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Dynamic Keynesian Model of Economic Growth with Memory and Lag

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Abstract: A mathematical model of economic growth with fading memory and continuous distribution of delay time is suggested. This model can be considered as a generalization of the standard Keynesian macroeconomic model. To take into account the memory and gamma-distributed lag we use the Abel-type integral and integro-differential operators with the confluent hypergeometric Kummer function in the kernel. These operators allow us to propose an economic accelerator, in which the memory and lag are taken into account. The fractional differential equation, which describes the dynamics of national income in this generalized model, is suggested. The solution of this fractional differential equation is obtained in the form of series of the confluent hypergeometric Kummer functions. The asymptotic behavior of national income, which is described by this solution, is considered.

Keywords: fractional differential equations; fractional derivative; Abel-type integral; time delay; distributed lag; gamma distribution; macroeconomics; Keynesian model

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1. Introduction

Advanced mathematical methods of fractional calculus [1–5] are a powerful tool for describing the fading memory and spatial non-locality. Fractional derivatives and integrals of non-integer order have different applications in natural and social sciences [6,7].

In this article, we suggest a generalization of one of the most famous models of economic growth, which is associated with the founder of modern macroeconomic theory, John M. Keynes [8–10]. In the suggested generalization, we take into account two types of phenomena: (I) long memory with power-law fading and (II) continuously distributed lag with gamma distribution of delay time.

The continuously distributed lag has been considered in economics starting with the works of Michal A. Kalecki [11] and Alban W.H. Phillips [12,13]. The macrodynamic models of business cycles, where the continuous uniform distribution of delay time is used, were considered by Michal A. Kalecki in 1935 [11], (see also Section 8.4 of [14], (pp. 251–254)). The economic growth models with continuously distributed lag were proposed by Alban W.H. Phillips [12,13] in 1954. In his works, the distribution of delay time has been described by exponential distribution. Operators with continuously distributed lag were considered by Roy G.D. Allen [14] (pp. 23–29), in 1956.

The time delay (lag) is caused by finite speeds of processes, i.e., the change of one variable does not lead to instant changes of another variable. Therefore, the distributed lag (time delay) cannot

be considered as a long memory in processes. For example, in physics, the retarded potential of the electromagnetic field is well known. The change in the value of the electromagnetic field at the point of observation r_1 is delayed with respect to the change in the sources of the field located at the point r_2 at the time $t = |r_1 r_2| / c$, where c is the speed of propagation of disturbances. It is known that the processes of propagation of the electromagnetic field in a vacuum do not mean the presence of memory in this process.

Long memory has been considered in economics starting with the works of C.W.J. Granger [15–18]. For the first time, the importance of long-range time dependence in economic data was recognized by C.W.J. Granger [15,16] in 1964, 1966. The long-range time dependencies have empirically observed in economics [19–21]. For these dependencies, the correlations between values of variables decay to zero more slowly than it can be expected from independent variables or variables from classical Markov model and autoregressive moving average (ARMA) model [19–25]. An interpretation of these dependencies between variables is that this process has a long memory. In economics, long memory was first related to fractional differencing and integrating by C.W.J. Granger, R. Joyeux [26] and J.R.M. Hosking [27] by using the discrete time approach. Granger, Joyeux, and Hosking independently proposed the so-called autoregressive fractional integrated moving average models (ARFIMA models). These models use the difference operator $\Delta^d := (1 - L)^d$, where L is the lag operator $LX(t) = X(t - 1)$, and d is the order of the fractional differencing if $d > 0$ (fractional integrating if $d < 0$), which need not be an integer. In papers [28,29], we noted that the operator Δ^d coincides with the Grunwald–Letnikov fractional difference $\Delta_\tau^\alpha := (1 - T_\tau)^\alpha$ of order $\alpha = d$ and the unit step $\tau = 1$, where T_τ is the translation (shift) operator that is defined [1], (pp. 95–96), by the expression $(T_\tau Y)(t) = Y(t - \tau)$, where $\tau > 0$ is the delay time. The Grunwald–Letnikov fractional differences were proposed over a hundred and fifty years ago. In mathematics these fractional differences are actively used (for example, see [1], (pp. 371–387), and [3], (pp. 43–62) and [4], (pp. 121–123)). Due to the historical circumstances, the description of processes with memory in economics was based on the Granger–Joyeux approach and models with discrete time only. The continuous time form of economic models with memory was practically not considered and advanced mathematical methods of fractional calculus were not applied in mathematical economics.

An application of advanced mathematical methods of fractional calculus in Keynesian economic models with continuous time was proposed by authors [30,31] in 2016 (see also [32–34]). The fractional differential equations of the dynamic Keynesian model with power-law memory and their solutions have been considered in [30–34]. Continuously distributed lag was not discussed in these works.

For macroeconomics, it is important to simultaneously take into account lagging and memory phenomena. In this article, we consider memory with power-law fading and lag with gamma distribution of delay time. The memory is described by the Riemann–Liouville fractional integrals and the Caputo fractional derivatives. The distributed lag is described by the translation T_τ , in which the delay time $\tau > 0$ is considered as a random variable that is distributed by probability law (distribution) on positive semiaxis. The composition of these operators is represented as the Abel-type integral and integro-differential operators with the confluent hypergeometric Kummer function in the kernel. Using these operators, we propose the fractional differential equation for the generalized dynamic Keynesian model that describes the fractional dynamics of national income. We obtain a solution for this equation that describes the macroeconomic growth with power-law fading memory and gamma distribution of delay time.

2. Standard Dynamic Keynesian Model

In macroeconomic growth models, two types of variables are used [14,35,36]. First, exogenous variables are considered as independent quantities that are external to the considered economic model. Secondly, endogenous variables are internal variables that are formed within the model. The endogenous variables are described as functions of exogenous variables. In models with continuous time, all these variables are considered as functions of time t .

Let us consider the standard dynamical Keynesian model with continuous time. In the Keynesian model, the following variables are used to describe the dynamics of the revenue and expenditure parts of the economy: $Y(t)$ is a national income; $G(t)$ is the government expenditure; $C(t)$ describes the consumption expenditure; $I(t)$ describes the investment expenditure, $E(t)$ is a total expenditure, i.e., $E(t)$ is defined as the sum of all expenditures:

$$E(t) = C(t) + I(t) + G(t). \quad (1)$$

In dynamic equilibrium, we have

$$Y(t) = E(t). \quad (2)$$

In this case, the balance equation establishes the equality of the national income to the sum of all expenditures

$$Y(t) = C(t) + I(t) + G(t). \quad (3)$$

In the Keynesian model, it is assumed that the consumption expenditure in period t depends on the income level in the same period. The consumption expenditure $C(t)$ is regarded as an endogenous variable equal to the amount of domestic consumption of some part of the national income and final consumption independent of income. As a result, the consumption expenditure $C(t)$ is described by the linear equation of the economic multiplier

$$C(t) = m(t)Y(t) + b(t), \quad (4)$$

where $m(t)$ is the multiplier factor that describes the marginal propensity to consume ($0 < m(t) < 1$), and the function $b(t) > 0$ describes the autonomous consumption that does not depend on income. The expression $m(t)Y(t)$ describes the part of consumption that depends on income.

In the static model, the investment expenditure and government expenditure are considered as exogenous variables. In the dynamic Keynesian model, the investment expenditure $I(t)$ is treated as endogenous and it is assumed to depend on the level of income [35], (pp. 95–97). The investment expenditure $I(t)$ is determined by the rate of change of the national income. This assumption is described by the equation of the economic accelerator

$$I(t) = v(t)Y^{(1)}(t), \quad (5)$$

where $v(t)$ is the rate of acceleration, which characterizes the level of technology and state infrastructure, and $Y^{(1)}(t) = dY(t)/dt$ is the first-order derivative of the income function $Y(t)$ with respect to the time variable.

In the Keynesian model, government expenditure $G(t)$, the propensity to consume $m(t)$, the rate of acceleration $v(t)$, and the autonomous consumption $b(t)$ are exogenous variables that are specified as external to the model and characterize the functioning and development of the economy. These variables, as functions of time, are assumed to be given.

