}(U, Y) + \mathcal{H}^+_{\varrho_b}(Y, V)]. \end{aligned}$$

□

Following lemma is essential:

Lemma 1. *Let (M, ϱ_b) be weak partial b-metric space with $s \geq 1$ and $\mathcal{T} : M \to CB^{\varrho_b}(M)$ be a multivalued mapping. If $\{u_n\}$ is a sequence in M such that $u_n \in \mathcal{T}u_{n-1}$ and*

$$\varrho_b(u_n, u_{n+1}) \leq \lambda \varrho_b(u_{n-1}, u_n)$$

for each where $\lambda \in (0, 1)$, then $\{u_n\}$ is Cauchy.

Proof. Let $u_0 \in M$ and $u_n \in \mathcal{T}u_{n-1}$ for all $n \in \mathbb{N}$. We divide the proof into two cases:
Case I. Let $\lambda \in [0, \frac{1}{s})$ $(s > 1)$. By the hypotheses, we have

$$\varrho_b(u_n, u_{n+1}) \leq \lambda \varrho_b(u_{n-1}, u_n) \leq \lambda^2 \varrho_b(u_{n-2}, u_{n-1}) \leq \cdots \leq \lambda^n \varrho_b(u_0, u_1).$$

Thus, for $n > m$, we have

$$\begin{aligned}
\varrho_b(u_m, u_n) &\leq s[\varrho_b(u_m, u_{m+1}) + \varrho_b(u_{m+1}, u_n)] \\
&\leq s\varrho_b(u_m, u_{m+1}) + s^2[\varrho_b(u_{m+1}, u_{m+2}) + \varrho_b(u_{m+2}, u_n)] \\
&\leq s\varrho_b(u_m, u_{m+1}) + s^2\varrho_b(u_{m+1}, \varrho_{um+2}) + s^3[\varrho_b(u_{m+2}, u_{m+3}) + \varrho_b(u_{m+3}, u_n)] \\
&\leq s\varrho_b(u_m, u_{m+1}) + s^2\varrho_b(u_{m+1}, u_{m+2}) + s^3\varrho_b(u_{m+2}, u_{m+3}) \\
&\quad + \cdots + s^{n-m-1}\varrho_b(u_{n-2}, u_{n-1}) + s^{n-m-1}\varrho_b(u_{n-1}, u_n) \\
&\leq s\lambda^m \varrho_b(u_0, u_1) + s^2\lambda^{m+1}\varrho_b(u_0, u_1) + s^3\lambda^{m+2}\varrho_b(u_0, u_1) \\
&\quad + \cdots + s^{n-m-1}\lambda^{n-2}\varrho_b(u_0, u_1) + s^{n-m-1}\lambda^{n-1}\varrho_b(u_0, u_1) \\
&\leq s\lambda^m \left(1 + (s\lambda) + (s\lambda)^2 + \cdots + (s\lambda)^{n-m-2} + \frac{(s\lambda)^{n-m-1}}{s}\right)\varrho_b(u_0, u_1) \\
&\leq s\lambda^m \left(\frac{1}{1 - s\lambda} + \frac{(s\lambda)^{n-m-1}}{s}\right)\varrho_b(u_0, u_1) \\
&= \left(\frac{s\lambda^m}{1 - s\lambda} + (s\lambda)^{n-1}\right)\varrho_b(u_0, u_1) \to 0\,(n, m \to \infty).
\end{aligned}$$

Using (1) and the definition of ϱ_b^s, we get that $\varrho_b^s(u_m, u_n) \leq \varrho_b(u_m, u_n)$ tends to 0 as m, n tends $to + \infty$ which implies that $\{u_n\}$ is Cauchy in b-metric space (M, ϱ_b^s). Since (M, ϱ_b) is complete, therefore (M, ϱ_b^s) is a complete b-metric space. Consequently, the sequence $\{u_n\}$ converges to a point (say) $u^* \in M$ w.r.t b-metric ϱ_b^s, that is, $\lim_{n \to +\infty} \varrho_b^s(u_n, u^*) = 0$. Again, from (1) we get

$$\varrho_b(u^*, u^*) = \lim_{n \to +\infty} \varrho_b(u_n, u^*) = \lim_{n \to +\infty} \varrho_b(u_n, u_n) = 0.$$

Thus $\{u_n\}$ is a Cauchy sequence in (M, ϱ_b).

