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Article

A Coupled System of Fractional Difference Equations with Nonlocal Fractional Sum Boundary Conditions on the Discrete Half-Line

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Abstract: In this article, we propose a coupled system of fractional difference equations with nonlocal fractional sum boundary conditions on the discrete half-line and study its existence result by using Schauder's fixed point theorem. An example is provided to illustrate the results.

Keywords: existence; coupled system of fractional difference equations; fractional sum; discrete half-line

MSC: 39A05; 39A12

1. Introduction

Recently, many mathematicians and researchers have extensively studied fractional difference calculus since this subject can be used for describing many problems of real-world phenomena such as mechanical, control systems, flow in porous media, and electrical networks (see [1,2] and the references therein). The basic definitions and properties of fractional difference calculus are given in the book [3]. The applications and developments of the theory can be found in [4–47] and the references cited therein. For example, Ferreira [20] studied the fractional difference equation of order less than one. Goodrich [22] presented the fractional difference equation of order $1 < \alpha \leq 2$ with a constant boundary condition. Chen et al. [28] proposed the initial value problem of order less than one. Chen and Zhou [29] studied the antiperiodic boundary value problem of order $1 < \alpha \leq 2$. Sitthiwirattham et al. [38] initiated the study of the fractional sum boundary value problem of order $1 < \alpha \leq 2$. Sitthiwirattham [40] proposed the sequential fractional difference equation with the fractional sum boundary condition. We observe that these research works are fractional problems containing only one equation.

The study of coupled systems of fractional differential equations is an important topic in this area (see [48–53] and the references cited therein), and a recent example of the application of systems of fractional difference equations is [54].

For the boundary value problems for systems of discrete fractional equations, there are some studies in this area (see [55–60] and the references cited therein).

Pan et al. [55] proposed the system of discrete fractional difference equations as given by:

$$\begin{aligned} -\Delta^{\nu} y_1(t) &= f\big(y_1(t+\nu), y_2(t+\mu-1)\big), \\ -\Delta^{\mu} y_2(t) &= g\big(y_1(t+\nu), y_2(t+\mu-1)\big), \end{aligned} \tag{1}$$

for $t \in \mathbb{N}_{0,b+1} := \{0, 1, 2, ..., b+1\}$, with the difference boundary conditions:

$$y_1(\nu - 2) = \Delta y_1(\nu + b) = 0,$$
$$y_2(\mu - 2) = \Delta y_2(\mu + b) = 0, \tag{2}$$

where $b \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$; $1 < \mu, \nu \leq 2$; $0 < \beta \leq 1$; and $f, g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions. Δ^ν and Δ^μ are fractional difference operator of order ν and μ, respectively.

In 2015, Goodrich [58] discussed the coupled system of discrete fractional difference equations:

$$\begin{aligned} -\Delta^{-\nu} x(t) &= \lambda_1 f\big(t + \nu - 1, y(t + \mu - 1)\big), \ t \in \mathbb{N}_{0,b+1}, \\ -\Delta^{-\mu} y(t) &= \lambda_2 g\big(t + \mu - 1, y(t + \nu - 1)\big), \end{aligned} \tag{3}$$

with the nonlinearities satisfying no growth conditions:

$$\begin{aligned} x(\nu - 2) &= H_1\left(\textstyle\sum_{i=1}^{n} a_i y(\xi_i)\right), & x(\nu + b + 1) &= 0, \\ y(\mu - 2) &= H_2\left(\textstyle\sum_{j=1}^{m} b_j x(\zeta_i)\right), & y(\mu + b + 1) &= 0, \end{aligned} \tag{4}$$

where $1 < \nu \leq 2$; $1 < \mu \leq 2$; $\lambda_1, \lambda_2 > 0$; $\{a_i\}_{i=1}^{n}, \{b_j\}_{j=1}^{m} \subseteq (0, \infty)$; and $H_1, H_2 : [0, \infty) \to [0, \infty)$ are continuous functions.

In this paper, we considered the coupled system of fractional difference equations:

$$\begin{cases} \Delta^{\alpha_1} u_1(t) = F_1\big(t + \alpha_1 - 1, t + \alpha_2 - 1, \Delta^{\beta_1} u_1(t + \alpha_1 - \beta_1), u_2(t + \alpha_2 - 1)\big), \\ \Delta^{\alpha_2} u_2(t) = F_2\big(t + \alpha_1 - 1, t + \alpha_2 - 1, \Delta^{\beta_2} u_2(t + \alpha_2 - \beta_1), u_1(t + \alpha_1 - 1)\big), \end{cases} \tag{5}$$

for $t \in \mathbb{N}_0$, subject to the nonlocal fractional sum boundary conditions on the discrete half-line $\mathbb{N}_0$:

$$\begin{cases} u_1(\alpha_1 - 2) = \phi_1(u_1, u_2), \\ u_2(\alpha_2 - 2) = \phi_2(u_1, u_2), \\ \lim_{t \to \infty} u_1(t + \alpha_1 - 2) = \lambda_2 \Delta^{-\theta_2} g_2(\eta_2 + \theta_2) u_2(\eta_2 + \theta_2), \\ \lim_{t \to \infty} u_2(t + \alpha_2 - 2) = \lambda_1 \Delta^{-\theta_1} g_1(\eta_1 + \theta_1) u_1(\eta_1 + \theta_1). \end{cases} \tag{6}$$

For $i = 1, 2$, $\alpha_i \in (1, 2]$; $\nu_i, \gamma_i, \theta_i \in (0, 1]$; $\beta_i \in (\alpha_i - 1, \alpha_i)$; $\lambda_1, \lambda_2 > 0$, and $\eta_i \in \mathbb{N}_{\alpha_i - 1, T + \alpha_i - 1}$ are given constants; $F_i \in C\left(\mathbb{N}_{\alpha_1 - 2} \times \mathbb{N}_{\alpha_2 - 2} \times \mathbb{R}^2, \mathbb{R}\right)$ and $g_i \in C\left(\mathbb{N}_{\alpha_i - 2, T + \alpha_i}, \mathbb{R}^+\right)$ are given functions; $\phi_i(u_1, u_2)$ are given functionals; and $\Delta^{-\theta_i}$ are fractional sums of order θ_i.

The goal of this study is to show the existence of solutions of the governing problems (5) and (6). The paper is structured as follows. Some definitions and basic lemmas are recalled in Section 2. In Section 3, we prove the existence of solutions of the boundary value problem (5) by employing Schauder's fixed point theorem. Finally, we present an example to illustrate our result in the last section.

2. Preliminaries

In what follows, the notation, definitions, and lemmas used in the main results are given.

Definition 1. *The generalized falling function is defined by* $t^{\underline{\alpha}} := \dfrac{\Gamma(t+1)}{\Gamma(t+1-\alpha)}$*, for any t and α for which the right-hand side is defined. If $t + 1 - \alpha$ is a pole of the Gamma function and $t + 1$ is not a pole, then* $t^{\underline{\alpha}} = 0$.

Lemma 1. *[4] Assume the falling factorial functions are well defined. If $t \leq r$, then $t^{\underline{\alpha}} \leq r^{\underline{\alpha}}$ for any $\alpha > 0$.*

Definition 2. *For $\alpha > 0$ and f defined on $\mathbb{N}_a$, the α-order fractional sum of f is defined by:*

$$\Delta^{-\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t-\sigma(s))^{\underline{\alpha-1}} f(s),$$

where $t \in \mathbb{N}_{a+\alpha}$ and $\sigma(s) = s+1$.

Definition 3. *For $\alpha > 0$ and f defined on $\mathbb{N}_a$, the α-order Riemann–Liouville fractional difference of f is defined by:*

$$\Delta^{\alpha} f(t) := \Delta^N \Delta^{-(N-\alpha)} f(t) = \frac{1}{\Gamma(-\alpha)} \sum_{s=a}^{t+\alpha} (t-\sigma(s))^{\underline{-\alpha-1}} f(s),$$

where $t \in \mathbb{N}_{a+N-\alpha}$ and $N \in \mathbb{N}$ are chosen so that $0 \le N-1 < \alpha \le N$.

Lemma 2. *[4] Let $0 \le N-1 < \alpha \le N$. Then,*

$$\Delta^{-\alpha}\Delta^{\alpha} y(t) = y(t) + C_1 t^{\underline{\alpha-1}} + C_2 t^{\underline{\alpha-2}} + \ldots + C_N t^{\underline{\alpha-N}},$$

for some $C_i \in \mathbb{R}$, with $1 \le i \le N$.

The following lemma deals with the linear variant of the boundary value problems (5) and (6) and gives a representation of the solution.

Lemma 3. *Let $\alpha_i \in (1,2]$, $\theta_i \in (0,1]$, $\lambda_1, \lambda_2 > 0$ and $\eta_i \in \mathbb{N}_{\alpha_i-1,T+\alpha_i-1}$ be given constants, $k_i \in C\left(\mathbb{N}_{\alpha_i-2}, \mathbb{R}\right)$ and $g_i \in C\left(\mathbb{N}_{\alpha_i-2,T+\alpha_i}, \mathbb{R}^+\right)$ given functions, and $\phi_i(u_1,u_2)$ given functionals. For each $i,j \in \{1,2\}$ and $i \ne j$, then the problems:*

$$\Delta^{\alpha_i} u_i(t) = k_i(t+\alpha_i-1), \quad t \in \mathbb{N}_0, \tag{7}$$

$$u_i(\alpha_i-2) = \phi_i(u_1,u_2), \tag{8}$$

$$\lim_{t\to\infty} u_i(t+\alpha_i) = \lambda_j \Delta^{-\theta_j} g_j(\eta_j+\theta_j) u_j(\eta_j+\theta_j). \tag{9}$$

have the unique solutions:

$$\begin{aligned} u_1(t_1) &= t_1^{\underline{\alpha_1-1}} \left\{ \frac{\lambda_1}{\Lambda\Gamma(\theta_1)} \sum_{s=\alpha_1-2}^{\eta_1} (\eta_1+\theta_1-\sigma(s))^{\underline{\theta_1-1}} g_1(s) s^{\underline{\alpha_1-1}} \mathcal{P}(k_1,k_2) \right. \\ &\quad \left. - \frac{\lambda_2}{\Lambda\Gamma(\theta_2)} \sum_{s=\alpha_2-2}^{\eta_2} (\eta_2+\theta_2-\sigma(s))^{\underline{\theta_2-1}} g_2(s) s^{\underline{\alpha_2-1}} \mathcal{Q}(k_1,k_2) \right\} \\ &\quad + \frac{t_1^{\underline{\alpha_1-2}} \phi_1(u_1,u_2)}{\Gamma(\alpha_1)} + \frac{1}{\Gamma(\alpha_1)} \sum_{s=0}^{t_1-\alpha_1} (t_1-\sigma(s))^{\underline{\alpha_1-1}} k_1(s+\alpha_1-1), \;\; t_1 \in \mathbb{N}_{\alpha_1-2}, \end{aligned} \tag{10}$$

$$\begin{aligned} u_2(t_2) &= t_2^{\underline{\alpha_2-1}} \left\{ \frac{\lim_{t_2\to\infty} t_2^{\underline{\alpha_2-1}}}{\Lambda} \mathcal{P}(k_1,k_2) - \frac{\lim_{t_1\to\infty} t_1^{\underline{\alpha_1-1}}}{\Lambda} \mathcal{Q}(k_1,k_2) \right\} \\ &\quad + \frac{t_2^{\underline{\alpha_2-2}} \phi_2(u_1,u_2)}{\Gamma(\alpha_2)} + \frac{1}{\Gamma(\alpha_2)} \sum_{s=0}^{t_2-\alpha_2} (t_2-\sigma(s))^{\underline{\alpha_2-1}} k_2(s+\alpha_2-1), \;\; t_2 \in \mathbb{N}_{\alpha_2-2}, \end{aligned} \tag{11}$$

provided that both $u_1(t_1), u_2(t_2)$ are uniformly bounded on $\mathbb{N}_{\alpha_1-2}$ and $\mathbb{N}_{\alpha_2-2}$, respectively, and:

$$\Lambda = \frac{\lambda_2 \lim_{t_2\to\infty} t_2^{\underline{\alpha_2-1}}}{\Gamma(\alpha_2)} \sum_{s=\alpha_2-1}^{\eta_2} (\eta_2+\theta_2-\sigma(s))^{\underline{\theta_2-1}} g_2(s)\, s^{\underline{\alpha_2-1}} \tag{12}$$

$$- \frac{\lambda_1 \lim_{t_1\to\infty} t_1^{\underline{\alpha_1-1}}}{\Gamma(\alpha_1)} \sum_{s=\alpha_1-1}^{\eta_1} (\eta_1+\theta_1-\sigma(s))^{\underline{\theta_1-1}} g_1(s)\, s^{\underline{\alpha_1-1}}, \quad t_i \in \mathbb{N}_{\alpha_i-2},$$

$$\begin{aligned}
\mathcal{P}(k_1,k_2) &= \frac{\lim_{t_1\to\infty} t_1^{\underline{\alpha_1-2}} \phi_1(u_1,u_2)}{\Gamma(\alpha_1)} - \frac{\lambda_2\phi_2(u_1,u_2)}{\Gamma(\alpha_2)\Gamma(\theta_2)} \sum_{s=\alpha_2-2}^{\eta_2} (\eta_2+\theta_2-\sigma(s))^{\underline{\theta_2-1}} g_2(s)\, s^{\underline{\alpha_2-2}} \\
&+ \frac{1}{\Gamma(\alpha_1)} \lim_{t_1\to\infty} \sum_{s=0}^{t_1-\alpha_1} (t_1-\sigma(s))^{\underline{\alpha_1-1}} k_1(s+\alpha_1-1) - \frac{\lambda_2}{\Gamma(\alpha_2)\Gamma(\theta_2)} \times \\
&\sum_{\xi=\alpha_2}^{\eta_2} \sum_{s=0}^{\xi-\alpha_2} (\eta_2+\theta_2-\sigma(\xi))^{\underline{\theta_2-1}} (\xi-\sigma(s))^{\underline{\alpha_2-1}} g_2(s+\alpha_2-1) k_2(s+\alpha_2-1),
\end{aligned} \tag{13}$$

$$\begin{aligned}
\mathcal{Q}(k_1,k_2) &= \frac{\lim_{t_2\to\infty} t_2^{\underline{\alpha_2-2}} \phi_2(u_1,u_2)}{\Gamma(\alpha_2)} - \frac{\lambda_1\phi_1(u_1,u_2)}{\Gamma(\alpha_1)\Gamma(\theta_1)} \sum_{s=\alpha_1-2}^{\eta_1} (\eta_1+\theta_1-\sigma(s))^{\underline{\theta_1-1}} g_1(s)\, s^{\underline{\alpha_1-2}} \\
&+ \frac{1}{\Gamma(\alpha_2)} \lim_{t_2\to\infty} \sum_{s=0}^{t_2-\alpha_2} (t_2-\sigma(s))^{\underline{\alpha_2-1}} k_2(s+\alpha_2-1) - \frac{\lambda_1}{\Gamma(\alpha_1)\Gamma(\theta_1)} \times \\
&\sum_{\xi=\alpha_1}^{\eta_1} \sum_{s=0}^{\xi-\alpha_1} (\eta_1+\theta_1-\sigma(\xi))^{\underline{\theta_1-1}} (\xi-\sigma(s))^{\underline{\alpha_1-1}} g_1(s+\alpha_1-1) k_1(s+\alpha_1-1).
\end{aligned} \tag{14}$$

Proof. For each $i, j \in \{1,2\}$ and $i \neq j$, using Lemma 2 and the fractional sum of order $\alpha \in (1,2]$ for (7), we obtain:

$$u_i(t_i) = C_{1i} t_i^{\underline{\alpha_i-1}} + C_{2i} t_i^{\underline{\alpha_i-2}} + \frac{1}{\Gamma(\alpha_i)} \sum_{s=0}^{t_i-\alpha_i} (t_i-\sigma(s))^{\underline{\alpha_i-1}} k_i(s+\alpha_i-1), \tag{15}$$

for $t_i \in \mathbb{N}_{\alpha_i-2}$.