The purpose of the dynamic Keynesian model is to describe the behavior of the national income. For this, it is necessary to find the national income $Y(t)$, as a function of time t . Substituting the multiplier Equation (4) and the accelerator Equation (5) into the balance Equation (3), we obtain

$$Y(t) = m(t)Y(t) + b(t) + v(t)Y^{(1)}(t) + G(t). \quad (6)$$

This equation can be written in the form

$$\frac{dY(t)}{dt} - \frac{1 - m(t)}{v(t)}Y(t) = -\frac{G(t) + b(t)}{v(t)}. \quad (7)$$

Equation (7) of the dynamic Keynesian model is a non-homogeneous linear differential equation with a first-order derivative.

We see that the functions $G(t)$ and $b(t)$ are included in Equation (7) as a sum. This could be expected since $G(t)$ is an independent expenditure on investment, that is independent of income, and $b(t)$ is an independent expenditure on consumption, also not dependent on national income. From the point of view of the main purpose of the Keynesian model, which is to describe the dynamics of the national income, these two types of expenditure simply complement each other [35], (pp. 95–98). Therefore, it is convenient to use the sum

$$G_b(t) = G(t) + b(t), \tag{8}$$

which describes the independent expenditure. In this case, the consumption function $C(t) = m(t)Y(t)$ is that part of consumption that depends on income. All this allows us to write down the equation of the standard dynamic Keynesian model in the form

$$\frac{dY(t)}{dt} = \frac{1 - m(t)}{v(t)}Y(t) - \frac{G_b(t)}{v(t)}, \tag{9}$$

where $0 < 1 - m(t) < 1$, and $v(t) > 0$. Equation (9) is a non-homogeneous first-order differential equation that describes the standard dynamic Keynesian model, which does not take into account the effects of memory and delay.

3. Dynamic Keynesian Model with Memory

In the standard Keynesian model, Equation (9) implies an instantaneous change in the investment expenditure, when the rate of growth of national income changes. This means that the equation of this model does not take into account the effects of memory and delay. Mathematically, this is due to the fact that the standard model equation is a first-order differential equation. The derivative of the first order, which is used in accelerator Equations (5), implies an instantaneous change of the investment expenditure $I(t)$, when changing the rate of the national income $Y(t)$. Because of this, accelerator Equation (5) does not take into account memory and lag. Multiplier Equation (4) also assumes that the consumption expenditure $C(t)$ changes instantly when the national income changes. As a result, model Equation (9) can describe an economy, in which agents have no memory. This fact greatly limits the applicability of the standard model to describe the real processes in the economy. To expand the scope of the model, we should take into account that economic agents can remember the history of changes of the national income and the investment expenditure, because it affects the behavior of these agents.

Generalization of the standard Keynesian model, in which memory [37,38] is taken into account, was proposed by the authors [30,31]. Let us briefly describe this generalized model with memory. The equation of investment accelerator with memory [37,39] can be written as

$$I(t) = v(t) \int_0^t M(t - \tau) Y^{(n)}(\tau) d\tau, \tag{10}$$

where $M(t - \tau)$ is the memory function. Note that Equation (10) can also be used to describe the distributed lag. In this case, $M(t - \tau)$ is called the weighting function, which is interpreted as the probability density function. Equation (5) can be obtained from (10) in the case $M(t - \tau) = \delta(t - \tau)$ and $n = 1$. Substitution of the investment $I(t)$ in the form of expression (10), and the consumption expenditure (4) into balance Equation (3), we get

$$v(t) \int_0^t M(t - \tau) Y^{(n)}(\tau) d\tau = (1 - m(t))Y(t) - G_b(t), \tag{11}$$

where $G_b(t)$ is defined by (8). Equation (9) of the standard Keynesian model without memory and lag can be obtained from (11) by using $M(t - \tau) = \delta(t - \tau)$ and $n = 1$.

Equation (11) describes the fractional dynamics of the national income within the framework of the Keynesian model of growth with memory. If the parameters $m(t)$ and $v(t)$ are given, the growth of national income $Y(t)$ is conditioned by the behavior of the independent expenditure (8).

The memory with one-parameter power-law fading is described [37,39] by the function

$$M(t - \tau) = M_{RL}^{n-\alpha}(t - \tau) = \frac{1}{\Gamma(n - \alpha)}(t - \tau)^{n-\alpha-1}, \tag{12}$$

where $\Gamma(\alpha)$ is the gamma function and $n - 1 < \alpha \leq n$. Using (12), the accelerator with memory (10) is represented [37,39] as

$$I(t) = v(t) (D_{C,0+}^\alpha Y)(t), \tag{13}$$

where $(D_{C,0+}^\alpha Y)(t)$ is the Caputo fractional derivative [4,5]. In general, the rate of acceleration $v(t)$ depends on the parameter of memory fading, i.e., $v(t) = v(t, \alpha)$. The parameter $\alpha > 0$ is interpreted as a fading parameter of power-law memory [37]. The concept of an accelerator with memory [39] allows us to get the equation of the Keynesian model with power-law memory in the form of the fractional differential equation

$$(D_{C,0+}^\alpha Y)(t) = \frac{1 - m(t)}{v(t)} Y(t) - \frac{G_b(t)}{v(t)}, \tag{14}$$

where the Caputo fractional derivative can be represented by the Laplace convolution

$$(D_{C,0+}^\alpha Y)(t) = (M_{RL}^{n-\alpha} * Y^{(n)})(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - \tau)^{n-\alpha-1} Y^{(n)}(\tau) d\tau, \tag{15}$$

where $n = [\alpha] + 1$ for $\alpha \notin \mathbb{N}$ and $n = \alpha$ for $\alpha \in \mathbb{N}$, and the function $Y(\tau)$ has integer-order derivatives $Y^{(j)}(\tau)$, $j = 1, \dots, (n - 1)$, that are absolutely continuous.

The solution of Equation (14) with constant values of $m(t) = m$ and $v(t) = v$ has the form

$$Y(t) = \sum_{j=0}^{n-1} Y^{(j)}(0) t^j E_{\alpha, j+1} \left[\frac{1 - m}{v} t^\alpha \right] - \frac{1}{v} \int_0^t (t - \tau)^{\alpha-1} E_{\alpha, \alpha} \left[\frac{1 - m}{v} (t - \tau)^\alpha \right] G_b(t) d\tau, \tag{16}$$

where $n - 1 < \alpha \leq n$. Solution (16) of the fractional differential Equation (14) and its properties are described in [30,31] (see also [32–34]).

4. Memory and Lag by Abel-Type Integral and Derivative with Kummer Functions

The economic accelerator and multiplier with continuously distributed lag were proposed by Alban W.H. Phillips [12,13] in 1954 (see also Sections 3.4, 3.5 and 8.7 in [14]). The distribution of delay time has been described by the exponential distribution. In 1956, the operators with continuously distributed lag were considered by Roy G.D. Allen in the book [14], (pp. 23–29). In the general case, the distribution of delay time can be described by other probability distributions [40], not just exponential distributions. Note that the time delay is caused by finite speeds of processes. Therefore, the distributed lag (time delay) cannot be interpreted as a memory.

4.1. Fractional Integral with Memory and Lag

The translation operator T_τ is defined [1], (pp. 95–96), by the expression $(T_\tau Y)(t) = Y(t - \tau)$, where $\tau > 0$ is the delay time. In the general case, the delay time $\tau > 0$ can be considered as a random variable, which is distributed by probability law (distribution) on positive semiaxis [40]. The translation operator with the continuously distributed delay time can be defined [40] by the equation

$$(\mathbf{T}_M Y)(t) = \int_0^\infty M_T(\tau) (T_\tau Y)(t) d\tau = \int_0^\infty M_T(\tau) Y(t - \tau) d\tau, \tag{17}$$

where we assume that $Y(t)$ and $M_T(t)$ are piecewise continuous functions on \mathbb{R} and the integral $\int_0^\infty M_T(\tau) |Y(t - \tau)| d\tau$ converges. In Equation (17), the kernel $M_T(\tau)$ is the weighting function that satisfies the condition

$$M_T(\tau) \geq 0, \int_0^\infty M_T(\tau) d\tau = 1. \tag{18}$$

To take into account the distributed time delay and power-law fading memory, we can use a composition of the translation operator (17) and integration of non-integer order. The Riemann–Liouville fractional integral with a distributed time delay can be defined [40] by the equation

$$(\mathbf{I}_{T;RL,t_0}^\alpha Y)(t) = (\mathbf{T}_M(I_{RL,t_0}^\alpha Y))(t) = \int_0^\infty M_T(\tau) (I_{RL,t_0}^\alpha Y)(t - \tau) d\tau, \tag{19}$$

where $(I_{RL,t_0}^\alpha Y)(t)$ is the Riemann–Liouville fractional integral of the order $\alpha \in \mathbb{R}_+$, and $M_T(\tau)$ is the weighting function that satisfies the conditions (18). Here we can assume that $(I_{RL,t_0}^\alpha Y)(t)$ and $M_T(t)$ are piecewise continuous functions on \mathbb{R} such that $\int_0^\infty M_T(\tau) |(I_{RL,t_0}^\alpha Y)(t - \tau)| d\tau$ converges. The Riemann–Liouville fractional integral I_{RL,t_0}^α is defined [4], (p. 92), by the expression

$$(I_{RL,t_0}^\alpha Y)(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t - \tau)^{\alpha-1} Y(\tau) d\tau, \tag{20}$$

where $\alpha \geq 0$ is the order of the fractional integral, $\Gamma(\alpha)$ is the gamma function, and $\tau \in [t_0, t]$.

