mathematics

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New Integral Inequalities via the Katugampola Fractional Integrals for Functions Whose Second Derivatives Are Strongly η-Convex

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Received: 28 December 2018; Accepted: 12 February 2019; Published: 15 February 2019

Abstract: In this paper, we introduced some new integral inequalities of the Hermite–Hadamard type for functions whose second derivatives in absolute values at certain powers are strongly η-convex functions via the Katugampola fractional integrals.

Keywords: Hermite–Hadamard type inequality; strongly η-convex functions; Hölder's inequality; Power mean inequality; Katugampola fractional integrals; Riemann–Liouville fractional integrals; Hadamard fractional integrals

2010 MSC: 26A33; 26A51; 26D10; 26D15

1. Introduction

Let I be an interval in $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is said to be convex on I if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in I$ and $t \in [0,1]$. The following inequalities which hold for convex functions is known in the literature as the Hermite–Hadamard type inequality.

Theorem 1 ([1]). *If $f : [a,b] \to \mathbb{R}$ is convex on $[a,b]$ with $a < b$, then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

Many authors have studied and generalized the Hermite–Hadamard inequality in several ways via different classes of convex functions. For some recent results related to the Hermite–Hadamard inequality, we refer the interested reader to the papers [2–11].

In 2016, Gordji et al. [12] introduced the concept of η-convexity as follows:

Definition 1 ([12]). *A function $f : I \to \mathbb{R}$ is said to be η-convex with respect to the bifunction $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ if*

$$f(tx + (1-t)y) \leq f(y) + t\eta(f(x), f(y))$$

for all $x, y \in I$ and $t \in [0,1]$.

Remark 1. *If we take $\eta(x,y) = x - y$ in Definition 1, then we recover the classical definition of convex functions.*

In 2017, Awan et al. [13] extended the class of η-convex functions to the class of strongly η-convex functions as follows:

Definition 2 ([13])**.** *A function $f : I \to \mathbb{R}$ is said to be strongly η-convex with respect to the bifunction $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with modulus $\mu \geq 0$ if*

$$f(tx + (1-t)y) \leq f(y) + t\eta(f(x), f(y)) - \mu t(1-t)(x-y)^2$$

for all $x, y \in I$ and $t \in [0,1]$.

Remark 2. *If $\eta(x,y) = x - y$ in Definition 2, then we have the class of strongly convex functions.*

For some recent results related to the class of η-convex functions, we refer the interested reader to the papers [8,12–16].

Definition 3 ([17])**.** *The left- and right-sided Riemann–Liouville fractional integrals of order $\alpha > 0$ of f are defined by*

$$J^{\alpha}_{a+} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt$$

and

$$J^{\alpha}_{b-} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt$$

with $a < x < b$ and $\Gamma(\cdot)$ is the gamma function given by

$$\Gamma(x) := \int_0^{\infty} t^{x-1} e^{-t} dt, \quad Re(x) > 0$$

with the property that $\Gamma(x+1) = x\Gamma(x)$.

Definition 4 ([18])**.** *The left- and right-sided Hadamard fractional integrals of order $\alpha > 0$ of f are defined by*

$$H^{\alpha}_{a+} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{t}\right)^{\alpha-1} \frac{f(t)}{t} dt$$

and

$$H^{\alpha}_{b-} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \left(\ln \frac{t}{x}\right)^{\alpha-1} \frac{f(t)}{t} dt.$$

Definition 5. $X^p_c(a,b)$ $(c \in \mathbb{R},\ 1 \leq p \leq \infty)$ *denotes the space of all complex-valued Lebesgue measurable functions f for which $\|f\|_{X^p_c} < \infty$, where the norm $\|\cdot\|_{X^p_c}$ is defined by*

$$\|f\|_{X^p_c} = \left(\int_a^b |t^c f(t)|^p \frac{dt}{t}\right)^{1/p} \quad (1 \leq p < \infty)$$

and for $p = \infty$

$$\|f\|_{X_c^\infty} = ess \sup_{a \le t \le b} |t^c f(t)|.$$

In 2011, Katugampola [19] introduced a new fractional integral operator which generalizes the Riemann–Liouville and Hadamard fractional integrals as follows:

Definition 6. *Let $[a,b] \subset \mathbb{R}$ be a finite interval. Then, the left- and right-sided Katugampola fractional integrals of order $\alpha > 0$ of $f \in X_c^p(a,b)$ are defined by*

$$^\rho I_{a+}^\alpha f(x) := \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{t^{\rho-1}}{(x^\rho - t^\rho)^{1-\alpha}} f(t) dt$$

and

$$^\rho I_{b-}^\alpha f(x) := \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{t^{\rho-1}}{(t^\rho - x^\rho)^{1-\alpha}} f(t) dt$$

with $a < x < b$ and $\rho > 0$, if the integrals exist.

Remark 3. *It is shown in [19] that the Katugampola fractional integral operators are well-defined on $X_c^p(a,b)$.*

Theorem 2 ([19])**.** *Let $\alpha > 0$ and $\rho > 0$. Then for $x > a$*

1. $\lim_{\rho \to 1} {}^\rho I_{a+}^\alpha f(x) = J_{a+}^\alpha f(x)$,
2. $\lim_{\rho \to 0^+} {}^\rho I_{a+}^\alpha f(x) = H_{a+}^\alpha f(x)$.

Similar results also hold for right-sided operators.

For more information about the Katugampola fractional integrals and related results, we refer the interested reader to the papers [19–21]. Recently, Chen and Katugampola [20] introduced several integral inequalities of Hermite–Hadamard type for functions whose first derivatives in absolute value are convex functions via the Katugampola fractional integrals. We present two of their results here for the purpose of our discussion. The first result of importance to us employs the following lemma.

Lemma 1 ([20])**.** *Let $\alpha > 0$, $\rho > 0$ and $f : [a^\rho, b^\rho] \to \mathbb{R}$ be a differentiable mapping on (a^ρ, b^ρ) with $0 \le a < b$. Then the following equality holds if the fractional integrals exist:*

$$\begin{aligned} &\frac{f(a^\rho) + f(b^\rho)}{\alpha\rho} - \frac{\rho^{\alpha-1}\Gamma(\alpha)}{(b^\rho - a^\rho)^\alpha} \left[{}^\rho I_{a+}^\alpha f(b^\rho) + {}^\rho I_{b-}^\alpha f(a^\rho) \right] \\ &= \frac{b^\rho - a^\rho}{\alpha} \int_0^1 t^{\rho(\alpha+1)-1} \left[f'((1-t^\rho)a^\rho + t^\rho b^\rho) - f'(t^\rho a^\rho + (1-t^\rho) b^\rho) \right] dt. \end{aligned} \tag{1}$$

By using Lemma 1, the authors proved the following result.

