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Cayley Inclusion Problem Involving XOR-Operation

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Abstract: In this paper, we study an absolutely new problem, namely, the Cayley inclusion problem which involves the Cayley operator and a multi-valued mapping with XOR-operation. We have shown that the Cayley operator is a single-valued comparison and it is Lipschitz-type-continuous. A fixed point formulation of the Cayley inclusion problem is shown by using the concept of a resolvent operator as well as the Yosida approximation operator. Finally, an existence and convergence result is proved. An example is constructed for some of the concepts used in this work.

Keywords: Cayley; Convergence; Resolvent; XOR-operation; Yosida

MSC: 47H05; 49H10; 47J25

1. Introduction

It is well known that inclusion problems were introduced and studied as a generalization of equilibrium problems, which include a vast range of problems in analysis such as variational inequalities, vector optimization, game theory, fixed point problems, the Nash equilibrium problem, complementary problems, traffic equilibrium problems, economics, etc., see [1–3]. It is interesting to note that the term "Variational inclusion", is understood with different aspects in several works. That is, it means simply multi-valued variational inequalities in [4,5] and the problem of finding the zeros of maximal monotone mappings in [6–8], etc. Variational inclusions involving different kinds of operators are useful and have a wide range of applications in industry, mathematical finance, decision sciences, ecology, engineering sciences, etc., see [9–15].

Due to the fact that the projection methods cannot be used to solve variational inclusion problems, the resolvent operator methods came into the picture to solve them efficiently. It is also known that the monotone operators in abstract spaces can be regularized into single-valued Lipschitzian monotone operator through a process known as Yosida approximation, see [16–19].

The XOR-operation $\oplus$ is a binary operation and behaves like ADD operation, which is associative as well as commutative. XOR-operation depicts interesting facts and observations and forms various real time applications, i.e., data encryption, error detection in digital communication, parity check and helps to implement multi-layer perception in neural networks.

Many problems related to ordered variational inequalities and ordered equations were studied by H.G.Li together with his co-authors, see [20–24] and I.Ahmad with his co-authors, see [25,26]. Considering all the facts mentioned above, in this paper, we introduce and study a quite new and interesting problem which we call Cayley inclusion problem involving XOR-operation. The Cayley inclusion problem involves a Cayley operator and a multi-valued mapping. We have shown some properties of the Cayley operator, that is, it is single-valued, comparison as well as

Lipschitz-type-continuous. A fixed point formulation of the Cayley inclusion problem is given by using the concept of resolvent operator and Yosida approximation operator. An iterative algorithm is established and finally an existence and convergence result is proved for the Cayley inclusion problem involving XOR-operation. An example is constructed to illustrate some of the concepts used in this paper.

2. Preliminaries

Throughout this paper, we suppose that $\mathcal{H}$ is a real ordered Hilbert space endowed with a norm $\|.\|$ and an inner product $\langle\cdot,\cdot\rangle$, d is the metric induced by the norm $\|.\|$ and $2^{\mathcal{H}}$ is the family of all nonempty subsets of $\mathcal{H}$.

Now, we recall some known concepts are results which we need to prove the main result of this paper and can be found in [22–24,27,28].

Definition 1. *A nonempty closed convex subset C of $\mathcal{H}$ is said to be a cone, if*

(i) *for any $x \in C$ and $\lambda > 0$, then $\lambda x \in C$,*
(ii) *for any $x, \in C$ and $-x \in C$, then $x = 0$.*

Definition 2. *A cone C is said to be normal if and only if, there exist a constant $\lambda_N > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq \lambda_N\|y\|$, where λ_N is the normal constant of C.*

Definition 3. *Let C be a cone. For arbitrary element $x, y \in \mathcal{H}$, $x \leq y$ holds if and only if, $x - y \in C$. The relation " $\leq$ " in $\mathcal{H}$ is called partial ordered relation.*

Definition 4. *For arbitrary elements $x, y \in \mathcal{H}$, if $x \leq y$ (or $y \leq x$) holds, then x and y are said to be comparable to each other (denoted by $x \propto y$).*

Definition 5. *For arbitrary elements $x, y \in \mathcal{H}$, $lub\{x, y\}$ and $glb\{x, y\}$ means least upper bound and greatest upper bound of the set $\{x, y\}$. Suppose $lub\{x, y\}$ and $glb\{x, y\}$ exist, then some binary operations are defined as follows:*

(i) $x \vee y = lub\{x, y\}$,
(ii) $x \wedge y = glb\{x, y\}$,
(iii) $x \oplus y = (x - y) \vee (y - x)$,
(iv) $x \odot y = (x - y) \wedge (y - x)$.

The operations $\vee$, $\wedge$, $\oplus$ and $\odot$ are called OR, AND, XOR and XNOR operations, respectively.