However, in order for definition (19) to be correct, it needs to take into account that the fractional integral (20) is defined for the case $t > t_0$ [4], (p. 69). By virtue of this, we can consider the two following cases.

For $t_0 = -\infty$, Equation (20) defines the Liouville fractional integral $(I_{L,+}^\alpha Y)(t)$, where $\tau \in (-\infty, t]$. Then the Liouville fractional integration with continuously distributed lag is defined in the form

$$(\mathbf{I}_{T,L}^\alpha Y)(t) = (\mathbf{T}_M(I_{L,+}^\alpha Y))(t) = \int_0^\infty M_T(\tau) (I_{L,+}^\alpha Y)(t - \tau) d\tau. \tag{21}$$

Considering that integral (20) is defined for the case $t > t_0 = 0$ [4], (p. 69), we can assume that $(I_{RL,0+}^\alpha Y)(t) = 0$ for $t < t_0 = 0$ in expression (20) with $t_0 = 0$. Here the function $Y(\tau)$ is defined on the finite interval $[t_0, t_1]$ with $0 = t_0 < t_1 < \infty$, [4], (p. 69). This allows us to use the upper limit $t > 0$ instead of infinity in Equation (19), such that

$$(\mathbf{I}_{T;RL,0+}^\alpha Y)(t) = (\mathbf{T}_M(I_{RL,0+}^\alpha Y))(t) = \int_0^t M_T(\tau) (I_{RL,0+}^\alpha Y)(t - \tau) d\tau. \tag{22}$$

As a result, we can define the fractional integration with the gamma-distributed lag in the form

$$(\mathbf{I}_{T;C,0+}^{\lambda,a;\alpha} Y)(t) = (M_T^{\lambda,a}(\tau) * (I_{RL,0+}^\alpha Y))(t) = \int_0^t M_T^{\lambda,a}(\tau) (I_{RL,0+}^\alpha Y)(t - \tau) d\tau, \tag{23}$$

where $M_T(\tau) = M_T^{\lambda,a}(\tau)$ is the probability density function of the gamma distribution

$$M_T^{\lambda,a}(\tau) = \begin{cases} \frac{\lambda^a \tau^{a-1}}{\Gamma(a)} \exp(-\lambda \tau) & \text{if } \tau > 0, \\ 0 & \text{if } \tau \leq 0 \end{cases} \tag{24}$$

with the shape parameter $a > 0$ and the rate parameter $\lambda > 0$. If $a = 1$, the function (24) describes the exponential distribution.

In economics, the gamma distribution (24) is applied to take into account waiting times, when there is a sharp increase in the average delay time. For example, this distribution is used to describe delays orders in queues, delays in payments, and to take into account the likelihood of risk events. The distribution also describes the time of receipt of the order for the enterprise, the service life of device components, and time between store visits.

Substitution of (24) into (23) gives the Riemann–Liouville fractional integral with gamma distribution of delay time in the form of the Laplace convolution of memory and weighting functions

$$(I_{T;C;0+}^{\lambda,a;\alpha} Y)(t) = \int_0^t M_T^{\lambda,a}(\tau)(I_{RL,0+}^{\alpha} Y)(t - \tau) d\tau = (M_T^{\lambda,a} * (M_{RL}^{\alpha} * Y))(t), \tag{25}$$

where $M_{RL}^{\alpha}(t) = (t - \tau)^{\alpha-1} / \Gamma(\alpha)$ is the kernel of the Riemann–Liouville fractional integral (20). The associativity of the Laplace convolution gives

$$(M_T^{\lambda,a} * (M_{RL}^{\alpha} * Y^{(n)}))(t) = (M_{TRL}^{\lambda,a;\alpha} * Y^{(n)})(t), \tag{26}$$

where $M_{TRL}^{\lambda,a;\alpha}(t)$ is the memory-and-lag function

$$M_{TRL}^{\lambda,a;\alpha}(t) = (M_T^{\lambda,a} * M_{RL}^{\alpha})(t). \tag{27}$$

This allows us to represent operator (25) in the form

$$(I_{T;C;0+}^{\lambda,a;\alpha} Y)(t) = (I_M^{\lambda,a}(I_{RL,0+}^{\alpha} Y))(t) = \int_0^t M_{TRL}^{\lambda,a;\alpha}(t - \tau) Y(\tau) d\tau, \tag{28}$$

where $M_{TRL}^{\lambda,a;\alpha}(t - \tau)$ is defined by Equation (27). Let us obtain an explicit form of the memory-and-lag function $M_{TRL}^{\lambda,a;\alpha}(t)$. For this purpose, we can use Equation 2.3.6.1 of [41], (p. 324), that has the form

$$\int_0^t (t - \tau)^{\alpha-1} \tau^{\beta-1} \exp(-\lambda \tau) d\tau = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)} t^{\alpha+\beta-1} F_{1,1}(\beta; \alpha + \beta; -\lambda t), \tag{29}$$

where $Re(\alpha) > 0, Re(\beta) > 0$ and $F_{1,1}(a; b; z)$ is the confluent hypergeometric Kummer function. The function $F_{1,1}(a; b; z)$ can be defined (see [42], (p. 115), and [4], (pp. 29–30)) by the equation

$$F_{1,1}(a; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c - a)} \int_0^1 t^{a-1} (1 - t)^{c-a-1} \exp(zt) dt = \sum_{k=0}^{\infty} \frac{\Gamma(a + k)\Gamma(c)}{\Gamma(a)\Gamma(c + k)} \frac{z^k}{k!}, \tag{30}$$

where $a, z \in \mathbb{C}, Re(c) > Re(a) > 0$ such that $c \neq 0, -1, -2, \dots$. Series (30) is absolutely convergent for all $z \in \mathbb{C}$.

Using Equation (29), the memory-and-lag function (27) can be written as

$$M_{TRL}^{\lambda,a;\alpha}(t) = \frac{\lambda^a \Gamma(a)}{\Gamma(a + n - \alpha)} t^{\alpha+a-1} F_{1,1}(a; a + \alpha; -\lambda t) \tag{31}$$

that defines the kernel of operator (28). In general, the function (31) can be interpreted as a new memory function. Note that equality $F_{1,1}(a; c; z) = \Gamma(c) E_{1,c}^a(z)$ (see Equation 5.1.18 of [43]) allows us to represent the memory kernel (31) through the three parameter Mittag–Leffler functions [43].

As a result, the Riemann–Liouville fractional integral with gamma distribution of delay time can be represented [40] by the equation

$$(I_{T;RL;0+}^{\lambda,a;\alpha} Y)(t) = \frac{\lambda^a \Gamma(a)}{\Gamma(a + \alpha)} \int_0^t (t - \tau)^{\alpha+a-1} F_{1,1}(a; a + \alpha; -\lambda(t - \tau)) Y(\tau) d\tau, \tag{32}$$

where $\alpha > 0$ is the order of integration and the parameters $a > 0, \lambda > 0$ describe the shape and rate of the gamma distribution, respectively.

It is known (see Section 37 of [2] and [44]), the integral operators of the form

$$(A_\alpha Y)(t) = \int_0^t (t - \tau)^{\alpha-1} K(t, \tau) Y(\tau) d\tau \tag{33}$$

are called the Abel-type integral operators. For example, we can consider the confluent hypergeometric Kummer function $F_{1,1}(\beta; \alpha; \lambda(t - \tau))$ as the kernel $K(t, \tau)$ of the operator (33). It is known that the Abel-type (AT) fractional integral operator with Kummer function in the kernel (see equation 37.1 in [1], (p. 731), and [45]) is defined by the equation

$$(I_{AT;0+}^{\alpha,\beta,\lambda} Y)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} F_{1,1}(\beta; \alpha; \lambda(t - \tau)) Y(\tau) d\tau. \tag{34}$$

In paper [45], the integral operator (34) is denoted as $K_0(\beta, \alpha, \lambda)$.

