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Topology Structure Implied in β-Hilbert Space, *Heisenberg* Uncertainty Quantum Characteristics and Numerical Simulation of the *DE* Algorithm

Kaiguang Wang [1,2,†] and Yuelin Gao [2,*,†]

1 School of Mathematics and Information Science, North Minzu University, Yinchuan 750021, China; wkg13759842420@foxmail.com

2 Ningxia Province Key Laboratory of Intelligent Information and Data Processing, Yinchuan 750021, China

* Correspondence: gaoyuelin@263.net; Tel.: +86-183-2997-0138 or +86-139-9510-0900

† These authors contributed equally to this work.

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Abstract: The differential evolutionary (DE) algorithm is a global optimization algorithm. To explore the convergence implied in the *Hilbert* space with the parameter β of the DE algorithm and the quantum properties of the optimal point in the space, we establish a control convergent iterative form of a higher-order differential equation under the conditions of $P_{-\varepsilon}$ and analyze the control convergent properties of its iterative sequence; analyze the three topological structures implied in *Hilbert* space of the single-point topological structure, branch topological structure, and discrete topological structure; and establish and analyze the association between the *Heisenberg* uncertainty quantum characteristics depending on quantum physics and its topological structure implied in the β-Hilbert space of the DE algorithm as follows: The speed resolution Δ_v^2 of the iterative sequence convergent speed and the position resolution $\Delta_{x_\beta^\varepsilon}$ of the global optimal point with the swinging range are a pair of conjugate variables of the quantum states in β-Hilbert space about eigenvalues $\lambda_i \in \mathbb{R}$, corresponding to the uncertainty characteristics on quantum states, and they cannot simultaneously achieve bidirectional efficiency between convergent speed and the best point precision with any procedural improvements. Where $\lambda_i \in \mathbb{R}$ is a constant in the β-Hilbert space. Finally, the conclusion is verified by the quantum numerical simulation of high-dimensional data. We get the following important quantitative conclusions by numerical simulation: except for several dead points and invalid points, under the condition of spatial dimension, the number of the population, mutated operator, crossover operator, and selected operator are generally decreasing or increasing with a variance deviation rate $+0.50$ and the error of less than ± 0.5; correspondingly, speed changing rate of the individual iterative points and position changing rate of global optimal point β exhibit a inverse correlation in β-Hilbert space in the statistical perspectives, which illustrates the association between the *Heisenberg* uncertainty quantum characteristics and its topological structure implied in the β-Hilbert space of the DE algorithm.

Keywords: DE algorithm; β-Hilbert space; topology structure; quantum uncertainty property; numerical simulation

MSC: 81S10; 65L07; 46B28; 90C59; 54A05

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1. Introduction

The differential evolutionary (DE) algorithm [1–3] is a global optimization algorithm with iterative search used to generate mutative individuals by differential operation, proposed by Storn and Price in 1995 to solve Chebyshev inequalities, which adopts floating-point vector coding to search in continuous space [4–6], is simple to operate and easy to achieve and offers better convergence, stronger robustness and other global optimization advantages [7–11]. In general, the minimization optimization problem of the *DE* algorithm is expressed as follows:

$$\min\ f(X_i^t + P_{-\varepsilon}), X_i^t = \{x_{ij}^t | i = 1, 2, \cdots, NP; j = 1, 2, \cdots, D\} \tag{1}$$

$$s.t.\ a_{ij} \leq x_{ij}^t \leq b_{ij}, i = 1, 2, \cdots, NP; j = 1, 2, \cdots, D \tag{2}$$

where the dimension (D) is the dimension of the decisional variable, number of population (NP) is the population size, $f(X_i + p_\varepsilon)$ is the fitness function, and $P_{-\varepsilon}$ is the individual perturbation variable with the relative error ε in the population, which is generally an infinitesimal and indicates the adjustable range of the optimal value when affected by some conditions. Conveniently, we assume that the perturbation variable $P_{-\varepsilon}$ of all individuals is the same when the external environment features perturbation. A larger perturbation variable $P_{-\varepsilon}$ indicates that the *DE* algorithm has a higher discrete degree for population individuals when generally approaching the optimal value.

A smaller perturbation variable $P_{-\varepsilon}$ indicates that the individual is less discrete when generally approaching the optimal value. Here, we assume that the infinitesimal has a fixed value, $X_i(i = 1, 2, \cdots, NP)$, is a D-dimensional vector, $x_{ij}(i = 1, 2, \cdots, NP; j = 1, 2, \cdots, D)$ is the jth components of the ith individual, and a_{ij}, b_{ij} are the upper bound and the lower bounds of the optimization range, respectively.

We are interested in the convergence of the *DE* algorithm in the optimization process and the spatial topological structure of the population in a closed ecological population [12,13], that is, the association between the population iterative sequence and population spatial topological structure. In this paper, the population is a closed ecological population, which generates an association of one-to-one correspondence between it and the population; thus, the population can be analyzed by the equivalent to the mathematical closed set. The assumptions are valid in theory. For the study of the dynamics of the *DE* algorithm, previous work [4] has analyzed the dynamics and behavior of the algorithm and provides a new direction for the dynamics of the algorithm. Numerical simulation of the route optimization and convergent problem of the *DE* algorithm has been performed [5], including studies of the convergence based on dynamics studies. Other researchers [6] have drawn comparisons regarding the convergence of various algorithm benchmark problems, and we can look at the corresponding relationship between convergence and the parameters. A parametric scheme for the algorithm dynamics research is provided for the *DE* algorithm to search and optimize the properties in the β-Hilbert space, and the study of the dynamic conditions of the *DE* algorithm is also performed.

In general, the spatial topology of a closed population is often associated with a composite operator topology on a defined function space [12,13]. One earlier study result is the isolated point theorem of the composite operator on H^2 given by Berkson [14], and MacCluer [15] and Shapiro J H [16] promote this conclusion. For the bounded analytic function space H^∞ on a unit circle or unit ball, previous work [16–18] studied the topology structure of $\mathcal{C}(H^\infty)$ and proved that the isolated composite operator of the operator topology on H^∞ is also isolated under the condition of essential norm topology. We now address the spatial topology implied in the limit point β of the convergent iterative sequence concerning the *DE* algorithm in the composite complete *Hilbert* space. Furthermore, the quantum characteristics of the *Heisenberg* uncertainty principle implied in *Hilbert* space or *Fock* [14,19–27] of the *DE* algorithm are one of the central issues studied in this paper. First, we solve the following problems:

1. The continuity of the closed population in the condition of $P_{-\varepsilon}$ and the control convergent properties of its iterative sequence;

2. The topological structure implied in the *Hilbert* space of the *DE* algorithm;

3. The *Heisenberg* uncertainty quantum characteristics implied in the β-Hilbert space of the *DE* algorithm;

4. High-dimensional numerical simulation of the quantum characteristics of the *DE* algorithm to determine the association between this algorithm and its topological structure.

2. Preparatory Knowledge

2.1. Basic Steps of the DE Algorithm

The basic operating principle of the DE algorithm is described as follows [1,4,7].

2.1.1. Initial Population

Let the population of the DE algorithm be $X(t)$; then, the population individuals can be expressed as

$$X_i^t = (x_{i1}^t, x_{i2}^t, \cdots, x_{iD}^t), i = 1, 2, \cdots, NP \tag{3}$$

where t is the evolutionary generation and NP is the population size.

Initial population: Determine the dimension D of the optimization problem. The maximum evolutionary generation is T, and the population size is NP. Let the initial value of the optimal vector be

$$X_i^0 = (x_{i1}^0, x_{i2}^0, \cdots, x_{iD}^0) \tag{4}$$

$$x_{ij}^0 = a_{ij} + rand(0,1) \cdot (b_{ij} - a_{ij}), i = 1, 2, \cdots, NP; j = 1, 2, \cdots, D \tag{5}$$

where, the range of individual variables is $a_{ij}, b_{ij} LeqR$, because of the randomness of iterative individuals in optimization process and real number coding for individuals.

2.1.2. Mutation Operation

The individual mutated component of the DE algorithm is the differential vector of the parental individuals, and the number of differential mutated individuals per time is derived from the two individual components $(x_{i_1}^t, x_{i_2}^t)$ in the tth generation parental population individuals, where $i_1, i_2 \in NP$. Then, the differential vector is defined as $D_{i_{1,2}} = (x_{i_1}^t - x_{i_2}^t)$. For any vector individual X_i^t, the mutation operation is defined as

$$V_i^{t+1} = X_{i_3}^t + F \cdot (X_{i_1}^t - X_{i_2}^t) \tag{6}$$

where $NP \geq 4$ is the population size, F is the contraction factor, and $i_1, i_2, i_3 \in \{1, 2, \cdots, NP\}$ and i_1, i_2, i_3 are mutually different so that we can obtain a mutated individual by differential operation by randomly selecting non-zero different vectors in the population, and the mutated individuals realize the possibility of adjusting the diversity of the population.