Theorem 3 ([20])**.** *Let $f : [a^\rho, b^\rho] \to \mathbb{R}$ be a differentiable mapping on (a^ρ, b^ρ) with $0 \le a < b$. If $|f'|$ is convex on $[a^\rho, b^\rho]$, then the following inequality holds:*

$$\left| \frac{f(a^\rho) + f(b^\rho)}{2} - \frac{\rho^\alpha \Gamma(\alpha+1)}{2(b^\rho - a^\rho)^\alpha} \left[{}^\rho I_{a+}^\alpha f(b^\rho) + {}^\rho I_{b-}^\alpha f(a^\rho) \right] \right| \le \frac{b^\rho - a^\rho}{2(\alpha+1)} \left[|f'(a^\rho)| + |f'(b^\rho)| \right].$$

The second result of importance to us also uses the following lemma.

Lemma 2 ([20]). *Let $\alpha > 0$, $\rho > 0$ and $f : [a^\rho, b^\rho] \to \mathbb{R}$ be a differentiable mapping on (a^ρ, b^ρ) with $0 \le a < b$. Then the following equality holds if the fractional integrals exist:*

$$\begin{aligned} &\frac{f(a^\rho)+f(b^\rho)}{2} - \frac{\rho^\alpha \Gamma(\alpha+1)}{2(b^\rho - a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a+} f(b^\rho) + {}^\rho I^\alpha_{b-} f(a^\rho)\right] \\ &= \frac{b^\rho - a^\rho}{2}\int_0^1 [(1-t^\rho)^\alpha - t^{\rho\alpha}]t^{\rho-1} f'(t^\rho a^\rho + (1-t^\rho)b^\rho)dt. \end{aligned} \tag{2}$$

By using Lemma 2, the authors proved the following result.

Theorem 4 ([20]). *Let $f : [a^\rho, b^\rho] \to \mathbb{R}$ be a differentiable mapping on (a^ρ, b^ρ) with $0 \le a < b$. If $|f'|$ is convex on $[a^\rho, b^\rho]$, then the following inequality holds:*

$$\begin{aligned} &\left|\frac{f(a^\rho)+f(b^\rho)}{2} - \frac{\rho^\alpha \Gamma(\alpha+1)}{2(b^\rho - a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a+} f(b^\rho) + {}^\rho I^\alpha_{b-} f(a^\rho)\right]\right| \\ &\qquad \le \frac{b^\rho - a^\rho}{2\rho(\alpha+1)}\left(1 - \frac{1}{2^\alpha}\right)\left[|f'(a^\rho)| + |f'(b^\rho)|\right]. \end{aligned}$$

Remark 4. *It is important to note that Lemmas 1 and 2 are corrected versions of [20] (Lemma 2.4 and Equation (14)).*

Our purpose in this paper is to provide some new estimates for the right hand side of the inequalities in Theorems 3 and 4 for functions whose second derivatives in absolute value at some powers are strongly η-convex.

2. Main Results

To prove the main results of this paper, we need the following lemmas which are extensions of Lemmas 1 and 2 for the second derivative case of the function f.

Lemma 3. *Let $\alpha > 0$, $\rho > 0$ and $f : [a^\rho, b^\rho] \to \mathbb{R}$ be a twice differentiable mapping on (a^ρ, b^ρ) with $0 \le a < b$. Then the following equality holds if the fractional integrals exist:*

$$\begin{aligned} &\frac{f(a^\rho)+f(b^\rho)}{\alpha\rho} - \frac{\rho^{\alpha-1} \Gamma(\alpha)}{(b^\rho - a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a+} f(b^\rho) + {}^\rho I^\alpha_{b-} f(a^\rho)\right] \\ &= \frac{(b^\rho - a^\rho)^2}{\alpha(\alpha+1)}\left[\int_0^1 \left[1 - t^{\rho(\alpha+1)}\right] t^{\rho-1} f''((1-t^\rho)a^\rho + t^\rho b^\rho)dt \right. \\ &\qquad \left. - \int_0^1 t^{\rho(\alpha+2)-1} f''(t^\rho a^\rho + (1-t^\rho)b^\rho)dt\right]. \end{aligned} \tag{3}$$

Proof. Let

$$I_1 = \int_0^1 \left[1 - t^{\rho(\alpha+1)}\right] t^{\rho-1} f''((1-t^\rho)a^\rho + t^\rho b^\rho)dt$$

and

$$I_2 = \int_0^1 t^{\rho(\alpha+2)-1} f''(t^\rho a^\rho + (1-t^\rho)b^\rho)dt.$$

By using integration by parts we have that

$$\begin{aligned}
I_1 &= \int_0^1 \left[1 - t^{\rho(\alpha+1)}\right] t^{\rho-1} f''((1-t^\rho)a^\rho + t^\rho b^\rho)dt \\
&= \frac{1}{\rho(b^\rho - a^\rho)}[1 - t^{\rho(\alpha+1)}] f'((1-t^\rho)a^\rho + t^\rho b^\rho)\Big|_0^1 \\
&\quad + \frac{\rho(\alpha+1)}{\rho(b^\rho - a^\rho)} \int_0^1 t^{\rho(\alpha+1)-1} f'((1-t^\rho)a^\rho + t^\rho b^\rho)dt \\
&= -\frac{1}{\rho(b^\rho - a^\rho)} f'(a^\rho) + \frac{(\alpha+1)}{(b^\rho - a^\rho)} \int_0^1 t^{\rho(\alpha+1)-1} f'((1-t^\rho)a^\rho + t^\rho b^\rho)dt.
\end{aligned} \tag{4}$$

By a similar argument, one gets:

$$I_2 = -\frac{1}{\rho(b^\rho - a^\rho)} f'(a^\rho) + \frac{(\alpha+1)}{(b^\rho - a^\rho)} \int_0^1 t^{\rho(\alpha+1)-1} f'(t^\rho a^\rho + (1-t^\rho)b^\rho)dt. \tag{5}$$