By using the boundary condition (8), we find that:

$$C_{2i} = \frac{\phi_i(u_1,u_2)}{\Gamma(\alpha_i)}. \tag{16}$$

Then, for $t_i \in \mathbb{N}_{\alpha_i-2}$, we have:

$$\begin{aligned}
u_i(t_i) &= C_{1i} t_i^{\underline{\alpha_i-1}} + \frac{\phi_i(u_1,u_2)}{\Gamma(\alpha_i)} t_i^{\underline{\alpha_i-2}} \\
&+ \frac{1}{\Gamma(\alpha_i)} \sum_{s=0}^{t_i-\alpha_i} (t_i-\sigma(s))^{\underline{\alpha_i-1}} k_i(s+\alpha_i-1).
\end{aligned} \tag{17}$$

Taking the fractional sum of order $0 < \theta_i \leq 1$ for (17), we obtain:

$$\begin{aligned}
&\Delta^{-\theta_i} u_i(t_i) \\
= &\frac{C_{1i}}{\Gamma(\theta_i)} \sum_{s=\alpha_i-2}^{t_i} (t_i+\theta-\sigma(s))^{\underline{\theta_i-1}} g_i(s)\, s^{\underline{\alpha_i-1}} + \frac{\phi_i(u_1,u_2)}{\Gamma(\alpha_i)} \times \\
&\sum_{s=\alpha_i-2}^{t_i} (t_i+\theta_i-\sigma(s))^{\underline{\theta_i-1}} g_i(s)\, s^{\underline{\alpha_i-2}} + \frac{1}{\Gamma(\theta_i)\Gamma(\alpha_i)} \sum_{\xi=\alpha_i}^{t_i} \sum_{s=0}^{\xi-\alpha_i} (t_i+\theta_i-\sigma(\xi))^{\underline{\theta_i-1}} \times \\
&(\xi-\sigma(s))^{\underline{\alpha_i-1}} g_i(s+\alpha_i-1)\, k_i(s+\alpha_i-1),
\end{aligned} \tag{18}$$

for $t_i \in \mathbb{N}_{\alpha_i-2}$.

Employing the boundary condition (9), this implies that:

$$
\begin{aligned}
&C_{11}\lim_{t_1\to\infty} t_1^{\underline{\alpha_1-1}} + \frac{\phi_1(u_1,u_2)}{\Gamma(\alpha_1)}\lim_{t_1\to\infty} t_1^{\underline{\alpha_1-2}} + \frac{\lim_{t_1\to\infty}}{\Gamma(\alpha_1)}\sum_{s=0}^{t_1-\alpha_1}(t_i-\sigma(s))^{\underline{\alpha_1-1}}k_1(s+\alpha_1-1)\\
=\;& \frac{\lambda_2 C_{12}}{\Gamma(\theta_2)}\sum_{s=\alpha_2-1}^{\eta_2}(\eta_2+\theta_2-\sigma(s))^{\underline{\theta_2-1}}g_2(s)\,s^{\underline{\alpha_2-1}}\\
&+\frac{\lambda_2\phi_2(u_1,u_2)}{\Gamma(\alpha_2)\Gamma(\theta_2)}\sum_{s=\alpha_2-2}^{\eta_2}(\eta_2+\theta_2-\sigma(s))^{\underline{\theta_2-1}}g_2(s)\,s^{\underline{\alpha_2-2}}\\
&+\frac{\lambda_2}{\Gamma(\alpha_2)\Gamma(\theta_2)}\sum_{\xi=\alpha_2}^{\eta_2}\sum_{s=0}^{\xi-\alpha_2}(\eta_2+\theta_2-\sigma(\xi))^{\underline{\theta_2-1}}(\xi-\sigma(s))^{\underline{\alpha_2-1}}g_2(s+\alpha_2-1)\,k_2(s+\alpha_2-1),
\end{aligned}
\tag{19}
$$

and:

$$
\begin{aligned}
&C_{12}\lim_{t_2\to\infty} t_2^{\underline{\alpha_2-1}} + \frac{\phi_2(u_1,u_2)}{\Gamma(\alpha_2)}\lim_{t_2\to\infty} t_2^{\underline{\alpha_2-2}} + \frac{1}{\Gamma(\alpha_2)}\lim_{t_2\to\infty}\sum_{s=0}^{t_2-\alpha_2}(t_2-\sigma(s))^{\underline{\alpha_2-1}}k_2(s+\alpha_2-1)\\
=\;& \frac{\lambda_1 C_{11}}{\Gamma(\theta_1)}\sum_{s=\alpha_1-1}^{\eta_1}(\eta_1+\theta_1-\sigma(s))^{\underline{\theta_1-1}}g_1(s)\,s^{\underline{\alpha_1-1}}\\
&+\frac{\lambda_1\phi_1(u_1,u_2)}{\Gamma(\alpha_1)\Gamma(\theta_1)}\sum_{s=\alpha_1-2}^{\eta_1}(\eta_1+\theta_1-\sigma(s))^{\underline{\theta_1-1}}g_1(s)\,s^{\underline{\alpha_1-2}}\\
&+\frac{\lambda_1}{\Gamma(\alpha_1)\Gamma(\theta_1)}\sum_{\xi=\alpha_1}^{\eta_1}\sum_{s=0}^{\xi-\alpha_1}(\eta_1+\theta_1-\sigma(\xi))^{\underline{\theta_1-1}}(\xi-\sigma(s))^{\underline{\alpha_1-1}}g_1(s+\alpha_1-1)\,k_1(s+\alpha_1-1).
\end{aligned}
\tag{20}
$$

After solving the system of Equations (19) and (20), we obtain:

$$
\begin{aligned}
C_{11} \;=\;& \frac{\lambda_1}{\Lambda\Gamma(\theta_1)}\sum_{s=\alpha_1-2}^{\eta_1}(\eta_1+\theta_1-\sigma(s))^{\underline{\theta_1-1}}g_1(s)s^{\underline{\alpha_1-1}}\mathcal{P}(k_1,k_2)\\
&-\frac{\lambda_2}{\Lambda\Gamma(\theta_2)}\sum_{s=\alpha_2-2}^{\eta_2}(\eta_2+\theta_2-\sigma(s))^{\underline{\theta_2-1}}g_2(s)s^{\underline{\alpha_2-1}}\mathcal{Q}(k_1,k_2),
\end{aligned}
\tag{21}
$$

and:

$$
C_{12} \;=\; \frac{\lim_{t_2\to\infty} t_2^{\underline{\alpha_2-1}}}{\Lambda}\mathcal{P}(k_1,k_2) - \frac{\lim_{t_1\to\infty} t_1^{\underline{\alpha_1-1}}}{\Lambda}\mathcal{Q}(k_1,k_2),
\tag{22}
$$

where $\Lambda, \mathcal{P}(k_1,k_2)$ and $\mathcal{Q}(k_1,k_2)$ are defined as (12)–(14), respectively. □

The following lemma deals with the solutions $u_i(t_i)$, $i = 1,2$ of the problems (7)–(9), and $\Delta^{\beta_i}u_i(t_i-\beta_i+1)$ are uniformly bounded on $\mathbb{N}_{\alpha_i-2}$, $\beta_i \in (\alpha_i-1,\alpha_i)$.

Lemma 4. *For each $i,j\in\{1,2\}$ and $i\neq j$, let $k_i \in C\left(\mathbb{N}_{\alpha_i-2},\mathbb{R}\right)$ and $g_i\in C\left(\mathbb{N}_{\alpha_i-2},\mathbb{R}^+\right)$ be given functions, $\phi_i(u_1,u_2)$ be given functionals, $\rho_i > \max\{\beta_i-\alpha_i\}$, $\beta_i\in(\alpha_i-1,\alpha_i)$, and $0<g_i\leq g_i(s_i)\leq G_i$, for each $s_i\in\mathbb{N}_{\alpha_i-2,T+\alpha_i}$.*

The solution $u_i(t_i)$ of the problems (7)–(9) and $\Delta^{\beta_i}u_i(t_i-\beta_i+1)$ are uniformly bounded on $\mathbb{N}_{\alpha_i-2}$, if and only if $u_i(t_i)$ and $\Delta^{\beta_i}u_i(t_i-\beta_i+1)$ satisfy the following properties:

(A_1) *There exist constants $M_1,N_1,m_1,n_1>0$ such that, for u_1 and $\Delta^{\beta_1}u_1$,*

$$|k_i(t_i)| \leq M_1\,e^{-m_1(2t_1+t_2)},$$

$$|\phi_i(u_1,u_2)| \le N_1(t_1+\rho_1)^{\underline{\rho_1}}\, e^{-n_1(t_1+1)}.$$

(A_2) *There exist constants* $M_2, N_2, m_2, n_2 > 0$ *such that, for* u_2 *and* $\Delta^{\beta_2}u_2$,

$$|k_i(t_i)| \le M_2\,(t_2+\rho_2)^{\underline{\rho_2}}\, e^{-m_2(t_1+2t_2)},$$
$$|\phi_i(u_1,u_2)| \le N_2(t_1+\rho_1)^{\underline{\rho_1}}\left[(t_2+\rho_2)^{\underline{\rho_2}}\right]^2 e^{-n_2(t_2+1)}.$$

(A_3) *There exist constants* $\Omega_i > 0,\ i = 1,2$ *such that,*

$$\left\{\frac{(t_1-\alpha_1+1)^{\underline{2-\alpha_1}}(t_2-\alpha_2+1)^{\underline{2-\alpha_2}}}{1+(t_1+\rho_1)^{\underline{\rho_1}}(t_2+\rho_2)^{\underline{\rho_2}}}\right\}\left|u_i(t_i)\right| < \Omega_i,$$
$$\left\{\frac{(t_1-\alpha_1+1)^{\underline{2-\alpha_1}}(t_2-\alpha_2+1)^{\underline{2-\alpha_2}}}{1+(t_1+\rho_1)^{\underline{\rho_1}}(t_2+\rho_2)^{\underline{\rho_2}}}\right\}\left|\Delta^{\beta_i}u_i(t_i)\right| < \Omega_i.$$

Proof. Firstly, taking the fractional difference of order $\alpha_i - 1 < \beta_i < \alpha_i,\ i = 1,2$ for (10) and (11), we obtain:

$$\begin{aligned}
&\Delta^{\beta_1}u_1(t_1)\\
&= \frac{1}{\Gamma(-\beta_1)}\sum_{s=\alpha_1-1}^{t_1+1}(t_1-\beta_1+1-\sigma(s))^{\underline{-\beta_1-1}}s^{\underline{\alpha_1-1}} \times \left\{\frac{\lambda_1}{\Lambda\Gamma(\theta_1)}\times\right.\\
&\sum_{s=\alpha_1-2}^{\eta_1}(\eta_1+\theta_1-\sigma(s))^{\underline{\theta_1-1}}g_1(s)s^{\underline{\alpha_1-1}}\mathcal{P}(k_1,k_2) - \frac{\lambda_2}{\Lambda\Gamma(\theta_2)}\times\\
&\left.\sum_{s=\alpha_2-2}^{\eta_2}(\eta_2+\theta_2-\sigma(s))^{\underline{\theta_2-1}}g_2(s)s^{\underline{\alpha_2-1}}\,\mathcal{Q}(k_1,k_2)\right\} \qquad (23)\\
&+\frac{\phi_1(u_1,u_2)}{\Gamma(-\beta_1)\Gamma(\alpha_1)}\sum_{s=\alpha_1-2}^{t_1+1}(t_1-\beta_1+1-\sigma(s))^{\underline{-\beta_1-1}}s^{\underline{\alpha_1-2}}\\
&+\frac{1}{\Gamma(-\beta_1)\Gamma(\alpha_1)}\sum_{\xi=\alpha_1}^{t_1+1}\sum_{s=0}^{\xi-\alpha_1}(t_1-\beta_1+1-\sigma(s))^{\underline{-\beta_1-1}}(\xi-\sigma(s))^{\underline{\alpha_1-1}}k_1(s+\alpha_1-1),
\end{aligned}$$

and:

$$\begin{aligned}
&\Delta^{\beta_2}u_2(t_2)\\
&= \frac{1}{\Gamma(-\beta_2)}\sum_{s=\alpha_2-1}^{t_2+1}(t_2-\beta_2+1-\sigma(s))^{\underline{-\beta_2-1}}s^{\underline{\alpha_2-1}} \times \left\{\frac{\lim_{t_2\to\infty} t_2^{\underline{\alpha_2-1}}}{\Lambda}\mathcal{P}(k_1,k_2)\right. \qquad (24)\\
&\left.-\frac{\lim_{t_1\to\infty} t_1^{\underline{\alpha_1-1}}}{\Lambda}\mathcal{Q}(k_1,k_2)\right\} + \frac{\phi_2(u_1,u_2)}{\Gamma(-\beta_2)\Gamma(\alpha_2)}\sum_{s=\alpha_2-2}^{t_2+1}(t_2-\beta_1+1-\sigma(s))^{\underline{-\beta_2-1}}s^{\underline{\alpha_2-2}}\\
&+\frac{1}{\Gamma(-\beta_2)\Gamma(\alpha_2)}\sum_{\xi=\alpha_2}^{t_2+1}\sum_{s=0}^{\xi-\alpha_2}(t_2-\beta_2+1-\sigma(s))^{\underline{-\beta_2-1}}(\xi-\sigma(s))^{\underline{\alpha_2-1}}k_2(s+\alpha_2-1).
\end{aligned}$$