2.1.3. Crossover Operation

First, the test individual U_i^{t+1} is generated by crossing the target vector individual X_i^t and the mutated individual V_i^{t+1} in the population. To maintain population diversity, we can conduct crossover and selection operations for the mutated individual V_i^{t+1} and the target vector individual X_i^t by introducing the crossover probability CR and the random function $rand(0,1)$ to ensure that at least one of the test individuals U_i^{t+1} is contributed by the mutated individuals V_i^{t+1}. For other loci points, we can determine the contribution of certain sites of the test individual U_i^{t+1} that are determined by the mutation vector individuals V_i^{t+1} and target vector individual components (x_i^t) that are determined by the crossover probability. The experimental equation of the crossover operation is as follows:

$$(u_{ij}^{t+1}) = \begin{cases} (v_{ij}^{t+1}), if\ rand_j[0,1] \leq CR\ or\ j = j_{rand}, \\ (x_{ij}^t), otherwise. \\ i = 1, 2, \cdots, NP; j = 1, 2, \cdots, D \end{cases} \tag{7}$$

where $rand_j[0,1], CR \in (0,1)$ is the crossover probability above the formula (7). The larger the value of CR is, the greater the probability of generating new vector individuals by locating the crossover operation of different loci points for vector individuals in the population. When $CR = 0$, $U_i^{t+1} = X_i^t$, it indicates that no crossover occurred, which is beneficial to maintain the diversity of the population and the ability of global searching. When $CR = 1$, $U_i^{t+1} = V_i^{t+1}$, it indicates that crossover operations must occur at certain loci points, which helps maintain global searching and speed up convergence. $CR = 1\ or\ 1$ represent the two extreme cases of crossover operation. $j = j_{rand}$ is a randomly selected loci point used to ensure that the test individuals U_i^t obtain at least one genetic locus of occurring mutation from the mutated individuals V_i^t and to ensure that the mutated individuals V_i^{t+1}, the target vector individuals X_i^t, and the test individuals U_i^{t+1} are different from each other, which indicates that this operation is an effective action in populations.

2.1.4. Selection Operation

The selection operation of the DE algorithm is a selected mechanism based on the greedy algorithm that the test individual U_i^{t+1} is generated by the mutation and selection operations, and the target vector individual X_i^t conducts competition and selection. If the fitness value of U_i^{t+1} is better than the fitness value of X_i^t, then U_i^{t+1} is inherited to the next generation as the best individual in the first iteration; otherwise, X_i^t remains in the next generation. The selection effect of the selection operator in the population is described by the following equation:

$$X_i^{t+1} = \begin{cases} U_i^{t+1}, if\ f(U_i^{t+1}) \le f(X_i^t) \\ X_i^t, otherwise \end{cases}, i = 1, 2, \cdots, NP \tag{8}$$

2.1.5. Compact Operator and *Fock* Space

Let H and L be the separable *Hilbert* space and $B(H, L)$ be the whole bounded linear operators from H to L; if the mapping $T(S)$ of the unit ball S of X in T satisfies relative compactness in Y, then $\forall T \in B(X, Y)$ is compact. In addition, the essential norm $\|T\|_e$ of operator $T \in B(X, Y)$ is the operator norm distance of all compact operators from T to $B(H, L)$. We also have $\|T\|_e \le T$ and

$$\|T\|_e = \sup_{f_\varepsilon{}^n \in \mathcal{U}} (\limsup_{k \to \infty} \|T f_\varepsilon^n\|_l) \tag{9}$$

where $\mathcal{U}$ is all unit element sequences f_ε^n that are weakly convergent to 0.

Define the *Gaussian* measure dG on $\mathbb{C}^n$ as $d(G) = \frac{1}{\pi^n} e^{-|z|^2} dV(z), z \in \mathbb{C}^n$ where dV is the spatial measure on $\mathbb{C}^n$; then, *Fock* space $F^2 = F^2(\mathbb{C}^n)$ is the *Hilbert* space $L^2(G) \cap H(\mathbb{C}^n)$. Its inner product and norm are designated $\langle f, g \rangle = \int_{\mathbb{C}^n} f(z)\overline{g(z)} dG(z)$ and $\|f\|^2 = \int_{\mathbb{C}^n} |f(z)|^2 dG(z)$, respectively, where $f, g \in F^2$.

3. Continuity Structure of Closed Populations and Convergence of Iterative Sequences under P_ε

For any population existing in real space, the population individuals show discrete characteristics from the biological viewpoint but show continuous characteristics from a physical viewpoint in space. For the DE algorithm, the adaptive optimal individual in any population must be the limit value of the iterative sequence formed by all individuals in the population. Thus, an existing population perturbation $P_{-\varepsilon}$ is theoretically reasonable, which is described in the form of limitation as the following equation:

$$\min\ f(X_i^t + P_{-\varepsilon}), X_i^t = \{x_{ij}^t | i = 1, 2, \cdots, NP; j = 1, 2, \cdots, D\} \tag{10}$$

$$s.t.\ a_{ij} \le x_{ij}^t \le b_{ij}, i = 1, 2, \cdots, NP; j = 1, 2, \cdots, D \tag{11}$$

This formulation is equivalent to

$$\lim_{t\to+\infty} f((X_i^t)_\varepsilon) = \lim_{t\to+\infty} f_\varepsilon(X_i^t) = f_\varepsilon(X_i) \in (f(X_i) - \delta_\varepsilon, f(X_i) + \delta_\varepsilon) \quad (12)$$

$$s.t.\ a_{ij} \le x_{ij}^t \le b_{ij}, i = 1, 2, \cdots, NP; j = 1, 2, \cdots, D \quad (13)$$

where $f(X_i)$ is the optimal value of the DE algorithm as $t \to +\infty$ because the stability of the optimal value in space, $f_\varepsilon(X_i)$, must be between $(f_\varepsilon(X_i) - \delta_\varepsilon, f_\varepsilon(X_i) + \delta_\varepsilon)$, where δ_ε is the maximum range of the optimal value as being up and down. In the same population, there is only one optimal value, which inherits all the adaptive characteristics of population individuals in the space, and the fitness function $f_\varepsilon(X_i^t)$ corresponding to those individuals measures its adaptability in the population. We say that the former is an eigenvalue and that the latter is an eigenfunction. Then, we establish the continuity characteristic relationship and the uniform convergence of the iterative form of the population eigenvalue and eigenfunction.

3.1. Continuity Structure of the Closed Population Feature Quantity in Perturbation P_ε

Definition 1. *Assume that a population of size NP is the continuous real value of the complete real space $\mathbb{R}^+$, the population eigenvalue is $\lambda_k = X_i$, the population eigenfunction is $f((X_i^t)_\varepsilon)$, and $|\varepsilon| < \frac{1}{r}, r \in \mathbb{R}^+$, which is a convergent form that can converge in the perturbation variable P_ε with iteration numbers increasing. If*

$$\lim_{t\to+\infty} f((X_i^t)_\varepsilon) = \lim_{t\to+\infty} f_\varepsilon(X_i^t) = f_\varepsilon(X_i) \in (f(X_i) - \delta_\varepsilon, f(X_i) + \delta_\varepsilon) \quad (14)$$

$$s.t.\ a_{ij} \le x_{ij}^t \le b_{ij}, i = 1, 2, \cdots, NP; j = 1, 2, \cdots, D \quad (15)$$

then we find that $f((X_i^t)_\varepsilon)$ is continuous at the eigenvalue $\lambda_k = X_i$.

Property 1. *If $f((X_i^t)_\varepsilon)$ is continuous at the eigenvalue $\lambda_k = X_i$, then $f_\varepsilon(X_i) \in (f(X_i) - \delta_\varepsilon, f(X_i) + \delta_\varepsilon)$, that is, $f((X_i^t)_\varepsilon)$ is locally bounded.*

3.2. Uniform Convergence of the Differential Equation in Perturbation P_ε

In general, population individuals show discrete characteristics in space and continuous characteristics in time concerning the optimal process. Under the condition of the perturbation variable $P_{-\varepsilon}$, the convergent limit value is a bounded range, which is not a definite real value. To ensure that individuals can converge to a precise real value in the late iteration, the convergence of the differential equation must be uniformly converged under the condition of being the perturbation variable $P_{-\varepsilon}$ for all population individuals. First, we construct a continuous iterative form of error variable ε under the condition of perturbation P_ε:

$$\left\{\begin{array}{l} \varepsilon f_\varepsilon^{(n+1)''} p_1(x) f_\varepsilon^{(n+1)'} - q_\varepsilon(x, f_\varepsilon^{(n)}) = 0 \\ (0 < x < 1, q_\varepsilon(x, f_\varepsilon) = p_2(x) f_\varepsilon) \\ f_\varepsilon^{(n+1)}(0) = A, f_\varepsilon^{(n+1)}(1+\varepsilon) = B, (A, B \in \mathbb{R}^+, n = 1, 2, \cdots) \\ f_\varepsilon^{(0)} \in \mathbf{V} = \{v \in C^2[0,1]/v(0) = A, v(1+\varepsilon) = B\} \end{array}\right. \quad (16)$$

Second, we construct an approximate format (17) of $Il'InAM$ [28] of the perturbation error variable ε:

$$\left\{\begin{array}{l} r f_\varepsilon^{(n+1)l} + a(x) f_\varepsilon^{(n+1)l} = q_{\varepsilon(n+1)}(x, f_\varepsilon^{(n)}) \\ f_{\varepsilon 1}^{(n+1)l} = A, f''_{\varepsilon(n+1)l} = B \\ r = \frac{a(x)h}{2} cth \frac{a(x)h}{2\varepsilon} \end{array}\right. \quad (17)$$

where $a(x)$ is a real-valued function.