Lemma 1. *If $x \propto y$, then $lub\{x, y\}$ and $glb\{x, y\}$ exist, $x - y \propto y - x$ and $0 \leq (x - y) \vee (y - x)$.*

Lemma 2. *For any natural number n, $x \propto y_n$ and $y_n \to y^*$ as $n \to \infty$, then $x \propto y^*$.*

Proposition 1. *Let $\oplus$ be an XOR-operation and $\odot$ be an XNOR-operation. Then the following relations hold:*

(i) $x \odot x = 0, x \odot y = y \odot x = -(x \oplus y) = -(y \oplus x)$,
(ii) *if $x \propto 0$, then $-x \oplus 0 \leq x \leq x \oplus 0$,*
(iii) $(\lambda x) \oplus (\lambda y) = |\lambda|(x \oplus y)$,
(iv) $0 \leq x \oplus y$, *if $x \propto y$,*
(v) *if $x \propto y$, then $x \oplus y = 0$ if and only if $x = y$,*
(vi) $(x + y) \odot (u + v) \geq (x \odot u) + (y \odot v)$,
(vii) $(x + y) \odot (u + v) \geq (x \odot v) + (y \odot u)$,
$(viii)$ *if x, y and w are comparable to each other, then $(x \oplus y) \leq x \oplus w + w \oplus y$,*
(ix) $\alpha x \oplus \beta x = |\alpha - \beta|x = (\alpha \oplus \beta)x$, *if $x \propto 0$,* $x, y, u, v \in \mathcal{H}$ *and* $\alpha, \beta, \lambda \in \mathbb{R}$.

Proposition 2. *Let C be a normal cone in $\mathcal{H}$ with normal constant λ_N, then for each $x, y \in \mathcal{H}$, the following relations hold:*

(i) $\|0 \oplus 0\| = \|0\| = 0,$
(ii) $\|x \vee y\| \leq \|x\| \vee \|y\| \leq \|x\| + \|y\|,$
(iii) $\|x \oplus y\| \leq \|x - y\| \leq \lambda_N |x \oplus y\|,$
(iv) *if* $x \propto y$, *then* $\|x \oplus y\| = \|x - y\|.$

Definition 6. *Let $A : \mathcal{H} \to \mathcal{H}$ be a single-valued mapping.*

(i) *A is said to be a comparison mapping if $x \propto y$ then $A(x) \propto A(y)$, $x \propto A(x)$ and $y \propto A(y)$, for all $x, y \in \mathcal{H}$,*
(ii) *A is said to be strongly comparison mapping, if A is a comparison mapping and $A(x) \propto A(y)$ if and only if $x \propto y$, for all $x, y \in \mathcal{H}$.*

Definition 7. *A mapping $A : \mathcal{H} \to \mathcal{H}$ is said to be β-ordered comparison mapping, if A is comparison mapping and*

$$A(x) \oplus A(y) \leq \beta(x \oplus y), \; for \; 0 \leq \beta \leq 1, \; for \; all \; x, y \in \mathcal{H}.$$

Definition 8. *Let $M : \mathcal{H} \to 2^{\mathcal{H}}$ be a multi-valued mapping. Then*

(i) *M is said to be a comparison mapping, if for any $v_x \in M(x)$, $x \propto v_x$, and if $x \propto y$, then for any $v_x \in M(x)$ and any $v_y \in M(y)$, $v_x \propto v_y$, for all $x, y \in \mathcal{H}$,*
(ii) *a comparison mapping M is said to be α-non-ordinary difference mapping, if for each $x, y \in \mathcal{H}$, $v_x \in M(x)$ and $v_y \in M(y)$ such that*

$$(v_x \oplus v_y) \oplus \alpha(x \oplus y) = 0,$$

(iii) *a comparison mapping M is said to be γ-ordered rectangular, if there exists a constant $\gamma > 0$, and for any $x, y \in \mathcal{H}$, there exist $v_x \in M(x)$ and $v_y \in M(y)$ such that*

$$\langle v_x \odot v_y, -(x \oplus y) \rangle \geq \gamma \|x \oplus y\|^2,$$

holds.
(iv) *M is said to be weak comparison mapping, if for any $x, y \in \mathcal{H}$, $x \propto y$, then there exist $v_x \in M(x)$ and $v_y \in M(y)$ such that $x \propto v_x$, $y \propto v_y$ and $v_x \propto v_y$.*
(v) *M is said to be λ-weak ordered different comparison mapping, if there exist a constant $\lambda > 0$ such that for any $x, y \in \mathcal{H}$, there exist $v_x \in M(x)$, $v_y \in M(y)$, $\lambda(v_x - v_y) \propto (x - y)$ holds.*
(vi) *a weak comparison mapping $M : \mathcal{H} \to 2^{\mathcal{H}}$ is said to be a (γ, λ)-weak ordered rectangular different multi-valued mapping, if M is a γ-ordered rectangular and λ-weak ordered different comparison mapping and $[I + \lambda M](\mathcal{H}) = \mathcal{H}$, for $\lambda > 0$.*

Definition 9. *Let $M : \mathcal{H} \to 2^{\mathcal{H}}$ be a multi-valued mapping. The operator $R^M_{I,\lambda} : \mathcal{H} \to \mathcal{H}$ defined by*

$$R^M_{I,\lambda}(x) = [I + \lambda M]^{-1}(x), \; for \; all \; x \in \mathcal{H}, \tag{1}$$

is called the resolvent operator associated with M, where $\lambda > 0$ is a constant.