As a result, the Riemann–Liouville fractional integral with gamma-distributed lag (32) can be expressed through the AT fractional integral (34) by the equation

$$(I_{T;RL;0+}^{\lambda,a;\alpha} Y)(t) = \lambda^a \Gamma(a) (I_{AT;0+}^{\alpha+\lambda,a,-\lambda} Y)(t). \tag{35}$$

The AT fractional integral (34) can be represented as an infinite series of the Riemann–Liouville fractional integrals

$$I_{AT;0+}^{\alpha,\beta,\lambda} = \sum_{k=0}^{\infty} \frac{(\beta)_k}{k!} \lambda^k I_{RL;0+}^{\alpha+k}, \tag{36}$$

where $(\beta)_k = \Gamma(\beta + k)/\Gamma(\beta)$ is the Pochhammer symbol. Expression (36) is called the Neumann generalized series (see Equation 37.10 of [1], (p. 732)), which characterizes the structure of the AT fractional integral operator (34). Using (35) and (36), the Riemann–Liouville fractional integral with gamma-distributed lag (32) can be represented as the series

$$(I_{T;RL;0+}^{\lambda,a;\alpha} Y)(t) = \sum_{k=0}^{\infty} \frac{\Gamma(a+k)}{\Gamma(k+1)} (-1)^k \lambda^{k+a} I_{RL;0+}^{\alpha+a+k}. \tag{37}$$

Since the Riemann–Liouville fractional integrals (20) are bounded in $L_p(t_0, t_1)$, where $p \geq 1, t_1 < \infty$ (see the proof of Theorem 2.6 in [1], (pp. 48–51)), then the series (37) may be summed for $|\lambda| < I_{RL;a+L_p(t_0,t_1)}^{-1}$. After evaluating the sums one may remove this restriction on λ and on the sum (37), since the suggested fractional integral operators (32) are analytic functions with respect to λ (see [1], (p. 732)).

Using Equation (37) and the semigroup property of the Riemann–Liouville fractional integrals, we can get the semigroup property for the Riemann–Liouville fractional integral with gamma-distributed lag (32) in the form

$$I_{T;RL;0+}^{\lambda,a;\alpha} I_{T;RL;0+}^{\lambda,b;\beta} = B(\alpha, \beta) I_{T;RL;0+}^{\lambda,a+b;\alpha+\beta}, \tag{38}$$

where

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)} \tag{39}$$

is the beta function. Equality (38) directly follows from Equation 37.14 of [1], (p. 733).

Using Theorem 37.1 of [1], (p. 733), and Equation (35), we can state that the suggested fractional integral with lag (32) has the same range in $L_p(t_0, t_1)$ as the Riemann–Liouville fractional integrals and it is bounden from $L_p(t_0, t_1)$ onto $I_{RL;0+}^k [L_p(t_0, t_1)] \subset L_p(t_0, t_1)$.

Using the condition of the invertibility of the AT operators (34), which is described by Theorem 37.2 of [1], (p. 736), and Equation 37.32 of [1], (p. 735), we get that the solution of the fractional integral Equation

$$m(t)(I_{AT;0+}^{\alpha,\beta,\lambda} Y)(t) = X(t) \tag{40}$$

can be represented in the form

$$Y(t) = \frac{m(t)}{\lambda^a \Gamma(a)} e^{-\lambda t} D_{RL,0+}^a (e^{\lambda t} D_{RL,0+}^\alpha X)(t), \tag{41}$$

where $D_{RL,0+}^\alpha$ and $D_{RL,0+}^a$ are the Riemann–Liouville fractional derivatives of orders $\alpha > 0$ and $a > 0$ respectively. These derivatives can be defined by the Laplace convolution as $(D_{RL,0+}^\alpha X)(t) = D_t^n (M_{RL}^{n-\alpha} * X)(t)$, where $D_t^n = d^n / dt^n$, $n \in \mathbb{N}$.

Note that Equation (40) can be interpreted as an equation of economic multiplier with power-law memory and distributed lag [39], which is a generalization of the multiplier Equation (4), where $m(t)$ is the multiplier factor. In this case, Equation (41) can be considered as the equation of an economic accelerator with memory [39]. The equation of multiplier with memory is a reversible, such that the dual (inverse) equation describes an accelerator with memory (see Section 4 of [39]).

4.2. Fractional Derivative with Memory and Lag

Using the integral operator (32), we can define the fractional derivatives with continuously distributed lag [40]. For example, the Caputo fractional derivative with gamma-distributed lag is defined by the Laplace convolution in the form

$$(D_{T;C;0+}^{\lambda,a;\alpha} Y)(t) = \int_0^t M_T^{\lambda,a}(\tau) (D_{C,0+}^\alpha Y)(t - \tau) d\tau = (M_T^{\lambda,a} * (M_{RL}^{n-\alpha} * Y^{(n)}))(t). \tag{42}$$

The convolution is an associative operation that allows us to write

$$(M_T^{\lambda,a} * (M_{RL}^{n-\alpha} * Y^{(n)}))(t) = (M_{TRL}^{\lambda,a;n-\alpha} * Y^{(n)})(t), \tag{43}$$

where $M_{TRL}^{\lambda,a;n-\alpha}(t)$ is defined by Equation (31). This allows us to represent operators (42) in the form

$$(D_{T;C;0+}^{\lambda,a;\alpha} Y)(t) = \int_0^t M_{TRL}^{\lambda,a;n-\alpha}(t - \tau) Y^{(n)}(\tau) d\tau, \tag{44}$$

where $n - 1 < \alpha \leq n$. Using Equation (31), we have the representation of the kernel $M_{TRL}^{\lambda,a;n-\alpha}(t)$ in the form

$$M_{TRL}^{\lambda,a;n-\alpha}(t) = \frac{\lambda^a \Gamma(a)}{\Gamma(a + n - \alpha)} t^{a+n-\alpha-1} F_{1,1}(a; a + n - \alpha; -\lambda t). \tag{45}$$

As a result, the Caputo fractional derivative with gamma-distributed lag is represented [40] by the Equation

$$(D_{T;C;0+}^{\lambda,a;\alpha} Y)(t) = \frac{\lambda^a \Gamma(a)}{\Gamma(a + n - \alpha)} \int_0^t (t - \tau)^{n-\alpha+a-1} F_{1,1}(a; a + n - \alpha; -\lambda(t - \tau)) Y^{(n)}(\tau) d\tau, \tag{46}$$

where $n - 1 < \alpha \leq n$. Fractional differential operator (46) can be expressed through the Riemann–Liouville fractional integral (32) with gamma-distributed lag in the form

$$(D_{T;C;0+}^{\lambda,a;\alpha} Y)(t) = (I_{T;RL;0+}^{\lambda,a;n-\alpha} Y^{(n)})(t), \tag{47}$$

where $n - 1 < \alpha \leq n$.

Using the Laplace transform of the Caputo fractional derivative and the gamma distribution function, we get [40] the Laplace transform of the Caputo fractional derivative with gamma-distributed lag in the form

$$(\mathcal{L}(D_{T;C;0+}^{\lambda,a;\alpha} Y)(t))(s) = \frac{\lambda^a}{(s + \lambda)^a} \left(s^\alpha (\mathcal{L}Y)(s) - \sum_{j=0}^{n-1} s^{\alpha-j-1} Y^{(j)}(0) \right), \tag{48}$$

where $n - 1 < \alpha \leq n$.

Let us note that the operator (42) with $a = 1$ describes the Caputo fractional derivative with exponentially distributed lag [40]. Using the fact that the Caputo fractional derivatives with integer values $\alpha = n \in \mathbb{N}$ are the integer-order derivatives $(D_{C,0+}^\alpha Y)(t) = d^n Y(t)/dt^n$, the operator (42) with $a = 1$ and $\alpha = n$ describes the integer-order derivatives with the exponential distribution [40]. Note that these operators with exponentially distributed lag were defined in the works of Caputo and Fabrizio [46,47], where they have been misinterpreted as fractional derivatives of non-integer orders. We can state [40] that the derivative of integer order with exponentially distributed lag coincides with the Caputo–Fabrizio operator of the order $\beta = n - 1/(\lambda + 1)$, where λ is the rate parameter of the distribution and $n = [\beta] + 1$. As a result, the Caputo–Fabrizio operator can be interpreted as an integer-order derivative with the exponentially distributed lag.