Case II. Let $\lambda \in [\frac{1}{s}, 1)$ $(s > 1)$. In this case, we have $\lambda^n \to 0$ as $n \to \infty$, then there is $k \in \mathbb{N}$ such that $\lambda^k < \frac{1}{s}$. Thus, by Case-I, we have that

$$\{u_k, u_{k+1}, u_{k+2}, ..., u_{k+n}, ...\},$$

is a Cauchy sequence. Since

$$\{u_n\}_{n=0}^{\infty} = \{u_0, u_{1,}..., u_{k-1}\} \cup \{u_k, u_{k+1}, u_{k+2}, ..., u_{k+n}, ...\},$$

we obtain that $u_n \in \mathcal{T}^n u_0$, $n = 1, 2, ...$ is a Cauchy sequence in M. □

Definition 8. *Let (M, ϱ_b) be a complete weak partial b-metric space. A multivalued mapping $\mathcal{T} : M \to CB^{\varrho_b}(M)$ is called $\mathcal{H}^+_{\varrho_b}$-contraction if*

$(1')$ *for every* $s, t \in M$, $\exists\, k \in (0,1)$ *such that*

$$\mathcal{H}^{+}_{\varrho_b}(Ts\backslash\{s\}, Tt\backslash\{t\}) \leq k\varrho_b(s,t);$$

$(2')$ *for every* $s \in X$, t *in* Ts *and* $\epsilon > 0$, $\exists\, z$ *in* $\mathcal{T}t$ *such that*

$$\varrho_b(t,z) \leq \mathcal{H}^{+}_{\varrho_b}(\mathcal{T}s, \mathcal{T}t) + \epsilon.$$

4. Fixed Point Result

Our main result is the following:

Theorem 2. *Every* $\mathcal{H}^{+}_{\varrho_b}$*-type multivalued contraction on a complete weak partial b-metric space* (M, ϱ_b) *has a fixed point.*

Proof. Let $u_0 \in M$ be arbitrary. If $u_0 \in \mathcal{T}u_0$ then u_0 is the fixed point. Therefore, we assume that $u_0 \notin \mathcal{T}u_0$. Let $u_1 \in \mathcal{T}u_0$ and $u_0 \neq u_1$ such that $u_1 \notin \mathcal{T}u_1$. From $(2')$, we have $u_2 \in \mathcal{T}u_1$ such that $u_2 \neq u_1$ and

$$\varrho_b(u_1, u_2) \leq \mathcal{H}^{+}_{\varrho_{b_b}}(\mathcal{T}u_0, \mathcal{T}u_1) + \epsilon.$$

Continuing this process we get $u_{n+1} \in \mathcal{T}u_n$ such that $u_{n+1} \neq u_n$ and

$$\varrho_b(u_n, u_{n+1}) \leq \mathcal{H}^{+}_{\varrho_b}(\mathcal{T}u_{n-1}, \mathcal{T}u_n) + \epsilon. \tag{2}$$

Choosing $\epsilon = \left(\frac{1}{\sqrt{k}} - 1\right)\mathcal{H}^{+}_{\varrho_b}(\mathcal{T}u_{n-1}, \mathcal{T}u_n)$ in (2), we have

$$\varrho_b(u_n, u_{n+1}) \leq \mathcal{H}^{+}_{\varrho_b}(\mathcal{T}u_{n-1}, \mathcal{T}u_n) + \left(\frac{1}{\sqrt{k}} - 1\right)\mathcal{H}^{+}_{\varrho_b}(\mathcal{T}u_{n-1}, \mathcal{T}u_n) = \frac{1}{\sqrt{k}}\mathcal{H}^{+}_{\varrho_b}(\mathcal{T}u_{n-1}, \mathcal{T}u_n).$$

Thus

$$\sqrt{k}\varrho_b(u_n, u_{n+1}) \leq \mathcal{H}^{+}_{\varrho_b}(\mathcal{T}u_{n-1}, \mathcal{T}u_n) = \mathcal{H}^{+}_{\varrho_b}\left(\mathcal{T}u_{n-1}\backslash\{u_{n-1}\}, \mathcal{T}u_n\backslash\{u_n\}\right).$$

From $(1')$, we get

$$\sqrt{k}\varrho_b(u_n, u_{n+1}) \leq k\varrho_b(u_{n-1}, u_n) = (\sqrt{k})^2\varrho_b(u_{n-1}, u_n).$$