It is well known that the resolvent operator associated with M is single-valued.

Definition 10. *The Yosida approximation operator $J^M_{I,\lambda}$ associated with M is defined by*

$$J^M_{I,\lambda}(x) = \frac{1}{\lambda}[I - R^M_{I,\lambda}](x), \; for \; all \; x \in \mathcal{H}, \tag{2}$$

where $\lambda > 0$ is a constant.

Now we define the Cayley operator based on resolvent operator (1)

Definition 11. *The Cayley operator $C^M_{I,\lambda}$ of M is defined as:*

$$C^M_{I,\lambda}(x) = [2R^M_{I\lambda} - I](x), \ \ for\ all\ x \in \mathcal{H}, \tag{3}$$

where I is the identity operator.

Proposition 3. *Let $M : \mathcal{H} \to 2^{\mathcal{H}}$ is a γ-ordered rectangular multi-valued mapping. Then, the Yosida approximation operator $J^M_{I,\lambda}$ is single-valued, for $\lambda > 0$.*

Proof. For the proof we refer to [25]. □

Proposition 4. *Let $M : \mathcal{H} \to 2^{\mathcal{H}}$ is a γ-ordered rectangular multi-valued mapping. Then, the Cayley operator $C^M_{I,\lambda}$ associated with M is single valued, for $\gamma\lambda > 1$.*

Proof. Let $x, y \in C^M_{I,\lambda}(u)$. Then

$$\begin{aligned} x \in C^M_{I,\lambda}(u) &= (2R^M_{I,\lambda} - I)(u), \\ \frac{1}{2}(x+u) \in R^M_{I,\lambda}(u) &= [I + \lambda M]^{-1}(u), \\ i.e.,\ u &\in \frac{1}{2}(x+u)[I+\lambda M], \\ u &\in \frac{1}{2}(x+u) + \frac{1}{2}\lambda M(x+u), \\ 2u &\in (x+u) + \lambda M(x+u), \\ u - x &\in \lambda M(x+u). \end{aligned}$$

Thus $\frac{1}{\lambda}(u-x) \in M(z_1)$, where $z_1 = x+u$. Let $\frac{1}{\lambda}(u-x) = v_{z_1}$, then $v_{z_1} \in M(z_1)$. Similarly, for $y \in C^M_{I,\lambda}(u)$, we have $v_{z_2} \in M(z_2)$, where

$$v_{z_2} = \frac{1}{\lambda}(u-y)\ and\ z_2 = y+u.$$

Now, we evaluate $v_{z_1} \oplus v_{z_2}$ by using the values of v_{z_1} and v_{z_2} calculated above and using Proposition 1.

$$\begin{aligned} v_{z_1} \oplus v_{z_2} &= [\frac{1}{\lambda}(u-x) \oplus \frac{1}{\lambda}(u-y)] \\ &= \frac{1}{\lambda}[(u-x) \oplus (u-y)] \\ &\leq \frac{1}{\lambda}(x \oplus y). \end{aligned} \tag{4}$$

Since M is γ-ordered rectangular multi-valued mapping and using (4), we have

$$\begin{aligned} \gamma\|(x+u) \oplus (y+u)\|^2 &\leq \langle v_{z_1} \odot v_{z_2}, -[(x+u) \oplus (y+u)]\rangle \\ &= \langle v_{z_1} \oplus v_{z_2}, x \oplus y\rangle \\ &\leq \frac{1}{\lambda}\langle x \oplus y, x \oplus y\rangle, \end{aligned}$$

which implies that

$$\begin{aligned} \gamma\|x \oplus y\|^2 &\leq \frac{1}{\lambda}\|x \oplus y\|^2 \\ (\gamma\lambda - 1)\|x \oplus y \oplus\|^2 &\leq 0 \\ i.e., \;\; x \oplus y &= 0 \\ \text{which implies that } x &= y. \end{aligned}$$

Therefore, the Cayley operator $C_{I,\lambda}^M$ associated with M is single-valued. □

Proposition 5. *For any $x, y \in \mathcal{H}$, let $x \propto y$ and $R_{I,\lambda}^M(x) \propto R_{I,\lambda}^M(y)$.Then, the Cayley operator $C_{I,\lambda}^M$ associated with M is a comparison mapping.*

Proof. For any $x, y \in \mathcal{H}$, let $x \propto y$ then obviously $I(x) \propto I(y)$. As $R_{I,\lambda}^M$ is a comparison mapping, we have $R_{I,\lambda}^M(x) \propto R_{I,\lambda}^M(y)$. Thus, we have

$$[2R_{I,\lambda}^M - I](x) \propto [2R_{I,\lambda}^M - I](y).$$

i.e., we have

$$C_{I,\lambda}^M(x) \propto C_{I,\lambda}^M(y).$$

Therefore, the Cayley operator $C_{I,\lambda}^M$ associated with M is a comparison mapping. □

Lemma 3. *Let $M : \mathcal{H} \to 2^{\mathcal{H}}$ be a γ-ordered rectangular multi-valued mapping with respect to $R_{I,\lambda}^M$, for $\lambda > \frac{1}{\gamma}$. Then the following condition holds:*

$$\|R_{I,\lambda}^M(x) \oplus R_{I,\lambda}^M(y)\| \leq \theta\|x \oplus y\|, \textit{for all } x, y \in \mathcal{H}, \textit{ where } \theta = \frac{1}{\gamma\lambda - 1}.$$

That is, the resolvent operator $R_{I,\lambda}^M$ is Lipschitz-type-continuous.