Lemma 1. [29]. *For differential equations, we have the following:*

$$\begin{cases} \varepsilon f_\varepsilon'' + \alpha(x, f_\varepsilon, \varepsilon) f_\varepsilon' - \beta(x, f_\varepsilon, \varepsilon) = \gamma(x, f_\varepsilon, \varepsilon) \\ f_\varepsilon(a) = A(\varepsilon), f_\varepsilon(b) = B(\varepsilon), (a < x < b, a, b \in \mathbb{R}^+) \end{cases} \tag{18}$$

Let $f_\varepsilon(x)$ be its solution; then, the following conditions are satisfied:

(i) $\alpha(x, f_\varepsilon, \varepsilon)$ is only a symbolic expression;

(ii) If $|\alpha(x, f_\varepsilon| + \beta(x, f_\varepsilon \geq a \geq 0$, then $\|f_\varepsilon\|_\infty \leq \max(|A(\varepsilon)| + |B(\varepsilon)|) + \frac{1}{a}[(b-a) \times (b-a+1)]\|\gamma(x, f_\varepsilon, \varepsilon)\|_\infty$.

Lemma 2. [28]. *Assume that there exists a constant $C > 0$ that satisfies $\|a(x)\|_\infty \leq C, \|q_{\varepsilon(n+1)}(x, f_\varepsilon^{(n)})\|_\infty \leq C, \max\{|A|, |B|\} \leq C$; then, there exists a constant $M > 0$ related to only C that satisfies $\|f_\varepsilon^{(n)} - f_\varepsilon^{(n)h}\|_\infty \leq Mh$, where h is the divided grid spacing, $f_\varepsilon^{(n)}$ is the solution of (16), and $f_\varepsilon^{(n)l}$ is the solution of (17).*

Theorem 1. (Theorem of Uniform Convergence). *For (16), if the Lipschitz condition and Lemmas 1 and 2 are satisfied, then*

$$\|f_\varepsilon^{(n+1)l} - f_\varepsilon\|_\infty \leq \rho^{n+1}\|f_\varepsilon^{(0)l} - f_\varepsilon\|_\infty + \frac{M}{1-\rho} l \tag{19}$$

$$\|f_\varepsilon^{(n+1)l} - f_\varepsilon\|_\infty \leq \frac{1}{1-\rho}\|f_\varepsilon^{(n)l} - f_\varepsilon^{(n+1)l}\|_\infty + \frac{M}{1-\rho} l \tag{20}$$

where $\rho = \frac{3L}{a} < 1$,L is the Lipschitz constant.

Proof. Let $\bar{f}_\varepsilon^{(n+1)}$ be an iterative solution obtained by formulating $f_\varepsilon^{(n)l}$ as in (16). From Lemmas 1 and 2, we obtain

$$\begin{aligned} \|f_\varepsilon^{(n+1)l} - f_\varepsilon\|_\infty \quad & \leq \|f_\varepsilon^{(n+1)l} - \bar{f}_\varepsilon^{(n+1)}\|_\infty + \|\bar{f}_\varepsilon^{(n+1)} - f_\varepsilon\|_\infty \\ & \leq Ml + \rho\|f_\varepsilon^{(n)l} - f_\varepsilon\|_\varepsilon \\ & \leq \textstyle\sum_{k=0}^{n} Ml\rho^k + \rho^{n+1}\|f_\varepsilon^{(0)} - f_\varepsilon\|_\infty \\ & \leq \tfrac{Ml}{1-\rho} + \rho^{n+1}\|f_\varepsilon^{(0)} - f_\varepsilon\|_\infty \end{aligned} \tag{21}$$

and

$$\lim_{\rho \to 0} \frac{Ml}{1-\rho} + \rho^{n+1}\|f_\varepsilon^{(0)} - f_\varepsilon\|_\infty = Ml \tag{22}$$

$$\|f_\varepsilon^{(n+1)l} - f_\varepsilon\|_\infty \leq Ml \tag{23}$$

Thus, (19) is true. In addition,

$$\begin{aligned} \|f_\varepsilon^{(n)l} - f_\varepsilon\|_\infty \quad & \leq \|f_\varepsilon^{(n)l} - f_\varepsilon^{(n+1)l}\|_\infty + \|f_\varepsilon^{(n+1)l} - f_\varepsilon\|_\infty \\ & \leq \|f_\varepsilon^{(n)l} - f_\varepsilon^{(n+1)l}\|_\infty + Ml + \rho\|f_\varepsilon^{(n)l} - f_\varepsilon\|_\infty \end{aligned} \tag{24}$$

In addition to the above,

$$\lim_{\rho \to 0} \frac{1}{1-\rho}\|f_\varepsilon^{(n)l} - f_\varepsilon^{(n+1)l}\|_\infty + \frac{M}{1-\rho} l = Ml \tag{25}$$

$$\|f_\varepsilon^{(n)l} - f_\varepsilon\|_\infty \leq Ml \tag{26}$$

□

4. Topological Structure Implied in *Hilbert* Space of the *DE* Algorithm

4.1. Single-Point Topological Structure of Closed Populations in Hilbert Space

In the former part, we establish the nonlinear differential equation and its continuous iterative format according to the evolution process of the population, and we analyze the uniform convergence of the solution that illustrates the dynamical principle of population optimization in some way. In a closed ecological population of NP, which is necessarily bounded, we should further verify a situation logically if there exists an optimal solution under the condition of the perturbation variable $P_{-\varepsilon}$ after the population individuals are infinitely iterating. This is the single-point theorem that we introduce below. Since the closed population is a complete closed set under topological mapping, to analyze the topological properties conveniently, we introduce the inner product in the closed population so that the closed population is a *Hilbert* space. First, we introduce several lemmas.

Lemma 3. [30]. *The bounded set is a column compact set, and the arbitrary bounded closed set is a self-column compact set in* $\mathbb{R}^n$.

Lemma 4. [30]. *The arbitrary subset is a column compact set, and the arbitrary closed subspace is a self-column compact set in the column compact space.*

Lemma 5. [30]. *The column compact space must be a complete space.*

Lemma 6. (Brower Fixed-Point Theorem) [31]. *Let B be a closed unit ball, $T : B \to B$ be a continuous mapping, and $T(C)$ be column compact. Then, T must exist at a fixed point $x \in B$.*

Theorem 2. (Single-Point Theorem). *Let C be the closed population in* $\mathbf{R}$; *the mapping $T : C \to C$ is continuous. Then, there exists a single point in the closed population on C by the mapping T.*

Proof. To prove the theorem, we prove only that $T(C)$ is column-compact, as described in Lemma 6. □

Step 1 Because $T : C \to C$ is continuous and C is a compact set, we infer that T is uniformly continuous, that is, $\forall \varepsilon > 0, \exists \delta > 0$; then, $\|Tx - Tx^{'}\| < \varepsilon, \forall x, x^{'} \in C, \|x - x^{'}\| < \delta$. If not, the above indicates that $\exists \varepsilon_0 > 0, \forall n \in \mathbb{N}, \exists x_n, x_n^{'} \in C$ so that $\|x_n - x_n^{'}\| < \frac{1}{n}$, but $\|Tx_n - Tx_n^{'}\| \geq \varepsilon_0$. Because of C being a compact set, there exists a subsequence n_k so that $x_{n_k} \to x_0 \in C$. Since $\|x_{n_k} - x_{n_k}^{'}\| < \frac{1}{n_k} \to 0$, then $x_{n_k^{'}} \to x_0 \in C$. Since T is continuous, $Tx_{n_k} \to Tx_0, Tx_{n_k}^{'} \to Tx_0, (k \to \infty)$, which implies that $\|Tx_{n_k} - Tx_{n_k}^{'}\| \to 0, (k \to \infty)$, which contradicts $\|Tx_n - Tx_n^{'}\| \geq \varepsilon_0$.

Step 2 To prove that $T(C)$ is column-compact, we prove only that there is a limited ϵ net on $T(C)$ $\forall \varepsilon > 0$. First, from the step 1 proof, we have $\forall \varepsilon > 0, \exists \delta > 0$ so that $\|Tx - Tx^{'}\| < \varepsilon, \forall x, x^{'} \in C, \|x - x^{'}\| < \delta$. Second, due to C being a compact set, there is a limited ϵ net: $x_1, \cdots, x_n$ for $\delta > 0$. Third, we show that $\{Tx_1, \cdots, Tx_n\}$ is the limited ϵ net on $T(C)$. Actually, $\forall y \in TC, \exists x \in C$ so that $y = TC$. Let $\|x_i - x\| \leq \delta (1 \leq i \leq n)$ to obtain $\|Tx_i - Tx\| < \varepsilon$. In other words, the closed population has a single point on C by mapping T.