Using (4) and (5), we have

$$I_1 - I_2 = \frac{(\alpha+1)}{(b^\rho - a^\rho)} \int_0^1 t^{\rho(\alpha+1)-1} \left[f'((1-t^\rho)a^\rho + t^\rho b^\rho) - f'(t^\rho a^\rho + (1-t^\rho)b^\rho)\right]dt. \tag{6}$$

The desired identity in (3) follows from (6) by using (1) and rearranging the terms. □

Lemma 4. *Let $\alpha > 0, \rho > 0$ and $f : [a^\rho, b^\rho] \to \mathbb{R}$ be a twice differentiable mapping on (a^ρ, b^ρ) with $0 \le a < b$. Then the following equality holds if the fractional integrals exist:*

$$\begin{aligned}
&\frac{f(a^\rho) + f(b^\rho)}{2} - \frac{\rho^\alpha \Gamma(\alpha+1)}{2(b^\rho - a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a+} f(b^\rho) + {}^\rho I^\alpha_{b-} f(a^\rho)\right] \\
&= \frac{(b^\rho - a^\rho)^2}{2(\alpha+1)} \int_0^1 [1 - (1-t^\rho)^{\alpha+1} - t^{\rho(\alpha+1)}] t^{\rho-1} f''(t^\rho a^\rho + (1-t^\rho)b^\rho)dt.
\end{aligned} \tag{7}$$

Proof. We start by considering the following computation which is a direct application of integration by parts.

$$\begin{aligned}
&\int_0^1 [1 - (1-t^\rho)^{\alpha+1} - t^{\rho(\alpha+1)}] t^{\rho-1} f''(t^\rho a^\rho + (1-t^\rho)b^\rho)dt \\
&= \frac{1}{\rho(a^\rho - b^\rho)}[1 - (1-t^\rho)^{\alpha+1} - t^{\rho(\alpha+1)}] f'(t^\rho a^\rho + (1-t^\rho)b^\rho)\Big|_0^1 \\
&- \frac{\rho(\alpha+1)}{\rho(a^\rho - b^\rho)} \int_0^1 [(1-t^\rho)^\alpha - t^{\rho\alpha}] t^{\rho-1} f'(t^\rho a^\rho + (1-t^\rho)b^\rho)dt \\
&= \frac{(\alpha+1)}{(b^\rho - a^\rho)} \int_0^1 [(1-t^\rho)^\alpha - t^{\rho\alpha}] t^{\rho-1} f'(t^\rho a^\rho + (1-t^\rho)b^\rho)dt.
\end{aligned} \tag{8}$$

The intended identity in (7) follows from (8) by using (2) and rearranging the terms. □

We are now in a position to prove our main results.

Theorem 5. *Let $\alpha > 0$, $\rho > 0$ and $f : [a^\rho, b^\rho] \to \mathbb{R}$ be a twice differentiable mapping on (a^ρ, b^ρ) with $0 \le a < b$. If $|f''|^q$ is strongly η-convex with modulus $\mu \ge 0$ for $q \ge 1$, then the following inequality holds:*

$$\begin{aligned}
&\left| \frac{f(a^\rho) + f(b^\rho)}{2} - \frac{\rho^\alpha \Gamma(\alpha+1)}{2(b^\rho - a^\rho)^\alpha} \left[{}^\rho I^\alpha_{a+} f(b^\rho) + {}^\rho I^\alpha_{b-} f(a^\rho) \right] \right| \\
&\le \frac{(b^\rho - a^\rho)^2}{2(\alpha+1)} \left[\left(\frac{\alpha+1}{\alpha+2} \right)^{1-\frac{1}{q}} \left(\frac{\alpha+1}{\alpha+2} |f''(a^\rho)|^q \right.\right. \\
&\quad \left. + \frac{\alpha+1}{2(\alpha+3)} \eta\Big(|f''(b^\rho)|^q, |f''(a^\rho)|^q \Big) - \frac{\mu (b^\rho - a^\rho)^2 \left[(\alpha+1)^2 + 5(\alpha+1) \right]}{6(\alpha+4)(\alpha+3)} \right)^{\frac{1}{q}} \\
&\quad + \left(\frac{1}{\alpha+2} \right)^{1-\frac{1}{q}} \left(\frac{1}{\alpha+2} |f''(b^\rho)|^q + \frac{1}{\alpha+3} \eta\Big(|f''(a^\rho)|^q, |f''(b^\rho)|^q \Big) \right. \\
&\quad \left.\left. - \frac{\mu (b^\rho - a^\rho)^2}{(\alpha+3)(\alpha+4)} \right)^{\frac{1}{q}} \right].
\end{aligned}$$

Proof. Using Lemma 3, the well-known power mean inequality and the strong η-convexity of $|f''|^q$, we obtain