Proof. For the proof we refer to [25]. □

Lemma 4. *Let $M : \mathcal{H} \to 2^{\mathcal{H}}$ be a (γ, λ)-weak ordered rectangular different multi-valued mapping with respect to $R_{I,\lambda}^M$ and the resolvent operator $R_{I,\lambda}^M$ defined by (1) is θ-Lipschitz-type-continuous. Then, the Yosida approximation operator $J_{I,\lambda}^M$ defined by (2) is θ'-Lipschitz-type-continuous. i.e.,*

$$\|J_{I,\lambda}^M(x) \oplus J_{I,\lambda}^M(y)\| \leq \theta'\|x \oplus y\|, \textit{for all } x, y \in \mathcal{H}, \textit{where } \theta = \frac{1}{\gamma\lambda - 1}, \; \theta' = \frac{\gamma}{\gamma\lambda - 1} \textit{ and } \gamma\lambda > 1.$$

That is, the Yosida approximation operator $J_{I,\lambda}^M$ is Lipschitz-type-continuous.

Proof. For the proof we refer to [25]. □

Lemma 5. *Let $M : \mathcal{H} \to 2^{\mathcal{H}}$ be (γ, λ)-weak ordered rectangular different multi-valued mapping with respect to $R_{I,\lambda}^M$ and the resolvent operator $R_{I,\lambda}^M$ is θ-Lipschitz-type-continuous. Then, the Cayley operator $C_{I,\lambda}^M$ defined by (3) is $(2\theta + 1)$-Lipschitz-type-continuous. That is,*

$$\|C^M_{I,\lambda}(x) \oplus C^M_{I,\lambda}(y)\| \leq (2\theta+1)\|x \oplus y\|, \textit{for all } x, y \in \mathcal{H}, \textit{where } \theta = \frac{1}{\gamma\lambda - 1} \textit{ and } \gamma\lambda > 1.$$

Proof. Using Cauchy-Schwartz inequality and Proposition 1, we have

$$\begin{aligned}
\|C^M_{I,\lambda}(x) \oplus C^M_{I,\lambda}(y)\|^2 &= \langle C^M_{I,\lambda}(x) \oplus C^M_{I,\lambda}(y), C^M_{I,\lambda}(x) \oplus C^M_{I,\lambda}(y)\rangle \\
&= \langle (2R^M_{I,\lambda}(x) - I(x)) \oplus (2R^M_{I,\lambda}(y) - I(y)), \\
&\quad (2R^M_{I,\lambda}(x) - I(x)) \oplus (2R^M_{I,\lambda}(y) - I(y))\rangle \\
&= \langle (2R^M_{I,\lambda}(x) \oplus 2R^M_{I,\lambda}(y)) + (I(x) \oplus I(y)), \\
&\quad (2R^M_{I,\lambda}(x) \oplus 2R^M_{I,\lambda}(y)) + (I(x) \oplus I(y))\rangle \\
&= \langle 2R^M_{I,\lambda}(x) \oplus 2R^M_{I,\lambda}(y), 2R^M_{I,\lambda}(x) \oplus 2R^M_{I,\lambda}(y)\rangle \\
&\quad + \langle 2R^M_{I,\lambda}(x) \oplus 2R^M_{I,\lambda}(y), I(x) \oplus I(y)\rangle \\
&\quad + \langle I(x) \oplus I(y), 2R^M_{I,\lambda}(x) \oplus 2R^M_{I,\lambda}(y)\rangle \\
&\quad + \langle I(x) \oplus I(y), I(x) \oplus I(y)\rangle \\
&\leq 4\|R^M_{I,\lambda}(x) \oplus R^M_{I,\lambda}(y)\|^2 + 2\|R^M_{I,\lambda}(x) \oplus R^M_{I,\lambda}(y)\|\|x \oplus y\| \\
&\quad + 2\|R^M_{I,\lambda}(x) \oplus R^M_{I,\lambda}(y)\|\|x \oplus y\| + \|x \oplus y\|^2 \\
&= \Big(2\|R^M_{I,\lambda}(x) \oplus R^M_{I,\lambda}(y)\| + \|x \oplus y\|\Big)^2 \\
\textit{i.e.,}\ \|C^M_{I,\lambda}(x) \oplus C^M_{I,\lambda}(y)\| &\leq 2\|R^M_{I,\lambda}(x) \oplus R^M_{I,\lambda}(y)\| + \|x \oplus y\|.
\end{aligned}$$

Using the Lipschitz-type-continuity of the resolvent operator $R^M_{I,\lambda}$, we have

$$\begin{aligned}
\|C^M_{I,\lambda}(x) \oplus C^M_{I,\lambda}(y)\| &\leq 2\theta\|x \oplus y\| + \|x \oplus y\|, \\
\textit{i.e.,}\ \|C^M_{I,\lambda}(x) \oplus C^M_{I,\lambda}(y) &\leq (2\theta+1)\|x \oplus y\|, \textit{ where } \theta = \frac{1}{\gamma\lambda - 1} \textit{ and } \gamma\lambda > 1.
\end{aligned}$$

i.e., the Cayley operator $C^M_{I,\lambda}$ is Lipschitz-type-continuous. □

We construct the following example in support of some of the concepts used in this paper.