It is known that the complete space implied in the closed population includes only one single point that is considered the closed population optimal characteristic value according to the single-point theorem; then, the convergent iterative sequence generated by the algorithm itself can converge to a single point in the closed population. The theorem illustrates the inevitability of an existing optimal characteristic value in the complete closed population theoretically.

4.2. Branch Topological Structure of Closed Populations in Hilbert Space

There has been no definite research field focused on the route optimization branch theory of the closed population up until now. The single-point theorem indicates that there may be countless pieces of optimization routes, and it is not known how to associate the optimization routes with each other. However, it is certain that the different optimization routes are branched in *Hilbert* space implied in the closed population to generate the branch topology structure in *Hilbert* space, so that we can obtain the geometric structure of the closed population. First, we provide a fundamental theorem of *Fock* space F_m^2 [31,32] derived from *Hilbert* space; then, we can obtain the branch topological structure theorem of the *Hilbert* space implied in the closed population.

Theorem 3. *Let $\varphi : \mathbb{C} \to \mathbb{C}$ be an analytic mapping; for an arbitrary non-negative integer m, there exists the following:*

(a) C_φ is a bounded operator on F_m^2 if and only if $\varphi(z) = Az + B$. Here, $A \in \mathbb{M}_n, \|A\| \leq 1, B \in \mathbb{C}^n$, and when $\zeta \in \mathbb{C}^n$ and $|A\zeta| = |\zeta|$, $A\zeta \cdot \bar{B} = 0$.

(b) C_φ is a compact operator on F_m^2 if and only if $\varphi(z) = Az + B$. Here, $A \in \mathbb{M}_n, \|A\| < 1, B \in \mathbb{C}^n$.

We assume that for each positive integer k, we have $\mathbb{M}_k$ as the $k \times k$ complex matrix of the whole, which is equivalent to $A \in \mathbb{M}_k$ by a linear transformation $A : \mathbb{C}^k \to \mathbb{C}^k$.

Lemma 7. **[31,32]**. *Assume that $\varphi(z) = Az + B$ and $\psi(z) = A_1 z + B_1$ cause the composite operators C_φ and C_ψ to be bounded on F_m^2 if there exists $\zeta \in \mathbb{C}^n$ that satisfies $|A\zeta| = |\zeta|$, but $|A\zeta| \neq |A_1\zeta|$. Then, there exists a positive constant $C_e \in \mathbf{R}$ that satisfies $\|C_\varphi - C_\psi\| \geq C_e$.*

Lemma 8. **[31,32].** *Assume that $A \in \mathbb{M}_n, B \in \mathbb{C}^n$ causes C_{Az+B} to be bounded; then, C_{Az} and C_{Az+B} exist in the same path-connected branch of $\mathcal{C}_e(F_m^2)$.*

Theorem 4. (Theorem of a Branch Topological Structure). *Let C be the closed population in $\mathbf{R}$; the mapping $T : C \to C$ is continuous, and $\varphi(z) = Az + B$ and $\psi(z) = A_1 z + B_1$ cause the composite operators C_φ and C_ψ to be bounded on F_m^2. Then, the necessary and sufficient condition of C_φ and C_ψ belonging to the same path-connected branch in Hilbert space is that for all $\zeta \in \mathbb{C}^n$ satisfied by $|A\zeta| = |\zeta|$ or $|A\zeta| \neq |A_1\zeta|$, there generally exists $|A\zeta| = |A_1\zeta|$.*

Proof. If we have C_φ and C_ψ in the same path-connected branch of $\mathcal{C}_e(F_m^2)$, then there exists a limited quantity of composite operators $C_{\varphi_i}{}_{i=1}^{k+1}$ that satisfy $C_{\varphi_{k+1}} = C_\varphi, C_{\varphi_1} = C_\psi$, and $\|C_\varphi - C_\psi\|_e < \frac{C_e}{2}, C_e \in \mathbf{R}, \forall i = 1, 2, \cdots, k$. Let $\varphi_i(z) = A_i z + B_i, i = 1, 2, \cdots, k+1, A_{k+1} = A, B_{k+1} = A$; then, for all $\zeta \in \mathbb{C}^n$ satisfied by $|A\zeta| = |\zeta|$ and $|A\zeta| \neq |A_1\zeta|$, there generally exists $|A_{i+1}\zeta| = |A_i\zeta|$. Thus, the necessary of the theorem is satisfied. Otherwise, we need only consider the case of $\|A\| = \|A_1\| = 1$. For all $\zeta \in \mathbb{C}^n$ satisfied by $|A\zeta| = |\zeta|$ and $|A\zeta| \neq |A_1\zeta|$, let there generally exist $|A\zeta| = |A_1\zeta|$. According to Lemma 8, we can prove the conclusion as follows: if the norm $\|D\| < 1, 1 \leq k \leq n-1$ of the matrix $D \in \mathbb{M}_{n-k}$ and $P = \begin{pmatrix} E_K & O \\ O & O \end{pmatrix}$, $P_1 = \begin{pmatrix} E_K & O \\ O & D \end{pmatrix}$, then C_{P_z} and $C_{P_1 z}$ exist in the same path-connected branch of $\mathcal{C}_e(F_m^2)$. From singular value decomposition (SVD) of matrix D, we need to prove only that C_{Q_z} and $C_{Q_1 z}$ exist in the same path-connected branch of $\mathcal{C}_e(F_m^2)$, where $Q = \begin{pmatrix} E_K & O \\ O & O \end{pmatrix}$, $Q_1 = \begin{pmatrix} E_K & O \\ O & \Lambda \end{pmatrix}$, where Λ is a diagonal matrix and the ith diagonal element is the ith singular value $\sigma_{k+i}, 0 \leq \sigma_{k+i} < 1, 1 \leq i \leq n-k$ of D. For $z \in \mathbb{C}^n$ where $z = (z_k^{'}, z_{n-k}^{'})$, $z_k^{'} = (z_1, \cdots, z_1), z_{n-k}^{'} = (z_{k+1}, \cdots, z_n)$. Let $\varphi_t(z) = tQ_1 z + (1-t)Qz, y \in [0,1]$; then, $\varphi_t(z) = (z_k^{'}, t\Lambda z_{n-k}^{'})$. To prove that the route $t \mapsto C_{\varphi_t}$ is continuous under the essential norm, note that $(C_{\varphi_t} - C_{\varphi_s})f_\varepsilon(z) = f_\varepsilon(z_k^{'}, t\Lambda z_{n-k}^{'}) - f_\varepsilon(z_k^{'}, s\Lambda z_{n-k}^{'}) = \Sigma_l a_l (t_{\lfloor l \rfloor} - s_{\lfloor l \rfloor}) \sigma_{k+1}^{l_{k+1}} \cdots \sigma_n^{l_n} z_1^{l_1} \cdots z_n^{l_n}$, where $f_\varepsilon(z) = \Sigma_l a_l z_1^{l_1} \cdots z_n^{l_n}$, $\lfloor l \rfloor = k_{k+1} + \cdots + l_n$, then $|t^{\lfloor l \rfloor} - s^{\lfloor l \rfloor}| \leq \lfloor l \rfloor |t-s|, \forall t, s \in [0,1]$. Because of $\forall a \in [0,1)$, the function xa_x is bounded in $(0, \infty)$. Thus, there exists a positive constant M that satisfies $\lfloor l \rfloor \sigma_{k+1}^{l_{k+1}} \cdots \sigma_n^{l_n} \leq l_{k+1}\sigma_{k+1}^{l_{k+1}} + \cdots + l_n \sigma_n^{l_n} \leq M$. Consequently, $\|(C_{\varphi_t} - C_{\varphi_s})f_\varepsilon\|_m^2 \leq, M^2 |t-s|^2 \|f_\varepsilon\|_m^2$. Combined with (9), we obtain $\|C_{\varphi_t} - C_{\varphi_s}\|_e \leq M|t-s|$, and the theorem is proven. □

4.3. Discrete Topological Structure of Closed Populations in Hilbert Space

Theorem 5. *C_φ is discrete in Hilbert space implied in $\mathcal{C}_e(F_m^2)$ if and only if $\varphi(z) = Uz$, $\varphi(z) = Uz$, where U is the U-matrix.*

Proof. The adequacy of this theorem is obtained from Theorem 4; therefore, we prove only the necessity component of the theorem. If $A \in \mathbb{M}_n$ is a non-U-matrix and $\|A\| \leq 1$, from Lemma 8, we obtain that $\exists B \in \mathbb{C}^n \Rightarrow C_{Az+B}$ is bounded in F_m^2. Actually, we can consider the case of $\|A\| = 1$. Let the singular value decomposition (SVD) of A be $U\Lambda V$; then, Λ is a non-U-matrix. Furthermore, $\exists B' = 0 \in \mathbb{C}^n \Rightarrow C_{\Lambda z+B'}$ is bounded in F_m^2. Thus, C_φ is discrete in the *Hilbert* space as implied in $\mathcal{C}_e(F_m^2)$. □

