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Some New Applications of Weakly $\mathcal{H}$-Embedded Subgroups of Finite Groups

Li Zhang [1], **Li-Jun Huo** [2] **and Jia-Bao Liu** [1,*]

[1] School of Mathematics and Physics, Anhui Jianzhu University, Hefei 230601, China; zhang12@mail.ustc.edu.cn
[2] School of Mathematics and Statistics, Chongqing University of Technology, Chongqing 400054, China; huolj@cqut.edu.cn
* Correspondence: liujiabaoad@163.com

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Abstract: A subgroup H of a finite group G is said to be weakly $\mathcal{H}$-embedded in G if there exists a normal subgroup T of G such that $H^G = HT$ and $H \cap T \in \mathcal{H}(G)$, where H^G is the normal closure of H in G, and $\mathcal{H}(G)$ is the set of all $\mathcal{H}$-subgroups of G. In the recent research, Asaad, Ramadan and Heliel gave new characterization of p-nilpotent: *Let p be the smallest prime dividing $|G|$, and P a non-cyclic Sylow p-subgroup of G. Then G is p-nilpotent if and only if there exists a p-power d with $1 < d < |P|$ such that all subgroups of P of order d and pd are weakly $\mathcal{H}$-embedded in G.* As new applications of weakly $\mathcal{H}$-embedded subgroups, in this paper, (1) we generalize this result for general prime p and get a new criterion for p-supersolubility; (2) adding the condition "$N_G(P)$ is p-nilpotent", here $N_G(P) = \{g \in G | P^g = P\}$ is the normalizer of P in G, we obtain p-nilpotence for general prime p. Moreover, our tool is the weakly $\mathcal{H}$-embedded subgroup. However, instead of the normality of $H^G = HT$, we just need HT is S-quasinormal in G, which means that HT permutes with every Sylow subgroup of G.

Keywords: finite groups; weakly $\mathcal{H}$-embedded subgroups; p-supersolubility; p-nilpotence

1. Introduction

Throughout this paper, "G is a group" always means that "G is a finite group". For convenience, one can refer to [1–4] for the definitions and notions in the paper.

The T-groups are defined as the groups G in which normality is a transitive relation, that is, if $H \trianglelefteq K \trianglelefteq G$, then $H \trianglelefteq G$. In 2000, Bianchi Gillio Berta Mauri, Herzog and Verardi [5] proved a characterization of soluble T-groups by means of $\mathcal{H}$-subgroup: a subgroup H of a group G is called an $\mathcal{H}$-subgroup in G if $N_G(H) \cap H^g \leq H$, for every element $g \in G$, where $N_G(H) = \{x \in G | H^x = H\}$ is the normalizer of H in G. They proved that a group G is a supersolvable T-group if and only if every subgroup of G is an $\mathcal{H}$-subgroup of G. Later, except for the exploration of T-groups, $\mathcal{H}$-subgroups were widely used to character finite groups. Csörgö and Herzog [6] obtained that a group G is supersolvable if every cyclic subgroup of G of prime order or order 4 is an $\mathcal{H}$-subgroup. Asaad [7] proved that a group G is supersolvable if every maximal subgroup of every Sylow subgroup of G is an $\mathcal{H}$-subgroup. The set of all $\mathcal{H}$-subgroups of a group G is denoted by $\mathcal{H}(G)$. Moreover, Guo and Wei [8] gave new characterization of p-nilpotent or supersolvable by assuming some subgroups of G of the same order all belong to $\mathcal{H}(G)$, which provide a unified version of the results mentioned above if the order of G is odd. Moreover, Li, Zhao and Xu [9] considered the case when G is of even order.

Recently, Asaad et al. [10] introduced a new subgroup embedding property called weakly $\mathcal{H}$-subgroup, which generalizes both c-normality and $\mathcal{H}$-subgroup, called weakly $\mathcal{H}$-subgroup. Soon after, Asaad and Ramadan [11] gave the definition of weakly $\mathcal{H}$-embedded subgroup. Please note that *a subgroup H of G is said to be a weakly $\mathcal{H}$-embedded subgroup (weakly $\mathcal{H}$-subgroup) of G if there exists a normal subgroup T of G such that $H^G = HT$ $(G = HT)$ and $H \cap T \in \mathcal{H}(G)$, where H^G is the normal closure of H in G*. Clearly, c-normal subgroups, $\mathcal{H}$-subgroups and weakly $\mathcal{H}$-subgroups imply weakly $\mathcal{H}$-embedded subgroups. However, the converse does not hold in general, see [11] (Examples 1.3, 1.4 and 1.5).

In fact, these subgroups were widely used to investigate the structure of finite groups. As a result, many interesting results have been subsequently obtained, such as [7,10–13].

In the recent research about $\mathcal{H}$-subgroups, Asaad, Ramadan, and Heliel gave a new characterization of p-nilpotency.