Thus for all $n \in \mathbb{N}$,

$$\varrho_b(u_n, u_{n+1}) \leq \sqrt{k}\varrho_b(u_{n-1}, u_n). \tag{3}$$

Taking $\sqrt{k} = \lambda$, we obtained by Lemma 1 that $\{u_n\}$ is a Cauchy sequence. Since (M, ϱ_b) is complete. Therefore, there exists $u^* \in M$ such that $\lim_{n\to+\infty} u_n = u^*$. To show that $u^* \in \mathcal{T}$. On contrary suppose that $u^* \notin \mathcal{T}u^*$. Since

$$\begin{aligned}
\frac{1}{2}[\delta_{\varrho_b}(\mathcal{T}u_n, \mathcal{T}u^*) + \delta_{\varrho_b}(\mathcal{T}u^*, \mathcal{T}u_n)] &= \mathcal{H}^{+}_{\varrho_b}(\mathcal{T}u_n, \mathcal{T}u^*) \\
&= \mathcal{H}^{+}_{\varrho_b}(\mathcal{T}u_n\backslash\{u_n\}, \mathcal{T}u^*\backslash\{u^*\}) \\
&\leq k\varrho_b(u_n, u^*),
\end{aligned}$$

hence

$$\lim_{n\to+\infty}\inf[\delta_{\varrho_b}(\mathcal{T}u_n, \mathcal{T}u^*) + \delta_{\varrho_b}(\mathcal{T}u^*, \mathcal{T}u_n)] = 0.$$

Since

$$\lim_{n\to+\infty}\inf\delta_{\varrho_b}(\mathcal{T}u_n, \mathcal{T}u^*) + \lim_{n\to+\infty}\inf\delta_{\varrho_b}(\mathcal{T}u^*, \mathcal{T}u_n) \leq \lim_{n\to+\infty}\inf[\delta_{\varrho_b}(\mathcal{T}u_n, \mathcal{T}u^*) + \delta_{\varrho_b}(\mathcal{T}u^*, \mathcal{T}u_n)],$$

we have

$$\lim_{n\to+\infty}\inf \delta_{\varrho_b}(\mathcal{T}u_n,\mathcal{T}u^*)+\lim_{n\to+\infty}\inf \delta_{\varrho_b}(\mathcal{T}u^*,\mathcal{T}u_n)=0.$$

This implies that

$$\lim_{n\to+\infty}\inf \delta_{\varrho_b}(\mathcal{T}u_n,\mathcal{T}u^*)=0.$$

Since

$$\varrho_b(u^*,\mathcal{T}u^*)\le \delta_{\varrho_b}(\mathcal{T}u_n,\mathcal{T}u^*)+\varrho_b(u_{n+1},u^*),$$

therefore

$$\begin{aligned}\varrho_b(u^*,\mathcal{T}u^*) &\le \lim_{n\to+\infty}\inf[\delta_{\varrho_b}(\mathcal{T}u_n,\mathcal{T}u^*)+\varrho_b(u_{n+1},u^*)]\\ &= \lim_{n\to+\infty}\inf \delta_{\varrho_b}(\mathcal{T}u_n,\mathcal{T}u^*)+\lim_{n\to+\infty}\varrho_b(u_{n+1},u^*).\end{aligned}$$

This implies $\varrho_b(u^*,\mathcal{T}u^*)=0$, therefore from (1), we obtain

$$\varrho_b(u^*,u^*)=\varrho_b(u^*,\mathcal{T}u^*),$$

which implies $u^*\in\overline{\mathcal{T}u^*}=\mathcal{T}u^*$, as $\mathcal{T}u^*$ is closed. □

Example 2. *Consider a set $M=\{0,\frac{1}{2},1\}$ and $\varrho_b: M\times M\to\mathbb{R}^+$ a weak partial b-metric given by*

$$\varrho_b(u,v)=\frac{1}{2}|u-v|^2+\frac{1}{2}\max\{u,v\}\quad \text{for all}\quad u,v\in M.$$

Since $\varrho_b\left(\frac{1}{2},\frac{1}{2}\right)=\frac{1}{4}\neq 0$ and $\varrho_b(1,1)=\frac{1}{2}\neq 0$. Also

$$\begin{aligned}u\in\overline{\{0\}} &\Leftrightarrow \varrho_b(u,\{0\})=\varrho_b(u,u)\\ &\Leftrightarrow \frac{1}{2}u^2+\frac{1}{2}u=\frac{1}{2}u\Leftrightarrow u=0\\ &\Leftrightarrow u\in\{0\}.\end{aligned}$$