Example 1. *Let $C \subseteq \mathcal{H}$ be a normal cone with constant λ_N. Let $M : \mathcal{H} \to 2^{\mathcal{H}}$ be the multi-valued mapping defined by $M(x) = \{x + 1 : x \in \mathcal{H}\}$ and $x \propto y$.*

$$\textit{As } x \propto y, \textit{ clearly } M(x) \propto M(y)$$

That is, M is a comparison mapping.

$$\begin{aligned}
&\textit{Let } v_x = x + 1 \in M(x) \textit{ and } v_y = y + 1 \in M(y) \textit{ and} \\
\langle v_x \oplus v_y, -(x \oplus y)\rangle &= \langle v_x \oplus v_y, x \oplus y\rangle \\
&= \langle (x+1) \oplus (y+1), x \oplus y\rangle \\
&= \langle x \oplus y, x \oplus y\rangle \\
&= \|x \oplus y\|^2 \\
&\geq \frac{4}{5}\|x \oplus y\|^2, \forall x, y \in \mathcal{H}.
\end{aligned}$$

Thus, M is $\frac{4}{5}$-ordered rectangular mapping. Also it is easy to see that for $\lambda = 2$, M is 2-weak ordered different comparison mapping. Hence, M is $(\frac{4}{5}, 2)$-weak ordered rectangular different multi-valued mapping.

The resolvent operator defined by (1) is given by

$$R^M_{I,\lambda}(x) = \frac{(x-2)}{3}, \ for\ all\ x \in \mathcal{H}.$$

Also,

$$\|R^M_{I,\lambda}(x) \oplus R^M_{I,\lambda}(y)\| = \|\frac{(x-2)}{3} \oplus \frac{(y-2)}{3}\| = \frac{1}{3}\|x \oplus y\| \leq \frac{5}{3}\|x \oplus y\|.$$

That is, the resolvent operator $R^M_{I,\lambda}$ is $\frac{5}{3}$-Lipschitz-type-continuous.

In view of the above, the Cayley operator $C^M_{I,\lambda}$ defined by (3) is of the form:

$$C^M_{I,\lambda}(x) = \frac{(-x-4)}{3}, \ for\ all\ x \in \mathcal{H}.$$

It is easy to see that that the Cayley operator defined above is a comparison and single-valued mapping. Also,

$$\begin{aligned} \|C^M_{I,\lambda}(x) \oplus C^M_{I,\lambda}(y)\| = \|\frac{(-x-4)}{3} \oplus \frac{(-y-4)}{3}\| = \frac{1}{3}\|x \oplus y\| &\leq \frac{13}{3}\|x \oplus y\|, \\ \Rightarrow \|C^M_{I,\lambda}(x) \oplus C^M_{I,\lambda}(y)\| &\leq \frac{13}{3}\|x \oplus y\|, \ for\ all\ x, y \in \mathcal{H}. \end{aligned}$$

That is, the Cayley operator $C^M_{I,\lambda}$ is $\frac{13}{3}$-Lipschitz-type-continuous.

3. Formulation of The Problem and Existence of Solution

Let $\mathcal{H}$ be a real ordered Hilbert space. Let $M : \mathcal{H} \to 2^{\mathcal{H}}$ be the multi-valued mapping and $C^M_{I,\lambda}$ be the Cayley operator. We consider the following problem:

Find $x \in \mathcal{H}$ such that

$$0 \in C^M_{I,\lambda}(x) \oplus M(x). \tag{5}$$

We call Problem (5) a Cayley inclusion problem involving XOR-operation.

If $C = 0$, then the Problem (5) reduces to the problem of finding $x \in \mathcal{H}$ such that

$$0 \in M(x). \tag{6}$$

Problem (6) is a fundamental problem of inclusions in analysis and studied by Li et al. [22] and others.

The following Lemma is a fixed point formulation of Cayley inclusion Problem involving XOR-operation (5).