Theorem 1. ([12] Theorem A) *Let p be the smallest prime dividing $|G|$, and P a non-cyclic Sylow p-subgroup of G. Then G is p-nilpotent if and only if there exists a p-power d with $1 < d < |P|$ such that all subgroups of P of order d and pd are weakly $\mathcal{H}$-embedded in G.*

However, according to this result, some natural questions arise:

Problem 1. (1) *If delete the condition "p is the smallest prime dividing $|G|$", can we claim that G is p-supersoluble?*
(2) *Does there exist another condition to obtain p-nilpotence rather than "p is the smallest prime dividing $|G|$"?*
(3) *As we know, the condition that HT is the smallest normal subgroup of G containing H, is too strict. Can we replace it by a weaker embedding subgroup property?*

In this paper, we further explore weakly $\mathcal{H}$-embedded subgroups and pay attention to Problem 1. However, instead of the normality of HT, we just consider HT is S-quasinormal in G. As we know, a subgroup K is S-quasinormal in G, means that K permutes with every Sylow subgroup P of G, that is $KP = PK$. However, for convenience, we also called it a weakly $\mathcal{H}$-embedded subgroup, that is:

Definition 1. *A subgroup H of a group G is said to be weakly $\mathcal{H}$-embedded in G if there exists a normal subgroup T of G such that HT is S-quasinormal in G and $H \cap T \in \mathcal{H}(G)$.*

As an application of these subgroups, we give a positive answer to Problem 1 in the class of p-soluble groups, for detail:

Theorem 2. *Let E be a p-soluble normal subgroup of a group G such that G/E is p-supersoluble, where p is a prime divisor of $|E|$. Let P be a Sylow p-subgroup of E. Suppose that P has a subgroup D with $1 \leqslant |D| < |P|$ such that all subgroups of P of order $|D|$ and $p|D|$ are weakly $\mathcal{H}$-embedded in G. When $|D| = 1$ and P is a non-abelian 2-group, we further assume that all cyclic subgroups of P of order 4 are weakly $\mathcal{H}$-embedded in G. Then G is p-supersoluble.*

Moreover, to avoid the condition "p is the smallest prime dividing $|G|$" of Theorem 1, we further prove that the conclusion holds if this condition is replaced by "$N_G(P)$ is p-nilpotent". Consequently, we give an answer to Problem 1.

Theorem 3. *Let E be a normal subgroup of G such that G/E is p-nilpotent, and P be a non-cyclic Sylow p-subgroup of E, where p is a prime dividing $|E|$. Assume that $N_G(P)$ is p-nilpotent and P has a subgroup D with order $1 < |D| < |P|$ such that all subgroups of P of order $|D|$ and order $p|D|$ are weakly $\mathcal{H}$-embedded in G. Then G is p-nilpotent.*

In the second section, we list some lemmas which will be useful for the proofs of the above results. The proofs of Theorems 2 and 3 are put in the third section. Some previously known results are generalized by our theorems, and we list some in the fourth section.

2. Preliminaries

Lemma 1. (*see* ([1], Chapter 1) *or* ([3], Chapter 1, Lemmas 5.34 and 5.35)) *Assume that H, E are subgroups of G and $N \trianglelefteq G$.*

(1) *If H is S-quasinormal in G, then $H \cap E$ is S-quasinormal in E, and HN/N is S-quasinormal in G/N.*

(2) *Assume that H is a p-group. Then H is S-quasinormal in G if and only if $O^p(G) \leq N_G(H)$.*

(3) *The set of S-quasinormal subgroups of G is a sublattice of the subnormal subgroup lattice of G.*

(4) *If H is a p-group and H is subnormal in G, then $H \leq O_p(G)$.*

Lemma 2. ([11] Lemma 2.1) *Let H, N be subgroups of G satisfying $H \in \mathcal{H}(G)$ and $N \trianglelefteq G$. Then:*

(1) *If E is a subgroup of G containing H, then $H \in \mathcal{H}(E)$;*

(2) *If H is subnormal in G, then H is normal in G;*

(3) *Assume that $N \leq N_G(H)$. Then $NH \in \mathcal{H}(G)$;*

(4) *If E is a subgroup of G satisfying $N \leq E$, then $E \in \mathcal{H}(G)$ if and only if $E/N \in \mathcal{H}(G/N)$;*

(5) *If H is a p-group and $p \nmid |N|$, then $NH \in \mathcal{H}(G)$ and $HN/N \in \mathcal{H}(G/N)$.*

Lemma 3. *Let H be a weakly $\mathcal{H}$-embedded subgroup of a group G.*

(1) *Assume that E is a subgroup of G containing H. Then H is weakly $\mathcal{H}$-embedded in E.*

(2) *If N is a normal subgroup of G satisfying $N \leq H$, then H/N is weakly $\mathcal{H}$-embedded in G/N.*

(3) *Assume that H is a p-group and N a normal p'-subgroup of G. Then HN/N is weakly $\mathcal{H}$-embedded in G/N.*

Proof. By the hypothesis, G has a normal subgroup T such that HT is S-quasinormal in G and $H \cap T \in \mathcal{H}(G)$.

(1) Clearly, $T \cap E$ is a normal subgroup of E such that $H(T \cap E) = HT \cap E$ is S-quasinormal in E and $H \cap (T \cap E) = H \cap T \in \mathcal{H}(E)$(see Lemmas 1(1) and 2(1)). This shows that H is weakly $\mathcal{H}$-embedded in E.