Also

$$\begin{aligned}u\in\overline{\{0,1\}} &\Leftrightarrow \varrho_b(u,\{0,1\})=\varrho_b(u,u)\\ &\Leftrightarrow \min\left\{\frac{1}{2}u^2+\frac{1}{2}u,\frac{1}{2}|u-1|^2+\frac{1}{2}\max\{u,1\}\right\}=\frac{1}{2}u\\ &\Leftrightarrow u\in\{0,1\}\end{aligned}$$

and

$$\begin{aligned}u\in\overline{\left\{0,\frac{1}{2}\right\}} &\Leftrightarrow \varrho_b\left(u,\left\{0,\frac{1}{2}\right\}\right)=\varrho_b(u,u)\\ &\Leftrightarrow \min\left\{\frac{1}{2}u^2+\frac{1}{2}u,\frac{1}{2}\left|u-\frac{1}{2}\right|^2+\frac{1}{2}\max\left\{u,\frac{1}{2}\right\}\right\}=\frac{1}{2}u\\ &\Leftrightarrow u\in\left\{0,\frac{1}{2}\right\}.\end{aligned}$$

Hence, $\{0\}$, $\{0,1\}$ and $\left\{0,\frac{1}{2}\right\}$ are closed w.r.t weak partial b-metric ϱ_b.
Define $\mathcal{T}: X\to CB^{\varrho_b}(M)$ by

$$\mathcal{T}(0)=\{0\},\ \mathcal{T}\left(\frac{1}{2}\right)=\left\{0,\frac{1}{2}\right\}\quad \text{and}\quad \mathcal{T}(1)=\{0,1\}.$$

To show that for all $u, v \in M$, the contractive condition $(1')$ holds for all $k \in (0,1)$, we consider the following cases:

For $u = v = 0$, we have

$$\mathcal{H}^{+}_{\varrho_b}\left(\mathcal{T}(0)\backslash\{0\}, \mathcal{T}(0)\backslash\{0\}\right) = \mathcal{H}^{+}_{\varrho_b}\left(\{0\}\backslash\{0\}, \{0\}\backslash\{0\}\right) = \mathcal{H}^{+}_{\varrho_b}\left(\emptyset, \emptyset\right) = 0,$$

so $(1')$ satisfied.
For $u = 0, v = \frac{1}{2}$, we have

$$\mathcal{H}^{+}_{\varrho_b}\left(\mathcal{T}(0)\backslash\{0\}, \mathcal{T}\left(\frac{1}{2}\right)\backslash\left\{\frac{1}{2}\right\}\right) = \mathcal{H}^{+}_{\varrho_b}\left(\{0\}\backslash\{0\}, \left\{0, \frac{1}{2}\right\}\backslash\left\{\frac{1}{2}\right\}\right) = \mathcal{H}^{+}_{\varrho_b}\left(\emptyset, \{0\}\right) = 0,$$

so $(1')$ satisfied.
For $u = v = \frac{1}{2}$, we have

$$\begin{aligned}&\mathcal{H}^{+}_{\varrho_b}\left(\mathcal{T}\left(\frac{1}{2}\right)\backslash\left\{\frac{1}{2}\right\}, \mathcal{T}\left(\frac{1}{2}\right)\backslash\left\{\frac{1}{2}\right\}\right) = \mathcal{H}^{+}_{\varrho_b}\left(\left\{0, \frac{1}{2}\right\}\backslash\left\{\frac{1}{2}\right\}, \left\{0, \frac{1}{2}\right\}\backslash\left\{\frac{1}{2}\right\}\right)\\ &= \mathcal{H}^{+}_{\varrho_b}\left(\{0\}, \{0\}\right) = \varrho_b(0,0) = 0,\end{aligned}$$

so $(1')$ satisfied.
For $u = 0, v = 1$, we have

$$\mathcal{H}^{+}_{\varrho_b}\left(\mathcal{T}(0)\backslash\{1\}, \mathcal{T}(1)\backslash\{1\}\right) = \mathcal{H}^{+}_{\varrho_b}\left(\{0\}\backslash\{0\}, \{0,1\}\backslash\{0\}\right) = \mathcal{H}^{+}_{\varrho_b}\left(\emptyset, \{0\}\right) = 0,$$