If $u_i(t_i)$ and $\Delta^{\beta_i}u_i(t_i-\beta_i+1)$ are uniformly bounded on $\mathbb{N}_{\alpha_i-2}$, we have:

$$|\Lambda| \quad \le \quad \left|\frac{\lambda_2\lim_{t_2\to\infty} t_2^{\underline{\alpha_2-1}}}{\Gamma(\alpha_2)}\sum_{s=\alpha_2-1}^{\eta_2}(\eta_2+\theta_2-\sigma(s))^{\underline{\theta_2-1}}g_2(s)\,s^{\underline{\alpha_2-1}}\right.$$

$$
\left. - \frac{\lambda_1 \lim_{t_1 \to \infty} t_1^{\underline{\alpha_1 - 1}}}{\Gamma(\alpha_1)} \sum_{s=\alpha_1-1}^{\eta_1} (\eta_1 + \theta_1 - \sigma(s))^{\underline{\theta_1 - 1}} g_1(s)\, s^{\underline{\alpha_1 - 1}} \right| \tag{25}
$$

$$
\begin{aligned}
\leq \quad \max \Bigg\{ & \left| \lim_{t_2 \to \infty} t_2^{\underline{\alpha_2 - 1}} (\eta_2 + \theta_2 - \alpha_2)^{\underline{\theta_2 - 1}} G_2 \lambda_2 \mathcal{A}_2 \right|, \\
& \left| \lim_{t_1 \to \infty} t_1^{\underline{\alpha_1 - 1}} (\eta_1 + \theta_1 - \alpha_1)^{\underline{\theta_1 - 1}} G_1 \lambda_1 \mathcal{A}_1 \right| \Bigg\}.
\end{aligned}
$$

Furthermore, considering $u_1(t_i)$ and $\Delta^{\beta_1} u_i(t_i)$, we obtain:

$$
|k_i(t_i)| < \begin{cases} M_1\, e^{1-(t_1+t_2)} & ,\text{for } u_1(t_1) \\ M_1\, e^{-(2t_1+t_2)} & ,\text{for } \Delta^{\beta_1} u_1(t_1 - \beta_1 + 1) \\ M_2\, (t_2 + \rho_2)^{\underline{\rho_2}}\, e^{-(t_1+t_2)} & ,\text{for } u_2(t_2) \\ M_2\, (t_2 + \rho_2)^{\underline{\rho_2}}\, e^{-(t_1+2t_2)} & ,\text{for } \Delta^{\beta_2} u_2(t_2 - \beta_2 + 1) \end{cases} \tag{26}
$$

and:

$$
|\phi_i(t_1, t_2)| < \begin{cases} N_1\, (t_1 + \rho_1 + 1)^{\underline{\rho_1}} & ,\text{for } u_1(t_1) \\ N_1\, (t_1 + \rho_1)^{\underline{\rho_1}}\, e^{-(t_1+1)} & ,\text{for } \Delta^{\beta_1} u_1(t_1 - \beta_1 + 1) \\ N_2 (t_1 + \rho_1)^{\underline{\rho_1}} \left[(t_2 + \rho_2 + 1)^{\underline{\rho_2}} \right]^2 & ,\text{for } u_2(t_2) \\ N_2 (t_1 + \rho_1)^{\underline{\rho_1}} \left[(t_2 + \rho_2)^{\underline{\rho_2}} \right]^2 e^{-(t_2+1)} & ,\text{for } \Delta^{\beta_2} u_2(t_2 - \beta_2 + 1) \end{cases} \tag{27}
$$

where:

$$
\begin{aligned}
M_1 = \min \Bigg\{ & \frac{\lambda_1 G_1 (\eta_1 - \alpha_1 + \theta_1)^{\underline{\theta_1 - 1}} \mathcal{A}_1}{\lambda_2 g_2 \Gamma(\alpha_2) \mathcal{A}_2}, \frac{G_1 (\eta_1 - \alpha_1 + \theta_1)^{\underline{\theta_1 - 1}}}{g_1}, \Gamma(\alpha_1), \\
& \frac{G_1 (\eta_1 - \alpha_1 + \theta_1)^{\underline{\theta_1 - 1}}}{\lambda_2 g_1 g_2 \Gamma(\alpha_1) \mathcal{C}_2}, \frac{G_1 (\eta_1 - \alpha_1 + \theta_1)^{\underline{\theta_1 - 1}} \mathcal{A}_1}{\lambda_2 g_1 g_2 \Gamma(\alpha_2) \mathcal{A}_2 \mathcal{C}_1} \Bigg\},
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
M_2 = \min \Bigg\{ & \Gamma(\alpha_2), \lambda_2 G_2 (\eta_2 - \alpha_2 + \theta_2)^{\underline{\theta_2 - 1}} \mathcal{A}_2, G_1 (\eta_1 - \alpha_1 + \theta_1)^{\underline{\theta_1 - 1}} \mathcal{A}_1, \\
& \frac{G_1 (\eta_1 - \alpha_1 + \theta_1)^{\underline{\theta_1 - 1}} \mathcal{A}_1}{g_1 \mathcal{C}_1}, \frac{G_2 (\eta_2 - \alpha_2 + \theta_2)^{\underline{\theta_2 - 1}} \mathcal{A}_2}{g_2 \mathcal{C}_2} \Bigg\},
\end{aligned} \tag{29}
$$

$$
N_1 = \min \left\{ \Gamma(\theta_1), \Gamma(\theta_2), \Gamma(\alpha_1), \frac{\Gamma(\theta_1)}{\lambda_2 g_2 \Gamma(\alpha_1) \mathcal{B}_2}, \frac{G_1 (\eta_1 - \alpha_1 + \theta_1)^{\underline{\theta_1 - 1}} \mathcal{A}_1}{\lambda_2 g_1 g_2 \Gamma(\alpha_1 - 1) \mathcal{B}_1 \mathcal{A}_2} \right\}, \tag{30}
$$

$$
\begin{aligned}
N_2 = \min \Bigg\{ & \Gamma(\alpha_2), \lambda_2 G_2 \Gamma(\alpha_1) (\eta_2 - \alpha_2 + \theta_2)^{\underline{\theta_2 - 1}} \mathcal{A}_2, \lambda_1 G_1 \Gamma(\alpha_2) (\eta_1 - \alpha_1 + \theta_1)^{\underline{\theta_1 - 1}} \mathcal{A}_1, \\
& \frac{G_2 (\eta_2 - \alpha_2 + \theta_2)^{\underline{\theta_2 - 1}} \mathcal{A}_2}{g_2 \mathcal{B}_2}, \frac{G_1 (\eta_1 - \alpha_1 + \theta_1)^{\underline{\theta_1 - 1}} \mathcal{A}_1}{g_1 \mathcal{B}_1} \Bigg\}.
\end{aligned} \tag{31}
$$

Consequently, the conditions $(A1)$ and $(A2)$ hold.

We next show that the condition $(A3)$ holds. By using the conditions $(A1)$ and $(A2)$, we obtain:

$$
\begin{aligned}
\left| u_i(t_i) \right| \leq \; & t_1^{\underline{\alpha_1}} t_2^{\underline{\alpha_2}}\, \Omega_i < t_1^{\underline{\alpha_1 + \rho_1 - 1}} t_2^{\underline{\alpha_2 + \rho_2 - 1}}\, \Omega_i \\
& < \left[\frac{1 + t_1^{\underline{\rho_1 + 2}} t_2^{\underline{\rho_2 + 2}}}{(t_1 - \alpha_1 + 1)^{\underline{2 - \alpha_1}} (t_2 - \alpha_2 + 1)^{\underline{2 - \alpha_2}}} \right] \Omega_i
\end{aligned}
$$

and:

$$\begin{aligned}\left|\Delta^{\beta_i}u_i(t_i-\beta_i+1)\right| &\le (t_i-\alpha_i-\beta_i+1)^{\underline{-\beta_i}}\,t_j^{\underline{\alpha_j}}\,\Omega_i < \ t_i^{\underline{\alpha_i}}\,t_j^{\underline{\alpha_j}}\,\Omega_i \\ &< \left[\frac{1+t_1^{\underline{\rho_1+2}}t_2^{\underline{\rho_2+2}}}{(t_1-\alpha_1+1)^{\underline{2-\alpha_1}}(t_2-\alpha_2+1)^{\underline{2-\alpha_2}}}\right]\Omega_i,\ \ i\neq j=1,2\end{aligned}$$

where:

$$\begin{aligned}\Omega_1 = \max\Bigg\{ &\frac{N_1}{\Gamma(\theta_1)}+M_1+\frac{\lambda_2 G_2\Gamma(\alpha_1)(\eta_2-\alpha_2+\theta_2)^{\underline{\theta_2-1}}}{\Gamma(\theta_2)}\times \\ &\left(N_2\mathcal{B}_2+\frac{M_2\mathcal{C}_2}{\lambda_1 G_1(\eta_1-\alpha_1+\theta_1)^{\underline{\theta_1-1}}\mathcal{A}_1}\right), \\ &\frac{N_2}{\Gamma(\theta_2)}+M_2+\frac{\lambda_1 G_1\Gamma(\alpha_2)(\eta_1-\alpha_1+\theta_1)^{\underline{\theta_1-1}}}{\Gamma(\theta_1)}\times \\ &\left(N_1\mathcal{B}_1+\frac{M_1\mathcal{C}_1}{\lambda_2 G_2(\eta_2-\alpha_2+\theta_2)^{\underline{\theta_2-1}}\mathcal{A}_2}\right)\Bigg\} \\ &+\frac{1}{\Gamma(\alpha_1)}(M_1+N_1), \qquad (32)\\ \Omega_2 = \max\Bigg\{ &\frac{1}{G_2(\eta_2-\alpha_2+\theta_2)^{\underline{\theta_2-1}}\mathcal{A}_2}\left[\frac{M_1+N_1}{\Gamma(\alpha_1)\lambda_2}+\frac{1}{\Gamma(\theta_2)\mathcal{A}_2}\left(N_2\mathcal{B}_2+M_2\mathcal{C}_2\right)\right], \\ &\frac{1}{G_1(\eta_1-\alpha_1+\theta_1)^{\underline{\theta_1-1}}\mathcal{A}_1}\left[\frac{M_2+N_2}{\Gamma(\alpha_2)\lambda_1}+\frac{1}{\Gamma(\theta_1)\mathcal{A}_1}\left(N_1\mathcal{B}_1+M_1\mathcal{C}_1\right)\right]\Bigg\} \\ &+\frac{1}{\Gamma(\alpha_2)}(M_2+N_2), \qquad (33)\end{aligned}$$

with

$$\mathcal{A}_i = {}_2F_1\big(\alpha_i,\alpha_i-\eta_i-1;\alpha_i-\eta_i-\theta_i;1\big) \qquad (34)$$

$$\mathcal{B}_i = {}_2F_1\big(\alpha_i-1,\alpha_i-\eta_i-1;\alpha_i-\eta_i-\theta_i-1;1\big) \qquad (35)$$

$$\mathcal{C}_i = {}_2F_1\big(\alpha_i+1,\alpha_i-\eta_i;\alpha_i-\eta_i-\theta_i+1;1\big). \qquad (36)$$

Therefore, the condition $(A3)$ holds.

Finally, if the conditions $(A1)$–$(A3)$ hold, it is clear that $u_i(t_i)$ and $\Delta^{\beta_i}u_i(t_i-\beta_i+1)$ are uniformly bounded on $\mathbb{N}_{\alpha_i-2}$. Our proof is complete. □

We next provide the following theorems used for proving the existence result for the problems (5) and (6).

Theorem 1. *(Arzelá–Ascoli theorem [61])*

A set of functions in $C[a,b]$ with the sup norm is relatively compact if and only if it is uniformly bounded and equicontinuous on $[a,b]$.

Theorem 2. *[61] If a set is closed and relatively compact, then it is compact.*

Theorem 3. *(Schauder's fixed point theorem [61])*

If S is a convex compact subset of a normed space, every continuous mapping of S into itself has a fixed point.