(2) Consider the normal subgroup TN/N of G/N. Please note that $N \leq N_G(H \cap T)$, so $(H \cap T)N \in \mathcal{H}(G)$ by Lemma 2(3). Furthermore, we have that $(H/N)(TN/N) = HT/N$ is S-quasinormal in G/N and

$$(H/N) \cap (TN/N) = (H \cap T)N/N \in \mathcal{H}(G/N)$$

(see Lemmas 1(1) and 2(4)). By the definition, H/N is weakly $\mathcal{H}$-embedded in G/N.

(3) By Lemma 1(1), the normal subgroup TN/N of G/N such that $(HN/N)(TN/N) = HTN/N$ is S-quasinormal in G/N. Please note that

$$(|HN \cap T : H \cap T|, |HN \cap T : N \cap T|) = (|N \cap HT|, |H \cap NT|) = 1,$$

so $HN \cap T = (H \cap T)(N \cap T)$. Combining with Lemma 2(5),

$$(HN/N) \cap (TN/N) = (HN \cap T)N/N = (H \cap T)N/N \in \mathcal{H}(G/N).$$

Hence HN/N is weakly $\mathcal{H}$-embedded in G/N. □

Recall that a class of groups $\mathfrak{F}$ is called a *formation* if for every group G, every homomorphic image of $G/G^{\mathfrak{F}}$ belongs to $\mathfrak{F}$, where $G^{\mathfrak{F}} = \bigcap\{N \trianglelefteq G | G/N \in \mathfrak{F}\}$. Furthermore, a formation $\mathfrak{F}$ is said to be *saturated* if $G \in \mathfrak{F}$ whenever $G/\Phi(G) \in \mathfrak{F}$. The intersection of all formations containing the set $\{G/O_{p',p}(G) | G \in \mathfrak{F}\}$ is denoted by $\mathfrak{F}(p)$, and $F(p)$ denotes the class of all groups G such that $G^{\mathfrak{F}(p)}$ is a p-group. Associated with a saturated formation $\mathfrak{F}$, there is a function f of the form $f : \mathbb{P} \rightarrow$ {group formations}, where $f(p) = F(p)$ for any prime p, which divides $|G|$ for some $G \in \mathfrak{F}$, and $f(p) = \varnothing$ otherwise. The function f is called *the canonical local satellite* of $\mathfrak{F}$. For more detail, please turn to ([3] P. 3) or ([2] Chap. IV, Theorem 3.7 and Definitions 3.9). Now we recall the subgroup $Z_{\mathfrak{F}}(G)$

of G, which is called the $\mathfrak{F}$-hypercenter of G. In fact, $Z_{\mathfrak{F}}(G)$ the product of all such normal subgroups N of G whose G-chief factors H/K satisfying $(H/K) \rtimes (G/C_G(H/K)) \in \mathfrak{F}$.

Lemma 4. *Let $\mathfrak{F}$ be a saturated formation and f the canonical local satellite of $\mathfrak{F}$. Let P be a normal p-subgroup of G. Then $P \leq Z_{\mathfrak{F}}(G)$ if and only if one of the following holds:*

(1) $G/C_G(P) \in f(p)$ (([3] Chap. 1, Lemma 2.26) *or* ([14] Lemma 2.14))*;*

(2) $P/\Phi(P) \leq Z_{\mathfrak{F}}(G/\Phi(P))$ ([15] Lemma 2.8)*.*

Lemma 5. ([1] Lemma 2.1.6) *If G is p-supersoluble and $O_{p'}(G) = 1$, then G has the unique Sylow p-subgroup.*

Lemma 6. ([2] Chap. A, Lemma 8.4) *Let N be a nilpotent normal subgroup of G and M a maximal subgroup of G such that $N \not\leq M$. Then $N \cap M$ is a normal subgroup of G.*

3. Proofs of Main Results

The following proposition plays an important role in the proof of Theorem 2.

Proposition 1. *Let P be a normal p-subgroup of a group G. Assume that P has a subgroup D satisfying $1 \leqslant |D| < |P|$, such that all subgroups of P of order $|D|$ and $p|D|$ are weakly $\mathcal{H}$-embedded in G. When $|D| = 1$ and P is a non-abelian 2-group, we further assume that all cyclic subgroups of P of order 4 are weakly $\mathcal{H}$-embedded in G. Then $P \leq Z_{\mathfrak{U}}(G)$.*

Proof. Assume by contradiction that (G, P) is a counterexample of minimal order $|G| + |P|$. We proceed via the following steps.

(1) *P is not a minimal normal subgroup of G.*

Assume that P is minimal normal in G. Let H be a subgroup of P of order $|D|$ or $p|D|$, which is normal in some Sylow subgroup of G. By the hypothesis, H is weakly $\mathcal{H}$-embedded in G. So G has a normal subgroup T such that HT is S-quasinormal in G and $H \cap T \in \mathcal{H}(G)$. Please note that $P \cap T$ is normal in G, so $P \cap T = 1$ or $P \cap T = P$ by the minimality of P. If $P \cap T = 1$, then $H = H(P \cap T) = P \cap HT$ is S-quasinormal in G. However, by the choice of H and Lemma 1(2), $H \trianglelefteq G$, a contradiction. So $P \leq T$. In this case, $H = H \cap T \in \mathcal{H}(G)$ and then $H \trianglelefteq G$ by the relationship $H \trianglelefteq P \trianglelefteq G$ and Lemma 2(2), which is impossible. Thus, P is not a minimal normal of G.