so $(1')$ satisfied.
For $u = \frac{1}{2}, v = 1$, we have

$$\begin{aligned}&\mathcal{H}^{+}_{\varrho_b}\left(\mathcal{T}\left(\frac{1}{2}\right)\backslash\left\{\frac{1}{2}\right\}, \mathcal{T}(1)\backslash\{1\}\right) = \mathcal{H}^{+}_{\varrho_b}\left(\left\{0, \frac{1}{2}\right\}\backslash\{0\}, \{0,1\}\backslash\{1\}\right) = \mathcal{H}^{+}_{\varrho_b}\left(\{0\}, \{0\}\right)\\ &= \varrho_b(0,0) = 0,\end{aligned}$$

so $(1')$ satisfied.
For $u = v = 1$, we have

$$\mathcal{H}^{+}_{\varrho_b}\left(\mathcal{T}(1)\backslash\{1\}, \mathcal{T}(1)\backslash\{1\}\right) = \mathcal{H}^{+}_{\varrho_b}\left(\{0,1\}\backslash\{1\}, \{0,1\}\backslash\{1\}\right) = \mathcal{H}^{+}_{\varrho_b}\left(\{0\}, \{0\}\right) = \varrho_b(0,0) = 0,$$

so $(1')$ satisfied.

Further, we show that for every $u \in M, v \in \mathcal{T}u$ and $\epsilon > 0$, $\exists\, w \in \mathcal{T}v$ such that

$$\varrho_b(v, w) \leq \mathcal{H}^{+}_{\varrho_b}\left(\mathcal{T}u, \mathcal{T}v\right) + \epsilon.$$

So,

(a) *If $u = 0, v \in \mathcal{T}(0) = \{0\}, \epsilon > 0, \exists\, w \in \mathcal{T}v = \{0\}$*

$$0 = \varrho_b(v, w) \leq \mathcal{H}^{+}_{\varrho_b}(\mathcal{T}v, \mathcal{T}u) + \epsilon.$$

(b) *If $u = \frac{1}{2}, v \in \mathcal{T}u = \mathcal{T}(\frac{1}{2}) = \{0, \frac{1}{2}\}$, for $v = 0, \epsilon > 0, \exists\, w \in \mathcal{T}v = \{0\}$ such that*

$$0 = \varrho_b(v, w) < \frac{3}{16} + \epsilon \leq \mathcal{H}^{+}_{\varrho_b}(\mathcal{T}v, \mathcal{T}u) + \epsilon$$

and for $v = \frac{1}{2}, \epsilon > 0, \exists\, w \in Tv = \{0, \frac{1}{2}\}$ *such that*

$$\frac{1}{4} = \varrho_b(v, w) < \frac{1}{4} + \epsilon \le \mathcal{H}^+_{\varrho_b}(\mathcal{T}v, \mathcal{T}u) + \epsilon.$$

(c) *If* $u = 1, v \in \mathcal{T}u = \mathcal{T}(1) = \{0, 1\}$, *for* $v = 0, \epsilon > 0, \exists\, w \in Tv = \{0\}$ *such that*

$$0 = \varrho_b(v, w) < \frac{3}{4} + \epsilon \le \mathcal{H}^+_{\varrho_b}(\mathcal{T}v, \mathcal{T}u) + \epsilon$$

and for $v = 1, \epsilon > 0, \exists\, w \in \mathcal{T}v = \{0, 1\}$ *such that*

$$\frac{1}{2} = \varrho_b(v, w) < \frac{1}{2} + \epsilon \le \mathcal{H}^+_{\varrho_b}(\mathcal{T}v, \mathcal{T}u) + \epsilon.$$

Thus condition $(2')$ *is satisfied.*

Hence Theorem 2 can be applied and we conclude that $u \in \{0, \frac{1}{2}, 1\}$ *is fixed points of* $\mathcal{T}$.