The proposed operator (42) in the form (46) can be interpreted as a new generalized operator with the memory function given by the confluent hypergeometric function. The generalized fractional derivative (46) can be used to simultaneously account of long memory with power-law fading and distributed lag with the gamma distribution of delay time. In the next section, we describe a macroeconomic model with memory and distributed lag by using this proposed operator.

5. Fractional Differential Equation of a Keynesian Model with Memory and Lag

Let us take into account that the relationship between the investment expenditure $I(t)$ and the national income $Y(t)$ depends on memory and lag effects. For the case of the power-law memory and gamma distribution of the delay time, we can use the generalized accelerator equation

$$I(t) = v(t) (D_{T;C;0+}^{\lambda,a;\alpha} Y)(t), \tag{49}$$

where $D_{T;C;0+}^{\lambda,a;\alpha}$ is the Caputo fractional derivative with distributed lag given by (46). Equation (49) describes the economic accelerator that takes into account the power-law fading memory and the gamma-distributed lag.

Substituting expressions (49) and (4) into balance Equation (3), we obtain the fractional differential equation of the Keynesian model with power-law memory and gamma distribution of delay time in the form

$$(D_{T;C;0+}^{\lambda,a;\alpha} Y)(t) = \frac{1 - m(t)}{v(t)} Y(t) - \frac{G_b(t)}{v(t)}, \tag{50}$$

where α is the parameter of memory fading, $a > 0$ is the shape parameter and $\lambda > 0$ is the rate parameter of the gamma distribution of the delay time.

Let us consider the case where $v(t)$ and $m(t)$ are constant quantities. Then the Keynesian model with one-parameter power-law memory and gamma-distributed lag is described by the fractional differential equation

$$(D_{T;C;0+}^{\lambda,a;\alpha} Y)(t) = \omega Y(t) + F(t), \tag{51}$$

where $\omega = (1 - m)/v$ and $F(t) = -v^{-1}G_b(t)$.

The general solution of Equation (51) can be written as

$$Y(t) = Y_0(t) + Y_F(t), \tag{52}$$

where $Y_0(t)$ is the solution of Equation (51) with $F(t) = 0$, i.e., the homogeneous equation

$$(D_{T;C;0+}^{\lambda,a;\alpha} Y)(t) = \omega Y(t), \tag{53}$$

and $Y_F(t)$ is particular solution of (51) that can be represented in the form

$$Y_F(t) = \int_0^t G_\alpha[t - \tau] F(\tau) d\tau, \tag{54}$$

where $G_\alpha[t - \tau]$ is the generalized Green function [4], (p.281,295). Equation (54) yields the solution $Y_F(t)$ for Equation (51) with initial conditions, $Y^{(j)}(0) = 0$ for all $j = 0, \dots, (n - 1)$.

Theorem 1. *The fractional differential Equation*

$$(D_{T;C;0+}^{\lambda,a;\alpha} Y)(t) = \omega Y(t) + F(t), \tag{55}$$

where $D_{T;C;0+}^{\lambda,a;\alpha}$ is the fractional derivative of order $\alpha > 0$ with gamma-distributed lag, in which $a > 0$ and $\lambda > 0$ are the shape and rate parameters of the gamma distribution respectively, has the solution

$$Y(t) = \sum_{j=0}^{n-1} S_{\alpha,a}^{\alpha-j-1} [\omega \lambda^{-a}, \lambda|t] Y^{(j)}(0) + \frac{1}{\omega} F(t) - \frac{1}{\omega} \int_0^t S_{\alpha,a}^{\alpha} [\omega \lambda^{-a}, \lambda|t - \tau] F(\tau) d\tau, \tag{56}$$

where $n = [\alpha] + 1$, and $S_{\alpha,\delta}^\gamma [\mu, \lambda|t]$ is the special function that is defined by the expression

$$S_{\alpha,\delta}^\gamma [\mu, \lambda|t] = - \sum_{k=0}^{\infty} \frac{t^{\delta(k+1) - \alpha k - \gamma - 1}}{\mu^{k+1} \Gamma(\delta(k+1) - \alpha k - \gamma)} F_{1,1}(\delta(k+1); \delta(k+1) - \alpha k - \gamma, -\lambda t), \tag{57}$$

where $F_{1,1}(a; b; z)$ is the confluent hypergeometric Kummer function (30).

Proof. The first step is to find a solution for the homogeneous Equation (53). Using the Laplace transform of Equation (53), we get

$$\frac{\lambda^a}{(s + \lambda)^a} \left(s^\alpha (\mathcal{L}Y)(s) - \sum_{j=0}^{n-1} s^{\alpha-j-1} Y^{(j)}(0) \right) = \omega (\mathcal{L}Y)(s). \tag{58}$$

Then we can write

$$(\mathcal{L}Y)(s) = \sum_{j=0}^{n-1} \frac{s^{\alpha-j-1}}{s^\alpha - \mu(s + \lambda)^a} Y^{(j)}(0), \tag{59}$$

where $\mu = \omega \lambda^{-a}$. Using Equation 5.4.9 of [48] in the form

$$\left(\mathcal{L}^{-1} \left(\frac{s^a}{(s + b)^c} \right) \right) (s) = \frac{1}{\Gamma(c - a)} t^{c-a-1} F_{1,1}(c; c - a, -bt), \tag{60}$$

where $Re(c - a) > 0$, we get [40] the Laplace transform of the function (57) (**Theorem 1**) as

$$\mathcal{L} (S_{\alpha,\delta}^\gamma [\mu, \lambda|t]) (s) = \frac{s^\gamma}{s^\alpha - \mu(s + \lambda)^\delta}. \tag{61}$$

Using Equation (61) the solution of the homogenous fractional differential Equation (53) has the form

$$Y_0(t) = \sum_{j=0}^{n-1} S_{\alpha,a}^{\alpha-j-1} [\omega \lambda^{-a}, \lambda|t] Y^{(j)}(0), \tag{62}$$

where $S_{\alpha,a}^{\alpha-j-1}[\omega \lambda^{-a}, \lambda | t]$ is defined by Equation (57).

The second step is to find a particular solution (54) of Equation (55). The Laplace transform of Equation (55) with conditions $Y^{(j)}(0) = 0$ for all $j = 0, \dots, (n - 1)$ gives the expression

$$\frac{\lambda^a}{(s + \lambda)^a} s^\alpha (\mathcal{L}Y)(s) = \omega (\mathcal{L}Y)(s) + (\mathcal{L}F)(s) \tag{63}$$

that can be rewritten in the form

$$(\mathcal{L}Y)(s) = \frac{(s + \lambda)^a}{\lambda^a s^\alpha - \omega (s + \lambda)^a} (\mathcal{L}F)(s). \tag{64}$$

The equality

$$\frac{(s + \lambda)^a}{\lambda^a s^\alpha - \omega (s + \lambda)^a} = -\frac{1}{\omega} + \frac{1}{\omega} \frac{s^\alpha}{s^\alpha - \mu (s + \lambda)^a}, \tag{65}$$

where $\mu = \omega \lambda^{-a}$, gives

$$(\mathcal{L}Y)(s) = -\frac{1}{\omega} (\mathcal{L}F)(s) + \frac{1}{\omega} \frac{s^\alpha}{s^\alpha - \mu (s + \lambda)^a} (\mathcal{L}F)(s). \tag{66}$$

Using (61) with $\delta = a, \gamma = \alpha$, we have

$$G_\alpha[t - \tau] = -\frac{1}{\omega} \delta(t - \tau) + \frac{1}{\omega} S_{\alpha,a}^\alpha [\mu, \lambda | t - \tau]. \tag{67}$$

As a result, we obtain

$$Y_F(t) = \frac{1}{\omega} F(t) - \frac{1}{\omega} \int_0^t S_{\alpha,a}^\alpha [\mu, \lambda | t - \tau] F(\tau) d\tau \tag{68}$$

that describes the particular solution of Equation (55).

Substitution of (62) and (68) into (52) gives (56).