(2) *If every maximal subgroup of P is weakly $\mathcal{H}$-embedded in G, then $P \leq Z_{\mathfrak{U}}(G)$.*

Let N be a minimal normal subgroup of G contained in P. By Lemma 3(2), $(G/N, P/N)$ satisfies the hypothesis. So, the choice of (G, P) implies that: (i) $P/N \leq Z_{\mathfrak{U}}(G/N)$; (ii) N is non-cyclic; (iii) N is the unique minimal normal subgroup of G contained in P. Now assume that $\Phi(P) = 1$. In this case, P is elementary abelian and $P = N \times B$, where B is a complement of N. Let N_1 be a maximal subgroup of N such that N_1 is normal in some Sylow p-subgroup G_p of G. Then $P_1 = N_1B$ is a maximal subgroup of P. By the hypothesis, G has a normal subgroup T such that P_1T is S-quasinormal in G and $P_1 \cap T \in \mathcal{H}(G)$. Please note that $P \cap T$ is a normal subgroup of G contained in P, so $N \leq P \cap T$ or $P \cap T = 1$ by (iii). First, assume that $N \leq T$. Then $1 < N_1 \leq P_1 \cap T$. However, $P_1 \cap T \trianglelefteq G$ by the relationship $P_1 \cap T \trianglelefteq P \trianglelefteq G$ and Lemma 2(2). Thus, the uniqueness of N deduces that $N \leq P_1 \cap T \leq P_1$, a contradiction. Secondly, if $P \cap T = 1$, then $P_1 = P_1(P \cap T) = P \cap P_1T$ is S-quasinormal in G, moreover $P_1 \cap N = N_1$ is S-quasinormal in G by Lemma 1(3). Hence Lemma 1(2) and the choice of N_1 imply that $N_1 \trianglelefteq G$, a contradiction. The above shows that $\Phi(P) \neq 1$ and consequently, $N \leq \Phi(P)$. Furthermore, $P/\Phi(P) \leq Z_{\mathfrak{U}}(G/\Phi(P))$. However, we have $P \leq Z_{\mathfrak{U}}(G)$ by Lemma 4. This contradiction shows that (2) holds.

(3) *If every cyclic subgroup of P of order p or 4 (when P is a non-abelian 2-group) is weakly $\mathcal{H}$-embedded in G, then $P \leq Z_{\mathfrak{U}}(G)$.*

If P is not a non-abelian 2-group, then we use Ω to denote the subgroup $\Omega_1(P)$ of P. Otherwise, $\Omega = \Omega_2(P)$.

Let R be a normal subgroup of G such that P/R is a G-chief factor. Obviously, R satisfies the hypothesis. So $R \leq Z_{\mathfrak{U}}(G)$ and P/R is non-cyclic by the choice of (G, P). Moreover, for any normal subgroup L of G satisfying $L < P$, we have $L \leq R$. In fact, if $L \nleq R$, then similarly $L \leq Z_{\mathfrak{U}}(G)$, and $P = RL \leq Z_{\mathfrak{U}}(G)$, a contradiction. Now, assume that $\Omega \leq R$. Then $\Omega \leq Z_{\mathfrak{U}}(G)$. From Lemma 4 and ([16] Lemma 2.4), it follows that $G/C_G(\Omega) \in F(p)$ and $C_G(\Omega)/C_G(P) \in \mathfrak{N}_p$, where F is the canonical local satellite of $\mathfrak{U}$ and $\mathfrak{N}_p$ is the class of p-groups. Consequently, $G/C_G(P) \in \mathfrak{N}_p F(p) = F(p)$, and thereby $P \leq Z_{\mathfrak{U}}(G)$ by Lemma 4 again. This contradiction shows that $\Omega = P$.

Let L/R be a minimal subgroup of $Z(G_p/R) \cap P/R$ and $x \in L \setminus R$, where G_p is a Sylow p-subgroup of G. Then $H = \langle x \rangle$ has order p or 4 and $L = HR$. By the hypothesis, H is weakly $\mathcal{H}$-embedded in G, so G has a normal subgroup T such that HT is S-quasinormal in G and $H \cap T \in \mathcal{H}(G)$. Please note that $P \cap T \trianglelefteq G$. Combining with the above result, we have $P \cap T = P$ or $P \cap T \leq R$. If $P \cap T = P$, that is, $P \leq T$, then $H = H \cap T \in \mathcal{H}(G)$. Moreover, the relationship $H \trianglelefteq \trianglelefteq P \trianglelefteq G$ and Lemma 2(2) deduce $H \trianglelefteq G$. By the choice of H, we have $P/R = L/R$ is cyclic, which is a contradiction. Now assume that $P \cap T \leq R$. Then

$$L/R = HR/R = H(P \cap T)R/R = P/R \cap HTR/R$$

is S-quasinormal in G/R by Lemma 1(3). From Lemma 1(2) and the choice of L/R, it follows that $L/R \trianglelefteq G/R$, which also shows that $P/R = L/R$, a contradiction. This completes the proof of (3).

(4) $p < |D| < \frac{|P|}{p^2}$ (it follows directly from (2) and (3)).

(5) $\Phi(P) = 1$.