3. Main Result

In this section, we aim to establish the existence result for the problems (5) and (6). To accomplish this, we let $\mathcal{C}_i = C(\mathbb{N}_{\alpha_i-2}, \mathbb{R})$ be a Banach space of all functions on $\mathbb{N}_{\alpha_i-2}$, for each $i, j \in \{1, 2\}$ and $i \neq j$. Obviously, the product spaces:

$$\mathcal{U}_i = \Big\{(u_1, u_2) \in \mathcal{C}_1 \times \mathcal{C}_2 : \ \Delta^{\beta_i} u_i(t_i - \beta_i + 1) \in \mathcal{C}_i \text{ and } \chi|u_j(t_j)|, \\ \chi|\Delta^{\beta_i} u_i(t_i - \beta_i + 1)| \ \text{ are bounded on } \mathbb{N}_{\alpha_j-2}, \mathbb{N}_{\alpha_i-2}, \text{respectively,}\Big\}$$

is also the Banach space endowed with the norm defined by:

$$\|(u_1, u_2)\|_{\mathcal{U}_i} = \|\Delta^{\beta_i} u_i\|_{\mathcal{C}_i} + \|u_j\|_{\mathcal{C}_j},$$

where:

$$\|\Delta^{\beta_i} u_i\|_{\mathcal{C}_i} = \max_{t_i \in \mathbb{N}_{\alpha_i-2}} \chi\Big|\Delta^{\beta_i} u_i(t_i - \beta_i + 1, t_j)\Big| \ \text{ and } \ \|u_j\|_{\mathcal{C}_j} = \max_{t_j \in \mathbb{N}_{\alpha_j-2}} \chi|u_j(t_i, t_j)|,$$

with for $\rho_i > \max\{\beta_i - \alpha_i\}$ and $\beta_i \in (\alpha_i - 1, \alpha_i)$,

$$\chi = \frac{(t_1 - \alpha_1 + 1)^{\underline{2-\alpha_1}}(t_2 - \alpha_2 + 1)^{\underline{2-\alpha_2}}}{1 + t_1^{\underline{\rho_1+2}} t_2^{\underline{\rho_2+2}}}. \tag{37}$$

Let $\mathcal{U} = \mathcal{U}_1 \cap \mathcal{U}_2$; clearly, the space $\Big(\mathcal{U}, \|(u_1, u_2)\|_{\mathcal{U}}\Big)$ is the Banach space with the norm:

$$\|(u_1, u_2)\|_{\mathcal{U}} = \max\left\{\|(u_1, u_2)\|_{\mathcal{U}_1}, \ \|(u_1, u_2)\|_{\mathcal{U}_2}\right\}.$$

Next, we define the operator $\mathcal{F} : \mathcal{U} \to \mathcal{U}$ by:

$$\big(\mathcal{F}(u_1, u_2)\big)(t_1, t_2) \quad = \quad \Big(\big(\mathcal{F}_1(u_1, u_2)\big)(t_1, t_2), \big(\mathcal{F}_2(u_1, u_2)\big)(t_1, t_2)\Big), \tag{38}$$

and:

$$\begin{aligned}\big(\mathcal{F}_1(u_1, u_2)\big)(t_1, t_2) = \frac{t_1^{\underline{\alpha_1-1}}}{\Lambda}\Bigg\{&\frac{\lambda_1}{\Gamma(\theta_1)}\sum_{s=\alpha_1-2}^{\eta_1}(\eta_1 + \theta_1 - \sigma(s))^{\underline{\theta_1-1}} g_1(s) s^{\underline{\alpha_1-1}} \mathcal{P}(F_1, F_2) \\ &- \frac{\lambda_2}{\Gamma(\theta_2)}\sum_{s=\alpha_2-2}^{\eta_2}(\eta_2 + \theta_2 - \sigma(s))^{\underline{\theta_2-1}} g_2(s) s^{\underline{\alpha_2-1}} \mathcal{Q}(F_1, F_2)\Bigg\} \\ &+ \frac{t_1^{\underline{\alpha_1-2}}\phi_1(u_1, u_2)}{\Gamma(\alpha_1)} + \frac{1}{\Gamma(\alpha_1)}\sum_{s=\alpha_1-1}^{t_1-1}(t_1 + \alpha_1 - 1 - \sigma(s))^{\underline{\alpha_1-1}} \times \\ &F_1\big(s, t_2, \Delta^{\beta_1} u_1(s - \beta_1 + 1), u_2(t_2)\big), \ \ t_i \in \mathbb{N}_{\alpha_i-2},\end{aligned} \tag{39}$$

$$\begin{aligned}\big(\mathcal{F}_2(u_1, u_2)\big)(t_1, t_2) = \frac{t_2^{\underline{\alpha_2-1}}}{\Lambda}\Bigg\{&\lim_{t_2\to\infty} t_2^{\underline{\alpha_2-1}} \mathcal{P}(F_1, F_2) - \lim_{t_1\to\infty} t_1^{\underline{\alpha_1-1}} \mathcal{Q}(F_1, F_2)\Bigg\} \\ &+ \frac{t_2^{\underline{\alpha_2-2}}\phi_2(u_1, u_2)}{\Gamma(\alpha_2)} + \frac{1}{\Gamma(\alpha_2)}\sum_{s=\alpha_2-1}^{t_2-1}(t_2 + \alpha_2 - 1 - \sigma(s))^{\underline{\alpha_2-1}} \times \\ &F_2\big(t_1, s, u_1(t_1), \Delta^{\beta_2} u_2(s - \beta_2 + 1)\big), \ \ t_i \in \mathbb{N}_{\alpha_i-2},\end{aligned} \tag{40}$$

where Λ is defined as (12), and:

$$
\begin{aligned}
\mathcal{P}(F_1,F_2) = & \frac{\lim\limits_{t_1\to\infty} t_1^{\underline{\alpha_1-2}}\phi_1(u_1,u_2)}{\Gamma(\alpha_1)} - \frac{\lambda_2\phi_2(u_1,u_2)}{\Gamma(\alpha_2)\Gamma(\theta_2)}\sum_{s=\alpha_2-2}^{\eta_2}(\eta_2+\theta_2-\sigma(s))^{\underline{\theta_2-1}}g_2(s)\,s^{\underline{\alpha_2-2}} \\
& + \frac{1}{\Gamma(\alpha_1)}\lim_{t_1\to\infty}\sum_{s=\alpha_1-1}^{t_1-1}(t_1+\alpha_1-1-\sigma(s))^{\underline{\alpha_1-1}}F_1\big(s,t_2,\Delta^{\beta_1}u_1(s-\beta_1+1),u_2(t_2)\big) \\
& - \frac{\lambda_2}{\Gamma(\alpha_2)\Gamma(\theta_2)}\sum_{\xi=\alpha_2}^{\eta_2}\sum_{s=\alpha_2-1}^{\xi-1}(\eta_2+\theta_2-\sigma(\xi))^{\underline{\theta_2-1}}(\xi+\alpha_2-1-\sigma(s))^{\underline{\alpha_2-1}}\times \\
& g_2(s)F_2\big(t_1,s,u_1(t_1),\Delta^{\beta_2}u_2(s-\beta_2+1)\big),
\end{aligned} \tag{41}
$$

$$
\begin{aligned}
\mathcal{Q}(F_1,F_2) = & \frac{\lim\limits_{t_2\to\infty} t_2^{\underline{\alpha_2-2}}\phi_2(u_1,u_2)}{\Gamma(\alpha_2)} - \frac{\lambda_1\phi_1(u_1,u_2)}{\Gamma(\alpha_1)\Gamma(\theta_1)}\sum_{s=\alpha_1-2}^{\eta_1}(\eta_1+\theta_1-\sigma(s))^{\underline{\theta_1-1}}g_1(s)\,s^{\underline{\alpha_1-2}} \\
& + \frac{1}{\Gamma(\alpha_2)}\lim_{t_2\to\infty}\sum_{s=\alpha_2-1}^{t_2-1}(t_2+\alpha_2-1-\sigma(s))^{\underline{\alpha_2-1}}F_2\big(t_1,s,u_1(t_1),\Delta^{\beta_2}u_2(s-\beta_2+1)\big) \\
& - \frac{\lambda_1}{\Gamma(\alpha_1)\Gamma(\theta_1)}\sum_{\xi=\alpha_1}^{\eta_1}\sum_{s=\alpha_1-1}^{\xi-1}(\eta_1+\theta_1-\sigma(\xi))^{\underline{\theta_1-1}}(\xi+\alpha_1-1-\sigma(s))^{\underline{\alpha_1-1}}\times \\
& g_1(s)F_1\big(s,t_2,\Delta^{\beta_1}u_1(s-\beta_1+1),u_2(t_2)\big).
\end{aligned} \tag{42}
$$

We next make the following assumptions:

(H_1) There exist positive numbers ${}_{ip}\rho_2 \in (-1,\rho_2)$ and $M_{ip}, m_{ip} > 0$ ($i = 1,2$ and $p = 1,2,3$) such that, for each $t_i \in \mathbb{N}_{\alpha_i-2}$ and $v_i \in \mathbb{R}$,

$$
\begin{aligned}
& \left|F_i\left(t_1,t_2,\frac{1}{\chi}v_1,\frac{1}{\chi}v_2\right) - M_{i1}\,(t_2+{}_{i1}\rho_2)^{{}_{i1}\rho_2}\,e^{-m_{i1}(t_1+t_2)}\right| \\
& \qquad \le M_{i2}\,(t_2+{}_{i2}\rho_2)^{{}_{i2}\rho_2}\,e^{-m_{i2}(t_1+t_2)}\,|v_1| \\
& \qquad\quad + M_{i3}\,(t_2+{}_{i3}\rho_2)^{{}_{i3}\rho_2}\,e^{-m_{i3}(t_1+t_2)}\,|v_2|.
\end{aligned}
$$

(H_2) There exist positive numbers ${}_{ip}\tilde{\rho}_i \in (-1,\rho_i)$ and $N_{ip}, n_{ip} > 0$ ($i = 1,2$ and $p = 1,2,3$) such that, for $v_i \in \mathcal{C}_i$,

$$
\begin{aligned}
& \left|\phi_i\left(\frac{1}{\chi}v_1,\frac{1}{\chi}v_2\right) - N_{i1}\,(t_1+{}_{i1}\tilde{\rho}_1)^{{}_{i1}\tilde{\rho}_1}\left[(t_2+{}_{i1}\tilde{\rho}_2)^{{}_{i1}\tilde{\rho}_2}\right]^2 e^{-n_{i1}(t_1+t_2)}\right| \\
& \qquad \le N_{i2}\,(t_1+{}_{i2}\tilde{\rho}_1)^{{}_{i2}\tilde{\rho}_1}\left[(t_2+{}_{i2}\tilde{\rho}_2)^{{}_{i2}\tilde{\rho}_2}\right]^2 e^{-n_{i2}(t_1+t_2)}\,\|v_1\| \\
& \qquad\quad + N_{i3}\,(t_1+{}_{i3}\tilde{\rho}_1)^{{}_{i3}\tilde{\rho}_1}\left[(t_2+{}_{i3}\tilde{\rho}_2)^{{}_{i3}\tilde{\rho}_2}\right]^2 e^{-n_{i3}(t_1+t_2)}\,\|v_2\|.
\end{aligned}
$$

(H_3) $g_i \le g_i(\eta_i)$ for all $\eta_i \in \mathbb{N}_{\alpha_i-1,T+\alpha_i-1}$.

Lemma 5. *Suppose that (H_1)–(H_3) hold. Then, the fixed point of $\mathcal{F}$ coincides with the solution of the problems (5) and (6), and $\mathcal{F}:\mathcal{U}\to\mathcal{U}$ is completely continuous.*

Proof. Let $(u_1,u_2)\in\mathcal{U}$, for each $i,j\in\{1,2\}$ and $i\neq j$. By the above assumptions (H_1) and (H_2), it follows that:

$$
\left|F_i\left(t_1,t_2,\Delta^{\beta_i}u_i(t_i-\beta_i+1),u_j(t_j)\right)\right|
$$

$$
\begin{aligned}
&= \left| F_i \left(t_1, t_2, \frac{1}{\chi} \left[\chi \Delta^{\beta_i} u_i(t_i - \beta_i + 1) \right], \frac{1}{\chi} \left[\chi u_j(t_j) \right] \right) \right| \\
&\leq \quad M_{i1} \left(t_2 + {}_{i1}\rho_2 \right)^{\underline{i1\rho_2}} e^{-m_{i1}(t_1+t_2)} + M_{i2} \left(t_2 + {}_{i2}\rho_2 \right)^{\underline{i2\rho_2}} e^{-m_{i2}(t_1+t_2)} \|\Delta^{\beta_i} u_i\|_{\mathcal{C}_i} \\
&\quad + M_{i3} \left(t_2 + {}_{i3}\rho_2 \right)^{\underline{i3\rho_2}} e^{-m_{i3}(t_1+t_2)} \|u_j\|_{\mathcal{C}_j},
\end{aligned} \tag{43}
$$

$$
\begin{aligned}
\text{and} \quad \left| \phi_i \left(u_1, u_2 \right) \right| &= \left| \phi_i \left(\frac{1}{\chi} \left[\chi u_1 \right], \frac{1}{\chi} \left[\chi u_2 \right] \right) \right| \\
&\leq \quad N_{i1} \left(t_1 + {}_{i1}\tilde{\rho}_1 \right)^{\underline{i1\tilde{\rho}_1}} \left[\left(t_2 + {}_{i1}\tilde{\rho}_2 \right)^{\underline{i1\tilde{\rho}_2}} \right]^2 e^{-n_{i1}(t_1+t_2)} \\
&\quad + N_{i2} \left(t_1 + {}_{i2}\tilde{\rho}_1 \right)^{\underline{i2\tilde{\rho}_1}} \left[\left(t_2 + {}_{i2}\tilde{\rho}_2 \right)^{\underline{i2\tilde{\rho}_2}} \right]^2 e^{-n_{i2}(t_1+t_2)} \|u_1\|_{\mathcal{C}_1} \\
&\quad + N_{i3} \left(t_1 + {}_{i3}\tilde{\rho}_1 \right)^{\underline{i3\tilde{\rho}_1}} \left[\left(t_2 + {}_{i3}\tilde{\rho}_2 \right)^{\underline{i3\tilde{\rho}_2}} \right]^2 e^{-n_{i3}(t_1+t_2)} \|u_2\|_{\mathcal{C}_2}.
\end{aligned} \tag{44}
$$

The rest of the proof follows from Lemmas 3 and 4. This implies that the fixed point of $\mathcal{F}$ coincides with the solution of the problems (5) and (6).

To show that $\mathcal{F}$ is completely continuous, we organize the proof as the following four steps.

Step I. $\mathcal{F}$ is well defined and maps bounded sets into bounded sets.