This ends the proof. \square

As a result, the solution of Equation (55) of the Keynesian model with one-parameter power-law memory and gamma-distributed lag for constant $v(t) = v$ and $m(t) = m$ is described by the expression

$$Y(t) = \sum_{j=0}^{n-1} S_{\alpha,a}^{\alpha-j-1}[\omega \lambda^{-a}, \lambda | t] Y^{(j)}(0) - \frac{1}{1 - m} G_b(t) + \frac{1}{1 - m} \int_0^t S_{\alpha,a}^\alpha [\omega \lambda^{-a}, \lambda | t - \tau] G_b(\tau) d\tau, \tag{69}$$

where $G_b(\tau) = G(\tau) + g(\tau)$ and $\omega = (1 - m)/v$.

6. Asymptotic Behavior of National Income Growth with Memory and Lag

In economic theory, the important concept of the Harrod’s warranted rate of growth [14] is used, which is also called the technological growth rate. The warranted growth rate describes the growth when the following two conditions are satisfied. The first condition is the constancy of the structure of the economy. This condition means that the parameters of the model do not change over time. In the Keynesian model, we should consider the parameters $(t), b(t),$ and $m(t)$ as constant quantities. The second condition is the absence of external influences. This condition means the absence of exogenous variables. In the Keynesian model, we should consider the case of the absence of the independent expenditure, i.e., $G_b(t) = 0$. Mathematically the warranted growth rate can be obtained from the asymptotic expression of the solution of the homogeneous differential equation of the macroeconomic model. In the standard Keynesian model, the solution of Equation (9) with $G_b(t) = 0$ has the form $Y(t) = Y(0) \exp(\omega t)$. Therefore, the warranted growth rate of this model is described by the value $\omega = (1 - m)/v$, when $G_b(t) = 0$.

The Keynesian model with memory was suggested by authors [30,31] in 2016 (see also [32–34]). For this growth model, the fractional differential equation, its solution, and properties have been described. We proved [32,33] that the warranted growth rate with memory is equal to the value $\omega_{\text{eff}}(\alpha) = \omega^{1/\alpha}$, where $\alpha > 0$ is a parameter of power-law memory fading and ω is the rate of growth without memory ($\alpha = 1$). The warranted growth rates of models with memory do not coincide with the growth rates of the standard Keynesian model. The memory effects can significantly change the growth rates of the economy [32–34] and lead to new types of behavior for the same parameters of the economic model. The principles of changing growth rates by memory have been proposed in [32,33]. The memory effects can both increase and decrease the warranted growth rates in comparison with the standard Keynesian model. For the memory fading with $\alpha < 1$, we get a slowdown in the growth and decline of the economy. We can state that memory with $\alpha < 1$ leads to inhibition of economic growth or decline, i.e., we have stagnation of the economy if $\alpha < 1$. For the memory fading with $\alpha > 1$ we have an improvement in the economy. In this case, the memory effect leads either to the slowdown in the decline rate or to the replacement of the decline with growth, or to the increase in the growth rate.

To consider the warranted growth rate of national income for the Keynesian model with memory and distributed lag, we should obtain an asymptotic behavior of the solution (62) of homogenous model Equation (53). This solution is expressed by the function $S_{\alpha,\delta}^\gamma[\mu, \lambda|t]$ that is represented as an infinite sum (57) of the confluent hypergeometric Kummer function $F_{1,1}(a; c; z)$. We can use the asymptotic expression of the function $F_{1,1}(a; c; z)$ at infinity $z \rightarrow -\infty$ that is given in [4] (p. 29), in the form

$$F_{1,1}(a; c; z) = \frac{\Gamma(c)}{\Gamma(c-a)} e^{-i\pi a} z^{-a} \left(1 + O\left(\frac{1}{z}\right) \right), \tag{70}$$

where $z = -\lambda t < 0$. Therefore the asymptotic expression for $t \rightarrow \infty$ is

$$F_{1,1}(\delta(k+1); \delta(k+1) - \alpha k - \gamma, -\lambda t) = \frac{\Gamma(\delta(k+1) - \alpha k - \gamma)}{\Gamma(-\alpha k - \gamma)} (\lambda t)^{-\delta(k+1)} e^{-i\pi\delta(k+1)} \left(1 + O\left(\frac{1}{t}\right) \right). \tag{71}$$

Equation (71) leads to the asymptotic expression at infinity ($t \rightarrow \infty$) of the function (57) in the form

$$S_{\alpha,\delta}^\gamma[\mu, \lambda|t] = - \sum_{k=0}^{\infty} \frac{\lambda^{-\delta(k+1)} t^{-\alpha k - \gamma - 1}}{\mu^{k+1} \Gamma(-\alpha k - \gamma)} e^{-i\pi\delta(k+1)} \left(1 + O\left(\frac{1}{t}\right) \right) = - \frac{\lambda^{-\delta} t^{-\gamma-1}}{\mu^1 \Gamma(-\gamma)} e^{-i\pi\delta} \left(1 + O(t^{-\alpha-\gamma-1}) \right), \tag{72}$$

where $\alpha k + \gamma \neq 0, 1, 2, \dots$ for all integer k .

Solution (62) is expressed by the function $S_{\alpha,a}^{\alpha-j-1}[\mu, \lambda|t]$, where $\mu = \omega \lambda^{-a}$. Then using Equation (72), we get the asymptotic expression

$$S_{\alpha,a}^{\alpha-j-1}[\omega \lambda^{-a}, \lambda|t] = - \sum_{k=0}^{\infty} \frac{\lambda^{-a} t^{-\alpha(k+1)+j}}{\omega^{k+1} \Gamma(-\alpha(k+1) + j + 1)} e^{-i\pi a} \left(1 + O\left(\frac{1}{t}\right) \right). \tag{73}$$

For the gamma distribution with the integer shape parameter ($a = m \in \mathbb{N}$), which is called the Erlang distribution, expression (73) gives

$$S_{\alpha,m}^{\alpha-j-1}[\omega \lambda^{-a}, \lambda|t] = - \sum_{k=0}^{\infty} \frac{(-1)^m \lambda^{-m} t^{-\alpha(k+1)+j}}{\omega^{k+1} \Gamma(-\alpha(k+1) + j + 1)} \left(1 + O\left(\frac{1}{t}\right) \right), \tag{74}$$

where $e^{-i\pi m} = (-1)^m$ is used.

Using Equation 5.1.18 of [43], (p. 99), in the form $F_{1,1}(a; c; z) = \Gamma(c) E_{1,c}^a(z)$, we can get asymptotic expressions (72) and (74) by using asymptotic expressions of the three parameter Mittag–Leffler functions $E_{1,\delta(k+1)-\alpha k-\gamma}^{\delta(k+1)}(-\lambda t)$ (for example, see [49]). Using Equation (74) we can see that the series

(74) can be represented through the two-parameter Mittag–Leffler function with the negative first parameter [50] as

$$-\sum_{k=0}^{\infty} \frac{(-1)^m \lambda^{-m} t^{-\alpha(k+1)+j}}{\omega^{k+1} \Gamma(-\alpha(k+1) + j + 1)} = \frac{(-1)^m \lambda^{-m} t^j}{\Gamma(j + 1)} - (-1)^m \lambda^{-m} t^j E_{-\alpha, j+1}(t^{-\alpha}/\omega). \tag{75}$$

This allows us to state that the function $S_{\alpha, m}^{\alpha-j-1}[\omega \lambda^{-a}, \lambda|t]$ in the long time limit leads to a series, which can be interpreted as a two-parameter Mittag–Leffler function, which is analogous to the property of the three-parameter Mittag–Leffler function described in [51].

Equation (74) allows us to give the asymptotic expression

$$S_{\alpha, a}^{\alpha-j-1}[\omega \lambda^{-a}, \lambda|t] = -\frac{(-1)^m \lambda^{-m} t^{-\alpha+j}}{\omega^1 \Gamma(-\alpha + j + 1)} \left(1 + O(t^{-2\alpha+j})\right). \tag{76}$$

As a result, the asymptotic behavior of solution (69) with $G_b(t) = 0$ can be described by the Equation

$$Y(t) = \sum_{j=0}^{n-1} \frac{(-1)^{m+1} \lambda^{-m} t^{-\alpha+j}}{\omega \Gamma(-\alpha + j + 1)} Y^{(j)}(0) \left(1 + O(t^{-2\alpha+j})\right) \tag{77}$$

that characterizes warranted growth of national income, which is represented by solution (69) with $G_b(\tau) = 0$ and $a = m \in \mathbb{N}$. Equation (77) allows us to state that the warranted growth has the power-law form with the power $-\alpha + j$, where $j \in \{0, \dots, n - 1\}$ is the smallest value at which $Y^{(j)}(0) \neq 0$. As a result, the warranted growth of the national income with memory and distributed lag has the power-law type instead of the exponential type of growth with memory without time delay [32,33], where warranted growth rate is $\omega_{\text{eff}}(\alpha) = \omega^{1/\alpha}$. Therefore we can state that the distributed lag (time delay) suppresses the effects of fading memory.