Lemma 6. *The Cayley inclusion Problem (5) involving XOR-operation has a solution $x \in \mathcal{H}$ if and only if, it satisfies the following equation:*

$$x = R^M_{I,\lambda}\Big\{\lambda(J^M_{I,\lambda}(x) \oplus C^M_{I,\lambda}(x)) + R^M_{I,\lambda}(x)\Big\}. \tag{7}$$

Proof. From Equation (7), we have

$$x = R^M_{I,\lambda}\Big\{\lambda(J^M_{I,\lambda}(x) \oplus C^M_{I,\lambda}(x)) + R^M_{I,\lambda}(x)\Big\}.$$

Using the definition of resolvent operator and Yosida approximation operator, we obtain

$$\begin{aligned} x &= [I+\lambda M]^{-1}\Big\{\lambda(J_{I,\lambda}^{M}(x)\oplus C_{I,\lambda}^{M}(x))+R_{I,\lambda}^{M}(x)\Big\}, \\ x+\lambda M(x) &= \lambda.\frac{1}{\lambda}(I-R_{I,\lambda}^{M})(x)\oplus\lambda C_{I,\lambda}^{M}(x)+R_{I,\lambda}^{M}(x) \\ &= x-R_{I,\lambda}^{M}(x)\oplus\lambda C_{I,\lambda}^{M}(x)+R_{I,\lambda}^{M}(x), \\ \lambda M(x) &= \lambda C_{I,\lambda}^{M}(x), \end{aligned}$$

which implies that

$$0\in C_{I,\lambda}^{M}(x)\oplus M(x),$$

i.e., the required Cayley inclusion Problem involving XOR-operation (5). □

Based on Lemma 6, we define the following iterative algorithm for finding the solution of the Cayley inclusion problem involving XOR-operation (5).

Iterative Algorithm 1. *For initial element $x_0 \in \mathcal{H}$, compute the sequence $\{x_n\}$ by the following iterative scheme:*

$$x_{n+1}=(1-\alpha)x_n+\alpha R_{I,\lambda}^{M}\{\lambda(J_{I,\lambda}^{M}(x_n)\oplus C_{I,\lambda}^{M}(x_n))+R_{I,\lambda}^{M}(x_n)\},$$

where $\alpha\in[0,1]$, $\lambda>0$ is a constant and I is the identity operator.

Theorem 1. *Let $\mathcal{H}$ be a real ordered Hilbert space and C be a normal cone with normal constant λ_N with ordering " $\leq$ ". Let $M:\mathcal{H}\to 2^{\mathcal{H}}$ be γ-ordered rectangular, (γ,λ)-weak ordered rectangular different multi-valued mapping. Let $J_{I,\lambda}^{M}$ be the Yosida approximation operator defined by (2) and $C_{I,\lambda}^{M}$ be the Cayley operator defined by (3) such that both the operators are Lipschitz-type-continuous with constant $\theta^{'}$ and $(2\theta+1)$, respectively. Let $x_{n+1}\propto x_n$ and $J_{I,\lambda}^{M}(x)\oplus J_{I,\lambda}^{M}(y)\propto C_{I,\lambda}^{M}(x)\oplus C_{I,\lambda}^{M}(y)$, for all $x,y\in\mathcal{H}$, $n=0,1,2,3,\ldots$ such that the following condition is satisfied:*

$$\theta^{'}+2\theta<\frac{[1-\theta(1+\lambda)]}{\lambda\theta},\ \textit{where}\ \theta=\frac{1}{\gamma\lambda-1},\ \theta^{'}=\frac{\gamma}{\gamma\lambda-1}\ \textit{and}\ \gamma\lambda>1. \tag{8}$$

Then the sequence $\{x_n\}$ generated by the Algorithm 1 strongly converges to x^*, the solution of the Cayley inclusion Problem involving XOR-operation (5). In addition, for any $x_0\in\mathcal{H}$, the following condition holds:

$$\begin{aligned} \|x^*-x_0\| &\leq \frac{1+(\lambda_N-1)[(1-\alpha)+\alpha\theta[1+\lambda\theta^{'}+\lambda(2\theta+1)]}{1-[(1-\alpha)+\alpha\theta[1+\lambda\theta^{1}+\lambda(2\theta+1)]}\times \\ &\quad \alpha\|x_0+R_{I,\lambda}^{M}\{\lambda(J_{I,\lambda}^{M}(x_0)\oplus C_{I,\lambda}^{M}(x_0))+R_{I,\lambda}^{M}(x_0)\}\|. \end{aligned}$$

Proof. By using Algorithm 1 and Proposition 1, we have

$$\begin{aligned} 0 &\leq x_{n+1}\oplus x_n \\ &= \Big((1-\alpha)x_n+\alpha\Big[R_{I,\lambda}^{M}\{\lambda(J_{I,\lambda}^{M}(x_n)\oplus C_{I,\lambda}^{M}(x_n))+R_{I,\lambda}^{M}(x_n)\}\Big]\Big) \\ &\quad \oplus\Big((1-\alpha)x_{n-1}+\alpha\Big[R_{I,\lambda}^{M}\{\lambda(J_{I,\lambda}^{M}(x_{n-1})\oplus C_{I,\lambda}^{M}(x_{n-1}))+R_{I,\lambda}^{M}(x_{n-1})\}\Big]\Big) \\ &= (1-\alpha)(x_n\oplus x_{n-1})+\alpha\Big(\Big[R_{I,\lambda}^{M}\{\lambda(J_{I,\lambda}^{M}(x_n)\oplus C_{I,\lambda}^{M}(x_n))+R_{I,\lambda}^{M}(x_n)\}\Big] \\ &\quad \oplus\Big[R_{I,\lambda}^{M}\{\lambda(J_{I,\lambda}^{M}(x_{n-1})\oplus C_{I,\lambda}^{M}(x_{n-1}))+R_{I,\lambda}^{M}(x_{n-1})\}\Big]\Big). \end{aligned} \tag{9}$$