Theorem 6. (Theorem of Discrete Topological Structure). *Let C be the closed population in* **R**, *the mapping $f_\varepsilon : C \to C$ be continuous, and β be a single point as described in Theorem 2, which is the optimal feature value of the convergent iterative sequence on the closed population C. Then, the single point must be a discrete point. Now, we can transform the original closed population C into a Hilbert space with the discrete parameter β by topological mapping; specifically, it is the β-Hilbert space.*

5. Quantum Characteristics of the *Heisenberg* Uncertainty Principle in β-Hilbert Space

The *Heisenberg* uncertainty principle is a fundamental principle of quantum mechanics that fundamentally illustrates that the position and momentum of a particle cannot be measured simultaneously in a quantum system; its basic form is $\Delta x \Delta p \geq \frac{h}{2}$, where h is the reduced *Planck* constant. When the *DE* algorithm pushes a closed population individual optimal process in β-Hilbert space, it measures the population individual in β-Hilbert space by the mutation, crossover, and selection of basic operational operators, which can screen the optimal characteristic value x^*. If we regard the entire β-Hilbert space as a complete space with the best signal source β, where each individual exhibits the characteristics of a better signal, then the signal source screened by the *DE* algorithm is the best of all of better signals, that is, it is the best signal source. Then, the quantity of information carried by each individual is related to the frequency of the best source and the information quantities of the best source retained by individuals that are convergent in probability F in the optimal time. With the optimization time gradually lengthening, the quantity of high-quality information carried by each individual in the convergent iterative sequence is continuously accumulated and gradually approaches that of the best signal source. There are two situations. One is that when slower the convergent speed of the iterative convergent sequence is slower, the speed of the high-quality information quantities carried by individuals accelerates is also slower, but the positional accuracy ε between the best source and population individuals is generally shrinking. Another situation is that when the convergent speed of the iterative convergent sequence is faster, and the high-quality information quantities carried by individuals in the population is reduced due to the spatial probability distributing unevenly, such that the positional accuracy ε between the best source and population individuals generally increases. Now, we provide a concrete representation of the quantum characteristics of the *Heisenberg* uncertainty principle of the *DE* algorithm in β-Hilbert space.

Definition 2. [33]. *For a $2n \times 2n$ matrix in symplectic groups, $Q = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, the linear canonical transformation of $f(q') \in L^2(\mathbb{R}^n)$ is defined as $\hat{f}(q) = [C(M)f](q) = \int_{\mathbb{R}^n} C(M)(q,q')f(q')dq'$, where $C(M)(q,q') = \frac{e^{(-\frac{in\pi}{4})}}{(\sqrt{2\pi})^n\sqrt{\det(B)}} \cdot e^{i(\frac{q^\top DB^{-1}q}{2} - q^\top (B^\top)^{-1}q' + \frac{q'^\top B^{-1}Aq'}{2})}$. Its inverse transform is $f(q') = [C(M_{-1})\hat{f}](q') = \int_{\mathbb{R}^n} C(M_{-1})^*(Q,Q')\hat{f}(q)dq$.*

Definition 3. (One-Dimensional Uncertainty Principle) [34]. *If f is a continuous function in Hilbert space, then its speed resolution Δ_v^2 and position resolution Δ_x^2 in Hilbert space are defined as*

$$\Delta_v^2 = \int_{\mathbb{R}_n} (v - v_0)^2 |f(v)|^2 dv, \Delta_x^2 = \int_{\mathbb{R}_n} (x - x_0)^2 |\hat{f}(x)|^2 dx$$

where $v_0 = \int_{\mathbb{R}_n} v|f(v)|^2 dv, x_0 = \int_{\mathbb{R}_n} x|\hat{f}(x)|^2 dx$; then, the Heisenberg uncertainty principle of the one-dimensional β-Hilbert space is $\Delta_v^2 \cdot \Delta_x^2 \geq \frac{b^2}{4}$.

Definition 4. *Let f_ε be a continuous-differential function defined in β-Hilbert space; then, its speed resolution Δ_v^2 and position resolution $\Delta_{x_\beta^\varepsilon}^2$ space are defined as*

$$\Delta_v^2 = \int_{\mathbb{R}_n} (v - v_0)^\top (v - v_0) |f_\varepsilon(v)|^2 dv, \Delta_{x_\beta^\varepsilon}^2 = \int_{\mathbb{R}_n} (x - x_0)^\top |\hat{f}_\varepsilon(x)|^2 dx \tag{27}$$

where $v^\top = (v_1, v_2, \cdots, v_n)^\top, v_0^\top = (\int_{\mathbb{R}_n} v_1 |f(v)|^2 dv, \cdots, \int_{\mathbb{R}_n} v_n |f(v)|^2 dv)^\top$,
$x^\top = (x_1, x_2, \cdots, x_n)^\top, x_0^\top = (\int_{\mathbb{R}_n} x_1 |f(x)|^2 dx, \cdots, \int_{\mathbb{R}_n} x_n |f(x)|^2 dx)^\top$.

Theorem 7. *Let f_ε be a continuous-differential function defined in β-Hilbert space and $f_\varepsilon(v_1, \cdots, v_n) \in L_2(\mathbb{R}_n), M \in Sp(2n, \mathbb{R})$. When $\det(B) \neq 0$, then we have the following equation:*

$$\begin{aligned} \Delta_v^2 \cdot \Delta_{x_\beta^\varepsilon}^2 &= \frac{\int_{\mathbb{R}_n} (v-v_0)^\top (v-v_0) |f_\varepsilon(v)|^2 dv}{\int_{\mathbb{R}_n} |f_\varepsilon(v)|^2 dv} \cdot \frac{\int_{\mathbb{R}_n} (x-x_0)^\top |\hat{f}_\varepsilon(x)|^2 dx}{\int_{\mathbb{R}_n} |\hat{f}_\varepsilon(x)|^2 dx} \\ &\geq \left(\frac{\sqrt{\lambda_1}}{2} + \cdots + \frac{\sqrt{\lambda_n}}{2}\right) \end{aligned} \tag{28}$$

where $v_0^\top = (\int_{\mathbb{R}_n} v_1 |f(v)|^2 dv, \cdots, \int_{\mathbb{R}_n} v_n |f(v)|^2 dv)^\top, x_0^\top = (\int_{\mathbb{R}_n} x_1 |f(x)|^2 dx, \cdots, \int_{\mathbb{R}_n} x_n |f(x)|^2 dx)^\top$, and λ_i is an eigenvalue of $B^\top B$.

Proof. Under the conditions of $\det(B) \neq 0$, assume that $v_0 = 0, x_0 = 0$. Then, we can obtain by using a linear canonical transform [35] that

$$\begin{aligned} \Delta_{x_\beta^\varepsilon}^2 &= \int_{\mathbb{R}_n} x^\top x |\hat{f}_\varepsilon(u)|^2 dx \\ &= \int_{\mathbb{R}_n} x^\top x | \int_{\mathbb{R}_n} f_\varepsilon(u) \frac{e^{-\frac{in\pi}{4}}}{(\sqrt{2\pi})^n \sqrt{\det(B)}} e^{-ix^\top (B^\top)^{-1} u + i\frac{u^\top B^{-1} Au}{2}} du|^2 dx \end{aligned} \tag{29}$$

Let $t = B^{-1}x$; then, we can obtain from an integral transform that

$$\Delta_{x_\beta^\varepsilon}^2 = \int_{\mathbb{R}_n} t^\top B^\top Bt \frac{1}{(2\pi)^n} | \int_{\mathbb{R}_n} f_\varepsilon(u) e^{-it^\top u + i\frac{u^\top B^{-1} Au}{2}} du|^2 dt \tag{30}$$

Now, we set $\widetilde{f_\varepsilon(u)} = f_\varepsilon(u) e^{i\frac{u^\top B^{-1} Au}{2}}$; then, there exists

$$\Delta_{x_\beta^\varepsilon}^2 = \int_{\mathbb{R}_n} t^\top B^\top Bt \frac{1}{(2\pi)^n} | \int_{\mathbb{R}_n} \widetilde{f_\varepsilon(u)} e^{-it^\top u} du|^2 dt \tag{31}$$

and

$$\Delta_v^2 = \int_{\mathbb{R}_n} u^\top u |f_\varepsilon(u)|^2 du = \int_{\mathbb{R}_n} u^\top u \widetilde{f_\varepsilon(u)}|^2 du \tag{32}$$

Because $\det(B) \neq 0$ and because $B^\top B$ is a symmetric positive definite matrix, using matrix spectral decomposition, we find that the existing orthogonal matrix P satisfies

$$B^\top B = P^\top \Lambda P \tag{33}$$

where Λ is a diagonal matrix and where elements distributed on the diagonal are the eigenvalues of $B^\top B$; then, there exists

$$\Delta^2_{x^\varepsilon_\beta} = \int_{\mathbb{R}_n} t^\top P^\top \Lambda P t \frac{1}{(2\pi)^n} | \int_{\mathbb{R}_n} \widetilde{f_\varepsilon(u)} e^{-it^\top u} du|^2 dt \tag{34}$$