7. Conclusions

The standard Keynesian model [8–10,14] describes the dynamics of national income in the absence of long memory and distributed time delay. The Keynesian model with power-law memory has been suggested by authors [30,31]. The effects of continuously distributed lag are not considered in [30,31]. In this paper, we generalize the Keynesian model with memory by taking into account gamma distribution of delay time. To take into account the distributed lag, we use the operators that are compositions of the translation operator with distributed delay time and the fractional derivatives or integrals. These operators allow us to take into account the memory and lag in the economic accelerator. These operators are the Abel-type integro-differential operators with the confluent hypergeometric Kummer function in the kernel. The solution of the suggested fractional differential Equation, which describes the fractional dynamics of national income, has been suggested. The asymptotic behavior of economic processes with memory and distributed lag demonstrates power-law growth. In the absence of delays, the processes with fading memory demonstrate exponential growth. The warranted growth rate with memory [30–34] is equal to the value $\omega_{\text{eff}}(\alpha) = \omega^{1/\alpha}$, where $\alpha > 0$ is a memory fading parameter. Therefore effects of long memory can significantly accelerate the growth rate of the economy by several orders of magnitude [30–33]. Fading memory can lead to an increase in the growth rate in processes without lag [30–34]. The appearance of distributed lag does not accelerate growth due to the memory effect (see also [52,53]). Moreover, we can state that the lag can suppress the effect of fading memory. The distributed lag leads to slower growth. We assume that the suggested approach and model can be used for economic growth modeling by analogy with the computer simulation of the economy in [54–58]. It should also be noted that fractional differential Equations have been applied to describe power-law memory in continuous-time finance [59–72]. This fact allows us to assume that the proposed approach can be used to take into account the continuously distributed time delay and memory in financial processes with the waiting-time distribution [60,62].

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References

1. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Fractional Integrals and Derivatives Theory and Applications*; Gordon and Breach: New York, NY, USA, 1993; 1006p, ISBN 9782881248641.
2. Kiryakova, V. *Generalized Fractional Calculus and Applications*; Longman and J. Wiley: New York, NY, USA, 1994; 360p, ISBN 9780582219779.
3. Podlubny, I. *Fractional Differential Equations*; Academic Press: San Diego, CA, USA, 1998; 340p.
4. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006; 540p, ISBN 9780444518323.
5. Diethelm, K. *The Analysis of Fractional Differential Equations: An Application-Oriented Exposition Using Differential Operators of Caputo Type*; Springer: Berlin, Germany, 2010; 247p.
6. Mainardi, F. (Ed.) *Fractional Calculus Theory and Applications*; MDPI: Basel, Switzerland, 2018; 208p, ISBN 978-3-03897-207-5. Available online: <https://www.mdpi.com/books/pdfview/book/755> (accessed on 24 January 2019).
7. Tenreiro Machado, J.A. (Ed.) *Handbook of Fractional Calculus with Applications*; De Gruyter: Berlin, Germany, 2019; Volumes 1–8.
8. Keynes, J.M. *The General Theory of Employment, Interest and Money*; Macmillan: London, UK, 1936.
9. Keynes, J.M. *The General Theory of Employment, Interest and Money: With the Economic Consequences of the Peace*; Series: Classics of World Literature; Wordsworth Editions: Hertfordshire, UK, 2017; 576p, ISBN 978-1840227475.
10. Keynes, J.M. *The General Theory of Employment, Interest and Money: With the Economic Consequences of the Peace*; Palgrave Macmillan: London, UK, 2018; 438p.
11. Kalecki, M. A macrodynamic theory of business cycles. *Econometrica* **1935**, *3*, 327–344. [[CrossRef](#)]
12. Phillips, A.W. Stabilisation policy in a closed economy. *Econ. J.* **1954**, *64*, 290–323. [[CrossRef](#)]
13. Phillips, A.W. *A. W. H. Phillips Collected Works in Contemporary Perspective*; Leeson, R., Ed.; Cambridge University Press: Cambridge, UK, 2000; 515p, ISBN 9780521571357.
14. Allen, R.G.D. *Mathematical Economics*, 2nd ed.; Macmillan: London, UK, 1959; 812p.
15. Granger, C.W.J. *The Typical Spectral Shape of an Economic Variable*; Technical Report No. 11. January 30, 1964; Department of Statistics, Stanford University: Stanford, CA, USA, 1964; 21p, Available online: <https://statistics.stanford.edu/research/typical-spectral-shape-economic-variable> (accessed on 24 January 2019).
16. Granger, C.W.J. The typical spectral shape of an economic variable. *Econometrica* **1966**, *34*, 150–161. [[CrossRef](#)]
17. Granger, C.W.J. *Essays in Econometrics: Collected Papers of Clive W. J. Granger. Volume I. Spectral Analysis, Seasonality, Nonlinearity, Methodology, and Forecasting*; Ghysels, E., Swanson, N.R., Watson, M.W., Eds.; Cambridge University Press: Cambridge, UK; New York, NY, USA, 2001; 523p.
18. Granger, C.W.J. *Essays in Econometrics Collected Papers of Clive W.J. Granger. Volume II: Causality, Integration and Cointegration, and Long Memory*; Ghysels, E., Swanson, N.R., Watson, M.W., Eds.; Cambridge University Press: Cambridge, UK, 2001; 398p.
19. Bearn, J. *Statistics for Long-Memory Processes*; Capman and Hall: New York, NY, USA, 1994; 315p, ISBN 0-412-04901-5.
20. Palma, W. *Long-Memory Time Series: Theory and Methods*; Wiley-InterScience: Hoboken, NJ, USA, 2007; 304p, ISBN 978-0-470-11402-5.
21. Beran, J.; Feng, Y.; Ghosh, S.; Kulik, R. *Long-Memory Processes: Probabilistic Properties and Statistical Methods*; Springer: Berlin/Heidelberg, Germany; New York, NY, USA, 2013; 884p.
22. Robinson, P.M. (Ed.) *Time Series with Long Memory*; Series: Advanced Texts in Econometrics; Oxford University Press: Oxford, UK, 2003; 392p, ISBN 978-0199257300.
23. Teysiere, G.; Kirman, A.P. (Eds.) *Long Memory in Economics*; Springer: Berlin/Heidelberg, Germany, 2007; 390p.