Let $B_R = \left\{ (u_1, u_2) \in \mathcal{U} : \|(u_1, u_2)\|_{\mathcal{U}} \leq R \right\}$, then for $(u_1, u_2) \in \mathcal{U}$:

$$
\begin{aligned}
R \quad &\geq \quad \max \left\{ \|(u_1, u_2)\|_{\mathcal{U}_1}, \|(u_1, u_2)\|_{\mathcal{U}_2} \right\} \\
&= \quad \max \Bigg\{ \chi \Big[|\Delta^{\beta_1} u_1(t_1 - \beta_1 + 1)| + |u_2(t_2)| \Big], \\
&\qquad \chi \Big[|u_1(t_1)| + |\Delta^{\beta_2} u_2(t_2 - \beta_2 + 1)| \Big] \Bigg\}.
\end{aligned} \tag{45}
$$

By the definition of $\mathcal{F}$, we get $\mathcal{F}_i(u_1, u_2), \Delta^{\beta_i} \mathcal{F}_i(u_1, u_2) \in \mathcal{U}$. Therefore, (43) and (44) imply that:

$$
\begin{aligned}
&\left| F_i(t_1, t_2, \Delta^{\beta_i} u_i(t_i - \beta_i + 1), u_j(t_j)) \right| \\
&\qquad = \left| F_i \left(t_1, t_2, \frac{1}{\chi} \left[\chi \Delta^{\beta_i} u_i(t_i - \beta_i + 1) \right], \frac{1}{\chi} \left[\chi u_j(t_j) \right] \right) \right| \\
&\qquad \leq M_{i1} \left(t_2 + {}_{i1}\rho_2 \right)^{\underline{i1\rho_2}} e^{-m_{i1}(t_1+t_2)} + R M_{i2} \left(t_2 + {}_{i2}\rho_2 \right)^{\underline{i2\rho_2}} \times \\
&\qquad\quad e^{-m_{i2}(t_1+t_2)} + R M_{i3} \left(t_2 + {}_{i3}\rho_2 \right)^{\underline{i3\rho_2}} e^{-m_{i3}(t_1+t_2)},
\end{aligned} \tag{46}
$$

$$
\begin{aligned}
\text{and} \qquad \left| \phi_i \left(u_1, u_2 \right) \right| &= \left| \phi_i \left(\frac{1}{\chi} \left[\chi u_1 \right], \frac{1}{\chi} \left[\chi u_2 \right] \right) \right| \\
&\leq N_{i1} \left(t_1 + {}_{i1}\tilde{\rho}_1 \right)^{\underline{i1\tilde{\rho}_1}} \left[\left(t_2 + {}_{i1}\tilde{\rho}_2 \right)^{\underline{i1\tilde{\rho}_2}} \right]^2 e^{-n_{i1}(t_1+t_2)} \\
&\quad + R N_{i2} \left(t_1 + {}_{i2}\tilde{\rho}_1 \right)^{\underline{i2\tilde{\rho}_1}} \left[\left(t_2 + {}_{i2}\tilde{\rho}_2 \right)^{\underline{i2\tilde{\rho}_2}} \right]^2 e^{-n_{i2}(t_1+t_2)} \\
&\quad + R N_{i3} \left(t_1 + {}_{i3}\tilde{\rho}_1 \right)^{\underline{i3\tilde{\rho}_1}} \left[\left(t_2 + {}_{i3}\tilde{\rho}_2 \right)^{\underline{i3\tilde{\rho}_2}} \right]^2 e^{-n_{i3}(t_1+t_2)}.
\end{aligned} \tag{47}
$$

Let

$$
\begin{aligned}
\tilde{\Omega}_1 =: \; & \big(N_{11} + N_{12}R + N_{13}R \big) \tilde{\Omega}_{11} + \big(N_{21} + N_{22}R + N_{23}R \big) \tilde{\Omega}_{12} \\
& + \big(M_{11} + M_{12}R + M_{13}R \big) \tilde{\Omega}_{13} + \big(M_{21} + M_{22}R + M_{23}R \big) \tilde{\Omega}_{14} \\
\tilde{\Omega}_2 =: \; & \big(N_{11} + N_{12}R + N_{13}R \big) \tilde{\Omega}_{21} + \big(N_{21} + N_{22}R + N_{23}R \big) \tilde{\Omega}_{22} \\
& + \big(M_{11} + M_{12}R + M_{13}R \big) \tilde{\Omega}_{23} + \big(M_{21} + M_{22}R + M_{23}R \big) \tilde{\Omega}_{24}
\end{aligned}
$$

where:

$$
\begin{aligned}
\tilde{\Omega}_{11} = & \left[\max\left\{\frac{1}{\Gamma(\theta_1)}, \frac{\lambda_1 G_1 \Gamma(\alpha_2)}{\Gamma(\theta_1)}(\eta_1-\alpha_1+\theta_1)^{\underline{\theta_1-1}}\mathcal{B}_1\right\} + \frac{1}{\Gamma(\alpha_1)}\right], \\
\tilde{\Omega}_{12} = & \max\left\{\frac{1}{\Gamma(\theta_2)}, \frac{\lambda_2 G_2 \Gamma(\alpha_1)}{\Gamma(\theta_2)}(\eta_2-\alpha_2+\theta_2)^{\underline{\theta_2-1}}\mathcal{B}_2\right\}, \\
\tilde{\Omega}_{13} = & \left[\max\left\{1, \frac{\lambda_1 G_1 \Gamma(\alpha_2)}{\Gamma(\theta_1)}(\eta_1-\alpha_1+\theta_1-1)^{\underline{\theta_1-1}}\mathcal{C}_1\right\} + \frac{1}{\Gamma(\alpha_1)}\right], \\
\tilde{\Omega}_{14} = & \max\left\{1, \frac{\lambda_2 G_2 \Gamma(\alpha_1)}{\Gamma(\theta_2)}(\eta_2-\alpha_2+\theta_2-1)^{\underline{\theta_2-1}}\mathcal{C}_2\right\}, \\
\tilde{\Omega}_{21} = & \max\left\{\frac{1}{G_2\lambda_2\Gamma(\alpha_1)(\eta_2-\alpha_2+\theta_2)^{\underline{\theta_2-1}}\mathcal{A}_2}, \frac{\mathcal{B}_1}{\Gamma(\theta_1)\mathcal{A}_1}\right\}, \\
\tilde{\Omega}_{22} = & \left[\max\left\{\frac{1}{G_1\lambda_1\Gamma(\alpha_2)(\eta_1-\alpha_1+\theta_1)^{\underline{\theta_1-1}}\mathcal{A}_1}, \frac{\mathcal{B}_2}{\Gamma(\theta_2)\mathcal{A}_2}\right\} + \frac{1}{\Gamma(\alpha_2)}\right], \\
\tilde{\Omega}_{23} = & \max\left\{\frac{1}{G_2\lambda_2\Gamma(\alpha_1)(\eta_2-\alpha_2+\theta_2)^{\underline{\theta_2-1}}\mathcal{A}_2}, \frac{\mathcal{C}_1}{\Gamma(\theta_1)\mathcal{A}_1}\right\}, \\
\tilde{\Omega}_{24} = & \left[\max\left\{\frac{1}{G_1\lambda_1\Gamma(\alpha_2)(\eta_1-\alpha_1+\theta_1)^{\underline{\theta_1-1}}\mathcal{A}_1}, \frac{\mathcal{C}_2}{\Gamma(\theta_2)\mathcal{A}_2}\right\} + \frac{1}{\Gamma(\alpha_2)}\right].
\end{aligned}
$$

Hence, we obtain:

$$
\begin{aligned}
& \chi\left|\left(\mathcal{F}_1(u_1,u_2)\right)(t_1,t_2)\right| \\
\leq\ & \left(N_{11}+N_{12}R+N_{13}R\right)\left[\max\left\{\frac{1}{\Gamma(\theta_1)}, \frac{\lambda_1 G_1 \Gamma(\alpha_2)}{\Gamma(\theta_1)}(\eta_1-\alpha_1+\theta_1)^{\underline{\theta_1-1}}\mathcal{B}_1\right\} + \frac{1}{\Gamma(\alpha_1)}\right] \\
& + \left(N_{21}+N_{22}R+N_{23}R\right)\max\left\{\frac{1}{\Gamma(\theta_2)}, \frac{\lambda_2 G_2 \Gamma(\alpha_1)}{\Gamma(\theta_2)}(\eta_2-\alpha_2+\theta_2)^{\underline{\theta_2-1}}\mathcal{B}_2\right\} \\
& + \left(M_{11}+M_{12}R+M_{13}R\right)\left[\max\left\{1, \frac{\lambda_1 G_1 \Gamma(\alpha_2)}{\Gamma(\theta_1)}(\eta_1-\alpha_1+\theta_1-1)^{\underline{\theta_1-1}}\mathcal{C}_1\right\} + \frac{1}{\Gamma(\alpha_1)}\right] \\
& + \left(M_{21}+M_{22}R+M_{23}R\right)\max\left\{1, \frac{\lambda_2 G_2 \Gamma(\alpha_1)}{\Gamma(\theta_2)}(\eta_2-\alpha_2+\theta_2-1)^{\underline{\theta_2-1}}\mathcal{C}_2\right\} \\
=\ & \tilde{\Omega}_1,
\end{aligned}
\tag{48}
$$

and:

$$
\begin{aligned}
& \chi\left|\left(\mathcal{F}_2(u_1,u_2)\right)(t_1,t_2)\right| \\
\leq\ & \left(N_{11}+N_{12}R+N_{13}R\right)\max\left\{\frac{1}{G_2\lambda_2\Gamma(\alpha_1)(\eta_2-\alpha_2+\theta_2)^{\underline{\theta_2-1}}\mathcal{A}_2}, \frac{\mathcal{B}_1}{\Gamma(\theta_1)\mathcal{A}_1}\right\} \\
& + \left(N_{21}+N_{22}R+N_{23}R\right)\times \\
& \quad \left[\max\left\{\frac{1}{G_1\lambda_1\Gamma(\alpha_2)(\eta_1-\alpha_1+\theta_1)^{\underline{\theta_1-1}}\mathcal{A}_1}, \frac{\mathcal{B}_2}{\Gamma(\theta_2)\mathcal{A}_2}\right\} + \frac{1}{\Gamma(\alpha_2)}\right] \\
& + \left(M_{11}+M_{12}R+M_{13}R\right)\max\left\{\frac{1}{G_2\lambda_2\Gamma(\alpha_1)(\eta_2-\alpha_2+\theta_2)^{\underline{\theta_2-1}}\mathcal{A}_2}, \frac{\mathcal{C}_1}{\Gamma(\theta_1)\mathcal{A}_1}\right\} \\
& + \left(M_{21}+M_{22}R+M_{23}R\right)\times \\
& \quad \left[\max\left\{\frac{1}{G_1\lambda_1\Gamma(\alpha_2)(\eta_1-\alpha_1+\theta_1)^{\underline{\theta_1-1}}\mathcal{A}_1}, \frac{\mathcal{C}_2}{\Gamma(\theta_2)\mathcal{A}_2}\right\} + \frac{1}{\Gamma(\alpha_2)}\right]
\end{aligned}
$$

$$= \tilde{\Omega}_2. \tag{49}$$

Similarly, we have:

$$\chi\left|\Delta^{\beta_1}\left(\mathcal{F}_1(u_1,u_2)\right)(t_1-\beta_1+1,t_2)\right| < \tilde{\Omega}_1, \tag{50}$$

$$\chi\left|\Delta^{\beta_2}\left(\mathcal{F}_2(u_1,u_2)\right)(t_1,t_2-\beta_2+1)\right| < \tilde{\Omega}_2. \tag{51}$$

Therefore, $\mathcal{F}_i(u_1,u_2) \in \mathcal{U}$. This implies that $\mathcal{F}:\mathcal{U}\longrightarrow\mathcal{U}$ is well defined.

Furthermore, we obtain:

$$\begin{aligned}\|\mathcal{F}(u_1,u_2)\|_{\mathcal{U}_i} = \max\Big\{&\chi\left|\Delta^{\beta_i}\big(\mathcal{F}_i(u_1,u_2)\big)(t_i-\beta_i+1,t_j)\right| \\ &+\chi\left|\big(\mathcal{F}_j(u_1,u_2)\big)(t_i,t_j)\right| \text{ for } i,j\in\{1,2\},\ i\neq j\Big\}.\end{aligned} \tag{52}$$

Hence,

$$\|\mathcal{F}(u_1,u_2)\|_{\mathcal{U}} = \max\left\{\|\mathcal{F}(u_1,u_2)\|_{\mathcal{U}_1}, \|\mathcal{F}(u_1,u_2)\|_{\mathcal{U}_2}\right\} < \tilde{\Omega}_1+\tilde{\Omega}_2. \tag{53}$$

Thus, $\mathcal{F}$ maps bounded sets into bounded sets.

Step II. $\mathcal{F}$ is continuous.

Let $\epsilon > 0$ be given. Since F_i and ϕ_i are continuous, then F_i and ϕ_i are uniformly continuous. Therefore, there exists $\delta = \min\{\delta_i,\hat{\delta}_i\} > 0$ such that, for each $t_i \in \mathcal{N}_{\alpha_i-2}$, $u_i,v_i \in \mathcal{C}_i$ with $\max\Big\{\chi\left|\Delta^{\beta_i}u_i(t_i-\beta_i+1)-\Delta^{\beta_i}v_i(t_i)\right|+\chi|u_j(t_i)-v_j(t_j)|\Big\} < \delta_i$,

$$\begin{aligned}&\left|F_i(t_1,t_2,\Delta^{\beta_i}u_i(t_i-\beta_i+1),u_j(t_j)) - F_i\Big(t_1,t_2,\Delta^{\beta_i}v_i(t_i-\beta_i+1),v_j(t_j)\Big)\right| \\ =&\left|F_i\left(t_1,t_2,\frac{1}{\chi}\Big[\chi\Delta^{\beta_i}u_i(t_i-\beta_i+1)\Big],\frac{1}{\chi}\left[\chi u_j(t_j)\right]\right)\right. \\ &\left.-F_i\left(t_1,t_2,\frac{1}{\chi}\Big[\chi\Delta^{\beta_i}v_i(t_i-\beta_i+1)\Big],\frac{1}{\chi}\left[\chi v_j(t_j)\right]\right)\right| \\ <& \ 2M_{i1}+2M_{i2}R+2M_{i3}R < \frac{\epsilon}{4\Omega_i}.\end{aligned} \tag{54}$$

For each $u_i,v_i\in\mathcal{C}_i$ with $\left|u_i-v_i\right| < \hat{\delta}_i$,

$$\begin{aligned}\left|\phi_i(u_1,u_2)-\phi_i(v_1,v_2)\right| &= \left|\phi_i\left(\frac{1}{\chi}[\chi u_1],\frac{1}{\chi}[\chi u_2]\right)-\phi_i\left(\frac{1}{\chi}[\chi v_1],\frac{1}{\chi}[\chi v_2]\right)\right| \\ &< 2N_{i1}+2N_{i2}R+2N_{i3}R < \frac{\epsilon}{4\Omega_i}.\end{aligned} \tag{55}$$

Similar to Step I, we obtain:

$$\chi\left|\left(\mathcal{F}_i(u_1,u_2)\right)-\left(\mathcal{F}_i(v_1,v_2)\right)\right| < 2\tilde{\Omega}_i < \tfrac{\epsilon}{2}$$

and $$\chi\left|\Delta^{\beta_i}\left(\mathcal{F}_i(u_1,u_2)\right)-\Delta^{\beta_i}\left(\mathcal{F}_i(v_1,v_2)\right)\right| < 2\tilde{\Omega}_i < \tfrac{\epsilon}{2}.$$

Thus, we have:

$$\begin{aligned}&\|\mathcal{F}_i(u_1,u_2)-\mathcal{F}_i(v_1,v_2)\|_{\mathcal{U}_i} \\ =& \ \|\Delta^{\beta_i}\mathcal{F}_i(u_1,u_2)-\Delta^{\beta_i}\mathcal{F}_i(v_1,v_2)\|_{\mathcal{C}_i}+\|\mathcal{F}_j(u_1,u_2)-\mathcal{F}_j(v_1,v_2)\|_{\mathcal{C}_j} \\ <& \ 2(\tilde{\Omega}_1+\tilde{\Omega}_2) < \epsilon.\end{aligned} \tag{56}$$

This means that each $\mathcal{F}_i,\ i = 1,2$ is continuous. This shows $\mathcal{F}$ is continuous.