Let $\omega = Pt$, conduct an integral transformation for (34), and let $u = P^\top y$; then, there exists

$$\begin{aligned} \Delta^2_{x^\varepsilon_\beta} &= \int_{\mathbb{R}_n} \omega^\top \Lambda \omega \frac{1}{(2\pi)^n} | \int_{\mathbb{R}_n} \widetilde{f_\varepsilon(u)} e^{-i\omega^\top P u} du|^2 dt \\ &= \int_{\mathbb{R}_n} \omega^\top \Lambda \omega \frac{1}{(2\pi)^n} | \int_{\mathbb{R}_n} f_\varepsilon \widetilde{(P^\top} y) e^{-i\omega^\top y} dy|^2 dt \end{aligned} \tag{35}$$

and

$$\Delta^2_v = \int_{\mathbb{R}_n} u^\top u |\widetilde{f_\varepsilon(u)}|^2 du = \int_{\mathbb{R}_n} y^\top y | f_\varepsilon \widetilde{(P^\top} y)|^2 dy \tag{36}$$

Let $f_\varepsilon \widetilde{(P^\top} y) = h(y)$; then, there exists

$$\Delta^2_{x^\varepsilon_\beta} = \int_{\mathbb{R}_n} \omega^\top \Lambda \omega \frac{1}{(2\pi)^n} | \int_{\mathbb{R}_n} h(y) e^{-i\omega^\top y} dy|^2 dt, \Delta^2_v = \int_{\mathbb{R}_n} y^\top y |h(y)|^2 dy \tag{37}$$

Furthermore, there exists

$$\begin{aligned} \Delta^2_v \cdot \Delta^2_{x^\varepsilon_\beta} &= \int_{\mathbb{R}_n} y^\top y |h(y)|^2 dy \cdot \int_{\mathbb{R}_n} \omega^\top \Lambda \omega \frac{1}{(2\pi)^n} | \int_{\mathbb{R}_n} h(y) e^{-i\omega^\top y} dy|^2 dt \\ &= \int_{\mathbb{R}_n} (y_1^2 + \cdots + y_n^2) |h(y)|^2 dy \\ &\times \int_{\mathbb{R}_n} (\lambda_1 \omega_1^2 + \cdots + \lambda_n \omega_n^2) \frac{1}{(2\pi)^n} | \int_{\mathbb{R}_n} h(y) e^{-i\omega^\top y} dy|^2 | d\omega \\ &= \int_{\mathbb{R}_n} (y_1^2 + \cdots + y_n^2) |h(y)|^2 dy \times \int_{\mathbb{R}_n} (\lambda_1 |\omega_1 \\ & \int_{\mathbb{R}_n} h(y) e^{2\pi i\omega^\top y} dy|^2 + \cdots + \lambda_n |\omega_n \int_{\mathbb{R}_n} h(y) e^{2\pi i \omega^\top y} dy|^2) d\omega \end{aligned} \tag{38}$$

where λ_i is the ith eigenvalue of $B^\top B$. Let $h_i = \frac{\partial h(y)}{\partial y_i}$; then, from the *Fourier* transformation property [35], there exists

$$\begin{aligned} \Delta^2_v \cdot \Delta^2_{x^\varepsilon_\beta} &= \int_{\mathbb{R}_n} (y_1^2 + \cdots + y_n^2) |h(y)|^2 dy \\ &\times \int_{\mathbb{R}_n} (\lambda_1 | \int_{\mathbb{R}_n} h_1(y) e^{2\pi i \omega^\top y} dy|^2 + \cdots + \lambda_n | \int_{\mathbb{R}_n} h_n(y) e^{2\pi i\omega^\top y} dy|^2) d\omega \end{aligned} \tag{39}$$

From the *Cauchy* inequality, we know that

$$\begin{aligned} \Delta^2_v \cdot \Delta^2_{x^\varepsilon_\beta} &\geq ((\int_{\mathbb{R}_n} y_1^2 |h(y)|^2 dy \cdot \int_{\mathbb{R}_n} \lambda_1 | \int_{\mathbb{R}_n} h_1(y) e^{2\pi i\omega^\top y} dy|^2 dy)^{\frac{1}{2}} \\ &+ \cdots + (\int_{\mathbb{R}_n} y_n^2 |h(y)|^2 dy \cdot \int_{\mathbb{R}_n} \lambda_1 | \int_{\mathbb{R}_n} h_n(y) e^{2\pi i\omega^\top y} dy|^2 dy)^{\frac{1}{2}})^2 \end{aligned} \tag{40}$$

Then, using the *Cauchy* inequality of integral form, we know that

$$\begin{aligned} \Delta^2_v \cdot \Delta^2_{x^\varepsilon_\beta} &\geq (\int_{\mathbb{R}_n} (|y_1 h(y) \sqrt{\lambda_1} h_1^*(y)| + \cdots + |y_n h(y) \sqrt{\lambda_n} h_n^*(y)|) dy)^2 \\ &= (\sqrt{\lambda_1} \int_{\mathbb{R}_n} (|y_1 h(y) h_1^*(y)| dy + \cdots + \sqrt{\lambda_n} \int_{\mathbb{R}_n} |y_n h(y) h_n^*(y)| dy)^2 \end{aligned} \tag{41}$$

From the one-dimensional uncertainty principle, we obtain

$$\int_{\mathbb{R}_n} |y_1 h(y) \sqrt{\lambda_1} h_1^*(y)| dy \geq \frac{1}{2} \tag{42}$$

To summarize, we obtain

$$\Delta^2_v \cdot \Delta^2_{x^\varepsilon_\beta} \quad \geq (\frac{\sqrt{\lambda_1}}{2} + \cdots + \frac{\sqrt{\lambda_n}}{2})^2 \tag{43}$$

□

6. Numerical Simulation

The above theorem fundamentally illustrates the geometric association between the convergent speed of the iterative sequence concerning the DE algorithm and the global optimal point precision. Specifically, Δ_v and $\Delta_{x_\beta^\varepsilon}$ are a pair of conjugate variables with quantum states, where the convergent speed of the iterative sequence caused by any improvement of the algorithm and the numerical accuracy of the global optimal point cannot be satisfied simultaneously. The above is a notably important conclusion for the DE algorithm. We use the SFEM (segmentation finite element method) to conduct a simple segmentation operation for β-Hilbert space and form a *Riemannian* manifold (Regarding *Riemannian* manifolds [36,37] in the β-Hilbert space, here, we mainly apply the space cluster caused by the wide-area property of *Riemannian* manifolds in *Hilbert* space. Then, we can improve the efficiency of algorithm optimization due to using the space cluster. In addition, because the *Riemannian* manifolds are more beneficial to the spatial segmentation operation by preventing the generation of singular points in space so that some points are omitted in the optimal process, we also consider the space quantum properties of *Riemannian* manifolds in the β-Hilbert space. Applying the *Riemannian* manifolds is a purely scientific method of mathematical physics in *Hilbert* space and is not intended to involve theoretical analysis of *Riemannian* manifolds) in the β-Hilbert region. The three topological structures implied in the β-Hilbert space of the DE algorithm are conducted by the operation of high-dimensional numerical simulation of quantum states to obtain the data of Tables 1–5 (In the Tables 1–5, $* \cdots *$ is the strength of the variable; a larger number implies a greater strength of the variable. Speed resolution is labeled as SR, position resolution as PR, relevancy of the finite unit element [38,39] as finite relevancy (FR), space dimension as (Dim), number population as (NP), mutational operation as (F), crossed operation as (CR), and elected operation as (X). The single-point topological structure, branch topological structure and discrete topological structure are labeled as $(SPTS)$, (BTS), and (DTS), respectively), (variance deviation rate (VDR) = (sample variance (SV)-population variance (PV))/PV; relevancy coefficient of the finite unit element as FR′ = SR's VDR + PR's VDR. If $FR' \in (0,1)$, then FR is true value 1; If $FR' = 0$, then FR is partial truth value $1--$; If $FR' = 1$, then FR is absolute truth value $1++$; EB is the error bounds about iterative points in population), such that one can determine the association between quantum characteristics of the *Heisenberg* uncertainty principle implied in β-Hilbert space of the DE algorithm and its topological structure.

Table 1. Quantum simulation of high-dimensional data tables describing the three topological structures of the DE algorithm implied in β-Hilbert space about Dim.

(Dim)	$(SPTS)$			(BTS)			(DTS)		
	$SR(\%)$	$PR(\%)$	FR	$SR(\%)$	$PR(\%)$	FR	$SR(\%)$	$PR(\%)$	FR
10^2	9.500	0.400	1	6.660	0.558	1	8.214	0.325	1
10^3	9.230	0.422	1	7.242	0.500	1	8.225	0.214	1
10^4	8.471	0.500	1	8.011	0.500	1	9.535	0.110	1
10^5	7.620	0.558	1	9.763	0.500	1	9.774	0.012	1
10^6	6.101	0.660	1	9.763	0.500	1	9.896	0.011	1
10^7	5.310	0.793	1	9.880	0.500	1	9.977	0.010	1
SV	2.89009	0.02248	/	2.05701	0.00056	/	0.68463	0.01727	/
PV	2.40841	0.01874	/	1.71418	0.00047	/	0.57052	0.01439	/
DVR	+0.20	+0.32	1	+0.34	+0.19	1	+0.11	+0.16	1
EB	±0.5	±0.5	/	±0.5	±0.5	/	±0.5	±0.5	/

Table 2. Quantum simulation of high-dimensional data tables describing the three topological structures of the DE algorithm implied in β-Hilbert space about NP.