Suppose that $\Phi(P) > 1$. We compare the order of $\Phi(P)$ with $|D|$. First, assume that $|\Phi(P)| > |D|$. In this case, we have $\Phi(P) \leq Z_{\mathfrak{U}}(G)$ by the hypothesis and the choice of P. Let N be a minimal normal subgroup of G contained in $\Phi(P)$. Clearly, $|N| = p$ and by (4), P/N satisfies the hypothesis. Thus, $P/N \leq Z_{\mathfrak{U}}(G/N)$ and consequently $P \leq Z_{\mathfrak{U}}(G)$, a contradiction. So $|\Phi(P)| \leqslant |D|$. Please note that $P/\Phi(P)$ is elementary abelian, so we can easily prove that $P/\Phi(P)$ satisfies the hypothesis. Therefore, $P/\Phi(P) \leq Z_{\mathfrak{U}}(G/\Phi(P))$ and by Lemma 4, we further have $P \leq Z_{\mathfrak{U}}(G)$. This contradiction shows that $\Phi(P) = 1$.

(6) *Final contradiction.*

Let N be a minimal normal subgroup of G contained in P. Clearly, $N < P$. Compare the order of N with $|D|$. If $|D| < |N|$, then N satisfies the hypothesis and the choice of P implies that $N \leq Z_{\mathfrak{U}}(G)$. Consequently, $|N| = p$ and then $|D| = 1$, which contradicts (4). Thus, $|D| \geqslant |N|$. By (5), P is elementary abelian, and all subgroups of P/N of order $|D|/|N|$ and $p|D|/|N|$ are weakly $\mathcal{H}$-embedded in G (see Lemma 3(2)). Therefore $P/N \leq Z_{\mathfrak{U}}(G/N)$ by the choice of P. Please note that $|P/N| \geqslant |P|/|D| > p^2$. So there exists a normal subgroup E of G contained in P satisfying $N \leq E \leq P$ and $|P/E| = p$. Consider the subgroup E. Then $E \leq Z_{\mathfrak{U}}(G)$ by the hypothesis and the choice of P, which implies $|N| = p$. Combining with $P/N \leq Z_{\mathfrak{U}}(G/N)$, we finally obtain $P \leq Z_{\mathfrak{U}}(G)$, which is a contradiction. The final contradiction completes the proof of the proposition. □

Now we give the proof of Theorem 2:

Proof. Suppose that the assertion is false and consider a counterexample (G, E) with minimal $|G| + |E|$. We proceed via the following steps.

(1) $O_{p'}(E) = 1$.

Clearly, $(G/O_{p'}(E), E/O_{p'}(E))$ satisfies the hypothesis by Lemma 3(3). If $O_{p'}(E) > 1$, then the choice of G implies that $G/O_{p'}(E)$ is p-supersoluble. Furthermore, G is p-supersoluble, which is a contradiction. Thus, $O_{p'}(E) = 1$.

(2) $E = G$.

Suppose that $E < G$. Please note that Lemma 3(1) shows that (E, E) satisfies the hypothesis, so E is p-supersoluble. Combining (1) with Lemma 5, we have $P \trianglelefteq E$ and consequently, $P \trianglelefteq G$. From the hypothesis and Proposition 1, it follows that $P \leq Z_{\mathfrak{U}}(G)$. This result implies $E \leq Z_{p\mathfrak{U}}(G)$ and then G is p-supersoluble, which is a contradiction. Thus, $E = G$.

(3) *If every maximal subgroup of P is weakly $\mathcal{H}$-embedded in G, then G is p-supersoluble.*

Let N be a minimal normal subgroup of G. Since G is p-soluble and $O_{p'}(G) = 1$, $N \leq O_p(G)$. By Lemma 3(2), G/N satisfies the hypothesis, so: (i) G/N is p-supersoluble; (ii) $|N| > p$; (iii) N is the unique minimal normal subgroup of G. Obviously, $N \nleq \Phi(G)$, so there exists a maximal subgroup M of G such that $G = N \rtimes M$. By Lemma 6, $O_p(G) \cap M \trianglelefteq G$. So $O_p(G) \cap M = 1$ by the uniqueness of N, and then

$$O_p(G) = N(O_p(G) \cap M) = N.$$

On one hand, $O_p(G) \leq C_G(O_p(G))$ by the minimality of $O_p(G)$. On the other hand, since G is p-soluble and $O_{p'}(G) = 1$,

$$C_G(O_p(G)) = C_G(F(G)) \leq F(G) = O_p(G).$$

In general, $C_G(O_p(G)) = O_p(G)$. Now we show that $O_p(G) < P$. In fact, if $P \trianglelefteq G$, then $P \leq Z_{\mathfrak{U}}(G)$ by Proposition 1. Similar to step (2), it is impossible.