In order to prove that $\mathcal{F}$ maps bounded sets of $\mathcal{U} \subset \mathcal{C}_1 \times \mathcal{C}_2$ to relatively compact sets of $\mathcal{U} \subset \mathcal{C}_1 \times \mathcal{C}_2$, it suffices to show that both $\mathcal{F}_1$ and $\mathcal{F}_2$ map bounded sets to relatively compact sets. Let $\Theta_i \subset \mathcal{C}_i$, $i = 1,2$ be bounded sets and $\Theta_1 \times \Theta_2 \subset \mathcal{U}$. Recall that Θ_i are relatively compact if:

- both Θ_i are bounded,
- both $\chi\Theta_i$ are equicontinuous on any closed subintervals of $\mathbb{N}_{\alpha_i-2}$,
- both $\chi\Theta_i$ are equiconvergent as $t_i \to \infty$.

It has been shown from in Step I that both $\mathcal{F}_i$ are uniformly bounded. Now, we show that $\mathcal{F}_i$ maps bounded sets into equicontinuous sets of $\mathcal{U}$.

Step III. Both $\mathcal{F}_i : \Theta_1 \times \Theta_2 \to \mathcal{U}$ are equicontinuous on $\big([a_1,b_1] \cap \mathbb{N}_{\alpha_1-2}\big) \times \big([a_2,b_2] \cap \mathbb{N}_{\alpha_2-2}\big) := \mathcal{D}$.

For any $\epsilon > 0$, there exists $\delta > 0$ such that, for each $t_{i1}, t_{i2} \in \mathbb{N}_{\alpha_i-2} \cap [a_i,b_i]$,

$$\left|(t_{11}+\rho_1)^{\underline{\rho_1}}(t_{21}+\rho_2)^{\underline{\rho_2}} - (t_{12}+\rho_1)^{\underline{\rho_1}}(t_{22}+\rho_2)^{\underline{\rho_2}}\right| \le \frac{\epsilon}{2\max\{\tilde{\Omega}_1^* + \tilde{\Omega}_2^*\}} = \delta, \tag{57}$$

where:

$$\begin{aligned}
\tilde{\Omega}_1^* &= \Big([N_{11}+N_{12}R+N_{13}R] + [M_{11}+M_{12}R+M_{13}R]\Big)\left[\frac{1}{\Gamma(\theta_1)} + \frac{1}{\Gamma(\alpha_1)}\right] \\
&\quad + \Big([N_{11}+N_{12}R+N_{13}R]\mathcal{B}_1 + [M_{11}+M_{12}R+M_{13}R]\mathcal{C}_1\Big) \times \\
&\quad \frac{\lambda_1 G_1 \Gamma(\alpha_2)(\eta_1-\alpha_1+\theta_1)^{\underline{\theta_1-1}}}{\Gamma(\theta_1)\Gamma(\theta_2)} \\
&\quad + \Big([N_{21}+N_{22}R+N_{23}R] + [M_{21}+M_{22}R+M_{23}R]\Big)\frac{1}{\Gamma(\theta_2)} \\
&\quad + \Big([N_{21}+N_{22}R+N_{23}R]\mathcal{B}_2 + [M_{21}+M_{22}R+M_{23}R]\mathcal{C}_2\Big) \times \\
&\quad \frac{\lambda_2 G_2 \Gamma(\alpha_1)(\eta_2-\alpha_2+\theta_2)^{\underline{\theta_2-1}}}{\Gamma(\theta_1)\Gamma(\theta_2)},
\end{aligned} \tag{58}$$

$$\begin{aligned}
\tilde{\Omega}_2^* &= [N_{11}+N_{12}R+N_{13}R]\left[\frac{1}{\lambda_2 G_2(\eta_2-\alpha_2+\theta_2)^{\underline{\theta_2-1}}\mathcal{A}_2} + \frac{1}{\Gamma(\alpha_2)}\right] \\
&\quad + [N_{21}+N_{22}R+N_{23}R]\frac{1}{\lambda_1 G_1(\eta_1-\alpha_1+\theta_1)^{\underline{\theta_1-1}}\mathcal{A}_1} \\
&\quad + [M_{11}+M_{12}R+M_{13}R]\frac{\mathcal{B}_1}{\Gamma(\theta_1)\mathcal{A}_1} + [M_{21}+M_{22}R+M_{23}R]\frac{\mathcal{B}_2}{\Gamma(\theta_2)\mathcal{A}_2}.
\end{aligned} \tag{59}$$

Hence, for each $t_{i1}, t_{i2} \in \mathbb{N}_{\alpha_i-2} \cap [a_i,b_i]$, and $u_i \in \Theta_i$, we have:

$$\begin{aligned}
&\left|\chi(\mathcal{F}_1 u_1)(t_{11},t_{21}) - \chi(\mathcal{F}_1 u_1)(t_{12},t_{22})\right| \\
\le\ & \chi\left| t_{11}^{\underline{\alpha_1}}\left\{\left[\frac{|\phi_1(u_1,u_2)|}{\Gamma(\theta_1)} - \frac{|\phi_2(u_1,u_2)|}{\Gamma(\theta_2)} + \frac{|\phi_1(u_1,u_2)|}{\Gamma(\alpha_1)}\right] + \frac{1}{\Gamma(\theta_1)\Gamma(\theta_2)} \times \right.\right. \\
&\quad \left[\frac{\lambda_1\Gamma(\alpha_2)\mathcal{B}_1|\phi_1(u_1,u_2)|}{t_{21}^{\underline{\alpha_2-1}}\Gamma(\alpha_1)} + \frac{\lambda_2\Gamma(\alpha_1)\mathcal{B}_2|\phi_2(u_1,u_2)|}{t_{11}^{\underline{\alpha_1-1}}\Gamma(\alpha_2)}\right] + \frac{1}{\Gamma(\theta_1)\Gamma(\theta_2)} \times \\
&\quad + \left[\frac{\lambda_1\Gamma(\alpha_2)\mathcal{B}_1|F_1(t_1,t_2,\Delta^{\beta_1}u_1,u_2)|}{t_{22}^{\underline{\alpha_2-1}}\Gamma(\alpha_1)} + \frac{\lambda_2\Gamma(\alpha_1)\mathcal{B}_2|F_2(t_1,t_2,u_1,\Delta^{\beta_2}u_2)|}{t_{12}^{\underline{\alpha_1-1}}\Gamma(\alpha_2)}\right] \\
&\quad \left[\frac{|F_1(t_1,t_2,\Delta^{\beta_1}u_1,u_2)|}{\Gamma(\theta_1)} - \frac{|F_2(t_1,t_2,u_1,\Delta^{\beta_2}u_2)|}{\Gamma(\theta_2)} + \frac{|F_1(t_1,t_2,\Delta^{\beta_1}u_1,u_2)|}{\Gamma(\alpha_1)}\right]\right\}
\end{aligned}$$

$$
\begin{aligned}
&-t_{12}^{\alpha_1}\Bigg\{\left[\frac{|\phi_1(u_1,u_2)|}{\Gamma(\theta_1)}-\frac{|\phi_2(u_1,u_2)|}{\Gamma(\theta_2)}+\frac{|\phi_1(u_1,u_2)|}{\Gamma(\alpha_1)}\right]+\frac{1}{\Gamma(\theta_1)\Gamma(\theta_2)}\times\\
&\quad\left[\frac{\lambda_1\Gamma(\alpha_2)\mathcal{B}_1|\phi_1(u_1,u_2)|}{t_{21}^{\alpha_2-1}\Gamma(\alpha_1)}+\frac{\lambda_2\Gamma(\alpha_1)\mathcal{B}_2|\phi_2(u_1,u_2)|}{t_{11}^{\alpha_1-1}\Gamma(\alpha_2)}\right]+\frac{1}{\Gamma(\theta_1)\Gamma(\theta_2)}\times\\
&\quad+\left[\frac{\lambda_1\Gamma(\alpha_2)\mathcal{B}_1|F_1(t_1,t_2,\Delta^{\beta_1}u_1,u_2)|}{t_{22}^{\alpha_2-1}\Gamma(\alpha_1)}+\frac{\lambda_2\Gamma(\alpha_1)\mathcal{B}_2|F_2(t_1,t_2,u_1,\Delta^{\beta_2}u_2)|}{t_{12}^{\alpha_1-1}\Gamma(\alpha_2)}\right]\\
&\quad\left[\frac{|F_1(t_1,t_2,\Delta^{\beta_1}u_1,u_2)|}{\Gamma(\theta_1)}-\frac{|F_2(t_1,t_2,u_1,\Delta^{\beta_2}u_2)|}{\Gamma(\theta_2)}+\frac{|F_1(t_1,t_2,\Delta^{\beta_1}u_1,u_2)|}{\Gamma(\alpha_1)}\right]\Bigg\}\Bigg|\\
<&\ \left|(t_{11}+\rho_1)^{\underline{\rho_1}}(t_{21}+\rho_2)^{\underline{\rho_2}}-(t_{12}+\rho_1)^{\underline{\rho_1}}(t_{22}+\rho_2)^{\underline{\rho_2}}\right|\tilde{\Omega}_1^*\\
<&\ \frac{\epsilon}{2},
\end{aligned}
\tag{60}
$$

and:

$$
\begin{aligned}
&\left|\chi\left(\mathcal{F}_2u_2\right)(t_{11},t_{21})-\chi\left(\mathcal{F}_2u_2\right)(t_{12},t_{22})\right|\\
\leq&\ \chi\Bigg|t_{21}^{\alpha_2}\Bigg\{\left[\frac{t_{11}^{\alpha_1-1}|\phi_1(u_1,u_2)|}{\lambda_2G_2(\eta_2-\alpha_2+\theta_2)^{\underline{\theta_2-1}}\mathcal{A}_2}-\frac{t_{21}^{\alpha_2-1}|\phi_2(u_1,u_2)|}{\lambda_1G_1(\eta_1-\alpha_1+\theta_1)^{\underline{\theta_1-1}}\mathcal{A}_1}+\frac{|\phi_2(u_1,u_2)|}{\Gamma(\alpha_2)}\right]\\
&\quad+t_{21}^{\alpha_2-1}\left[\frac{\mathcal{B}_1|\phi_1(u_1,u_2)|}{\Gamma(\theta_1)\mathcal{A}_1}-\frac{\mathcal{B}_2|\phi_2(u_1,u_2)|}{\Gamma(\theta_2)\mathcal{A}_2}\right]+\left[\frac{t_{11}^{\alpha_1-1}|F_1(t_1,t_2,\Delta^{\beta_1}u_1,u_2)|}{\lambda_2G_2(\eta_2-\alpha_2+\theta_2)^{\underline{\theta_2-1}}\mathcal{A}_2}\right.\\
&\quad\left.-\frac{t_{21}^{\alpha_2-1}|F_2(t_1,t_2,u_1,\Delta^{\beta_2}u_2)|}{\lambda_1G_1(\eta_1-\alpha_1+\theta_1)^{\underline{\theta_1-1}}\mathcal{A}_1}+\frac{|F_2(t_1,t_2,u_1,\Delta^{\beta_2}u_2)|}{\Gamma(\alpha_2)}\right]\\
&\quad+t_{21}^{\alpha_2-1}\left[\frac{t_{11}^{\alpha_1}\mathcal{C}_1|F_1(t_1,t_2,\Delta^{\beta_1}u_1,u_2)|}{\Gamma(\theta_1)\mathcal{A}_1}-\frac{t_{21}^{\alpha_2}\mathcal{C}_2|F_2(t_1,t_2,u_1,\Delta^{\beta_2}u_2)|}{\Gamma(\theta_2)\mathcal{A}_2}\right]\Bigg\}\\
&-t_{22}^{\alpha_2}\Bigg\{\left[\frac{t_{11}^{\alpha_1-1}|\phi_1(u_1,u_2)|}{\lambda_2G_2(\eta_2-\alpha_2+\theta_2)^{\underline{\theta_2-1}}\mathcal{A}_2}-\frac{t_{21}^{\alpha_2-1}|\phi_2(u_1,u_2)|}{\lambda_1G_1(\eta_1-\alpha_1+\theta_1)^{\underline{\theta_1-1}}\mathcal{A}_1}+\frac{|\phi_2(u_1,u_2)|}{\Gamma(\alpha_2)}\right]\\
&\quad+t_{21}^{\alpha_2-1}\left[\frac{\mathcal{B}_1|\phi_1(u_1,u_2)|}{\Gamma(\theta_1)\mathcal{A}_1}-\frac{\mathcal{B}_2|\phi_2(u_1,u_2)|}{\Gamma(\theta_2)\mathcal{A}_2}\right]+\left[\frac{t_{11}^{\alpha_1-1}|F_1(t_1,t_2,\Delta^{\beta_1}u_1,u_2)|}{\lambda_2G_2(\eta_2-\alpha_2+\theta_2)^{\underline{\theta_2-1}}\mathcal{A}_2}\right.\\
&\quad\left.-\frac{t_{21}^{\alpha_2-1}|F_2(t_1,t_2,u_1,\Delta^{\beta_2}u_2)|}{\lambda_1G_1(\eta_1-\alpha_1+\theta_1)^{\underline{\theta_1-1}}\mathcal{A}_1}+\frac{|F_2(t_1,t_2,u_1,\Delta^{\beta_2}u_2)|}{\Gamma(\alpha_2)}\right]\\
&\quad+t_{21}^{\alpha_2-1}\left[\frac{t_{11}^{\alpha_1}\mathcal{C}_1|F_1(t_1,t_2,\Delta^{\beta_1}u_1,u_2)|}{\Gamma(\theta_1)\mathcal{A}_1}-\frac{t_{21}^{\alpha_2}\mathcal{C}_2|F_2(t_1,t_2,u_1,\Delta^{\beta_2}u_2)|}{\Gamma(\theta_2)\mathcal{A}_2}\right]\Bigg\}\\
<&\ \left|(t_{11}+\rho_1)^{\underline{\rho_1}}(t_{21}+\rho_2)^{\underline{\rho_2}}-(t_{12}+\rho_1)^{\underline{\rho_1}}(t_{22}+\rho_2)^{\underline{\rho_2}}\right|\tilde{\Omega}_2^*\\
<&\ \frac{\epsilon}{2}.
\end{aligned}
\tag{61}
$$