(NP)	$(SPTS)$			(BTS)			(DTS)		
	$SR(\%)$	$PR(\%)$	FR	$SR(\%)$	$PR(\%)$	FR	$SR(\%)$	$PR(\%)$	FR
10×10	8.687	0.793	1	5.243	0.021	1	8.688	0.029	1
10×20	8.756	0.660	1	7.242	0.021	1	8.744	0.025	1
10×30	8.863	0.558	1	9.011	0.020	1	8.880	0.013	1
10×40	8.880	0.500	1	9.763	0.015	1	8.863	0.009	1
10×50	9.000	0.500	1	9.841	0.010	1	9.010	0.005	1
10×60	9.010	0.500	1	9.865	0.010	1	9.101	0.004	1
SV	0.18447	0.01426	/	3.54164	0.00003	/	0.02428	0.00011	/
PV	0.15373	0.01188	/	2.95137	0.00002	/	0.02023	0.00009	/
DVR	+0.20	+0.20	1	+0.19	+0.50	1	+0.20	+0.22	1
EB	±0.5	±0.5	/	±0.5	±0.5	/	±0.5	±0.5	/

Table 3. Quantum simulation of high-dimensional data tables describing the three topological structures of the DE algorithm implied in β-Hilbert space about F.

(F)	$(SPTS)$			(BTS)			(DTS)		
	$SR(\%)$	$PR(\%)$	FR	$SR(\%)$	$PR(\%)$	FR	$SR(\%)$	$PR(\%)$	FR
0.2	6.786	0.660 ± 0.0001	1	2.652	2.558	1	11.749	0.005	1
0.3	6.020	0.877 ± 0.0001	1	2.641	2.560	1	11.744	0.009	1
0.4	6.020	0.966 ± 0.0001	1	2.633	2.559	1	11.126	0.010	1
0.5	5.852	1.101 ± 0.0001	1	2.620	2.660	1	9.535	0.010	1
0.6	5.633	1.210 ± 0.0001	1	2.619	2.676	1	9.535	0.015	1
0.7	5.330	1.220 ± 0.0001	1	2.618	2.881	1	9.535	0.055	1
SV	0.24052	0.04688	/	0.0002	0.0158	/	0.02428	0.00035	/
PV	0.20043	0.03907	/	0.00016	0.01316	/	0.02023	0.00029	/
DVR	+0.20	+0.19	1	+0.25	+0.20	1	+0.20	+0.20	1
EB	±0.5	±0.5	/	±0.5	±0.5	/	±0.5	±0.5	/

Table 4. Quantum simulation of high-dimensional data tables describing the three topological structures of the DE algorithm implied in β-Hilbert space about CR.

(CR)	$(SPTS)$			(BTS)			(DTS)		
	$SR(\%)$	$PR(\%)$	FR	$SR(\%)$	$PR(\%)$	FR	$SR(\%)$	$PR(\%)$	FR
0^{*}	0.430	0.991	$1++$	7.002	0.990	$1++$	/	0.000	$1--$
0^{**}	0.520	0.853	$1++$	7.242	0.960	$1++$	/	0.000	$1--$
0^{***}	0.522	0.793	$1++$	7.620	0.960	$1++$	/	0.000	$1--$
SV	0.00276	0.01031	/	0.09707	0.0003	/	/	0	/
PV	0.00184	0.00687	/	0.06471	0.0002	/	/	0	/
DVR	+0.50	+0.50	$1++$	+0.50	+0.50	$1++$	/	0	$1--$
EB	±0.5	±0.5	/	±0.5	±0.5	/	/	±0.5	/
1^{*}	1.233	0.099	1	9.855	0.010 ± 0.0001	1	11.144	0.397	1
1^{**}	1.122	0.124	1	9.676	0.500 ± 0.0001	1	11.126	0.542	1
1^{***}	1.010	0.500	1	9.110	0.500 ± 0.0001	1	10.250	0.633	1
SV	0.01243	0.05047	/	0.15124	0.08003	/	0.26116	0.01417	/
PV	0.00829	0.03364	/	0.10082	0.05336	/	0.1741	0.00944	/
DVR	+0.49	+0.49	1	+0.50	+0.03	1	+0.50	+0.20	1
EB	±0.5	±0.5	/	±0.5	±0.5	/	±0.5	±0.5	/

Table 5. Quantum simulation of high-dimensional data tables describing the three topological structures of the DE algorithm implied in β-Hilbert space about X.

(X)	$(SPTS)$			(BTS)			(DTS)		
	$SR(\%)$	$PR(\%)$	FR	$SR(\%)$	$PR(\%)$	FR	$SR(\%)$	$PR(\%)$	FR
0 *	0.523	0.796	$1++$	7.002	0.081	$1++$	/	0.000	$1--$
0 **	0.550	0.788	$1++$	7.242	0.055	$1++$	/	0.000	$1--$
0 ***	0.599	0.721	$1++$	7.620	0.001	$1++$	/	0.000	$1--$
SV	0.00035	0.0017	/	0.09707	0.00167	/	/	0	/
PV	0.00023	0.00113	/	0.06471	0.00111	/	/	0	/
VDR	+0.52	+0.50	$1++$	+0.50	+0.50	$1++$	/	0	$1--$
EB	±0.5	±0.5	/	±0.5	±0.5	/	/	±0.5	/
1 *	0.997	0.499	1	9.855	0.723	$1++$	11.144	0.551	$1++$
1 **	0.947	0.500	1	9.676	0.956	$1++$	11.126	0.640	$1++$
1 ***	0.930	0.110	1	9.110	0.990	$1++$	10.250	0.688	$1++$
SV	0.00121	0.05057	/	0.15124	0.02112	/	0.26116	0.00483	/
PV	0.00081	0.03371	/	0.10082	0.01408	/	0.1741	0.00322	/
VDR	+0.49	+0.50	1	+0.50	+0.50	$1++$	+0.50	+0.50	$1++$
EB	±0.5	±0.5	/	±0.5	±0.5	/	±0.5	±0.5	/

Because of the uncertainty of random algorithm, except for several dead points and invalid points, we conduct a quantitative analysis of Tables 1–5 to ensure the regularity of data analysis.

For **Dim**, we set the dimensions to increase with common ratio of 10. Firstly, we analyze the relationship between SR and PR about the SPTS: when the dimension increases, the SR decreases gradually with a variance deviation rate $+0.20$, then the SR of iterative individuals decreases gradually; and the PR increases gradually with a variance deviation rate $+0.32$, then the PR of iterative individual increases gradually. The above case shows that FR is true value 1, then SR and PR are completely inverse correlation.

Then, we analyze the relationship between SR and PR about the BTS: when the dimension increases, the SR increases gradually with a variance deviation rate $+0.34$, then the SR of iterative individuals increases gradually; and the PR decreases gradually with a variance deviation rate $+0.19$, then the PR of iterative individuals decreases gradually. The above case shows that FR is true value 1, then SR and PR are completely inverse correlation.

Finally, we analyze the relationship between SR and PR about the DTS: when the dimension increases, the SR increases gradually with a variance deviation rate $+0.11$, then the SR of iterative individuals increases gradually; and the PR decreases gradually with a variance deviation rate $+0.16$, then the PR of iterative individuals decreases gradually. The above case shows that FR is true value 1, then SR and PR are completely inverse correlation.

Similarly, we quantitatively analyze the **NP**. We set the NP to increase with tolerance of 100. Firstly, we analyze the relationship between SR and PR about the SPTS: when the NP increases, the SR increases gradually with a variance deviation rate $+0.20$, then the SR of iterative individuals increases gradually; and the PR decreases gradually with a variance deviation rate $+0.20$, then the PR of iterative individuals decreases gradually. The above case shows that FR is true value 1, then SR and PR are completely inverse correlation.

Then, we analyze the relationship between SR and PR about the BTS: when the NP increases, the SR increases gradually with a variance deviation rate $+0.19$, then the SR of iterative individuals increases gradually; and the PR decreases gradually with a variance deviation rate $+0.50$, then the PR of iterative individuals decreases gradually. The above case shows that FR is true value 1, then SR and PR are completely inverse correlation.

Finally, we analyze the relationship between SR and PR about the DTS: when the NP increases, the SR increases gradually with a variance deviation rate $+0.20$, then the SR of iterative individuals increases gradually; and the PR decreases gradually with a variance deviation rate $+0.22$, then the PR of iterative individuals decreases gradually. The above case shows that FR is true value 1, then SR and PR are completely inverse correlation.

Similarly, we quantitatively analyze the **F**. We set the F to increase with tolerance of 0.1. Firstly, we analyze the relationship between SR and PR about the SPTS: when the F increases, the SR decreases gradually with a variance deviation rate $+0.20$, then the SR of iterative individuals decreases gradually; and the PR increases gradually with a variance deviation rate $+0.19$, then the PR of iterative individual increases gradually. The above case shows that FR is true value 1, then SR and PR are completely inverse correlation.