5. Application

We now apply our main result to show the existence of solution of nonlinear integral inclusion of Volterra type. Suppose $l = (0, 1)$, and $M = C[l, \mathbb{R})$, the space of all continuous functions $f : l \to \mathbb{R}$. Consider weak partial b-metric on M by

$$\varrho_b(x, y) = \sup_{t \in l} e^{-\beta t} |x(t) - y(t)|^p + \alpha,$$

$\forall\, x, y \in C(l, \mathbb{R})$, $p > 1$ and $\alpha > 0$. We have $\varrho^s_b(x, y) = \sup_{t \in l} e^{-\beta t} |x(t) - y(t)|^p$, so by Definition 6, $(C(l, \mathbb{R}), \varrho_b)$ is complete partial b-metric space. Denote by $P_{cl}(\mathbb{R})$ the class of all nonempty closed subsets of $\mathbb{R}$.

Theorem 3. *Assume the integral equation inclusion of Volterra type*

$$y(t) \in f(t) + \int_0^t K(t, s, y(s))ds, \quad t \in l. \tag{4}$$

Suppose

(a) $K : l \times l \times \mathbb{R} \to P_{cl}(\mathbb{R})$ *is such that* $K_y(t, s) := K(t, s, y(s))$ *is continuous for all* $(t, s) \in l \times l$ *and* $y \in C(l, \mathbb{R})$;
(b) $f \in C(l, \mathbb{R})$;
(c) *for each* $t \in l$, *there exist* $y \in C(l, \mathbb{R})$, *such that*

$$\mathcal{H}^+_{\varrho_b}(K(t, x, y(x)), K(t, x, h(x))) \le \frac{1}{t^{p-1}} \left(\sup_{x \in l} |y(x) - h(x)|^p + \alpha \right),$$

for all $t, x \in l$ *and all* $y, h \in C(l, \mathbb{R})$.

Then there is at least one solution of (4) *in* $C(l, \mathbb{R})$.

Proof. Define $\mathcal{T} : C(l, \mathbb{R}) \to P_{cl}(C(l, \mathbb{R}))$ by

$$\mathcal{T}x(t) = \left\{ y \in C(l, \mathbb{R}) \text{ such that } y(t) \in f(t) + \int_0^t K(t, s, x(s))ds, t \in l \right\}$$

for each $x \in C(l, \mathbb{R})$. For each $K_x : l \times l \to P_{cl}(\mathbb{R})$ there exists $k_x : l \times l \to \mathbb{R}$ such that $k_x(t, s) \in K_x(t, s)$ for all $t, s \in l$. This implies that $f(t) + \int_0^t k_x(t, s)ds \in \mathcal{T}x$, and so $\mathcal{T}x \ne \emptyset$. It is easy to prove that $\mathcal{T}x$ is closed.

We show that $\mathcal{T}$ is $\mathcal{H}^{+}_{\varrho_b}$-type multivalued contraction. Let $u_1, u_2 \in C(l, \mathbb{R})$ and $y \in \mathcal{T}x$. Then $\exists$ $k_{u_1}(t,s) \in K_{u_1}(t,s)$, $t, s \in l$ such that $y(t) = f(t) + \int_0^t k_x(t,s)ds, t \in l$. Also by hypothesis (iii),

$$\mathcal{H}^{+}_{\varrho_b}(K(t,s,u_1(s)), K(t,s,u_2(s))) \leq \frac{1}{t^{p-1}} \left(\sup_{s \in l} |u_1(s) - u_2(s)|^p + \alpha \right) \ \ \forall\, t, s \in l.$$

Then there exist $g(t,s) \in K_{u_1}(t,s)$ such that

$$|k_{u_1}(t,s) - g(t,s)|^p + \xi \leq \frac{1}{t^{p-1}} \left[|u_1(s) - u_2(s)|^p + \alpha \right]$$

for all $t, s \in l$. Define a multivalued operator $Q(t,s)$ by

$$Q(t,s) = K_{u_2}(t,s) \cap \{\eta \in \mathbb{R}, |k_{u_1} - \eta|^p + \alpha \leq \frac{1}{t^{p-1}} |u_1(s) - u_2(s)|^p + \alpha\}$$

for all $t, s \in l$. Since Q is continuous operator, there exists a continuous operator $k_{u_2} : l \times l \to \mathbb{R}$ such that $k_{u_2}(t,s) \in Q(t,s)$ for all $t, s \in l$ and