Using Proposition 2, we calculate

$$\begin{aligned}
\|x_{n+1} \oplus x_n\| &\leq \lambda_N \|(1-\alpha)(x_n \oplus x_{n-1}) + \alpha\Big(\Big[R^M_{I,\lambda}\{\lambda(J^M_{I,\lambda}(x_n) \oplus C^M_{I,\lambda}(x_n)) + R^M_{I,\lambda}(x_n)\}\Big] \\
&\quad \oplus \Big[R^M_{I,\lambda}\{\lambda(J^M_{I,\lambda}(x_{n-1}) \oplus C^M_{I,\lambda}(x_{n-1})) + R^M_{I,\lambda}(x_{n-1})\}\Big]\Big)\| \\
&\leq \lambda_N(1-\alpha)\|x_n \oplus x_{n-1}\| + \lambda_N\alpha\|R^M_{I,\lambda}\{\lambda(J^M_{I,\lambda}(x_n) \oplus C^M_{I,\lambda}(x_n))\} \\
&\quad \oplus R^M_{I,\lambda}\{\lambda(J^M_{I,\lambda}(x_{n-1}) \oplus C^M_{I,\lambda}(x_{n-1}))\}\| + \lambda_N\alpha\|R^M_{I,\lambda}(x_n) \oplus R^M_{I,\lambda}(x_{n-1})\|.
\end{aligned}$$

As $R^M_{I,\lambda}$ is Lipschitz-type-continuous, we have

$$\begin{aligned}
\|x_{n+1} \oplus x_n\| &\leq \lambda_N(1-\alpha)\|x_n \oplus x_{n-1}\| + \lambda_N\alpha\lambda\theta\|[J^M_{I,\lambda}(x_n) \oplus C^M_{I,\lambda}(x_n)] \\
&\quad \oplus [J^M_{I,\lambda}(x_{n-1}) \oplus C^M_{I,\lambda}(x_{n-1})]\| + \lambda_N\alpha\theta\|x_n \oplus x_{n-1}\| \\
&\leq \Big(\lambda_N(1-\alpha) + \lambda_N\alpha\theta\Big)\|x_n \oplus x_{n-1}\| + \lambda_N\alpha\lambda\theta\|\Big(J^M_{I,\lambda}(x_n) \oplus J^M_{I,\lambda}(x_{n-1}\Big) \\
&\quad \oplus \Big(C^M_{I,\lambda}(x_n) \oplus C^M_{I,\lambda}(x_{n-1})\Big)\| \\
&\leq \Big(\lambda_N(1-\alpha) + \lambda_N\alpha\theta\Big)\|x_n \oplus x_{n-1}\| + \lambda_N\alpha\lambda\theta\|(J^M_{I,\lambda}(x_n) \oplus J^M_{I,\lambda}(x_{n-1})) \\
&\quad -(C^M_{I,\lambda}(x_n) \oplus C^M_{I,\lambda}(x_{n-1}))\|.
\end{aligned}$$

That is,

$$\begin{aligned}
\|x_{n+1} \oplus x_n\| &\leq \Big(\lambda_N(1-\alpha) + \lambda_N\alpha\theta\Big)\|x_n \oplus x_{n-1}\| + \lambda_N\alpha\lambda\theta\|J^M_{I,\lambda}(x_n) \oplus J^M_{I,\lambda}(x_{n-1})\| \\
&\quad +\lambda_N\alpha\lambda\theta\|C^M_{I,\lambda}(x_n) \oplus C^M_{I,\lambda}(x_{n-1})\|, \; \textit{where } \theta = \frac{1}{\gamma\lambda - 1} \textit{ and } \gamma\lambda > 1. \qquad (10)
\end{aligned}$$

Using the Lipschitz-type-continuity of Yosida approximation operator $J^M_{I,\lambda}$ and Cayley operator $C^M_{I,\lambda}$, we have

$$\begin{aligned}
\|x_{n+1} \oplus x_n\| &\leq \Big(\lambda_N(1-\alpha) + \lambda_N\alpha\theta\Big)\|x_n \oplus x_{n-1}\| + \lambda_N\alpha\lambda\theta\theta'\|x_n \oplus x_{n-1}\| \\
&\quad +\lambda_N\alpha\lambda\theta(2\theta+1)\|x_n \oplus x_{n-1}\| \\
&= \lambda_N\Big[(1-\alpha) + \alpha\theta + \alpha\lambda\theta\theta' + \alpha\lambda\theta(2\theta+1)\Big]\|x_n \oplus x_{n-1}\|, \\
\textit{i.e.}, \; \|x_{n+1} \oplus x_n\| &\leq \lambda_N\Big[(1-\alpha) + \alpha\theta(1 + \lambda\theta' + \lambda(2\theta+1))\Big]\|x_n \oplus x_{n-1}\|, \qquad (11)
\end{aligned}$$

where $\theta = \frac{1}{\gamma\lambda - 1}$, $\theta' = \frac{\gamma}{\gamma\lambda-1}$ *and* $\gamma\lambda > 1$.