Using the above symbol, $G = O_p(G) \rtimes M$ and then $P = O_p(G) \rtimes (P \cap M)$. Let P_1 be a maximal subgroup of P containing $P \cap M$. Then $P_1 \cap O_p(G) > 1$ and it is not normal in G. In fact, if $P_1 \cap O_p(G) \trianglelefteq G$, then $O_p(G) \leq P_1 \cap O_p(G) \leq P_1$ by the minimality of $O_p(G)$ and consequently, $P = P_1$, a contradiction. By the hypothesis, P_1 is weakly $\mathcal{H}$-embedded in G. So G has a normal subgroup T such that P_1T is S-quasinormal in G and $P_1 \cap T \in \mathcal{H}(G)$. If $T = 1$, then P_1 is S-quasinormal in G, which implies that $P_1 \leq O_p(G)$ by Lemma 1(3)(4) and then $O_p(G) = P$. However, it contradicts the above result. So, the uniqueness of $O_p(G)$ implies that $O_p(G) \leq T$. Next, we prove that

$$P_1 \cap O_p(G) \in \mathcal{H}(G).$$

First, we show that $N_G(P_1 \cap O_p(G)) = N_G(P_1 \cap T)$. On one hand, note that

$$P_1 \cap O_p(G) = (P_1 \cap T) \cap O_p(G),$$

so

$$P \leq N_G(P_1 \cap T) \leq N_G(P_1 \cap O_p(G)) < G.$$

On the other hand, $N_G(P_1 \cap O_p(G))$ is p-supersoluble by Lemma 3(1) and the relation

$$N_G(P_1 \cap O_p(G)) < G.$$

Please note that $C_G(O_p(G)) = O_p(G)$, so it is rather clear that $O_{p'}(N_G(P_1 \cap O_p(G))) = 1$. Thus, P is normal in $N_G(P_1 \cap O_p(G))$ by Lemma 5. At this moment, we have

$$P_1 \cap T \trianglelefteq P \trianglelefteq N_G(P_1 \cap O_p(G)),$$

and by Lemma 2(1),

$$P_1 \cap T \in \mathcal{H}(N_G(P_1 \cap O_p(G))).$$

Consequently, $P_1 \cap T \trianglelefteq N_G(P_1 \cap O_p(G))$ by Lemma 2(2), that is, $N_G(P_1 \cap O_p(G)) \leq N_G(P_1 \cap T)$. Together with the above proof, we finally obtain $N_G(P_1 \cap O_p(G)) = N_G(P_1 \cap T)$. Please note that $P_1 \cap T \in \mathcal{H}(G)$. So, for any element $g \in G$,

$$(P_1 \cap O_p(G))^g \cap N_G(P_1 \cap O_p(G)) = (P_1 \cap T)^g \cap O_p(G) \cap N_G(P_1 \cap T) \leq P_1 \cap T \cap O_p(G) = P_1 \cap O_p(G).$$

This shows that $P_1 \cap O_p(G) \in \mathcal{H}(G)$. By Lemma 2(2), we further have $P_1 \cap O_p(G) \trianglelefteq G$, a contradiction. This completes the proof of (3).

(4) *If every cyclic subgroup of P of order p or 4 (when P is a non-abelian 2-group) is weakly $\mathcal{H}$-embedded in G, then G is p-supersoluble.*

Let M be any proper subgroup of G and M_p a Sylow p-subgroup of M. Clearly, $(M_p)^g \leq P$ for some element $g \in G$. Then consider M^g, which has a Sylow p-subgroup $(M_p)^g$ contained in P. So, without loss of generality, assume that $M_p \leq P$. By Lemma 3(1), M satisfies the hypothesis, so the choice of G implies that M is p-supersoluble. As a result, G is a minimal non-p-supersoluble group.

By ([17] Theorem 1), $G^{\mathfrak{U}^p}\Phi(G)/\Phi(G)$ is the unique minimal normal subgroup of $G/\Phi(G)$, where $\mathfrak{U}^p$ is the class of all p-supersoluble groups. Clearly, $p \mid |G^{\mathfrak{U}^p}\Phi(G)/\Phi(G)|$, so $G^{\mathfrak{U}^p}\Phi(G)/\Phi(G)$ is a p-group and $G^{\mathfrak{U}^p}$ is solvable. From ([18] Theorem 3.4.2), it follows that $G^{\mathfrak{U}^p}$ is a p-group of exponent p or 4 (when $G^{\mathfrak{U}^p}$ is a non-abelian 2-group). By the hypothesis, every cyclic subgroup of $G^{\mathfrak{U}^p}$ of order p is weakly $\mathcal{H}$-embedded in G. When $G^{\mathfrak{U}^p}$ is a non-abelian 2-group, clearly, P is also a non-abelian 2-group, so every cyclic subgroup of $G^{\mathfrak{U}^p}$ of order 4 is also weakly $\mathcal{H}$-embedded in G in this case. Hence, we have $G^{\mathfrak{U}^p} \leq Z_{\mathfrak{U}}(G)$ by Proposition 1, and then G is p-supersoluble, a contradiction. So (4) holds.

(5) $p < |D| < \frac{|P|}{p^2}$ (It follows directly from (3) and (4)).

(6) $p > 2$ (It follows directly from (2), (5) and Theorem 1).