Then, we analyze the relationship between SR and PR about the BTS: when the F increases, the SR decreases gradually with a variance deviation rate $+0.25$, then the SR of iterative individuals decreases gradually; and the PR increases gradually with a variance deviation rate $+0.20$, then the PR of iterative individual increases gradually. The above case shows that FR is true value 1, then SR and PR are completely inverse correlation.

Finally, we analyze the relationship between SR and PR about the DTS: when the F increases, the SR decreases gradually with a variance deviation rate $+0.20$, then the SR of iterative individuals decreases gradually; and the PR increases gradually with a variance deviation rate $+0.20$, then the PR of iterative individual increases gradually. The above case shows that FR is true value 1, then SR and PR are completely inverse correlation.

Similarly, we quantitatively analyze the **CR**. We divide the CR into two cases that the one is absolute crossover and the other is non-crossover, which are represented by '1' and '0' respectively. Firstly, we analyze the relationship between SR and PR about the SPTS: under the condition of the latter,when the intensity of CR increases gradually, the SR increases gradually with a variance deviation rate $+0.50$, then the SR of iterative individuals increases gradually; and the PR decreases gradually with a variance deviation rate $+0.50$, then the PR of iterative individuals decreases gradually. The above case shows that FR is absolute true value $1++$, then SR and PR are absolutely and completely inverse correlation. Under the condition of the former, when the intensity of CR increases gradually, the SR decreases gradually with a variance deviation rate $+0.49$, then the SR of iterative individuals decreases gradually; and the PR increases gradually with a variance deviation rate $+0.49$, then the PR of iterative individual increases gradually. The above case shows that FR is true value 1, then SR and PR are completely inverse correlation.

Then, we analyze the relationship between SR and PR about the BTS: under the condition of the latter,when the intensity of CR increases gradually, the SR increases gradually with a variance deviation rate $+0.50$, then the SR of iterative individuals increases gradually; and the PR decreases gradually with a variance deviation rate $+0.50$, then the PR of iterative individuals decreases gradually. The above case shows that FR is absolute true value $1++$, then SR and PR are absolutely and completely inverse correlation. Under the condition of the former, when the intensity of CR increases gradually, the SR decreases gradually with a variance deviation rate $+0.50$, then the SR of iterative individuals decreases gradually; and the PR increases gradually with a variance deviation rate $+0.03$, then the PR of iterative individual increases gradually. The above case shows that FR is true value 1, then SR and PR are completely inverse correlation.

Finally, we analyze the relationship between SR and PR about the DTS: under the condition of the latter, when the intensity of CR increases gradually, the variance deviation rate of SR does not exist, and the variance deviation rate of PR is 0, which shows that there are been no change in the individual diversity of the original population, and SR and PR are no change. The above case shows that FR is partial truth value $1--$, then SR and PR are relatively inverse correlation. Under the condition of the former, when the intensity of CR increases gradually, the SR decreases gradually with a variance deviation rate $+0.50$, then the SR of iterative individuals decreases gradually; and the PR increases gradually with a variance deviation rate $+0.20$, then the PR of iterative individual increases gradually. The above case shows that FR is true value 1, then SR and PR are completely inverse correlation.

Similarly, we quantitatively analyze the **X**. We divide the CR into two cases that the one is absolute choice and the other is non-choice, which are represented by '1' and '0' respectively. Firstly, we analyze the relationship between SR and PR about the SPTS: under the condition of the latter, when the intensity of X increases gradually, the SR increases gradually with a variance deviation rate $+0.52$, then the SR of iterative individuals increases gradually; and the PR decreases gradually with a variance deviation rate $+0.50$, then the PR of iterative individuals decreases gradually. The above case shows that FR is absolute true value $1++$, then SR and PR are absolutely and completely inverse correlation. Under the condition of the former, when the intensity of X increases gradually, the SR decreases gradually with a variance deviation rate $+0.49$, then the SR of iterative individuals decreases gradually; and the PR increases gradually with a variance deviation rate $+0.50$, then the PR of iterative individual increases gradually. The above case shows that FR is true value 1, then SR and PR are completely inverse correlation.

Then, we analyze the relationship between SR and PR about the BTS: under the condition of the latter, when the intensity of X increases gradually, the SR increases gradually with a variance deviation rate $+0.50$, then the SR of iterative individuals increases gradually; and the PR decreases gradually with a variance deviation rate $+0.50$, then the PR of iterative individuals decreases gradually. The above case shows that FR is absolute true value $1++$, then SR and PR are absolutely and completely inverse correlation. Under the condition of the former, when the intensity of X increases gradually, the SR decreases gradually with a variance deviation rate $+0.50$, then the SR of iterative individuals decreases gradually; and the PR increases gradually with a variance deviation rate $+0.50$, then the PR of iterative individual increases gradually. The above case shows that FR is absolute true value $1++$, then SR and PR are absolutely and completely inverse correlation.

Finally, we analyze the relationship between SR and PR about the DTS: under the condition of the latter, when the intensity of X increases gradually, the variance deviation rate of SR does not exist, and the variance deviation rate of PR is 0, which shows that there are been no change in the individual diversity of the original population, and SR and PR are no change. The above case shows that FR is partial truth value $1--$, then SR and PR are relatively inverse correlation. Under the condition of the former, when the intensity of X increases gradually, the SR decreases gradually with a variance deviation rate $+0.50$, then the SR of iterative individuals decreases gradually; and the PR increases gradually with a variance deviation rate $+0.50$, then the PR of iterative individual increases gradually. The above case shows that FR is absolute true value $1++$, then SR and PR are absolutely and completely inverse correlation.

We conduct a qualitative analysis of Tables 1–5 as well. Except for several dead points and invalid points, under the condition of spatial dimension, the number of the population, mutated operator, crossover operator, and selected operator are generally decreasing or increasing; correspondingly, the speed changing rate of individual iterative points and the position changing rate of global optimal point β exhibit a inverse correlation in β-Hilbert space, which illustrates the association between the *Heisenberg* uncertainty quantum characteristics and its topological structure implied in the β-Hilbert space of the *DE* algorithm. Specifically, the association of the convergent iterative sequence and the global optimal point precision is a pair of conjugate variables on the quantum states in β-Hilbert space with the uncertainty characteristics on quantum states. It is fundamentally explained that any improvement in the algorithm cannot pursue the bidirectional efficiency between the convergent speed and the optimal point precision.

7. Conclusions

This paper mainly discusses the continuity structure of closed populations and the control convergent properties of the iterative sequences of the *DE* algorithm under the condition of $P_{-\varepsilon}$, establishes and analyzes the single-point topological structure, branch topological structure, and discrete topological structure implied in β-Hilbert space of the *DE* algorithm, verifies the association between the *Heisenberg* uncertainty quantum characteristics and its topological structure implied in the β-Hilbert space of the *DE* algorithm, and obtains the specific directions of the quantum uncertainty characters of the *DE* algorithm in β-Hilbert space by quantum simulation of high-dimensional data. The findings are that the speed resolution Δ_v^2 of the iterative sequence convergent speed and the position resolution $\Delta_{x_\beta^\varepsilon}$ of the global optimal point with the swinging range are a pair of conjugate variables of the quantum states in β-Hilbert space, corresponding to uncertainty characteristics of quantum states; they cannot simultaneously achieve bidirectional efficiency between the convergent speed and the best point precision with any procedural improvements. Because they are geometric features of *Riemannian* manifolds in the view of operator optimization in *Hilbert* space theoretically, however, which is only a theoretical guess, the quantum characters of the pair of conjugate variables in the *Riemannian* space require further exploration.

We all know that the most important theoretical research of meta-heuristic algorithm is how to balance the convergence speed and accuracy of the iterative points better to ensure that the iterative process is more efficient, when the iterative points approaches the global optimal point. We get the quantum uncertainty properties of the *DE*algorithm in the *beta-Hilbert*space by theoretical analysis. In the future, we will discuss the quantum estimation form and its asymptotic estimation form between convergent speed and convergent accuracy of iterative points by numerical simulation, which will lay a solid mathematical foundation for the convergent mechanism of meta-heuristic algorithm. Our second work in the future is to study the computational structure and physical structure of differential evolution algorithm, including computational complexity, spatial complexity, time tensor expansion, convergent analysis, quantum transformation state structure, *Heisenberg* uncertainty quantum state, dynamic torque analysis and so on, which will become the physical basis of the convergent theory of meta-heuristic algorithm.

Author Contributions: The first author has solved the following problems: 1. The continuity of the closed population in the condition of $P_{-\varepsilon}$ and the control convergent properties of its iterative sequence; 2. The topological structure implied in the *Hilbert* space of the *DE* algorithm; 3. The *Heisenberg* uncertainty quantum characteristics implied in the β-Hilbert space of the *DE* algorithm; And the same time,the first author implements numerical simulation of quantum inequalities for differential evolutionary algorithm. The correspondent Author reviewed the logical structure and data specification of the paper, and gave academic guidance to the first and second parts.

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