$$\begin{aligned}
&\left| \frac{f(a^\rho) + f(b^\rho)}{\alpha\rho} - \frac{\rho^{\alpha-1} \Gamma(\alpha)}{(b^\rho - a^\rho)^\alpha} \left[{}^\rho I^\alpha_{a+} f(b^\rho) + {}^\rho I^\alpha_{b-} f(a^\rho) \right] \right| \\
&\le \frac{(b^\rho - a^\rho)^2}{\alpha(\alpha+1)} \left[\int_0^1 \left[1 - t^{\rho(\alpha+1)} \right] t^{\rho-1} \left| f''((1-t^\rho)a^\rho + t^\rho b^\rho) \right| dt \right. \\
&\qquad \left. + \int_0^1 t^{\rho(\alpha+2)-1} \left| f''(t^\rho a^\rho + (1-t^\rho) b^\rho) \right| dt \right] \\
&\le \frac{(b^\rho - a^\rho)^2}{\alpha(\alpha+1)} \left[\left(\int_0^1 \left[1 - t^{\rho(\alpha+1)} \right] t^{\rho-1} dt \right)^{1-\frac{1}{q}} \right. \\
&\qquad \times \left(\int_0^1 \left[1 - t^{\rho(\alpha+1)} \right] t^{\rho-1} \left| f''((1-t^\rho)a^\rho + t^\rho b^\rho) \right|^q dt \right)^{\frac{1}{q}} \\
&\qquad \left. + \left(\int_0^1 t^{\rho(\alpha+2)-1} dt \right)^{1-\frac{1}{q}} \left(\int_0^1 t^{\rho(\alpha+2)-1} \left| f''(t^\rho a^\rho + (1-t^\rho) b^\rho) \right| dt \right)^{\frac{1}{q}} \right] \\
&\le \frac{(b^\rho - a^\rho)^2}{\alpha(\alpha+1)} \left[\left(\int_0^1 \left[1 - t^{\rho(\alpha+1)} \right] t^{\rho-1} dt \right)^{1-\frac{1}{q}} \right. \\
&\qquad \times \left(\int_0^1 \left[1 - t^{\rho(\alpha+1)} \right] t^{\rho-1} \Big(|f''(a^\rho)|^q + t^\rho \eta\Big(|f''(b^\rho)|^q, |f''(a^\rho)|^q \Big) \right. \\
&\qquad\qquad \left. - \mu t^\rho (1-t^\rho)(b^\rho - a^\rho)^2 \Big) dt \right)^{\frac{1}{q}} \\
&\qquad + \left(\int_0^1 t^{\rho(\alpha+2)-1} dt \right)^{1-\frac{1}{q}} \left(\int_0^1 t^{\rho(\alpha+2)-1} \Big(|f''(b^\rho)|^q \right. \\
&\qquad \left.\left. + t^\rho \eta\Big(|f''(a^\rho)|^q, |f''(b^\rho)|^q \Big) - \mu t^\rho (1-t^\rho)(b^\rho - a^\rho)^2 \Big) dt \right)^{\frac{1}{q}} \right] \\
&= \frac{(b^\rho - a^\rho)^2}{\alpha(\alpha+1)} \left[\left(\int_0^1 \left[1 - t^{\rho(\alpha+1)} \right] t^{\rho-1} dt \right)^{1-\frac{1}{q}} \left(|f''(a^\rho)|^q \int_0^1 \left[1 - t^{\rho(\alpha+1)} \right] t^{\rho-1} dt \right. \right. \\
&\qquad + \eta\Big(|f''(b^\rho)|^q, |f''(a^\rho)|^q \Big) \int_0^1 \left[1 - t^{\rho(\alpha+1)} \right] t^{2\rho-1} dt \\
&\qquad \left. - \mu (b^\rho - a^\rho)^2 \int_0^1 \left[1 - t^{\rho(\alpha+1)} \right] t^{2\rho-1} (1-t^\rho) dt \right)^{\frac{1}{q}}
\end{aligned}$$

$$
\begin{aligned}
&+\left(\int_0^1 t^{\rho(\alpha+2)-1}dt\right)^{1-\frac{1}{q}}\left(|f''(b^\rho)|^q\int_0^1 t^{\rho(\alpha+2)-1}dt\right.\\
&\qquad+\eta\left(|f''(a^\rho)|^q,|f''(b^\rho)|^q\right)\int_0^1 t^{\rho(\alpha+3)-1}dt\\
&\qquad\left.-\mu(b^\rho-a^\rho)^2\int_0^1 t^{\rho(\alpha+3)-1}(1-t^\rho)dt\right)^{\frac{1}{q}}\Bigg].
\end{aligned}
$$

The desired inequality follows from the above estimation and observing that:

$$
\int_0^1\left[1-t^{\rho(\alpha+1)}\right]t^{\rho-1}dt=\frac{\alpha+1}{\rho(\alpha+2)},\ \int_0^1\left[1-t^{\rho(\alpha+1)}\right]t^{2\rho-1}dt=\frac{\alpha+1}{2\rho(\alpha+3)},
$$

$$
\int_0^1\left[1-t^{\rho(\alpha+1)}\right]t^{2\rho-1}(1-t^\rho)dt=\frac{(\alpha+1)^2+5(\alpha+1)}{6\rho(\alpha+4)(\alpha+3)},\ \int_0^1 t^{\rho(\alpha+2)-1}dt=\frac{1}{\rho(\alpha+2)}
$$

$$
\int_0^1 t^{\rho(\alpha+3)-1}dt=\frac{1}{\rho(\alpha+3)}\ \text{and}\ \int_0^1 t^{\rho(\alpha+3)-1}(1-t^\rho)dt=\frac{1}{\rho(\alpha+3)(\alpha+4)}.
$$

This completes the proof of Theorem 5. □

Corollary 1. *Let $\alpha>0$, $\rho>0$ and $f:[a^\rho,b^\rho]\to\mathbb{R}$ be a twice differentiable mapping on (a^ρ,b^ρ) with $0\le a<b$. If $|f''|^q$ is convex for $q\ge1$, then the following inequality holds:*

$$
\begin{aligned}
&\left|\frac{f(a^\rho)+f(b^\rho)}{2}-\frac{\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho-a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a+}f(b^\rho)+{}^\rho I^\alpha_{b-}f(a^\rho)\right]\right|\\
&\le\frac{(b^\rho-a^\rho)^2}{2(\alpha+1)}\left[\left(\frac{\alpha+1}{\alpha+2}\right)^{1-\frac{1}{q}}\left(\frac{(\alpha+1)(\alpha+4)}{2(\alpha+2)(\alpha+3)}|f''(a^\rho)|^q+\frac{\alpha+1}{2(\alpha+3)}|f''(b^\rho)|^q\right.\right.\\
&\qquad\left.\left.+\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}}\left(\frac{1}{(\alpha+2)(\alpha+3)}|f''(b^\rho)|^q+\frac{1}{\alpha+3}|f''(a^\rho)|^q\right)^{\frac{1}{q}}\right]\right..
\end{aligned}
$$

Proof. The result follows directly from Theorem 5 if we take $\eta(x,y)=x-y$ and $\mu=0$. □