(7) *$O_p(G)$ is the unique minimal normal subgroup of G and $G/O_p(G)$ is p-supersoluble.*

Let N be a minimal normal subgroup of G. Clearly, $N \leq O_p(G)$. If $|N| > |D|$, then $N \leq Z_{\mathfrak{U}}(G)$ by Proposition 1, which shows that $|N| = p$ and $|D| = 1$, a contradiction. So, we have $|N| \leqslant |D|$. Please note that $p > 2$, so it is easy to show that $(G/N, P/N)$ satisfies the hypothesis. Thus, the choice of G implies that: G/N is p-supersoluble; $|N| > p$; N is the unique minimal normal subgroup of G. Since $N \nleq \Phi(G)$, there exists a maximal subgroup M of G such that $G = N \rtimes M$. By Lemma 6, $O_p(G) \cap M \trianglelefteq G$, so $O_p(G) \cap M = 1$ and $O_p(G) = N(O_p(G) \cap M) = N$. Thus, (7) holds.

(8) *Final contradiction.*

Let R be a normal subgroup of G such that $O_p(G) \leq R \leq G$ and G/R is a G-chief factor. Please note that $G/O_p(G)$ is p-supersoluble. So $|G/R| = p$ or $p \nmid |G/R|$. First, assume that $|G/R| = p$. Then $|P : P \cap R| = p$ and by (6), R satisfies the hypothesis of the theorem. So R is p-supersoluble. Please note that $O_{p'}(R) \leq O_{p'}(G) = 1$. Together with Lemma 5, R has the unique Sylow p-subgroup $P \cap R$, and furthermore, $P \cap R \trianglelefteq G$. By (6), $P \cap R$ satisfies the hypothesis of Proposition 1. Thus, $P \cap R \leq Z_{\mathfrak{U}}(G)$, that is, $R \leq Z_{p\mathfrak{U}}(G)$, which deduces that G is p-supersoluble, a contradiction. Then assume that $p \nmid |G/R|$, that is $P \leq R$. In this case, R satisfies the hypothesis and so R is p-supersoluble by the choice of G. Similarly, we have $O_{p'}(R) = 1$ and by Lemma 5, $P \trianglelefteq R$, which implies that $P \trianglelefteq G$. By Proposition 1, $P \leq Z_{\mathfrak{U}}(G)$ and consequently, G is p-supersoluble, a contradiction. The final contradiction completes the proof of the theorem. □

Next we give the proof of Theorem 3:

Proof. Suppose that the assertion is false and consider a counterexample G of minimal order. According to Theorem 1, we only need to consider that p is odd. We proceed via the following steps.

(1) $O_{p'}(E) = 1$.

If $O_{p'}(E) > 1$, then it is normal in G. Consider $\overline{G} = G/O_{p'}(E)$. Please note that $\overline{P}$ a Sylow p-subgroup of $\overline{E}$ and $N_{\overline{G}}(\overline{P}) = \overline{N_G(P)}$ is p-nilpotent. Moreover, by hypothesis and Lemma 3(3), all subgroups of $\overline{P}$ of order $|D|$ and order $p|D|$ are weakly $\mathcal{H}$-embedded in $\overline{G}$, that is $\overline{G}$ satisfies the hypothesis for G. Thus, the choice of G implies that $\overline{G}$ is p-nilpotent. Consequently, G is p-nilpotent, a contradiction. So $O_{p'}(E) = 1$.

(2) $E = G$.

By Lemma 3(1), all subgroups of P of order $|D|$ and order $p|D|$ are weakly $\mathcal{H}$-embedded in E. Since $N_E(P) = N_G(P) \cap E$, $N_E(P)$ is p-nilpotent. Then E satisfies the hypothesis. If $E < G$, then E is p-nilpotent by the choice of G. Let $E_{p'}$ be the normal p'-Hall subgroup of E. Clearly, $E_{p'} \trianglelefteq G$. So, by (1), $E_{p'} = 1$, that is, $E = P$. In this case, $G = N_G(P)$ is p-nilpotent. This contradiction shows that $E = G$.

(3) $O_p(G) > 1$.

Let $J(P)$ be the Thompson subgroup of P. Then clearly, $Z(J(P)) > 1$, $P \leq N_G(Z(J(P)))$ and $N_{N_G(Z(J(P)))}(P)$ is p-nilpotent. Assume that $N_G(Z(J(P))) < G$. Please note that $N_G(Z(J(P)))$ satisfies the hypothesis by Lemma 3(1). So, the choice of G implies that $N_G(Z(J(P)))$ is p-nilpotent. However, it contradicts ([19] Theorem 8.3.1). Thus, $N_G(Z(J(P))) = G$, that is $Z(J(P)) \trianglelefteq G$, which shows that (3) holds.

(4) *G is not p-soluble.*

Suppose that G is p-soluble. Then G is p-supersoluble by the Theorem 2. Please note that $O_{p'}(G) = 1$. So $P \trianglelefteq G$ by Lemma 5, which shows that $N_G(P) = G$ is p-nilpotent, a contradiction. Thus, (4) holds.