Similarly, for each $i,j\in\{1,2\}$ and $j\neq i$, we obtain:

$$
\left|\Delta^{\beta_i}\left(\mathcal{F}_i(u_1,u_2)\right)(t_{i1}-\beta_i+1,t_{j1})-\Delta^{\beta_i}\left(\mathcal{F}_i(u_1,u_2)\right)(t_{i2}-\beta_i+1,t_{j2})\right|<\frac{\epsilon}{2}.
\tag{62}
$$

Hence:

$$
\begin{aligned}
&\|\mathcal{F}_i(u_1,u_2)(t_{11},t_{21})-\mathcal{F}_i(u_1,u_2)(t_{12},t_{22})\|_{\mathcal{U}_i}\\
=&\ \|\Delta^{\beta_i}\mathcal{F}_i(u_1,u_2)(t_{i1}-\beta_i+1,t_{j1})-\Delta^{\beta_i}\mathcal{F}_i(u_1,u_2)(t_{i2}-\beta_i+1,t_{j2})\|_{\mathcal{C}_i}
\end{aligned}
$$

$$
\begin{aligned}
&+ \|\mathcal{F}_j(u_1,u_2)(t_{11},t_{21}) - \mathcal{F}_j(u_1,u_2)(t_{12},t_{22})\|_{\mathcal{C}_j} \\
<& \ \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \qquad (63)
\end{aligned}
$$

This implies that both $\mathcal{F}_1$ and $\mathcal{F}_2$ are equicontinuous on $\mathcal{D}$, which shows that $\mathcal{F}$ is equicontinuous on $\mathcal{D}$. Therefore, by the Arzelá–Ascoli theorem and Theorem 2, we can conclude that $\mathcal{F}$ is completely continuous.

Step IV. Both $\mathcal{F}_i : \Theta_1 \times \Theta_2 \to \mathcal{U}$ are equiconvergent as $t_1, t_2 \to \infty$.

By the assumption $(H1) - (H2)$, we obtain:

$$
\begin{aligned}
\chi\left| \left(\mathcal{F}_1(u_1,u_2)\right)(t_1,t_2)\right| &< \frac{\tilde{\Omega}_1^*}{t_1^{\alpha_1} t_2^{\alpha_2}} \to 0 \ \text{ uniformly in } \Theta_1 \times \Theta_2 \text{ as } t_1, t_2 \to \infty, \\
\chi\left| \left(\mathcal{F}_2(u_1,u_2)\right)(t_1,t_2)\right| &< \frac{\tilde{\Omega}_2^*}{t_1^{\alpha_1} t_2^{\alpha_2}} \to 0 \ \text{ uniformly in } \Theta_1 \times \Theta_2 \text{ as } t_1, t_2 \to \infty,
\end{aligned}
$$

where $\tilde{\Omega}_1^*, \tilde{\Omega}_2^*$ are defined as (58) and (59).

Furthermore, we have:

$$
\begin{aligned}
\chi\left|\Delta^{\beta_1}\left(\mathcal{F}_1(u_1,u_2)\right)(t_1-\beta_1+1,t_2)\right| &< \frac{\tilde{\Omega}_1^*}{t_1^{\alpha_1} t_2^{\alpha_2}} \to 0 \\
&\quad \text{uniformly in } \Theta_1 \times \Theta_2 \text{ as } t_1, t_2 \to \infty, \\
\chi\left|\Delta^{\beta_2}\left(\mathcal{F}_2(u_1,u_2)\right)(t_1,t_2-\beta_2+1)\right| &< \frac{\tilde{\Omega}_2^*}{t_1^{\alpha_1} t_2^{\alpha_2}} \to 0 \\
&\quad \text{uniformly in } \Theta_1 \times \Theta_2 \text{ as } t_1, t_2 \to \infty.
\end{aligned}
$$

Hence, both $\mathcal{F}_i$ are equiconvergent as $t_1, t_2 \to \infty$.

Consequently, from Step I–Step IV, we conclude that $\mathcal{F}$ is completely continuous. This complete the proof. □

Finally, we present the main result of the article. For the sake of convenience, we set:

$$
\begin{aligned}
\Psi_1 = &\left(N_{12}+N_{13}\right)\tilde{\Omega}_{11} + \left(N_{22}+N_{23}\right)\tilde{\Omega}_{12} + \left(M_{12}+M_{13}\right)\tilde{\Omega}_{13} \\
&+ \left(M_{22}+M_{23}\right)\tilde{\Omega}_{14}, \qquad (64)
\end{aligned}
$$

and (65)

$$
\begin{aligned}
\Psi_2 = &\left(N_{12}+N_{13}\right)\tilde{\Omega}_{21} + \left(N_{22}+N_{23}\right)\tilde{\Omega}_{22} + \left(M_{12}+M_{13}\right)\tilde{\Omega}_{23} \\
&+ \left(M_{22}+M_{23}\right)\tilde{\Omega}_{24}, \qquad (66)
\end{aligned}
$$

where $\tilde{\Omega}_{1p}, \tilde{\Omega}_{2p},\ p = 1,2,3,4$ are defined as (48) and (49).

Theorem 4. *Suppose that (H_1)–(H_2) hold. Then, the problems (5) and (6) has at least one solution if:*

$$\Psi_1 + \Psi_2 < 1. \qquad (67)$$

Proof. Under the Banach space $\mathcal{U}$ equipped with the norm $\|\cdot\|_{\mathcal{U}}$, we let:

$$
\begin{aligned}
&\omega_1(t_1,t_2) \\
= &\frac{t_1^{\alpha_1-1}}{\Lambda}\left\{\frac{\lambda_1 G_1 \Gamma(\alpha_1)(\eta_1-\alpha_1+\theta_1)^{\underline{\theta_1-1}}\mathcal{A}_1}{\Gamma(\theta_1)}\left[(t_1+\rho_1)^{\underline{\rho_1}} e^{-n_1(t_1+1)} \times\right.\right.
\end{aligned}
$$

$$\left(\frac{N_{11}t_1^{\underline{\alpha_1-2}}}{\Gamma(\alpha_1)} - \frac{N_{21}\lambda_2 G_2(\eta_2-\alpha_2+\theta_2)^{\underline{\theta_2-1}}\mathcal{B}_2}{\Gamma(\theta_2)}\right) + \frac{M_{11}e^{-m_1t_2}}{\Gamma(\alpha_1)}\times$$
$$\sum_{s=0}^{t_1-\alpha_1}(t_1-\sigma(s))^{\underline{\alpha_1-1}}e^{-2m_1s} - \frac{M_{21}G_2(\eta_2-\alpha_2+\theta_2-1)^{\underline{\theta_2-1}}e^{-m_1(2t_1+t_2)}}{\Gamma(\theta_2)}\Bigg]$$
$$+\frac{\lambda_2 G_2\Gamma(\alpha_2)(\eta_2-\alpha_2+\theta_2)^{\underline{\theta_2-1}}\mathcal{A}_2}{\Gamma(\theta_2)}\Bigg[(t_1+\rho_1)^{\underline{\rho_1}}e^{-n_1(t_1+1)}\times$$
$$\left(\frac{N_{21}t_2^{\underline{\alpha_2-2}}}{\Gamma(\alpha_2)} - \frac{N_{11}\lambda_1 G_1(\eta_1-\alpha_1+\theta_1)^{\underline{\theta_1-1}}\mathcal{B}_1}{\Gamma(\theta_1)}\right) + \frac{M_{21}e^{-2m_1t_1}}{\Gamma(\alpha_2)}\times$$
$$\sum_{s=0}^{t_2-\alpha_2}(t_2-\sigma(s))^{\underline{\alpha_2-1}}e^{-m_1s} - \frac{M_{11}G_1(\eta_1-\alpha_1+\theta_1-1)^{\underline{\theta_1-1}}e^{-m_1(2t_1+t_2)}}{\Gamma(\theta_1)}\Bigg]\Bigg\}$$
$$+\frac{N_{11}}{\Gamma(\alpha_1)}(t_1+\rho_1)^{\underline{\rho_1}}e^{-n_1(t_1+1)} + \frac{M_{11}e^{-m_1t_2}}{\Gamma(\alpha_1)}\sum_{s=0}^{t_1-\alpha_1}(t_1-\sigma(s))^{\underline{\alpha_1-1}}e^{-2m_1s}, \tag{68}$$

and

$$\omega_2(t_1,t_2)$$
$$= \frac{t_2^{\underline{\alpha_2-1}}}{\Lambda}\Bigg\{t_2^{\underline{\alpha_2-1}}\Bigg[(t_1+\rho_1)^{\underline{\rho_1}}\left[(t_2+\rho_2)^{\underline{\rho_2}}\right]^2 e^{-n_2(t_2+1)}\Bigg(\frac{N_{12}t_1^{\underline{\alpha_1-2}}}{\Gamma(\alpha_1)}$$
$$-\frac{N_{22}\lambda_2 G_2(\eta_2-\alpha_2+\theta_2)^{\underline{\theta_2-1}}\mathcal{B}_2}{\Gamma(\theta_2)}\Bigg) + \frac{M_{12}e^{-m_2t_2}}{\Gamma(\alpha_1)}(t_2+\rho_2)^{\underline{\rho_2}}\sum_{s=0}^{t_1-\alpha_1}(t_1-\sigma(s))^{\underline{\alpha_1-1}}\times$$
$$e^{-2m_2s} - \frac{M_{22}G_2(\eta_2-\alpha_2+\theta_2-1)^{\underline{\theta_2-1}}e^{-m_2(t_1+2t_2)}(t_2+\rho_2)^{\underline{\rho_2}}}{\Gamma(\theta_2)}\Bigg] + t_1^{\underline{\alpha_1-1}}\times$$
$$\Bigg[(t_1+\rho_1)^{\underline{\rho_1}}\left[(t_2+\rho_2)^{\underline{\rho_2}}\right]^2 e^{-n_2(t_2+1)}\left(\frac{N_{22}t_2^{\underline{\alpha_2-2}}}{\Gamma(\alpha_2)} - \frac{N_{12}\lambda_1 G_1(\eta_1-\alpha_1+\theta_1)^{\underline{\theta_1-1}}\mathcal{B}_1}{\Gamma(\theta_1)}\right)$$
$$+\frac{M_{22}e^{-m_2t_1}}{\Gamma(\alpha_2)}\sum_{s=0}^{t_2-\alpha_2}(t_2-\sigma(s))^{\underline{\alpha_2-1}}e^{-2m_2s}(t_2+\rho_2)^{\underline{\rho_2}}$$
$$-\frac{M_{22}G_1(\eta_1-\alpha_1+\theta_1-1)^{\underline{\theta_1-1}}e^{-m_2(t_2+2t_2)}}{\Gamma(\theta_1)}\Bigg] + \frac{N_{22}}{\Gamma(\alpha_2)}\left[(t_2+\rho_2)^{\underline{\rho_2}}\right]^2 e^{-n_2(t_2+1)}t_2^{\underline{\alpha_2-2}}\Bigg\}$$
$$+\frac{M_{22}e^{-m_2t_1}}{\Gamma(\alpha_2)}\sum_{s=0}^{t_2-\alpha_2}(t_2-\sigma(s))^{\underline{\alpha_2-1}}e^{-2m_2s}(t_2+\rho_2)^{\underline{\rho_2}}. \tag{69}$$

It is clear that $(\omega_1,\omega_2)\in\mathcal{U}$. For $\ell>0$, we define:

$$\Xi_\ell = \left\{(u_1,u_2)\in\mathcal{U} \;:\; \|(u_1,u_2)-(\omega_1,\omega_2)\|_{\mathcal{U}} \le \ell\right\}. \tag{70}$$

For $(u_1,u_2)\in\Xi_\ell$, we have:

$$\|(u_1,u_2)\|_{\mathcal{U}} \le \|(u_1,u_2)-(\omega_1,\omega_2)\|_{\mathcal{U}} + \|(\omega_1,\omega_2)\|_{\mathcal{U}} \le \ell + \|(\omega_1,\omega_2)\|_{\mathcal{U}}, \tag{71}$$
$$\|(u_1,u_2)\|_{\mathcal{U}} = \max\left\{\|(u_1,u_2)\|_{\mathcal{U}_1}, \|(u_1,u_2)\|_{\mathcal{U}_2}\right\} \le \ell + \|(\omega_1,\omega_2)\|_{\mathcal{U}}. \tag{72}$$

Using the conditions (H_1)–(H_2), together with the procedure employed in Lemma 5, we have:

$$\left|F_i\left(t_1,t_2,\Delta^{\beta_i}u_i,u_j\right) - M_{i1}\left(t_2+{}_{i1}\rho_2\right)^{\underline{i1\rho_2}}e^{-m_{i1}(t_1+t_2)}\right|$$
$$\le \Bigg\{M_{i2}\left(t_2+{}_{i2}\rho_2\right)^{\underline{i2\rho_2}}e^{-m_{i2}(t_1+t_2)}$$

$$+ M_{i3}\left(t_2 + {}_{i3}\rho_2\right)^{\underline{i3\rho_2}} e^{-m_{i3}(t_1+t_2)} \Big\} \|(u_1,u_2)\|_{\mathcal{U}}, \tag{73}$$

and:

$$\begin{aligned} &\left|\phi_i\left(u_1,u_2\right) - N_{i1}\left(t_1 + {}_{i1}\tilde{\rho}_1\right)^{\underline{i1\tilde{\rho}_1}}\left[\left(t_2 + {}_{i1}\tilde{\rho}_2\right)^{\underline{i1\tilde{\rho}_2}}\right]^2 e^{-n_{i1}(t_1+t_2)}\right| \\ = &\Big\{ N_{i2}\left(t_1 + {}_{i2}\tilde{\rho}_1\right)^{\underline{i2\tilde{\rho}_1}}\left[\left(t_2 + {}_{i2}\tilde{\rho}_2\right)^{\underline{i2\tilde{\rho}_2}}\right]^2 e^{-n_{i2}(t_1+t_2)} \\ &+ N_{i3}\left(t_1 + {}_{i3}\tilde{\rho}_1\right)^{\underline{i3\tilde{\rho}_1}}\left[\left(t_2 + {}_{i3}\tilde{\rho}_2\right)^{\underline{i3\tilde{\rho}_2}}\right]^2 e^{-n_{i3}(t_1+t_2)}\Big\} \|(u_1,u_2)\|_{\mathcal{U}}. \end{aligned} \tag{74}$$