Theorem 6. *Let $\alpha>0$, $\rho>0$ and $f:[a^\rho,b^\rho]\to\mathbb{R}$ be a twice differentiable mapping on (a^ρ,b^ρ) with $0\le a<b$. If $|f''|^q$ is strongly η-convex with modulus $\mu\ge0$ for $q>1$, then the following inequalities hold:*

$$
\begin{aligned}
&\left|\frac{f(a^\rho)+f(b^\rho)}{2}-\frac{\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho-a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a+}f(b^\rho)+{}^\rho I^\alpha_{b-}f(a^\rho)\right]\right|\\
&\le\frac{\rho(b^\rho-a^\rho)^2}{2(\alpha+1)}\left[\left(\frac{1}{\rho}\int_0^1\left[1-u^{\alpha+1}\right]^s du\right)^{\frac{1}{s}}\left(\frac{1}{\rho}|f''(a^\rho)|^q+\frac{1}{2\rho}\eta\left(|f''(b^\rho)|^q,|f''(a^\rho)|^q\right)\right.\right.\\
&\quad\left.-\frac{\mu(b^\rho-a^\rho)^2}{6\rho}\right)^{\frac{1}{q}}+\left(\frac{1}{s\rho(\alpha+2)-s+1}\right)^{\frac{1}{s}}\left(|f''(b^\rho)|^q+\frac{1}{\rho+1}\eta\left(|f''(a^\rho)|^q,|f''(b^\rho)|^q\right)\right.\\
&\quad\left.\left.-\frac{\mu\rho(b^\rho-a^\rho)^2}{(\rho+1)(2\rho+1)}\right)^{\frac{1}{q}}\right]
\end{aligned}
$$

$$\leq \frac{\rho(b^\rho-a^\rho)^2}{2(\alpha+1)}\left[\left(\frac{s(\alpha+1)}{\rho(s(\alpha+1)+1)}\right)^{\frac{1}{s}}\left(\frac{1}{\rho}|f''(a^\rho)|^q+\frac{1}{2\rho}\eta\left(|f''(b^\rho)|^q,|f''(a^\rho)|^q\right)\right.\right.$$
$$\left.-\frac{\mu(b^\rho-a^\rho)^2}{6\rho}\right)^{\frac{1}{q}}+\left(\frac{1}{s\rho(\alpha+2)-s+1}\right)^{\frac{1}{s}}\left(|f''(b^\rho)|^q+\frac{1}{\rho+1}\eta\left(|f''(a^\rho)|^q,|f''(b^\rho)|^q\right)\right.$$
$$\left.\left.-\frac{\mu\rho(b^\rho-a^\rho)^2}{(\rho+1)(2\rho+1)}\right)^{\frac{1}{q}}\right],$$

where $\frac{1}{s}+\frac{1}{q}=1$.

Proof. Using Lemma 3, the Hölder's inequality and the strong η-convexity of $|f''|^q$, we obtain

$$\left|\frac{f(a^\rho)+f(b^\rho)}{\alpha\rho}-\frac{\rho^{\alpha-1}\Gamma(\alpha)}{(b^\rho-a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a+}f(b^\rho)+{}^\rho I^\alpha_{b-}f(a^\rho)\right]\right|$$
$$\leq \frac{(b^\rho-a^\rho)^2}{\alpha(\alpha+1)}\left[\left(\int_0^1\left|1-t^{\rho(\alpha+1)}\right|^s t^{\rho-1}dt\right)^{\frac{1}{s}}\left(\int_0^1 t^{\rho-1}\left|f''((1-t^\rho)a^\rho+t^\rho b^\rho)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1 t^{s\rho(\alpha+2)-s}dt\right)^{\frac{1}{s}}\left(\int_0^1\left|f''(t^\rho a^\rho+(1-t^\rho)b^\rho)\right|^q dt\right)^{\frac{1}{q}}\right]$$
$$\leq \frac{(b^\rho-a^\rho)^2}{\alpha(\alpha+1)}\left[\left(\int_0^1\left|1-t^{\rho(\alpha+1)}\right|^s t^{\rho-1}dt\right)^{\frac{1}{s}}\left(\int_0^1 t^{\rho-1}\left(|f''(a^\rho)|^q\right.\right.\right.$$
$$\left.\left.+t^\rho\eta\left(|f''(b^\rho)|^q,|f''(a^\rho)|^q\right)-\mu t^\rho(1-t^\rho)(b^\rho-a^\rho)^2\right)dt\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1 t^{s\rho(\alpha+2)-s}dt\right)^{\frac{1}{s}}\left(\int_0^1\left(|f''(b^\rho)|^q\right.\right.$$
$$\left.\left.\left.+t^\rho\eta\left(|f''(a^\rho)|^q,|f''(b^\rho)|^q\right)-\mu t^\rho(1-t^\rho)(b^\rho-a^\rho)^2\right)dt\right)^{\frac{1}{q}}\right]$$
$$=\frac{(b^\rho-a^\rho)^2}{\alpha(\alpha+1)}\left[\left(\int_0^1\left|1-t^{\rho(\alpha+1)}\right|^s t^{\rho-1}dt\right)^{\frac{1}{s}}\left(|f''(a^\rho)|^q\int_0^1 t^{\rho-1}dt\right.\right.$$
$$\left.+\eta\left(|f''(b^\rho)|^q,|f''(a^\rho)|^q\right)\int_0^1 t^{2\rho-1}dt-\mu(b^\rho-a^\rho)^2\int_0^1 t^{2\rho-1}(1-t^\rho)dt\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1 t^{s\rho(\alpha+2)-s}dt\right)^{\frac{1}{s}}\left(|f''(b^\rho)|^q\int_0^1 1\,dt+\eta\left(|f''(a^\rho)|^q,|f''(b^\rho)|^q\right)\int_0^1 t^\rho dt\right.$$
$$\left.\left.-\mu(b^\rho-a^\rho)^2\int_0^1 t^\rho(1-t^\rho)dt\right)^{\frac{1}{q}}\right]$$
$$=\frac{(b^\rho-a^\rho)^2}{\alpha(\alpha+1)}\left[\left(\frac{1}{\rho}\int_0^1\left[1-u^{\alpha+1}\right]^s du\right)^{\frac{1}{s}}\left(\frac{1}{\rho}|f''(a^\rho)|^q+\frac{1}{2\rho}\eta\left(|f''(b^\rho)|^q,|f''(a^\rho)|^q\right)\right.\right.$$
$$\left.-\frac{\mu(b^\rho-a^\rho)^2}{6\rho}\right)^{\frac{1}{q}}+\left(\frac{1}{s\rho(\alpha+2)-s+1}\right)^{\frac{1}{s}}\left(|f''(b^\rho)|^q+\frac{1}{\rho+1}\eta\left(|f''(a^\rho)|^q,|f''(b^\rho)|^q\right)\right.$$
$$\left.\left.-\frac{\mu\rho(b^\rho-a^\rho)^2}{(\rho+1)(2\rho+1)}\right)^{\frac{1}{q}}\right].$$