(5) *Let N be a minimal normal subgroup of G contained in $O_p(G)$. Then $|N| > |D|$.*

If $|N| = |D|$, then every subgroup of P/N of order p is weakly $\mathcal{H}$-embedded in G/N by Lemma 3(2). Denote $\overline{G} = G/N$. Let $\overline{M}$ be a proper subgroup of $\overline{G}$ and $\overline{M_p}$ a Sylow p-subgroup of $\overline{M}$. Clearly, $\overline{M_p}^{\overline{g}} \leq \overline{P}$ for some $\overline{g} \in \overline{G}$. Now consider $\overline{M}^{\overline{g}}$, which has a Sylow p-subgroup $\overline{M_p}^{\overline{g}}$ contained in $\overline{P}$. Without loss of generality, we can assume that the Sylow p-subgroup $\overline{M_p}$ of $\overline{M}$ contains in $\overline{P}$. By Lemma 3(1), every cyclic subgroup of $\overline{M_p}$ of order p is weakly $\mathcal{H}$-embedded in $\overline{M}$. Moreover, $N_{\overline{M}}(\overline{P}) = \overline{N_M(P)}$ is p-nilpotent. So $\overline{M}$ satisfies the hypothesis, and the choice of G implies that $\overline{M}$ is p-nilpotent. Consequently, G is a minimal non-p-nilpotent group. However, in this case, G is soluble, which contradicts (4). Suppose that $|N| < |D|$. Then all subgroups of P/N of order $|D|/|N|$ and $p|D|/|N|$ are weakly $\mathcal{H}$-embedded in G/N by Lemma 3(2), that is G/N satisfies the hypothesis for G. SoSo, from the choice of G, we deduce that G/N is p-nilpotent. Similarly, G is p-soluble in this case, a contradiction. Thus, $|N| > |D|$.

(6) *Final contradiction.*

By (5), all subgroups of N of order $|D|$ and $p|D|$ are weakly $\mathcal{H}$-embedded in G. Then $N \leq Z_{\mathfrak{U}}(G)$ by Proposition 1. From this result, we deduce that $|N| = p$ and $|D| = 1$, that is, every subgroup of P of order p is weakly $\mathcal{H}$-embedded in G. Similarly, as the proof of (5), we can prove that in this case G is soluble, a contradiction. The final contradiction completes the proof. □

4. Some Applications

In this section, we list some applications of our results.

Corollary 1. *Let E be a normal subgroup of G. For every non-cyclic Sylow subgroup P of E, assume that P has a subgroup D such that $1 < |D| < |P|$ and all subgroups of P of order $|D|$ and $p|D|$ are weakly $\mathcal{H}$-embedded in G. Then $E \leq Z_{\mathfrak{U}}(G)$.*

Proof. Assume that p is the smallest prime divisor of $|E|$ and P is a Sylow p-subgroup of E. If P is cyclic, then E is p-nilpotent by the famous Burnside Theorem. Otherwise, by Lemma 3(1) and the hypothesis, all subgroups of P of order $|D|$ and $p|D|$ are weakly $\mathcal{H}$-embedded in E. So E is p-nilpotent by Theorem 1, and then E is soluble. By Lemma 3(1) again, we have that for any prime p dividing $|E|$, E satisfies the hypothesis of Theorem 2. So E is supersoluble. Let q be the maximal prime dividing $|E|$ and Q the unique Sylow q-subgroup of E. Clearly, $Q \trianglelefteq G$. Note that Q satisfies the hypothesis of Proposition 1, so $Q \leq Z_{\mathfrak{U}}(G)$. Now consider E/Q. By Lemma 3(3), E/Q satisfies the hypothesis of corollary. So $E/Q \leq Z_{\mathfrak{U}}(E/G)$ by induction. Therefore, $E \leq Z_{\mathfrak{U}}(G)$. □

Corollary 2. ([12]) *Assume that the Sylow subgroups of G are non-cyclic for all primes p dividing $|G|$. Assume further that for each such p there is a p-power d with $1 < d < |G|_p$ such that all subgroups of P of order d and pd are weakly $\mathcal{H}$-embedded in G, then G is supersoluble.*

Proof. Let p be the smallest prime dividing $|G|$. By Theorem 1, G is p-nilpotent. Consequently, G is soluble. From the Theorem 2, it follows that G is q-supersoluble, for any prime divisor q of $|G|$, that is, G is supersoluble. □

Corollary 3. ([10]) *Let P be a normal p-subgroup of a group G. If all maximal subgroups of P are weakly $\mathcal{H}$-subgroups in G, then $P \leq Z_{\mathfrak{U}}(G)$.*

Corollary 4. ([10]) *Let $\mathfrak{F}$ be a saturated formation containing the class of supersolvable groups $\mathfrak{U}$. A group G lies in $\mathfrak{F}$ if and only if it has a normal subgroup H such that $G/H \in \mathfrak{F}$ and all maximal subgroups of every Sylow subgroup of H (or $F^*(H)$) are weakly $\mathcal{H}$-subgroups in G.*

Corollary 5. *G is supersolvable, if one of the following holds:*

(1) *G has a normal subgroup H such that G/H is supersolvable and all maximal subgroups of every Sylow subgroup of H belong to $\mathcal{H}(G)$* [7];

(2) *all maximal subgroups of every Sylow subgroup of $F^*(G)$ belong to $\mathcal{H}(G)$* [7];

(3) *all maximal subgroups of every Sylow subgroup of a group G are weakly $\mathcal{H}$-subgroups in G* [10].

5. Conclusions

In this paper, we further explore weakly $\mathcal{H}$-embedded subgroups. As new applications, we generalize the characterization of p-nilpotent given by Asaad, Ramadan and Heliel and get a new criterion for p-supersolubility for general prime p. Moreover, adding condition "$N_G(P)$ is p-nilpotent", we obtain p-nilpotence for general prime p.

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