Since $x_{n+1} \propto x_n$, we have

$$\|x_{n+1} \oplus x_n\| = \|x_{n+1} - x_n\| \leq \lambda_N\Big[(1-\alpha) + \alpha\theta[1 + \lambda\theta' + \lambda(2\theta+1)]\Big]\|x_n - x_{n-1}\|.$$

Thus, we have

$$\begin{gathered}
\|x_{n+1} - x_n\| \leq \lambda_N\nu^n\|x_1 - x_0\|, \\
\textit{where } \nu = (1-\alpha) + \alpha\theta[1 + \lambda\theta' + \lambda(2\theta+1)].
\end{gathered}$$

Hence, for $m > n > 0$, we have

$$\|x_m - x_n\| \leq \sum_{i=n}^{m-1}\|x_{i+1} - x_i\| \leq \lambda_N\|x_1 - x_0\|\sum_{i=n}^{m-1}\nu^i. \qquad (12)$$

It follows from condition (8) that $0 < \nu < 1$, and thus $\|x_m - x_n\| \to 0,\ as\ n \to \infty$ and so $\{x_n\}$ is a Cauchy sequence in $\mathcal{H}$. Since $\mathcal{H}$ is complete $x_n \to x^* \in \mathcal{H},\ as\ n \to \infty$. Thus, we can write

$$\begin{aligned}
x^* &= \lim_{n\to\infty} x_{n+1} \\
&= \lim_{n\to\infty} \Big[(1-\alpha)x_n + \alpha[R^M_{I,\lambda}\{\lambda(J^M_{I,\lambda}(x_n) \oplus C^M_{I,\lambda}(x_n)) + R^M_{I,\lambda}(x_n)\}]\Big] \\
&= (1-\alpha)\lim_{n\to\infty} x_n + \alpha \lim_{n\to\infty} \Big[R^M_{I,\lambda}\{\lambda(J^M_{I,\lambda}(x_n) \oplus C^M_{I,\lambda}(x_n)) + R^M_{I,\lambda}(x_n)\}\Big] \\
&= (1-\alpha)x^* + \alpha\Big[R^M_{I,\lambda}\{\lambda(J^M_{I,\lambda}(\lim_{n\to\infty} x_n) \oplus C^M_{I,\lambda}(\lim_{n\to\infty} x_n)) + R^M_{I,\lambda}(\lim_{n\to\infty} x_n)\}\Big] \\
&= (1-\alpha)x^* + \alpha\Big[R^M_{I,\lambda}\{\lambda(J^M_{I,\lambda}(x^*) \oplus C^M_{I,\lambda}(x^*)) + R^M_{I,\lambda}(x^*)\}\Big].
\end{aligned}$$

It follows that x^* satisfies the Equation (7),

$$i.e.,\ x = R^M_{I,\lambda}\Big\{\lambda(J^M_{I,\lambda}(x) \oplus C^M_{I,\lambda}(x)) + R^M_{I,\lambda}(x)\Big\}.$$

By Lemma 6, x^* is a solution of Cayley inclusion problem involving XOR-operation (5). On the other hand, it follows that $R^M_{I,\lambda}\Big\{\lambda(J^M_{I,\lambda}(x) \oplus C^M_{I,\lambda}(x)) + R^M_{I,\lambda}(x)\Big\} \propto x^*,\ n = 0, 1, 2, \ldots$.

Using Lemma 1 and (12), we have

$$\begin{aligned}
\|x^* - x_0\| &= \lim_{n\to\infty} \|x_n - x^*\| \\
&\leq \lim_{n\to\infty} \sum_{i=1}^{n} \|x_{i+1} - x_i\| \leq \lim_{n\to\infty} \lambda_N \sum_{i=2}^{n} \nu^{i-1}\|x_1 - x_0\| + \|x_1 - x_0\| \\
&\leq \frac{1 + (\lambda_N - 1)[(1-\alpha) + \alpha\theta[1 + \lambda\theta' + \lambda(2\theta+1)]}{1 - [(1-\alpha) + \alpha\theta[1 + \lambda\theta' + \lambda(2\theta+1)]} \times \\
&\quad \alpha\|x_0 + R^M_{I,\lambda}\{\lambda J^M_{I,\lambda}(x_0) \oplus C^M_{I,\lambda}(x_0) + R^M_{I,\lambda}(x_0)\}\|.
\end{aligned}$$

This complete the proof. □

4. Conclusions

We have introduced and studied a new problem which involves a Cayley operator and a multi-valued mapping with XOR-operation in real ordered Hilbert space, called the Cayley Inclusion problem involving XOR-operation. A fixed point formulation of the Cayley inclusion problem involving XOR-operation is given by using the Yosida approximation operator and resolvent operator. Finally, an existence and convergence result is proved with some extra condition. An example is constructed to illustrate some of the concepts used in this paper.

We remark that our results may be extended in ordered Banach spaces and other higher dimensional spaces.

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