This proves the first inequality. To prove the second inequality, we observe that for any $A > B \geq 0$ and $s \geq 1$, we have $(A-B)^s \leq A^s - B^s$. Thus, it follows that $\left[1-u^{\alpha+1}\right]^s \leq 1-u^{s(\alpha+1)}$ for all $u \in [0,1]$. Hence, we have that

$$\int_0^1 \left[1-u^{\alpha+1}\right]^s du \leq \int_0^1 1-u^{s(\alpha+1)} du = \frac{s(\alpha+1)}{s(\alpha+1)+1}.$$

This completes the proof. □

Corollary 2. *Let $\alpha > 0$, $\rho > 0$ and $f : [a^\rho, b^\rho] \to \mathbb{R}$ be a twice differentiable mapping on (a^ρ, b^ρ) with $0 \leq a < b$. If $|f''|^q$ is convex for $q > 1$, then the following inequalities hold:*

$$\begin{aligned}
&\left|\frac{f(a^\rho)+f(b^\rho)}{2} - \frac{\rho^\alpha \Gamma(\alpha+1)}{2(b^\rho-a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a+} f(b^\rho) + {}^\rho I^\alpha_{b-} f(a^\rho)\right]\right| \\
&\leq \frac{\rho(b^\rho-a^\rho)^2}{2(\alpha+1)}\left[\left(\frac{1}{\rho}\int_0^1 \left|1-u^{\alpha+1}\right|^s du\right)^{\frac{1}{s}}\left(\frac{1}{2\rho}|f''(a^\rho)|^q + \frac{1}{2\rho}|f''(b^\rho)|^q\right)^{\frac{1}{q}}\right. \\
&\qquad \left. + \left(\frac{1}{s\rho(\alpha+2)-s+1}\right)^{\frac{1}{s}}\left(\frac{\rho}{\rho+1}|f''(b^\rho)|^q + \frac{1}{\rho+1}|f''(a^\rho)|^q\right)^{\frac{1}{q}}\right] \\
&\leq \frac{\rho(b^\rho-a^\rho)^2}{2(\alpha+1)}\left[\left(\frac{s(\alpha+1)}{\rho(s(\alpha+1)+1)}\right)^{\frac{1}{s}}\left(\frac{1}{2\rho}|f''(a^\rho)|^q + \frac{1}{2\rho}|f''(b^\rho)|^q\right)^{\frac{1}{q}}\right. \\
&\qquad \left. + \left(\frac{1}{s\rho(\alpha+2)-s+1}\right)^{\frac{1}{s}}\left(\frac{\rho}{\rho+1}|f''(b^\rho)|^q + \frac{1}{\rho+1}|f''(a^\rho)|^q\right)^{\frac{1}{q}}\right],
\end{aligned}$$

where $\frac{1}{s}+\frac{1}{q}=1$.

Proof. The result follows directly from Theorem 6 if we take $\eta(x,y) = x-y$ and $\mu = 0$. □

Theorem 7. *Let $\alpha > 0$, $\rho > 0$ and $f : [a^\rho, b^\rho] \to \mathbb{R}$ be a twice differentiable mapping on (a^ρ, b^ρ) with $0 \leq a < b$. If $|f''|^q$ is a strongly η-convex function on $[a^\rho, b^\rho]$ with modulus $\mu \geq 0$ for $q \geq 1$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{f(a^\rho)+f(b^\rho)}{2} - \frac{\rho^\alpha \Gamma(\alpha+1)}{2(b^\rho-a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a+} f(b^\rho) + {}^\rho I^\alpha_{b-} f(a^\rho)\right]\right| \\
&\leq \frac{(b^\rho-a^\rho)^2}{2\rho(\alpha+1)}\left(\frac{\alpha}{\alpha+2}\right)^{1-\frac{1}{q}}\left[\frac{\alpha}{\alpha+2}|f''(b^\rho)|^q\right. \\
&\qquad + \left(\frac{\alpha+1}{2(\alpha+3)} - \mathrm{B}(2,\alpha+2)\right)\eta\left(|f''(a^\rho)|^q, |f''(b^\rho)|^q\right) \\
&\qquad \left. - \mu(b^\rho-a^\rho)^2\left(\frac{1}{6} - 2\mathrm{B}(2,\alpha+3)\right)\right]^{\frac{1}{q}},
\end{aligned}$$

where $\mathrm{B}(\cdot,\cdot)$ denotes the beta function defined by $\mathrm{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$.

Proof. Using Lemma 4, the power mean inequality and the strong η-convexity of $|f''|^q$, we obtain