24. Baillie, R. Long memory processes and fractional integration in econometrics. *J. Econom.* **1996**, *73*, 5–59. [[CrossRef](#)]
25. Graves, T.; Gramacy, R.; Watkins, N.; Franzke, C. A Brief History of Long Memory: Hurst, Mandelbrot and the Road to ARFIMA, 1951–1980. *Entropy* **2017**, *19*, 437. [[CrossRef](#)]
26. Granger, C.W.J.; Joyeux, R. An introduction to long memory time series models and fractional differencing. *J. Time Ser. Anal.* **1980**, *1*, 15–39. [[CrossRef](#)]
27. Hosking, J.R.M. Fractional differencing. *Biometrika* **1981**, *68*, 165–176. [[CrossRef](#)]
28. Tarasov, V.E.; Tarasova, V.V. Long and short memory in economics: Fractional-order difference and differentiation. *IRA Int. J. Manag. Soc. Sci.* **2016**, *5*, 327–334. [[CrossRef](#)]
29. Tarasova, V.V.; Tarasov, V.E. Comments on the article «Long and short memory in economics: Fractional-order difference and differentiation». *Probl. Modern Sci. Educ.* **2017**, *113*, 26–28. [[CrossRef](#)]
30. Tarasova, V.V.; Tarasov, V.E. Keynesian model of economic growth with memory. *Econ. Manag. Probl. Solut.* **2016**, *58*, 21–29. (In Russian)
31. Tarasova, V.V.; Tarasov, V.E. Memory effects in hereditary Keynes model. *Probl. Modern Sci. Educ.* **2016**, *80*, 56–61. (In Russian) [[CrossRef](#)]
32. Tarasov, V.E.; Tarasova, V.V. Macroeconomic models with long dynamic memory: Fractional calculus approach. *Appl. Math. Comput.* **2018**, *338*, 466–486. [[CrossRef](#)]
33. Tarasov, V.E. Economic models with power-law memory. In *Handbook of Fractional Calculus with Applications; Volume 8: Applications in Engineering, Life and Social Sciences, Part B; Chapter 1*; De Gruyter: Berlin, Germany, 2019; ISBN 978-3-11-057092-2.
34. Tarasov, V.E. Self-organization with memory. *Commun. Nonlinear Sci. Numer. Simul.* **2019**, *72*, 240–271. [[CrossRef](#)]
35. Volgina, O.A.; Golodnaya, N.Y.; Odiako, N.N.; Shuman, G.I. *Mathematical Modeling of Economic Processes and Systems*, 3rd ed.; Kronus: Moscow, Russia, 2014; 200p, ISBN 978-5-406-03252-7.
36. Shone, R. *An Introduction to Economic Dynamics*, 2nd ed.; Cambridge University Press: Cambridge, UK, 2012; 224p, ISBN 9781139165020.
37. Tarasova, V.V.; Tarasov, V.E. Concept of dynamic memory in economics. *Commun. Nonlinear Sci. Numer. Simul.* **2018**, *55*, 127–145. [[CrossRef](#)]
38. Tarasov, V.E.; Tarasova, V.V. Criterion of existence of power-law memory for economic processes. *Entropy* **2018**, *20*, 414. [[CrossRef](#)]
39. Tarasova, V.V.; Tarasov, V.E. Accelerator and multiplier for macroeconomic processes with memory. *IRA Int. J. Manag. Soc. Sci.* **2017**, *9*, 86–125. [[CrossRef](#)]
40. Tarasov, V.E.; Tarasova, S.S. Fractional and integer derivatives with continuously distributed lag. *Commun. Nonlinear Sci. Numer. Simul.* **2019**, *70*, 125–169. [[CrossRef](#)]
41. Prudnikov, A.P.; Brychkov, Y.A.; Marichev, O.I. *Integrals and Series. Vol. 1. Elementary Functions*, 5th ed.; Taylor & Francis: London, UK, 2002; 798p, ISBN 2-88124-089-5.
42. Luke, Y.L. *The Special Functions and Their Approximations*; Academic Press: San Diego, CA, USA; New York, NY, USA, 1969; Volume 1, 348p, ISBN 0-12-459901-X.
43. Gorenflo, R.; Kilbas, A.A.; Mainardi, F.; Rogosin, S.V. *Mittag-Leffler Functions, Related Topics and Applications*; Springer: Berlin, Germany, 2014; 443p.
44. Gorenflo, R.; Vessella, S. *Abel Integral Equations: Analysis and Applications*; Springer: Berlin/Heidelberg, Germany, 1991; 215p, ISBN 3-540-53668-X.
45. Prabhakar, T.R. Some integral equations with Kummer's functions in the kernels. *Can. Math. Bull.* **1971**, *4*, 391–404. [[CrossRef](#)]
46. Caputo, M.; Fabrizio, M. A new definition of fractional derivative without singular kernel. *Prog. Fract. Differ. Appl.* **2015**, *1*, 73–85. [[CrossRef](#)]
47. Caputo, M.; Fabrizio, M. Applications of new time and spatial fractional derivatives with exponential kernels. *Prog. Fract. Differ. Appl.* **2016**, *2*, 1–11. [[CrossRef](#)]
48. Bateman, H. *Tables of Integral Transforms*; McGraw-Hill: New York, NY, USA, 1954; Volume I, 391p, ISBN 07-019549-8.
49. Mainardi, F.; Garrappa, R. On complete monotonicity of the Prabhakar function and non-Debye relaxation in dielectrics. *J. Comput. Phys.* **2015**, *293*, 70–80. [[CrossRef](#)]

50. Hanneken, J.W.; Narahari Achar, B.N.; Puzio, R.; Vaught, D.M. Properties of the Mittag–Leffler function for negative alpha. *Phys. Scr.* **2009**, *2009*, 014037. [[CrossRef](#)]
51. Sandev, T.; Metzler, R.; Tomovski, Z. Correlation functions for the fractional generalized Langevin equation in the presence of internal and external noise. *J. Math. Phys.* **2014**, *55*, 023301. [[CrossRef](#)]
52. Tarasov, V.E.; Tarasova, V.V. Phillips model with exponentially distributed lag and power-law memory. *Comput. Appl. Math.* **2019**, *38*, 13. [[CrossRef](#)]
53. Tarasov, V.E.; Tarasova, V.V. Harrod-Domar growth model with memory and distributed lag. *Axioms* **2019**, *8*, 9. [[CrossRef](#)]
54. Tejado, I.; Valerio, D.; Perez, E.; Valerio, N. Fractional calculus in economic growth modelling: The Spanish and Portuguese cases. *Int. J. Dyn. Control* **2015**, *5*, 208–222. [[CrossRef](#)]
55. Tejado, I.; Valerio, D.; Perez, E.; Valerio, N. Fractional calculus in economic growth modelling: The economies of France and Italy. In Proceedings of the International Conference on Fractional Differentiation and its Applications, Novi Sad, Serbia, 18–20 July 2016; Spasic, D.T., Grahovac, N., Zigic, M., Rapaic, M., Atanackovic, T.M., Eds.; pp. 113–123.
56. Tejado, I.; Perez, E.; Valerio, D. Fractional calculus in economic growth modelling of the group of seven. *SSRN Electron. J.* **2018**. [[CrossRef](#)]
57. Luo, D.; Wang, J.R.; Feckan, M. Applying fractional calculus to analyze economic growth modelling. *J. Appl. Math. Stat. Inform.* **2018**, *14*, 25–36. [[CrossRef](#)]
58. Tejado, I.; Perez, E.; Valerio, D. Economic growth in the European Union modelled with fractional derivatives: First results. *Bull. Pol. Acad. Sci. Tech. Sci.* **2018**, *66*, 455–465. [[CrossRef](#)]
59. Scalas, E.; Gorenflo, R.; Mainardi, F. Fractional calculus and continuous-time finance. *Physica A* **2000**, *284*, 376–384. [[CrossRef](#)]
60. Mainardi, F.; Raberto, M.; Gorenflo, R.; Scalas, E. Fractional calculus and continuous-time finance II: The waiting-time distribution. *Physica A* **2000**, *287*, 468–481. [[CrossRef](#)]
61. Laskin, N. Fractional market dynamics. *Physica A* **2000**, *287*, 482–492. [[CrossRef](#)]
62. Raberto, M.; Scalas, E.; Mainardi, F. Waiting-times and returns in high-frequency financial data: An empirical study. *Physica A* **2002**, *314*, 749–755. [[CrossRef](#)]
63. West, B.J.; Picozzi, S. Fractional Langevin model of memory in financial time series. *Phys. Rev. E* **2002**, *65*, 037106. [[CrossRef](#)] [[PubMed](#)]
64. Picozzi, S.; West, B.J. Fractional Langevin model of memory in financial markets. *Phys. Rev. E* **2002**, *66*, 046118. [[CrossRef](#)]
65. Scalas, E. The application of continuous-time random walks in finance and economics. *Physica A* **2006**, *362*, 225–239. [[CrossRef](#)]
66. Meerschaert, M.M.; Scalas, E. Coupled continuous time random walks in finance. *Physica A* **2006**, *370*, 114–118. [[CrossRef](#)]
67. Cartea, A.; Del-Castillo-Negrete, D. Fractional diffusion models of option prices in markets with jumps. *Physica A* **2007**, *374*, 749–763. [[CrossRef](#)]
68. Mendes, R.V. A fractional calculus interpretation of the fractional volatility model. *Nonlinear Dyn.* **2009**, *55*, 395–399. [[CrossRef](#)]
69. Blackledge, J. Application of the fractional diffusion equation for predicting market behavior. *Int. J. Appl. Math.* **2010**, *40*, 130–158.
70. Tenreiro Machado, J.; Duarte, F.B.; Duarte, G.M. Fractional dynamics in financial indices. *Int. J. Bifurc. Chaos* **2012**, *22*, 1250249. [[CrossRef](#)]
71. Keress, A.; Leonenko, N.; Sikorskii, A. Fractional Skellam processes with applications to finance. *Fract. Calc. Appl. Anal.* **2014**, *17*, 532–551. [[CrossRef](#)]
72. Korbel, J.; Luchko, Yu. Modeling of financial processes with a space-time fractional diffusion equation of varying order. *Fract. Calc. Appl. Anal.* **2016**, *19*, 1414–1433. [[CrossRef](#)]