Therefore, we obtain:

$$\chi\left|\left(\mathcal{F}_i(u_i,u_j)\right)(t_1,t_2) - \omega_i(t_1,t_2)\right| \leq \Big(\ell + \|(\omega_1,\omega_2)\|_{\mathcal{U}}\Big)\Psi_i. \tag{75}$$

Furthermore, we have:

$$\begin{aligned} &\chi\left|\Delta^{\beta_i}\left(\mathcal{F}_i(u_i,u_j)\right)(t_i-\beta_i+1,t_j) - \Delta^{\beta_i}\omega_i(t_i-\beta_i+1,t_j)\right| \\ \leq &\Big(\ell + \|(\omega_1,\omega_2)\|_{\mathcal{U}}\Big)\Psi_i. \end{aligned} \tag{76}$$

Hence, it follows that:

$$\left\|\left(\mathcal{F}_i(u_1,u_2)\right) - \omega_i\right\|_{\mathcal{U}_i} \leq \Big(\ell + \|(\omega_1,\omega_2)\|_{\mathcal{U}}\Big)2\Psi_i. \tag{77}$$

Therefore,

$$\left\|\left(\mathcal{F}(u_1,u_2)\right) - (\omega_1,\omega_2)\right\|_{\mathcal{U}} \leq \Big(\ell + \|(\omega_1,\omega_2)\|_{\mathcal{U}}\Big) 2\max\{\Psi_1,\Psi_2\}. \tag{78}$$

Choosing:

$$\ell \geq \frac{\left\|(\omega_1,\omega_2)\right\|_{\mathcal{U}} 2\max\{\Psi_1,\Psi_2\}}{1 - 2\max\{\Psi_1,\Psi_2\}}, \tag{79}$$

and for $(u_1,u_2) \in \Xi_\ell$, we consequently obtain:

$$\left\|\left(\mathcal{F}(u_1,u_2)\right) - (\omega_1,\omega_2)\right\|_{\mathcal{U}} \leq \ell. \tag{80}$$

From the Schauder fixed point theorem, this implies that $\mathcal{F}$ has a fixed point $(u_1,u_2) \in \Xi_\ell$, which is a bounded solution of the problems (5) and (6). The proof is complete. □

4. Example

In order to illustrate our result, we consider the following fractional sum boundary value problem:

$$\begin{aligned} \Delta^{\frac{3}{2}} u_1(t) \quad = \quad & \frac{2}{3}\left(t + \frac{4}{3}\right) e^{-(12t+5)} + \frac{\left(t+\frac{5}{3}\right)^{\frac{4}{3}} e^{-\left(25t+\frac{31}{3}\right)} u_2\left(t+\frac{1}{3}\right)}{4000\left(t+\frac{301}{3}\right)^2\left(1+\cos^2 u_2\pi\right)} \\ & + \frac{\left(t+\frac{11}{6}\right)^{\frac{3}{2}} e^{-\left[(12t+5)+\left(t+\frac{1}{2}\right)\pi\right]} \Delta^{\frac{1}{3}} u_1\left(t+\frac{4}{3}\right)}{5000e + 10\cos^2\left(t+\frac{1}{2}\right)\pi}, \quad t \in \mathbb{N}_0 \end{aligned}$$

$$\begin{aligned}
\Delta^{\frac{4}{3}}u_2(t) &= \frac{1}{2}\left(t+\frac{4}{3}\right)e^{-(12t+5)} + \frac{\left(t+\frac{5}{3}\right)^{\frac{4}{3}}e^{-\left(25t+\frac{21}{3}\right)}u_1\left(t+\frac{1}{2}\right)}{1000\left(e^{\left(t+\frac{1}{2}\right)}+10\right)^2} \\
&\quad + \frac{\left(t+\frac{11}{6}\right)^{\frac{3}{2}}e^{-(12t+5)}\arctan\left(\cos^2\left(t+\frac{1}{3}\right)\pi\right)\Delta^{\frac{3}{4}}u_2\left(t_2+\frac{7}{12}\right)}{1000\pi\left(t+\frac{10}{3}\right)^2}, \quad t\in\mathbb{N}_0 \\
u_1\Big(-\frac{1}{2}\Big) &= \phi_1(u_1,u_2) = \frac{|u_1|}{2000e^3}\cos^2|\pi u_1| + \frac{|u_2||u_2^2+2|^{1-|u_2^2+2|}}{4000\pi^2(u_2^2+e)}, \\
u_2\Big(-\frac{2}{3}\Big) &= \phi_2(u_1,u_2) = \frac{|u_2|}{5000e^2}\sin^2|\pi u_2| + \frac{|u_1||u_1^2+3|^{1-|u_1^2+3|}}{2000\pi(u_1^2+\pi)}, \\
\lim_{t\longrightarrow\infty} u_1\Big(t-\frac{1}{2}\Big) &= \frac{1}{2}\Delta^{-\frac{1}{4}}\Big(12e+\cos(4)\Big)^2 u_2(4) \\
\lim_{t\longrightarrow\infty} u_2\Big(t-\frac{2}{3}\Big) &= \frac{3}{4}\Delta^{-\frac{2}{3}}\left(10e-\sin\left(\frac{15}{4}\right)\right)^3 u_1\Big(\frac{15}{4}\Big).
\end{aligned} \tag{81}$$

Here, $\alpha_1 = \frac{3}{2}$, $\alpha_2 = \frac{4}{3}$, $\beta_1 = \frac{1}{3}$, $\beta_2 = \frac{3}{4}$, $\gamma_1 = \frac{3}{4}$, $\gamma_2 = \frac{5}{6}$, $\theta_1 = \frac{1}{4}$, $\theta_2 = \frac{2}{3}$, $\eta_1 = \frac{7}{2}$, $\eta_2 = \frac{10}{3}$, $\lambda_1 = \frac{1}{2}$, $\lambda_2 = \frac{3}{4}$, $T = 4$, $g_1(t_1) = (10e - \sin t_1)^3$, $g_2(t_2) = (12e + \cos t_2)^2$, and:

$$\begin{aligned}
F_1\left(t_1,t_2,\Delta^{\frac{1}{3}}u_1\left(t_1+\tfrac{2}{3}\right),u_2(t_2)\right) &= \tfrac{2}{3}\left(t_2+1\right)e^{-6(t_1+t_2)} + \frac{\left(t_2+\frac{4}{3}\right)^{\frac{4}{3}}e^{-[12(t_1+t_2)+t_2]}u_2(t_2)}{4000(t_2+10)^2(1+\cos^2 u_2\pi)} \\
&\quad + \frac{\left(t_2+\frac{3}{2}\right)^{\frac{3}{2}}e^{-[6(t_1+t_2)+t_1\pi]}\Delta^{\frac{1}{3}}u_1\left(t_1+\frac{2}{3}\right)}{5000e+10\cos^2 t_1\pi} \\
F_2\left(t_1,t_2,\Delta^{\frac{3}{4}}u_2\left(t_2+\tfrac{1}{4}\right),u_1(t_1)\right) &= \tfrac{1}{2}\left(t_2+1\right)e^{-6(t_1+t_2)} + \frac{\left(t_2+\frac{4}{3}\right)^{\frac{4}{3}}e^{-[12(t_1+t_2)+t_1]}u_1(t_1)}{1000(e^{t_1}+10)^2} \\
&\quad + \frac{\left(t_2+\frac{3}{2}\right)^{\frac{3}{2}}e^{-6(t_1+t_2)}\arctan\left(\cos^2 t_2\pi\right)\Delta^{\frac{3}{4}}u_2\left(t_2+\frac{1}{4}\right)}{1000\pi(t_2+3)^2}.
\end{aligned}$$

Choose $\rho_1 = 1$, $\rho_2 = 2$, ${}_{i1}\rho_1 = {}_{i1}\tilde{\rho}_1 = \frac{1}{2}$, ${}_{i2}\rho_1 = {}_{i2}\tilde{\rho}_1 = \frac{2}{3}$, ${}_{i3}\rho_1 = {}_{i3}\tilde{\rho}_1 = \frac{3}{4}$ ${}_{i1}\rho_2 = 1$, ${}_{i2}\rho_2 = \frac{3}{2}$, ${}_{i3}\rho_2 = \frac{4}{3}$, $m_{i1} = n_{i1} = 6$, $m_{i2} = n_{i2} = 6$, $m_{i3} = n_{i3} = 12$, where $\rho_i > \max\{\beta_1-\alpha_1, \beta_2-\alpha_2\}$, ${}_{ip}\rho_1 \in (-1,1)$, ${}_{ip}\rho_2 \in (-1,2)$ for $i = 1,2$ and $p = 1,2,3$.

Let $t_1 \in \mathbb{N}_{-\frac{1}{2},\frac{11}{2}}$, $t_2 \in \mathbb{N}_{-\frac{2}{3},\frac{16}{3}}$ and $\chi = \frac{\left(t_1-\frac{1}{2}\right)^{1/2}\left(t_2-\frac{1}{3}\right)^{2/3}}{1+t_1^3t_2^4}$. Since:

$$\begin{aligned}
&\left|F_1\left(t_1,t_2,\frac{1}{\chi}\Delta^{\frac{1}{3}}u_1,\frac{1}{\chi}u_2\right) - \frac{2}{3}\left(t_2+1\right)e^{-6(t_1+t_2)}\right| \\
\leq\ & \frac{1}{361000}\left(t_2+\frac{4}{3}\right)^{\frac{4}{3}}e^{-12(t_1+t_2)}|u_2| + \frac{1}{5000e^{10}+10}\left(t_2+\frac{3}{2}\right)^{\frac{3}{2}}e^{-6(t_1+t_2)}\left|\Delta^{\frac{1}{3}}u_1\right|, \\
&\left|F_2\left(t_1,t_2,\frac{1}{\chi}\Delta^{\frac{3}{4}}u_2,\frac{1}{\chi}u_1\right) - \frac{1}{2}\left(t_2+1\right)e^{-6(t_1+t_2)}\right| \\
\leq\ & \frac{1}{121000}\left(t_2+\frac{4}{3}\right)^{\frac{4}{3}}e^{-12(t_1+t_2)}|u_2| + \frac{9}{196000}\left(t_2+\frac{3}{2}\right)^{\frac{3}{2}}e^{-6(t_1+t_2)}\left|\Delta^{\frac{3}{4}}u_2\right|,
\end{aligned}$$

we find that (H_1) holds with $M_{11} = 0.666$, $M_{12} = 9.080$, $M_{13} = 2.770$ and $M_{21} = 0.500$, $M_{22} = 0.000046$, $M_{23} = 0.0000083$.

Furthermore, we obtain:

$$\left|\phi_1\left(\frac{1}{\chi}u_1,\frac{1}{\chi}u_2\right) - \frac{2}{5}\left(t_1+\frac{1}{2}\right)^{\frac{1}{2}}\left(t_2+\frac{1}{2}\right)e^{-6(t_1+t_2)}\right|$$

$$\le \frac{1}{2000e^3}\left(t_1+\frac{2}{3}\right)^{\frac{2}{3}}\left(t_2+\frac{2}{3}\right)^{\frac{4}{3}}e^{-6(t_1+t_2)}\|u_1\|+\frac{1}{4000\pi^2}\left(t_1+\frac{3}{4}\right)^{\frac{3}{4}}\left(t_2+\frac{3}{4}\right)^{\frac{3}{2}}\times$$
$$e^{-3(t_1+t_2)}\|u_2\|,$$
$$\left|\phi_2\left(\frac{1}{\chi}u_1,\frac{1}{\chi}u_2\right)-\frac{5}{6}\left(t_1+\frac{1}{2}\right)^{\frac{1}{2}}\left(t_2+\frac{1}{2}\right)e^{-6(t_1+t_2)}\right|$$
$$\le \frac{1}{2000\pi}\left(t_1+\frac{2}{3}\right)^{\frac{2}{3}}\left(t_2+\frac{2}{3}\right)^{\frac{4}{3}}e^{-6(t_1+t_2)}\|u_1\|+\frac{1}{500e^2}\left(t_1+\frac{3}{4}\right)^{\frac{3}{4}}\left(t_2+\frac{3}{4}\right)^{\frac{3}{2}}\times$$
$$e^{-3(t_1+t_2)}\|u_2\|,$$

and $(10e-1)^3 < g_1(t_1) < (10e+1)^3$ and $(12e-1)^2 < g_2(t_2) < (12e+1)^2$.

Thus, $(H_2),(H3)$ hold with $N_{11}=0.4$, $N_{12}=0.0000249$, $N_{13}=0.0000253$, $N_{21}=0.833$, $N_{22}=0.000159$, $N_{23}=0.000271$, $g_1=17949.37$, $g_2=999.79$, $G_1=22384.80$, and $G_2=1130.26$.

Finally, we find that:

$$\tilde{\Omega}_{11}=2313.238,\ \tilde{\Omega}_{12}=319.647,\ \tilde{\Omega}_{13}=27053.522,\ \tilde{\Omega}_{14}=2597.063,$$
$$\tilde{\Omega}_{21}=0.0212,\ \tilde{\Omega}_{22}=1.1403,\ \tilde{\Omega}_{23}=0.0198,\ \tilde{\Omega}_{24}=1.5093.$$

Therefore, we have:

$$\Psi_1+\Psi_2=0.00057+0.4692=0.4698<1.$$

Hence, by Theorem 4, this boundary value problem has at least one solution. □

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