$$
\begin{aligned}
&\left|\frac{f(a^\rho)+f(b^\rho)}{2}-\frac{\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho-a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a+}f(b^\rho)+{}^\rho I^\alpha_{b-}f(a^\rho)\right]\right| \\
&\le \frac{(b^\rho-a^\rho)^2}{2(\alpha+1)}\int_0^1\left|1-(1-t^\rho)^{\alpha+1}-t^{\rho(\alpha+1)}\right|t^{\rho-1}\left|f''(t^\rho a^\rho+(1-t^\rho)b^\rho)\right|dt \\
&\le \frac{(b^\rho-a^\rho)^2}{2(\alpha+1)}\left(\int_0^1\left[1-(1-t^\rho)^{\alpha+1}-t^{\rho(\alpha+1)}\right]t^{\rho-1}dt\right)^{1-\frac{1}{q}} \\
&\qquad\times\left(\int_0^1\left[1-(1-t^\rho)^{\alpha+1}-t^{\rho(\alpha+1)}\right]t^{\rho-1}\left|f''(t^\rho a^\rho+(1-t^\rho)b^\rho)\right|^q dt\right)^{\frac{1}{q}} \\
&\le \frac{(b^\rho-a^\rho)^2}{2(\alpha+1)}\left(\int_0^1\left[1-(1-t^\rho)^{\alpha+1}-t^{\rho(\alpha+1)}\right]t^{\rho-1}dt\right)^{1-\frac{1}{q}} \\
&\qquad\times\left(\int_0^1\left[1-(1-t^\rho)^{\alpha+1}-t^{\rho(\alpha+1)}\right]t^{\rho-1}\Big(|f''(b^\rho)|^q+t^\rho\eta\left(|f''(a^\rho)|^q,|f''(b^\rho)|^q\right)\right. \\
&\qquad\qquad\left.-\mu t^\rho(1-t^\rho)(b^\rho-a^\rho)^2\Big)dt\right)^{\frac{1}{q}} \\
&\le \frac{(b^\rho-a^\rho)^2}{2(\alpha+1)}\left(\int_0^1\left[1-(1-t^\rho)^{\alpha+1}-t^{\rho(\alpha+1)}\right]t^{\rho-1}dt\right)^{1-\frac{1}{q}} \\
&\qquad\times\left(|f''(b^\rho)|^q\int_0^1\left[1-(1-t^\rho)^{\alpha+1}-t^{\rho(\alpha+1)}\right]t^{\rho-1}dt\right. \\
&\qquad\qquad+\eta\left(|f''(a^\rho)|^q,|f''(b^\rho)|^q\right)\int_0^1\left[1-(1-t^\rho)^{\alpha+1}-t^{\rho(\alpha+1)}\right]t^{2\rho-1}dt \\
&\qquad\qquad\left.-\mu(b^\rho-a^\rho)^2\int_0^1\left[1-(1-t^\rho)^{\alpha+1}-t^{\rho(\alpha+1)}\right]t^{2\rho-1}(1-t^\rho)dt\right)^{\frac{1}{q}}.
\end{aligned}
$$

The desired result follows from the above inequality and using the following computations:

$$
\begin{aligned}
\int_0^1\left[1-(1-t^\rho)^{\alpha+1}-t^{\rho(\alpha+1)}\right]t^{\rho-1}dt &= \frac{1}{\rho}\int_0^1\left[1-(1-u)^{\alpha+1}-u^{\alpha+1}\right]du \\
&= \frac{\alpha}{\rho(\alpha+2)},
\end{aligned}
$$

$$
\begin{aligned}
\int_0^1\left[1-(1-t^\rho)^{\alpha+1}-t^{\rho(\alpha+1)}\right]t^{2\rho-1}dt &= \frac{1}{\rho}\int_0^1\left[1-(1-u)^{\alpha+1}-u^{\alpha+1}\right]u\,du \\
&= \frac{1}{\rho}\left(\frac{1}{2}-\mathrm{B}(2,\alpha+2)-\frac{1}{\alpha+3}\right) \\
&= \frac{1}{\rho}\left(\frac{\alpha+1}{2(\alpha+3)}-\mathrm{B}(2,\alpha+2)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^1\left[1-(1-t^\rho)^{\alpha+1}-t^{\rho(\alpha+1)}\right]t^{2\rho-1}(1-t^\rho)dt &= \frac{1}{\rho}\int_0^1\left[1-(1-u)^{\alpha+1}-u^{\alpha+1}\right]u(1-u)\,du \\
&= \frac{1}{\rho}\left(\frac{1}{6}-2\mathrm{B}(2,\alpha+3)\right).
\end{aligned}
$$

This completes the proof of the theorem. □

Corollary 3. *Let $\alpha > 0$, $\rho > 0$ and $f : [a^\rho, b^\rho] \to \mathbb{R}$ be a twice differentiable mapping on (a^ρ, b^ρ) with $0 \le a < b$. If $|f''|^q$ is convex for $q \ge 1$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{f(a^\rho)+f(b^\rho)}{2} - \frac{\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho-a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a+} f(b^\rho) + {}^\rho I^\alpha_{b-} f(a^\rho)\right]\right| \\
&\le \frac{(b^\rho-a^\rho)^2}{2\rho(\alpha+1)}\left(\frac{\alpha}{\alpha+2}\right)^{1-\frac{1}{q}}\left[\left(\mathrm{B}(2,\alpha+2) - \frac{\alpha^2+3\alpha-2}{2(\alpha+2)(\alpha+3)}\right)|f''(b^\rho)|^q\right. \\
&\qquad\left. + \left(\frac{\alpha+1}{2(\alpha+3)} - \mathrm{B}(2,\alpha+2)\right)|f''(a^\rho)|^q\right]^{\frac{1}{q}}.
\end{aligned}$$

Proof. The result follows directly from Theorem 7 if we take $\eta(x,y) = x - y$ and $\mu = 0$. □

Theorem 8. *Let $\alpha > 0$, $\rho > 0$ and $f : [a^\rho, b^\rho] \to \mathbb{R}$ be a twice differentiable mapping on (a^ρ, b^ρ) with $0 \le a < b$. If $|f''|^q$ is a strongly η-convex function on $[a^\rho, b^\rho]$ with modulus $\mu \ge 0$ for $q > 1$, then the following inequalities hold:*

$$\begin{aligned}
&\left|\frac{f(a^\rho)+f(b^\rho)}{2} - \frac{\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho-a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a+} f(b^\rho) + {}^\rho I^\alpha_{b-} f(a^\rho)\right]\right| \\
&\le \frac{(b^\rho-a^\rho)^2}{2\rho(\alpha+1)}\left(\int_0^1\left[1-(1-u)^{\alpha+1}-u^{\alpha+1}\right]^s du\right)^{\frac{1}{s}} \\
&\qquad\times\left(|f''(b^\rho)|^q + \frac{1}{2}\eta\left(|f''(a^\rho)|^q, |f''(b^\rho)|^q\right) - \frac{\mu\rho(b^\rho-a^\rho)^2}{2(\rho+1)(2\rho+1)}\right)^{\frac{1}{q}} \\
&\le \frac{(b^\rho-a^\rho)^2}{2\rho(\alpha+1)}\left(\frac{s(\alpha+1)-1}{s(\alpha+1)+1}\right)^{\frac{1}{s}}\left(|f''(b^\rho)|^q + \frac{1}{2}\eta\left(|f''(a^\rho)|^q, |f''(b^\rho)|^q\right)\right. \\
&\qquad\left. - \frac{\mu\rho(b^\rho-a^\rho)^2}{2(\rho+1)(2\rho+1)}\right)^{\frac{1}{q}},
\end{aligned}$$

where $\dfrac{1}{s} + \dfrac{1}{q} = 1$.