$$h(t) = f(t) + \int_0^t k_{u_2}(t,s)ds \in f(t) + \int_0^t K(t,s,u_2(s))ds.$$

Therefore, let $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

$$\begin{aligned}
\varrho_b(y(t), \mathcal{T}u_2(t)) &\leq \varrho_b(y(t), h(t)) \\
&= \sup_{t \in l} e^{-\beta t} |y(t) - h(t)|^p + \alpha \\
&= \sup_{t \in l} e^{-\beta t} \left| \int_0^t [k_{u_1}(t,s) - k_{u_2}(t,s)] ds \right|^p + \alpha \\
&\leq \sup_{t \in l} e^{-\beta t} \left[\left(\int_0^t ds \right)^{\frac{1}{q}} \left(\int_0^t |k_{u_1}(t,s) - k_{u_2}(t,s)|^p ds \right)^{\frac{1}{p}} \right]^p + \alpha \\
&\leq \sup_{t \in l} e^{-\beta t} \, (t)^{\frac{p}{q}} \left(\int_0^t |k_{u_1}(t,s) - k_{u_2}(t,s)|^p ds \right) + \alpha \\
&= \sup_{t \in l} e^{-\beta t} \, (t)^{p-1} \left(\int_0^t e^{\beta s} e^{-\beta s} |k_{u_1}(t,s) - k_{u_2}(t,s)|^p ds \right) + \alpha \\
&= \sup_{t \in l} e^{-\beta t} \, (t)^{p-1} \left(\int_0^t e^{\beta s} e^{-\beta s} |k_{u_1}(t,s) - k_{u_2}(t,s)|^p ds \right) + \alpha \\
&= e^{-\beta t} \, (t)^{p-1} \left(\int_0^t \left(e^{\beta s} \sup_{t \in l} \{ e^{-\beta s} |k_{u_1}(t,s) - k_{u_2}(t,s)|^p + \alpha \} - \alpha \right) ds \right) + \alpha \\
&\leq e^{-\beta t} \, (t)^{p-1} \left(\int_0^t \left(e^{\beta s} \sup_{t \in l} \{ \frac{1}{t^{p-1}} |u_1(t) - u_2(t)|^p + \alpha \} - \alpha \right) ds \right) + \alpha \\
&= e^{-\beta t} \, (t)^{p-1} \frac{1}{t^{p-1}} \varrho_b(u_1(t), u_2(t)) \int_0^t e^{\beta s} ds - e^{-\beta t} \, (t)^{p-1} \int_0^t \alpha ds + \alpha \\
&= e^{-\beta t} \varrho_b(u_1(t), u_2(t)) (e^{\beta t} - 1) - e^{-\beta t} \, (t)^{p-1} \alpha t + \alpha \\
&= (1 - e^{-\beta t}) \varrho_b(u_1(t), u_2(t)) + (1 - e^{-\beta t} t^p) \alpha \\
&\leq (1 - e^{-\beta t}) \varrho_b(u_1(t), u_2(t)) \\
&= k.\varrho_b(u_1(t), u_2(t)),
\end{aligned}$$

where $k = (1 - e^{-\beta t}) < 1$. Since $y(t)$ is arbitrary, we have

$$\delta_{\varrho_b}(\mathcal{T}u_1, \mathcal{T}u_2) \leq k.\varrho_b(u_1, u_2). \tag{5}$$

Similarly, we get

$$\delta_{\varrho_b}(\mathcal{T}u_2, \mathcal{T}u_1) \leq k.\varrho_b(u_2, u_1). \tag{6}$$

From (5) and (6), we get

$$\mathcal{H}^+_{\varrho_b}(\mathcal{T}u_1, \mathcal{T}u_2) = k.\frac{\delta_{\varrho_b}(\mathcal{T}u_1, \mathcal{T}u_2) + \delta_{\varrho_b}(\mathcal{T}u_2, \mathcal{T}u_1))}{2} \leq k.\varrho_b(u_2, u_1).$$

Hence, $\mathcal{T}$ is $\mathcal{H}^+_{\varrho_b}$-type multivalued contraction. Thus all the assertions of Theorem 2 are satisfied and hence (4) has a solution. □

6. Conclusions

In this paper, we present the concept of weak partial *b*-metric spaces with their topology and weak partial Hausdorff *b*-metric spaces and generalized the famous Nadler's theorem in weak partial *b*-metric space by using weak partial Hausdorff *b*-metric spaces. We give an example to show the validity and an application to nonlinear Volterra integral inclusion for the usability of our result.

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Abbreviations

The following abbreviations are used in this manuscript:

MDPI	Multidisciplinary Digital Publishing Institute
DOAJ	Directory of open access journals
TLA	Three letter acronym
LD	linear dichroism

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