Proof. Using Lemma 4, the Hölder's inequality and the strong η-convexity of $|f''|^q$, we obtain

$$\begin{aligned}
&\left|\frac{f(a^\rho)+f(b^\rho)}{2} - \frac{\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho-a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a+} f(b^\rho) + {}^\rho I^\alpha_{b-} f(a^\rho)\right]\right| \\
&\le \frac{(b^\rho-a^\rho)^2}{2(\alpha+1)}\int_0^1\left|1-(1-t^\rho)^{\alpha+1}-t^{\rho(\alpha+1)}\right|t^{\rho-1}\left|f''(t^\rho a^\rho + (1-t^\rho)b^\rho)\right|dt \\
&\le \frac{(b^\rho-a^\rho)^2}{2(\alpha+1)}\left(\int_0^1\left[1-(1-t^\rho)^{\alpha+1}-t^{\rho(\alpha+1)}\right]^s t^{\rho-1}dt\right)^{\frac{1}{s}} \\
&\qquad\times\left(\int_0^1 t^{\rho-1}\left|f''(t^\rho a^\rho + (1-t^\rho)b^\rho)\right|^q dt\right)^{\frac{1}{q}} \\
&\le \frac{(b^\rho-a^\rho)^2}{2(\alpha+1)}\left(\int_0^1\left[1-(1-t^\rho)^{\alpha+1}-t^{\rho(\alpha+1)}\right]^s t^{\rho-1}dt\right)^{\frac{1}{s}} \\
&\qquad\times\left(\int_0^1 t^{\rho-1}\left(|f''(b^\rho)|^q + t^\rho\eta\left(|f''(a^\rho)|^q, |f''(b^\rho)|^q\right)\right.\right. \\
&\qquad\left.\left. - \mu t^\rho(1-t^\rho)(b^\rho-a^\rho)^2\right)dt\right)^{\frac{1}{q}}
\end{aligned}$$

$$\leq \frac{(b^\rho - a^\rho)^2}{2(\alpha+1)} \left(\int_0^1 \left[1 - (1-t^\rho)^{\alpha+1} - t^{\rho(\alpha+1)} \right]^s t^{\rho-1} dt \right)^{\frac{1}{s}}$$
$$\times \left(|f''(b^\rho)|^q \int_0^1 t^{\rho-1} dt + \eta\left(|f''(a^\rho)|^q, |f''(b^\rho)|^q\right) \int_0^1 t^{2\rho-1} dt \right.$$
$$\left. - \mu(b^\rho - a^\rho)^2 \int_0^1 t^{2\rho-1}(1-t^\rho) dt \right)^{\frac{1}{q}},$$

where

$$\int_0^1 \left[1 - (1-t^\rho)^{\alpha+1} - t^{\rho(\alpha+1)} \right]^s t^{\rho-1} dt = \frac{1}{\rho} \int_0^1 \left[1 - (1-u)^{\alpha+1} - u^{\alpha+1} \right]^s du,$$

$$\int_0^1 t^{\rho-1}\, dt = \frac{1}{\rho}, \quad \int_0^1 t^{2\rho-1} dt = \frac{1}{2\rho} \text{ and } \int_0^1 t^{2\rho-1}(1-t^\rho) dt = \frac{1}{2(\rho+1)(2\rho+1)}.$$

This proves the first inequality. Using a similar argument as in the proof of Theorem 6, we obtain

$$\begin{aligned} \int_0^1 \left[1 - (1-u)^{\alpha+1} - u^{\alpha+1} \right]^s du &\leq \int_0^1 1 - (1-u)^{s(\alpha+1)} - u^{s(\alpha+1)} du \\ &= 1 - \frac{2}{s(\alpha+1)+1} \\ &= \frac{s(\alpha+1)-1}{s(\alpha+1)+1}. \end{aligned}$$

This completes the proof of the theorem. □

Corollary 4. *Let $\alpha > 0$, $\rho > 0$ and $f : [a^\rho, b^\rho] \to \mathbb{R}$ be a twice differentiable mapping on (a^ρ, b^ρ) with $0 \leq a < b$. If $|f''|^q$ is convex for $q > 1$, then the following inequalities hold:*

$$\begin{aligned} &\left| \frac{f(a^\rho) + f(b^\rho)}{2} - \frac{\rho^\alpha \Gamma(\alpha+1)}{2(b^\rho - a^\rho)^\alpha} \left[{}^\rho I^\alpha_{a+} f(b^\rho) + {}^\rho I^\alpha_{b-} f(a^\rho) \right] \right| \\ &\leq \frac{(b^\rho - a^\rho)^2}{2\rho(\alpha+1)} \left(\int_0^1 \left[1 - (1-u)^{\alpha+1} - u^{\alpha+1} \right]^s du \right)^{\frac{1}{s}} \\ &\qquad \times \left(\frac{1}{2}|f''(b^\rho)|^q + \frac{1}{2}|f''(a^\rho)|^q \right)^{\frac{1}{q}} \\ &\leq \frac{(b^\rho - a^\rho)^2}{2\rho(\alpha+1)} \left(\frac{s(\alpha+1)-1}{s(\alpha+1)+1} \right)^{\frac{1}{s}} \\ &\qquad \times \left(\frac{1}{2}|f''(b^\rho)|^q + \frac{1}{2}|f''(a^\rho)|^q \right)^{\frac{1}{q}}, \end{aligned}$$

where $\frac{1}{s} + \frac{1}{q} = 1$.

Proof. The result follows directly from Theorem 8 if we take $\eta(x,y) = x - y$ and $\mu = 0$. □

3. Conclusions

Four main results related to the Hermite–Hadamard inequality via the Katugampola fractional integrals involving strongly η-convex functions have been introduced. Similar results via the Riemann–Liouville and Hadamard fractional integrals could be derived as particular cases by taking $\rho \to 1$ and $\rho \to 0^+$, respectively. Several other interesting results can be obtained by considering different bifunctions η and/or the modulus μ as well as different values for the parameters α and ρ.

Author Contributions: S.K., E.R.N and A.M.T contributed equally to this work.

Funding: This research received no external funding.

Acknowledgments: We kindly appreciate the efforts of the anonymous referees for their valuable comments and suggestions.

Conflicts of Interest: The authors declare no conflict of interest.

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